FUZZY SMIRNOV DESCRIPTIVE PROXIMITY MEASURE. APPROXIMATE CLOSENESS OF IMAGE OBJECT SHAPES IN TRIANGULATIONS OF FINITE, BOUNDED SHAPES

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FUZZY SMIRNOV DESCRIPTIVE PROXIMITY MEASURE.
APPROXIMATE CLOSENESS OF IMAGE OBJECT SHAPES IN
TRIANGULATIONS OF FINITE, BOUNDED SHAPES

M.Z. AHMAD\textsuperscript{α} AND J.F. PETERS\textsuperscript{β}

Dedicated to J.H.C. Whitehead and Som Naimpally

Abstract. This article introduces descriptive Smirnov fuzzy proximity useful
in the study of the closeness of spatially separated but descriptively close
surface shapes. In addition, a corresponding measure is given. In addition,
we establish useful results regarding the behavior of the new descriptive fuzzy
proximity measure. The focus here is on triangular fuzzy numbers due to their
favorable arithmetical properties. Applications of this work are given in terms
of the study of approximate closeness of object shapes found in single digital
images and in video frames.

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1. INTRODUCTION

This paper introduces descriptive Smirnov fuzzy proximity and is correspond-
ing measure, which is extension of the fuzzy Lodato-Smithov proximity measure
introduced in [8, §9.4, p. 268], which is a measure of proximity δ introduced by
Ju. M. Smirnov [9]. In addition, descriptive Smirnov fuzzy proximity offers a
substantive extension descriptive proximity [?], which is an extension of classical

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proximity [5, 7]. Basically, a **proximity measure** is a measure of the closeness of a pair of nonempty sets. Notice that a fuzzy proximity measure is not a distance metric but instead a proximity measure is an **set inclusion** measure, *i.e.*, a measure of the extent (degree) that one set is included in another set.

There are many different forms of fuzzy proximity, *e.g.*, A.P. Sostak [10, §3.1.1], Y.-M. Liu and M.-K. Luo [6, §13.2], J. Brennan and E. Martin [§3, p. 945][2], V. Çetkin, A.P. Sostak and H. Aygün [3], K.C. Chattopadhyay, H. Hazra and S.K. Samanta [4].

By contrast with the earlier work on the use of the Lodato-Smirnov proximity measure in measuring the closeness of image object shapes Delaunay triangulations of both single digital images and in video frames.

2. Preliminaries

**Fuzzy sets** generalize the notion of membership(ε), as defined in classical set theory. A **fuzzy set** is a pair \((A, \mu_A)\), which consists of a nonempty set \(A\) and an associated **membership function** \(\mu_A : A \rightarrow [0, 1]\). The inclusion of an element \(x\) is not a binary notion. Based on the values of the membership function \(\mu_A\), an element can be excluded from \(A\) \((\mu_A(x) = 0), \) partially in \(A \ (0 < \mu_A(x) < 1)\) or fully included in \(A \ (\mu_A(x) = 1)\).

**Fuzzy logic** is a multi-valued logic such that instead of a proposition \(x\) being true \((x = 1)\) or false \((x = 0)\), the proposition can also be partially true \((0 < x < 1)\). This form of logic works with the membership values of elements, mimicking Boolean logic. As a result \(x \land y = \min(x, y)\) and \(x \lor y = \max(x, y)\), provided \(0 \leq x, y \leq 1\) are fuzzy logic variables.

We define an \(\alpha\)-**level set**, associated with a fuzzy set as \(A_\alpha = \{x \in A : \mu_A(x) \geq \alpha\}\). \(A_\alpha\) is a classical or a crisp set that is useful in defining operations of fuzzy sets. Moreover, the notion of **convexity** for a fuzzy set \((A, \mu_A)\) can be defined as \(\forall x, y \in A \ \forall \alpha \in [0, 1] \ s.t. \mu_A(\lambda x + (1-\lambda)y) \geq \lambda \mu_A(x) + (1-\lambda) \mu_A(y)\).

**Fuzzy numbers** generalize the reals \(\mathbb{R}\). It is defined as a fuzzy set \(\tilde{A} = (A, \mu_A)\) satisfying the following properties:

1°: \(\tilde{A}\) is a convex fuzzy set

2°: \(\tilde{A}\) is a normalized fuzzy set *i.e.* \(\forall x \in A : \max(\mu_A) = 1\)

3°: \(\mu_A\) is a piecewise continuous function

4°: \(A \subset \mathbb{R}\)

In this paper we restrict ourselves to triangular fuzzy numbers(TFNs), which have a membership function that is shaped like a triangle. Thus, for a TFN \((A, \mu_A)\) such that \(A = [a_1, a_3] \subset \mathbb{R}\), and

\[
\mu_A(x) = \begin{cases} 
0 & x < a_1 \\
\frac{x-a_1}{a_2-a_1} & a_1 < x < a_2 \\
\frac{a_3-x}{a_3-a_2} & a_2 < x < a_3 \\
1 & x > a 
\end{cases}
\]

where \(x \in A\) and \(a_1 \leq a_2 \leq a_3\). A TFN can be compactly represented as \([a_1, a_2, a_3]\), where \(a_1, a_2\) are the endpoints and \(a_3\) is the peak *i.e.* \(\mu_A(a_2) = 1\).
We denote the set of TFNs defined over the set $A$ as $\mathcal{F}(A)$. In this paper we will only consider $\mathcal{F}([0,1])$. Let us define the addition of such numbers and their scalar multiplication. Let $\hat{A} = (A, \mu_A), \hat{B} = (B, \mu_B)$ be two fuzzy numbers then addition yields $(Z, \mu_Z)$ defined as

