

A data-driven Fourier transform like method

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Signal decomposition: Fixed *universal* set of atoms

$x(\cdot)$: Time domain \longrightarrow $\hat{x}(\cdot)$: Transform domain

$$\hat{x}(\boldsymbol{\alpha}) = \begin{cases} \int x(t)\Phi(t, \boldsymbol{\alpha})dt \\ \sum_n x(n)\Phi(n, \boldsymbol{\alpha}) \end{cases}$$

$\Phi(n, \boldsymbol{\alpha})$:

$e^{i\omega t}$	$\boldsymbol{\alpha} = \omega$	Fourier
$g(t-u)e^{i\omega t}$	$\boldsymbol{\alpha} = (\omega, u)$	Gabor
$\psi\left(\frac{t-u}{s}\right)$	$\boldsymbol{\alpha} = (s, u)$	Wavelet
...		

Pros:

- $\Phi(\cdot, \alpha)$: physical meaning
- $\{\Phi(\cdot, \alpha_j)\}$: well understood mathematical properties

⇒ nice interpretation of the decomposition

Cons: $\{\Phi(\cdot, \alpha_j)\}$ not always adapted to the signal x .

- Fourier analysis: smooth and stationary signals
- Time-Frequency analysis:
 - atoms are localized in time and frequency
 - Heisenberg-Gabor uncertainty principle

Signals with different high energy structures are problematic (S. Mallat)

Data-driven decomposition

Several decompositions:

- PCA
- Koopman operator (Approximation of its eigenmodes)
 - Dynamic Mode Decomposition [P. Schmid, 08]
 - Moment method for Spectrum estimation [M. Korda, 18]
 - etc
- Empirical Mode Decomposition [N. E. Huang et al., 98]

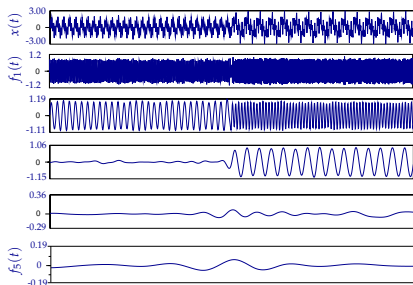
... Data-driven decomposition

Empirical Mode Decomposition (EMD):

- 1 Input: $x(t)$
- 2 Identify all extrema of $x(t)$
- 3 Interpolate between minima (resp. maxima), to obtain the lower (resp. upper) envelope $\ell(t)$ (resp. $u(t)$)
- 4 Compute the mean
 $m(t) = (\ell(t) + u(t))/2$
- 5 Extract the detail
 $v(t) = x(t) - m(t)$
- 6 Iterate 2-5 on $v(t)$ until
 $\int v(t)dt = 0$
- 7 Extract the i^{th} *Intrinsic Mode Function* $f_i(t) = v(t)$
- 8 Iterate on the residual $m(t)$

[N. E. Huang et al., 98]

[G. Rilling et al., 03]



Signal decomposition: data-driven atoms

Objectives:

- Let y be a given signal: the ad-hoc model
- Decompose x on a set of atoms constructed from y

$x(\cdot)$: Time domain \xrightarrow{y} $\hat{x}_y(\cdot)$: Transform domain

$$\hat{x}_y(\boldsymbol{\alpha}) = \begin{cases} \int x(t)\Phi_y(t, \boldsymbol{\alpha})dt \\ \sum_n x(n)\Phi_y(n, \boldsymbol{\alpha}) \end{cases}$$

For $y = x$, Φ_x is intrinsically adapted to x .

- self-adaptability of the EMD
- mathematical formalism of the traditional signal decomposition methods

Data-driven Fourier-like decomposition

Program:

- 1 To a given signal y , associate a (Reproducing Kernel) Hilbert space \mathcal{H}_y .
The space \mathcal{H}_y will represent the ad-hoc model.
- 2 Then identify \mathcal{H}_y as the image of $L_2(\mathbb{R}, d\sigma)$ (for some σ),
by a linear integral transform.
- 3 Define \hat{x}_y by the corresponding image of $x \in L_2(\mathbb{R}, d\sigma)$

Items 2 and 3 were already in the de Branges's program in the early 60s.

So the only missing step is the Hilbert space-valued map $y \mapsto \mathcal{H}_y$

De Branges spaces

Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$, $\alpha(t), \beta(t), \gamma(t)$ a.c. and $m'(t) > 0$ a.e. $t > 0$. (1)

Define $X(t, z) = \begin{bmatrix} A(t, z) \\ B(t, z) \end{bmatrix}$ $z \in \mathbb{C}$, by $\begin{cases} \frac{d}{dt} X(t, z) = zJm'(t)X(t, z) \\ A(0, z) = 1, B(0, z) = 0 \end{cases}$ where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Also define $\mathcal{E}(t, z) = A(t, z) - \frac{i}{