$$Z = \bigcup \{x + y : \forall x, y \text{ s.t. } x \in A, y \in Y\}$$

$$\mu_Z(z) = \vee(\mu_A(x) \wedge \mu_B(y)),$$

for $z = x + y$

where $\vee, \wedge$ are fuzzy logical OR, AND respectively. This can be understood rather easily in terms of $\alpha$–intervals. For a TFN $A = [a_1, a_2, a_3]$ defined over $\mathbb{R}^+$, $A_\alpha = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_1)\alpha + a_3]$. Substituting $\alpha = 0$ we get $[a_1, a_3]$ the end points and substituting $\alpha + 1$ yields the peak point $a_2$. If $\hat{A}, \hat{B}$ are two fuzzy numbers with $\alpha$–intervals $[a_1^\alpha, a_2^\alpha], [b_1^\alpha, b_2^\alpha]$, then the $\alpha$–interval for $\hat{Z} = \hat{A} + \hat{B}$ is $Z_\alpha = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha]$. Substituting $\alpha = 0, 1$ yield the end and peak points thus completely specifying $\hat{Z}$. It should be noted that sum of TFNs is itself a triangular fuzzy number.

For $k \in \mathbb{R}$, the scalar multiplication for TFNs can be defined in terms of their $\alpha$–intervals. If $[a_1^\alpha, a_2^\alpha]$ is $\alpha$–interval for $\hat{A}$ then $[ka_1^\alpha, ka_2^\alpha]$ is the corresponding $\alpha$–interval for $k\hat{A}$. Moreover, if $[a_1, a_2, a_3]$ is a TFN then upon scalar multiplication with $k \in \mathbb{R}$ it becomes $[ka_1, ka_2, ka_3]$.

**Cell complex** is a space in which the subsets are glued at their boundaries. In this study we consider planar complexes in which there are three different types of cells(subsets), namely 0–cell (point), 1–cell (line) and 2–cell(triangles). It must be noted that we are talking of shapes(lines and triangles) in a topological rather than a geometric sense. We will consider a CW topology on the cell complex [11, §5, p. 223]. For this purpose we define the closure of set $A$ in a space $X$ as $clA = \{q \in X : \exists r \text{ s.t. } B_r(q) \subset X\}$. Here, $B_r(x)$ is a ball of radius $r$ centered on $x$.

In a Hausdorff space two distinct points are separated by their respective neighborhoods i.e. $\forall x, y \in X \exists U, V \subset X \text{ s.t. } x \in U, y \in V \text{ and } U \cap V = \varnothing$. A CW complex $K$ is a Hausdorff space with a decomposition, that satisfies the following conditions

1. **Closure finiteness**: closure of each cell, $cl\sigma^n, \sigma^n \subset K$, intersects a finite number of other cells
2. **Weak Topology**: $A \subset K$ is closed, provided $A \cap cl\sigma^n \neq \emptyset$ is closed for all $\sigma^n \subset K$.

An $n$–cell is denoted as $\sigma^n$. The union of all $\sigma^j \subset K, j \leq n$ is termed the $n$–skeleton $K^n$.

A descriptive set is a set $A$ paired with a region based probe function $\phi_A : 2^A \rightarrow \mathbb{R}$, which assigns to subsets of $A$ a description. We generally assume that $\phi$ is a finite-valued function and to distinguish $\emptyset$ we define $\phi(\emptyset) = \infty$. We define the notion of a descriptive intersection as $A \cap B = \{X \subset A \cup B : \phi(x) \in \phi(A) \text{ and } (x \in \phi(B))\}$. We define this notion in an equivalent way which will help us later on in generalizing this concept.

**Definition 1.** Let $A, B \subset X$ be subsets of a space $X$ and $\phi : 2^X \rightarrow \mathbb{R}$ be a region-based probe function. Then

$$A \cap \Phi B = \{x \in A \cup B : \prod_{a_i \in A} |\phi(x) - \phi(a_i)| + \prod_{b_j \in B} |\phi(x) - \phi(b)| = 0\}$$

is the descriptive intersection.
Let us extend this definition to the notion of an approximate descriptive intersection which has a tolerance value associated to it.

**Definition 2.** Let \( A, B \subset X \) be the subsets of a space \( X \), \( \phi : 2^X \to \mathbb{R} \) be a region-based probe function and \( \gamma \in \mathbb{Z}^* \). Then

\[
\begin{align*}
t(x, \gamma) &= \begin{cases} 
 x & x > \gamma \\
 0 & x \leq \gamma 
\end{cases} \\
A \bigcap_{\Phi, \gamma} B = \{ x \in A \cup B : \prod_{a \in A} t(|\phi(x) - \phi(a_i)|, \gamma) + \prod_{b \in B} t(|\phi(x) - \phi(b_i)|, \gamma) = 0 \}
\end{align*}
\]

is an approximate descriptive intersection.

Fiber bundle \((E, B, \pi, F)\) is a structure in which \( \pi : E \to B \) is a continuous surjection from total space to base space and \( F \subset E \). Such a structure generalizes the notion of a product space and satisfies the local trivialization property. This property states that a small neighborhood \( \pi^{-1}(U) \subset E \) is homeomorphic to \( U \times F \), where \( \pi^{-1} \) is the section. We construct a descriptively CW complex as \( (K_\Phi, K, \pi, \phi(\sigma^n)) \), where \( \sigma^n \) is a \( n \)-cell in a CW complex \( K \). Probe function \( \phi \) is the section of the bundle as it is homeomorphic to \( \pi \) over a small neighborhood \( U \subset B \).

Hyper-connectedness defines proximity relations (near or far) for collections of subsets in a space. The axioms of different categories of hyper-connectedness are given in [1, §2]. To summarize the concepts we can consider that Lodato(\( \delta^k \)) hyper-connectedness implies that subsets have a non-empty intersection while strong(\( \delta^k \)) hyper-connectedness implies non-empty intersection of interiors of the subsets. Descriptive(\( \delta^k \)) hyper-connectedness implies a non-empty descriptive intersection.

Let \( \{A_1, \ldots, A_n\} \) such that \( A_i \subset X \), then

\[
\delta^n(A_1, \ldots, A_n) = \begin{cases} 
 0, & A_1, \ldots, A_n \text{ are near} \\
 1, & A_1, \ldots, A_n \text{ are far}
\end{cases}
\]

Next, we give the axioms for fuzzy extensions of the Lodato(\( \delta \)) and strong(\( \delta \)) hyper-connectedness, proposed in an earlier work. Let \( \{A_i\}_{i \in \mathbb{Z}}, B, C \subset X \), and for set \( F \), the set of all the \( n \)-permutations is \( S(F) \) where \( n = |F| \). Lodato hyper-connectedness \( \delta : 2^X \times 2^X \to \mathcal{F}(\{0, 1\}) \) satisfies the following axioms:

**fhP1:** \( \forall A_k \subset X : \delta^k(A_1, \ldots, A_k) = [0, 1, 1] \) if any \( A_1, \ldots, A_k = \emptyset \)

**fhP2:** \( \forall Y \in S(\{A_1, \ldots, A_k\}) : \delta^k(A_1, \ldots, A_k) = [0, x, 1] \iff \delta^k(Y) = [0, 0, 1] \) for \( 0 \leq x \leq 1 \)

**fhP3:** \( \bigcap_{i=1}^k A_i \neq \emptyset \iff \delta^k(A_1, \ldots, A_k) = [0, x, 1] \) for \( 0 \leq x < 1 \)

**fhP4:** \( \delta^k(A_1, \ldots, A_k, B \cup C) = [0, x, 1] \iff \delta^k(A_1, \ldots, A_k, B) = [0, x, 1] \) or \( \delta^k(A_1, \ldots, A_k, C) = [0, x, 1] \) for \( 0 \leq x < 1 \)

**fhP5:** \( \delta^k(A_1, \ldots, A_k, B) = [0, x, 1] \) and \( b \in B : \delta^2(\{b\}, C) = [0, 0, 1] \iff \delta^k(A_1, \ldots, A_k, C) = [0, x, 1] \) for \( 0 \leq x < 1 \)

**fhP6:** \( \forall A \subset X, \delta^1(A) = [0, 0, 1] \), a constant map

Let \( \{A_i\}_{i \in \mathbb{Z}}, \{B_j\}_{j \in \mathbb{Z}} \) be a family of subsets and \( x, \{y_i\}_{i \in \mathbb{Z}} \in X \) are points in the space \( X \). The interior of a set \( A \) is \( int(A) \). Strong hyper-connectedness \( \delta : 2^X \times 2^X \to \mathcal{F}(\{0, 1\}) \) follows the following axioms:
Let’s define the same notion in a similar fashion to def. fsnhN1):

∀k ∈ X, δk(A1, ..., Ak) = [0, 1] if any A1, ..., Ak = ∅

(fsnhN2): ∀Y ∈ S(\{A1, ..., Ak\}) : δk(A1, ..., Ak) = [0, x, 1] ⇔ δk(Y) = [0, x, 1] for 0 ≤ x ≤ 1

(fsnhN3): δk(A1, ..., Ak) = [0, x, 1] ⇒ ∩i Ai ≠ ∅ for 0 ≤ x < 1

(fsnhN4): ∀\{Bi\}k∈\mathbb{N} : δk(A1, ..., Ak−1, Bi) = [0, x, 1] ⇒ δk(A1, ..., Ak−1, \bigcupi Bi) = [0, y, 1] for 0 ≤ x, y < 1

(fsnhN5): ∩k=1i A_i ≠ ∅ ⇒ δk(A1, ..., Ak) = [0, x, 1] for 0 ≤ x < 1

(fsnhN6): x ∈ ∩k=1i A_i ⇒ δk(x, A1, ..., Ak−1) = [0, x, 1] for 0 ≤ x < 1

(fsnhN7): δk({x_1}, ..., {x_k}) = [0, y, 1] for 0 ≤ y < 1 ⇔ x_1 = x_2 = ... = x_k

(fsnhN8): ∃A ⊆ X, δ(A) = [0, 0, 1], a constant map

3. Main Results

In this section we start by extending the notion of descriptive(δφ) hyper-connectedness as defined in [1, §. 2].

Definition 3. Let \{Ai\}i∈\mathbb{N}, B ⊆ X be sets in space X and φ : 2^X → \mathbb{R} be a region-based probe function. For a set F, let S(F) be the set of all n-permutations of F, where n = |F|. Then, fuzzy descriptive hyper-connectedness δφ : 2^X × ... × 2^X → \mathbb{R}([0, 1]), follows the following axioms:

(dhP1): ∀A_k ∈ X, δ_k(A1, ..., Ak) = [0, 1, 1] if any A1, ..., Ak = ∅

(dhP2): ∀Y ∈ S(\{A1, ..., Ak\}) : δ_k(A1, ..., Ak) = [0, x, 1] ⇔ δ_k(Y) = [0, x, 1] for 0 ≤ x < 1

(dhP3): ∩k=1i A_i ≠ ∅ ⇒ δ_k(A1, ..., Ak) = [0, x, 1] for 0 ≤ x < 1

(dhP4): δ_k(A1, ..., Ak−1, B) = [0, x, 1] and ∀b ∈ B s.t. δ_k(\{b\}, C) = [0, 0, 1] ⇒ δ_k(A1, ..., Ak−1, C) = [0, x, 1] for 0 ≤ x, x < 1

(dhP5): ∃A ⊆ X, δ_k(A) = [0, 0, 1], a constant map

We now extend the notion of an approximate descriptive intersection to multiple sets, here the utility of defs. 1, 2 come to light. We first define the notion of descriptive intersection using binary logic. For A1, ..., An ⊆ X we define \bigcap_\Phi \{A1, A2, ..., An\} = \{ x ∈ A1 ∪ A2 ∪ ... ∪ An : φ(x) ∈ φ(A1) ∧ ... ∧ φ(x) ∈ φ(A_n) \}. Now let’s define the same notion in a similar fashion to def. 1.

Definition 4. Let A1, A2, ..., An be subsets of space X and φ : 2^X → \mathbb{R} be a region-based probe function. Then,

\bigcap_\Phi \{A1, A2, ..., Ak\} = \{ x ∈ A1 ∪ A2 ∪ ... ∪ Ak : \sum_{i=1}^{k} \prod_{a_j ∈ A_i} |φ(x) - φ(a_j)| = 0 \}

is the descriptive intersection.

In a similar manner we define the approximate descriptive intersection (def. 2) for multiple sets.
Definition 5. Let \(A_1, A_2, \ldots, A_n\) be subsets of space \(X\), \(\phi : 2^X \rightarrow \mathbb{R}\) be a region-based probe function and \(\gamma \in \{\mathbb{R}^+ \setminus +\infty\}\). Then,

\[
t(x, \gamma) = \begin{cases} x & x > \gamma \\ 0 & x \leq \gamma \end{cases}
\]

\[\bigcap_{\Phi, \gamma} \{A_1, A_2, \ldots, A_k\} = \{x \in A_1 \cup A_2 \cup \ldots \cup A_k : \sum_{i=1}^{k} \prod_{A_i} t(\phi(x) - \phi(a^i_j), \gamma) = 0\}\]

is the approximate descriptive intersection.

It must be noted that the approximate descriptive intersection\(\bigcap_{\Phi, \gamma}\) is a more general notion. By setting \(\gamma = 0\) we get the descriptive intersection\(\bigcap_{\Phi}\).

Theorem 1. Let \(A_1, A_2, \ldots, A_n\) be subsets of space \(X\), \(\phi : 2^X \rightarrow \mathbb{R}\) be a region based probe function and \(\gamma \in \{\mathbb{R}^+ \setminus +\infty\}\). Then,

\[
\bigcap_{\Phi} \{A_1, \ldots, A_n\} \Leftrightarrow \bigcap_{\Phi, \gamma} \{A_1, \ldots, A_n\} \text{ for } \gamma = 0
\]

Proof. Substituting \(\gamma = 0\) in the definition of \(\bigcap_{\Phi, \gamma}\) (def. 5) yields \(\bigcap_{\Phi}\) (def. 4). Hence, proved.

Building on the notion of approximate descriptive intersection we introduce an extension of the fuzzy descriptive hyper-connectedness\(\delta_\Phi\). We consider an approximate fuzzy descriptive hyper-connectedness\(\delta_{\Phi, \gamma}\), where \(\gamma \in \{\mathbb{R}^+ \setminus +\infty\}\). It can be seen that axiom \(dhP4\) in def. 3 is a transitive relation. In case of the approximate version of \(\delta_\Phi\) we need to modify this axiom to accommodate propagation of uncertainties in composition of relations.

Definition 6. Let \(\{A_i\}_{i \in \mathbb{N}}, B \subset X\) be sets in space \(X\), \(\phi : 2^X \rightarrow \mathbb{R}\) be a region-based probe function and \(\gamma \in \{\mathbb{R}^+ \setminus +\infty\}\) be the tolerance value. Moreover, for a set \(F\), let \(S(F)\) be the set of all \(n\)-permutations of \(F\) with \(n = |F|\). Then the approximate fuzzy descriptive hyper connectedness, \(\delta_{\Phi, \gamma} : 2^X \times \ldots \times 2^X \rightarrow \mathbb{P}(\{0, 1\})\), follows following axioms:

\((adhP1.)\) \(\forall A_k \subset X, k \delta_{\Phi, \gamma}(A_1, \ldots, A_k) = [0, 1, 1]\) if any \(A_1, \ldots, A_k = \emptyset\)

\((adhP2.)\) \(\forall Y \in S(\{A_1, \ldots, A_k\}) : k \delta_{\Phi, \gamma}(A_1, \ldots, A_k) = [0, x, 1] \iff k \delta_{\Phi, \gamma}(Y) = [0, x, 1] \text{ for } 0 \leq x < 1\)

\((adhP3.)\) \(\bigcap_{i=1}^{k} A_i \neq \emptyset \Rightarrow k \delta_{\Phi, \gamma}(A_1, \ldots, A_k) = [0, x, 1] \text{ for } 0 \leq x < 1\)

\((adhP4.)\) \(k \delta_{\Phi, \gamma}(A_1, \ldots, A_k, B) = [0, x, 1]\) and \(\forall b \in B \text{ s.t. } k \delta_{\Phi, \gamma_2}(\gamma_2) = [0, 0, 1] \Rightarrow k \delta_{\Phi, \gamma_3}(A_1, \ldots, A_k, C) = [0, x, 1] \text{ for } 0 \leq x, x < 1\) and \(\gamma_3 = \gamma_1 + \gamma_2\)

\((adhP5.)\) \(\forall A \subset X, 1 \delta_{\Phi, \gamma}(A) = [0, 0, 1]\), a constant map

Based on fuzzy descriptive hyper-connectedness\(\delta_\Phi\) and its approximate version\(\delta_{\Phi, \gamma}\) we can define a more general class of non-commutative proximities.
**Definition 7.** Let $\delta_\Phi : 2^X \times 2^X \rightarrow \mathfrak{P}([0,1])$ be a relation defined on a space $X$. If it satisfied all the axioms $\text{adhP}1 - 5$ except $\text{adhP}2$, as defined in Def. 3, then it is a non-commutative fuzzy descriptive hyper-connectedness.

**Definition 8.** Let $\delta : 2^X \times 2^X \rightarrow \mathfrak{P}([0,1])$ be relation defined on a space $X$. If it satisfies all the axioms $\text{adhP}1 - 5$ except $\text{adhP}2$, as defined in def. 6, then it is a non-commutative approximate fuzzy descriptive hyper-connectedness.

Let us now look at some important results related to fuzzy descriptive($\delta_\Phi$) hyper-connectedness. We start by defining a fuzzy descriptive analogue of the Smirnov similarity measure, which is defined as

$$\delta(|\bigcap \{A_1, A_2, \ldots, A_n\}|/|X|, 1], \quad k > 1$$

is a descriptive Smirnov fuzzy similarity measure.

We demonstrate that the measure defined in def. 9 satisfies the axioms of fuzzy descriptive($\delta_\Phi$) hyper-connectedness as per def. 3.

**Theorem 2.** Descriptive Smirnov fuzzy similarity measure, $\eta^k_\Phi$ as defined in def. 9 is a fuzzy descriptive hyper-connectedness as per def. 3.

**Proof.** We prove this statement by showing that $\eta^k_\Phi$ defined in def. 9 satisfies the axioms of descriptive hyper-connectedness stated in def. 3.

$$(\text{dhP}1):$$ the basic assumption is $\phi(\emptyset) = \infty$, substituting this in def. 1 yields $\emptyset$ for any $A_1, \ldots, A_k$. This means that $\emptyset \in \bigcup \{A_1, \ldots, A_k\}$. Substituting this in def. 9 gives $[0, x, 1]$ where $0 \leq x < 1$

$$(\text{dhP}3):$$ $\bigcap_{i=1}^k A_i$ means that there is an element $x$ in each $A_i$. Thus for this $x$, $\prod_{a_j \in A_j} |\phi(x) - \phi(a_j)| = 0$. This means that $x \in \bigcap \{A_1, \ldots, A_k\}$. Substituting this in def. 9 gives $[0, x, 1]$ where $0 \leq x < 1$

$$(\text{dhP}4):$$ $\eta^k_\Phi(A_1, \ldots, A_{k-1}, B) = [0, x, 1]$ means that there is some $x \in A_1 \cup \ldots \cup A_{k-1}$ and some $b \in B$ such that $\prod_{c \in C} |\phi(x) - \phi(b)| = 0$. Since for all $b \in B$ it is assumed that $\eta^2_\Phi(b, C) = [0, 0, 1]$, which means that $\forall b \in B : \prod_{c \in C} |\phi(b) - \phi(c)| = 0$. Hence, established that $|\phi(x) - \phi(b)| = 0$ and $|\phi(b) - \phi(c)| = 0$, be the transitivity of equality we see that $|\phi(x) - \phi(c)| = 0$. Thus $\prod_{c \in C} |\phi(x) - \phi(c)| = 0$, thus stating that $x \in \bigcap \{A_1, \ldots, A_k, C\}$. Substituting this in def. 9 yields $[0, \hat{x}, 1]$ where $0 \leq \hat{x} < 1$.

$$(\text{dhP}5):$$ by def. 9 $\eta^k_\Phi = [0, 0, 1]$. 

\qed
Definition 10. Let \( \{A_i\}_{i \in \mathbb{Z}} \subset X \) be a family of subsets in the space \( X, \mathcal{P}([0,1]) \) be set of TFNs defined over the interval \([0,1]\) and \( \gamma \in \mathbb{R}^+ \cup \{\infty\} \). Then \( \frac{k}{\Phi, \gamma} : 2^X \times \cdots \times 2^X \to \mathcal{P}([0,1]) \), defined as
\[
\frac{k}{\Phi, \gamma}(A_1, \cdots, A_k) = \begin{cases} 
[0, 1] - \frac{|\cap_{i=1}^k \{A_1, A_2, \cdots, A_n\}|}{|X|}, & k > 1 \\
[0, 1], & k = 1
\end{cases}
\]
is an approximate descriptive Smirnov fuzzy similarity measure.

We demonstrate that the measure defined in def. 10 satisfies the axioms for approximate fuzzy descriptive (\( \delta^\Phi_\gamma \)) hyper-connectedness as defined in def. 6.

Theorem 3. Approximate descriptive Smirnov fuzzy similarity measure, \( \frac{k}{\Phi, \gamma} \) as defined in def. 10 is an approximate fuzzy descriptive hyper-connectedness as defined in def. 6.

Proof. We prove this statement by showing that \( \frac{k}{\Phi, \gamma} \) as defined in def. 10 conforms to the axioms of approximate fuzzy descriptive hyper-connectedness in def. 6.

(adhP1): as one of the basic assumptions is \( \phi(\emptyset) = \infty \) substituting this in def. 5 we can see that it yields \( \emptyset \) if any of \( A_1, \cdots, A_k \) is an empty set. Substituting this into definition of the similarity measure yields \([0, 1, 1]\).

(adhP2): \( \cap \) is associative and commutative because its constituents i.e. the union \( \Phi, \gamma \) \( \cup \) is associative and commutative over sets and addition, multiplication are associative and commutative over the \( \mathbb{R} \).

(adhP3): \( \cap_{i=1}^k A_i \) means that there is an element \( x \) for which \( \prod_{a_j \in A_i} t(||\phi(x) - \phi(a_j)||, \gamma) = 0 \) for all \( A_i \). Thus, \( x \in \cap \{A_1, \cdots, A_k\} \). Using this we can see that \( \frac{k}{\Phi, \gamma}(A_1, \cdots, A_k) = [0, x, 1] \) where \( 0 \leq x < 1 \)

(adhP4): \( \frac{k}{\Phi, \gamma}(A_1, \cdots, A_{k-1}, B) = [0, x, 1] \) means that there is some \( x \in A_1 \cup \cdots \cup A_{k-1} \) and some \( b_i \in B \) for which \( \prod_{b_i \in B} t(||\phi(x) - \phi(b)||, \gamma_1) = 0 \). Since, for all \( b \in B \) it assumed that \( \frac{k}{\Phi, \gamma_1}(b, C) = [0, 0, 1] \), which means that \( \forall b \in B : \prod_{c \in C} t(||\phi(b) - \phi(c)||, \gamma_2) = 0 \). This establishes that \( ||\phi(x) - \phi(b)|| \leq \gamma_1 \) and \( ||\phi(b) - \phi(c)|| \leq \gamma_2 \) and the transitivity of \( \leq \) gives us that \( ||\phi(x) - \phi(c)|| \leq \gamma_1 + \gamma_2 \). Thus, yielding \( \prod_{c \in C} t(||\phi(x) - \phi(c)||, \gamma_1 + \gamma_2) \), thus stating that \( x \in \cap \{A_1, \cdots, A_{k-1}, C\} \). Substituting this in def. 10 yields \([0, x, 1] \) where \( 0 \leq x < 1 \).

(adhP5): by def. 10 \( \frac{1}{\Phi, \gamma} = [0, 0, 1] \)

\( \square \)
Let us show that fuzzy descriptive hyper-connectedness is sub case of approximate fuzzy descriptive hyper-connectedness.

**Theorem 4.** Let \( \{ A_i \} \in \mathcal{A} \subset X \) be an indexed family of subsets, \( \delta_\Phi \) be the fuzzy descriptive hyper-connectedness and \( \delta_\Phi \) be the approximate fuzzy descriptive hyper-connectedness. Then,

\[
\eta_\Phi^k(A_1, \ldots, A_i) \leftrightarrow \eta_\Phi^\gamma \text{ for } \gamma = 0
\]

**Proof.** From thm. 1 it is established that for \( \gamma = 0, \bigcap_{i=1}^\Phi \{ A_1, \ldots, A_k \} \leftrightarrow \bigcap_{i=1}^\Phi \{ A_1, \ldots, A_k \} \). Substituting this in the definition of \( \eta_\Phi \) def. 10 yields def. 9, the definition of \( \eta_\Phi^k \).

We now establish the relationships between different fuzzy hyper-connectedness namely Lodato(\( \delta_\Phi \)), strong(\( \delta_\Phi \)) and descriptive(\( \delta_\Phi \)).

**Theorem 5.** Let \( \{ A_i \} \in \mathcal{A} \subset X \) be a family of subsets be space and let \( \delta_\Phi \) be the lodato, \( \delta_\Phi \) be strong and \( \delta_\Phi \) be the descriptive hyper-connectedness.

1°: \( \delta_\Phi^k(A_1, \ldots, A_k) = [0, x, 1] \Rightarrow \delta_\Phi^k(A_1, \ldots, A_k) = [0, x, 1] \) for \( 0 \leq x < 1 \)

2°: \( \delta_\Phi^k(A_1, \ldots, A_k) = [0, x, 1] \Rightarrow \delta_\Phi^k(A_1, \ldots, A_k) = [0, x, 1] \) for \( 0 \leq x < 1 \)

**Proof.** 1°: from axiom fsnhN3 that \( \delta_\Phi^k(A_1, \ldots, A_k) = [0, x, 1] \) for \( 0 \leq x < 1 \) implies \( \bigcap_{i=1}^\Phi A_i \neq \emptyset \). From axiom fhP3 this implies \( \delta_\Phi^k(A_1, \ldots, A_k) = [0, x, 1] \) for \( 0 \leq x < 1 \)

2°: from axiom fsnhN3 that \( \delta_\Phi^k(A_1, \ldots, A_k) = [0, x, 1] \) for \( 0 \leq x < 1 \) implies \( \bigcap_{i=1}^\Phi A_i \neq \emptyset \).

From axiom dhP3(def. 3) this implies \( \delta_\Phi^k(A_1, \ldots, A_k) = [0, x, 1] \) for \( 0 \leq x < 1 \)

Let us prove a summability relation that will allow us to extend hyper-connectedness relations from complexes to their union.

**Lemma 1.** Let \( (A, \eta_\Phi, (B, \eta_\Phi) \) be two hyper-connectedness CW complexes in 2-dimensional space, where \( \eta_\Phi, \eta_\Phi, \eta_\Phi, \eta_\Phi \) are the respective approximate fuzzy Smirnov measures(def. 10). It is assumed that \( \gamma \in \mathbb{R}^+ \) is same for \( A, B \).

Moreover, let \( B^k \) be the \( k \)-skeleton of \( B \). Then,

\[
\sum_{k=0}^2 \sum_{b_j \in B^k} \eta_\Phi(A, b_j) = [0, \sum_{k=0}^2 \sum_{b_j \in B^k} (1 - \frac{|A \cap b_j|}{|A|}), \sum_{k=0}^2 |B^k|]
\]

**Proof.** By definition each \( \eta_\Phi(A, b_j) \) is a TFN given by \([0, 1 - \frac{|A \cap b_j|}{|A|}, 1]\). Summation of TFNs yields a TFN. It can be seen from the rules of summation of TFNs
It is evident that $0 \leq \frac{|A \cap b_j|}{|A|} \leq 1$, hence $0 \leq \sum_{k=0}^{2} \sum_{j \in b_k \in B^k} (1 - \frac{|A \cap b_j|}{|A|}) \leq \sum_{k=0}^{2} |B^k|$. Thus, the result is a valid TFN.

Using this lemma we construct a measure $\zeta_{A \cup B, \gamma}$ that is not equivalent to the approximate fuzzy Smirnov measure.

**Theorem 6.** Let $(A, \eta_A, \Phi_A, \gamma)$, $(B, \eta_B, \Phi_B, \gamma)$ be two hyper-connectedness CW complexes in 2-dimensional space, where $\eta_{\Phi_A, \gamma}$, $\eta_{\Phi_B, \gamma}$ are the respective approximate fuzzy Smirnov measures (Def. 10). It is assumed that $\gamma \in \mathbb{R}^+ \setminus +\infty$ is same for $A, B$. Moreover, let $B^k$ be the $k$-skeleton of $B$. Then,

$$\zeta_{A \cup B, \gamma}(A, B) = \frac{1}{\sum_{k=0}^{2} |B^k|} \sum_{k=0}^{2} \sum_{j \in b_k \in B^k} \eta_{\Phi_A, \gamma}^2(A, b_j)$$

is a $\Phi_\gamma$ non-commutative approximate fuzzy descriptive hyper-connectedness.

**Proof.** We prove this statement axiom by axiom as follows,

(adhP1): it holds as the assumption is that $\phi(\emptyset) = \infty$, substituting this into Def. 10 yields $\emptyset$ if any $A_1, \ldots, A_k$ is an empty set. Substituting this into definition of $\sum_{k=0}^{2} \sum_{j \in b_k \in B^k} \eta_{\Phi_A, \gamma}(A, b_j)$ yields $\sum_{k=0}^{2} |B^k|$. Thus $\zeta_{A \cup B, \gamma}$ yields $[0, 1, 1]$.

(adhP2): It will only be commutative when $|A| = |B|$ and $\sum_{k=0}^{2} |A^k| = \sum_{k=0}^{2} |B^k|$

(adhP3): If one of the subcomplexes $b_j$ of $B$ shares an intersection with $A$ this implies $A \cap b_j \neq \emptyset$, thus $\eta_{\Phi_A, \gamma}(A, b_j)$ gives $[0, x, 1]$, where $0 \leq x < 1$.

This implies from definition $\zeta_{A \cup B, \gamma}(A, B)$ yields $[0, x, 1]$ for $0 \leq x < 1$

(adhP4): $\zeta_{A \cup B, \gamma}(A, B) = [0, x, 1]$ means that there is at least one subcomplex $b_j$ in $B$ for which $A \cap b_j \neq \emptyset$. This means that $\prod_{a \in A} |\phi(a) - \phi(b_j)| \leq \gamma_1$. Since, for all $b \in B$ it is assumed that $\eta_{\Phi_A, \gamma}^2(b, C) = [0, 0, 1]$, which implies $\forall b \in B: \prod_{c \in C} |\phi(b) - \phi(c)| \leq \gamma_2$, the transivity of $\leq$ gives us $|\phi(a) - \phi(c)| \leq \gamma_1 + \gamma_2$. Thus, $\prod_{c \in C} t(|\phi(x) - \phi(c)|, \gamma_1 + \gamma_2)$, stating $\bigcap_{b \in B, C} (A, C) \neq \emptyset$. Thus $\eta_{\Phi_A, \gamma}(A, C)$...
1.1: Original image

1.2: Hue dissimilarity

1.3: $\delta_{\Phi,0.01}$

1.4: $\delta_{\Phi,0.1}$

Figure 1. Fig. 1.1 represents a demo image included in MATLAB called peppers.png, where the black circle on the red bell pepper displays the reference pixel [216, 385]. The similarity between the pixels based on Hue is displayed in fig. 1.2. The approximate fuzzy descriptive hyper-connectedness of the image to the reference pixel using tolerance values $\gamma = 0.01$ and $\gamma = 0.1$ are displayed in Figs. 1.3 and 1.4 respectively.

\[
yields [0, x, 1] \text{ for some subcomplexes } c \in C. \text{ Hence } \zeta_{A \wedge C, \gamma} (A, C) \text{ is } [0, x, 1] \text{ where } 0 \leq x < 1.
\]

(adhP5): The axiom does not apply as we are specifically talking about the case $k = 2$

4. Applications

In this section we present applications of the concepts that have been defined in the sections above. We will present possible applications of approximate descriptive fuzzy hyper-connectedness($\delta_{\Phi,\gamma}$) to images and videos.

4.1. In Images. We begin by showing how $\delta_{\Phi,\gamma}$ presents itself in digital images.

In this section we are comparing individual pixels, but such an approach can be extended to the regions or groups of pixels. In the present case possible out puts of the fuzzy descriptive hyper-connectedness is either $[0, 0, 1]$ for nearness or $[0, 1, 1]$ for far pixels.

We begin by presenting a test image in the MATLAB software named peppers.png. The original image is displayed in fig. 1.1. We select a reference point
Fig. 2.1 represents the painting of the face of a young girl with a green headscarf, where the yellow circle displays the reference pixel [682, 1121]. The similarity between the pixels based on Hue is displayed in fig. 2.2. The approximate fuzzy descriptive hyper-connectedness of the image to the reference pixel using tolerance values $\gamma = 0.01$ and $\gamma = 0.1$ are displayed in Figs. 2.3 and 2.4 respectively.

Figure 1.2 represents the distance map of all the pixels from the reference pixel. This produces a visualization of the similarity within the image with reference to a particular pixel([216, 385]).

Now, upon this similarity measure we can construct an approximate fuzzy descriptive hyper-connectedness ($\delta$). We look at the hyper-connectedness using two different values of the tolerance parameter $\gamma$. We present $\delta_{\Phi,0.01}$ in figure 1.3, and $\delta_{\Phi,0.1}$ in figure 1.4. It can be observed that increasing the value of $\gamma$ the number of pixels that become similar to the reference pixels increase. This is the reason why it is termed the tolerance parameter. It must also be noted that when we vary the tolerance parameter it is possible for the whole image to be approximately hyper-connected to the reference pixel.

Let us present another image as shown in figure 2. It is an image of a portrait depicting the face of a young girl with a green headscarf. We again select a reference pixel [682, 1121] that is represented as a yellow dot on the head scarf in [216, 385] which is shown as the black circle on the red bell pepper in the image(fig. 1.1). We select the $\phi : 2^X \to \mathbb{R}$ as the hue value of each of the pixels. Figure 1.2 represents the distance map of all the pixels from the reference pixel. This produces a visualization of the similarity within the image with reference to a particular pixel([216, 385]).
3.1: Maximum cycle perimeter

3.2: relative similarity

3.3: Frame 241

3.4: Frame 250

**Figure 3.** The perimeter of the biggest cycle in each frame is plotted in figure 3.1. The relative similarity ($\eta_p(cyc_n, cyc_{250})$) measure with frame 250 as the reference is plotted in figure 3.2. We display the reference frame 250 in figure 3.4 and approximately descriptive hyper-connected $\frac{2}{\Phi_G}$ frame 241 in figure 3.3.

4.2. **In Videos.** In this section we will now focus on hyper-connectedness across different frames of a video. Before we embark on this study we would like to define a few structures that arise from the video frames by the notions of hyper-connectedness.

We construct the tessellation of the video frame based on corners detected using Harris features. The number of triangles connected to each of the vertices is its *degree*. The vertex with the maximal degree is called the *maximal nuclear cluster* (MNC). Each of the triangles in the MNC is a *spoke*. We can construct a cycle
by connecting the centroids of these spokes. Such a cycle is called the maximal centroidal cycle. For more information regarding such constructs and the algorithms associated with them we refer the reader to [1].

For this study we use a video of traffic. We extract cycles from each of the video frames. It is possible for each frame to have multiple MNCs and resulting maximal centroidal cycles. Thus, to simplify the comparison we only compare the cycles which have the biggest area or perimeter depending on the region-based probe function that we use. We use two different probe functions one maps each cycle to its perimeter and the other one maps each cycle to its area.

The first region-based probe function that we use calculates the perimeter of the cycle, which is simply the summation of the lengths of each of the links. We plot the perimeter of the biggest cycle for each of the frames in figure 3.1. The absence of car in the first 95 frames results in the zero values that we see for these.

For the sake of simplicity we assign a single real valued feature to each cycle. We could extend this analysis to accommodate vector valued features such as pixel values at the vertices of the cycle etc. For simplicity as we are only considering one cycle per frame we have only two values for the approximate fuzzy descriptive hyper-connectedness \( \delta \) i.e. \([0, 1, 1]\) if the cycles are far and \([0, 0, 1]\) if they are near (or hyper-connected).

We use the frame number 250 as the reference to construct a relative similarity metric for frame \( n \) to be \( \eta_p(cyc_n, cyc_{250}) = \frac{\text{per}(cyc_n) - \text{per}(cyc_{250})}{\text{per}(cyc_{250})} \). In this equation \( \text{per}(\cdot) \) is the function that returns the perimeter of an object and \( cyc_n \) is the maximal cycle in frame \( n \). This is displayed in figure 3.2. Using this relative similarity we can define a notion of approximate fuzzy descriptive hyper-connectedness \( \delta \), where \( \gamma \in \{ \mathbb{R} \setminus +\infty \} \) is the tolerance parameter.

\[
\delta_{\Phi, \gamma}(cyc_n, cyc_{250}) = \begin{cases} 
[0, 0, 1], & |\eta_p(cyc_n, cyc_{250})| \leq \gamma \\
[0, 1, 1], & |\eta_p(cyc_n, cyc_{250})| > \gamma 
\end{cases}
\]

We now display the frame 250 in which we can see the yellow cycle and the frame 241 that is similar to it. It must be noted that the perimeter of the yellow cycle in frame 250 is 281.75 units and the perimeter of yellow cycle in frame 241 is 280.79 units. The relative similarity between the two cycles is \(-0.0034\). Thus, the frames are hyper-connected i.e. \( \delta_{\Phi, \gamma}(cyc_n, cyc_{250}) = [0, 0, 1] \) for tolerance values equal to or greater than 0.0034.

The second probe function that we use in this study maps each cycle in the frame to its area. Similar to the previous case we only consider the biggest cycle in each frame in terms of its area. We plot the area of the maximal cycle in each frame in figure 4.1. We can construct the relative similarity measure between the frames based on the area of the biggest cycle extracted from them. We consider frame 250 as the reference. Then, the relative similarity measure value for frame \( n \) is defined as \( \eta_a(cyc_n, cyc_{250}) = \frac{\text{area}(cyc_n) - \text{area}(cyc_{250})}{\text{area}(cyc_{250})} \). The function \( \text{area}(\cdot) \) measures the area covered by an object and \( cyc_n \) is the maximal cycle in frame \( n \). Using this we can construct an approximate fuzzy descriptive hyper-connectedness \( \delta \) where
4.1: Maximum cycle area

4.2: relative similarity

4.3: Frame 239

4.4: Frame 250

FIGURE 4. The area of the biggest cycle in each frame is plotted in figure 4.1. The relative similarity($\eta_a(cyc_n, cyc_{250})$) measure with frame 250 as the reference is plotted in figure 4.2. We display the reference frame 250 in figure 4.4 and approximately descriptive hyper-connected $\frac{\delta}{\Phi, \gamma}$, frame 239 in figure 4.3.

$\gamma \in \{\mathbb{R} \setminus +\infty\}$ is the tolerance parameter.

$$\frac{\delta}{\Phi, \gamma} = \begin{cases} [0, 0, 1], & |\eta_a(cyc_n, cyc_{250})| \leq \gamma \\ [0, 1, 1], & |\eta_a(cyc_n, cyc_{250})| > \gamma \end{cases}$$

We now display the frame 250 in figure 4.4 in which the maximal cycle is shown in yellow color. The area covered by the cycle and its interior is 4,925.4 squared units. Next, we display frame 239 in which the maximal cycle is displayed in yellow color. The area covered by this cycle and its interior is 4933.9 squared units. Thus, the relative similarity between these cycles is 0.0017. These cycles are hyper-connected for tolerance values $\gamma >= 0.0017$.

5. Conclusions

In this paper we consider the extension of descriptive hyper-connectedness to yield fuzzy numbers instead of real number as is the case with previously defined notions. Moreover, we formulate the axioms for approximate version of the fuzzy descriptive hyper-connectedness. Moreover, we define descriptive Smirnov measures and its approximate version both of which are fuzzy number valued functions. These measure are shown to follow the axioms of fuzzy descriptive hyper-connectedness and the approximate version respectively. We also define a mechanism of extending
the measure defined over one CW complex to union of two complexes. In the application section we consider how such hyper-connectedness notions arise in digital images and videos.

References


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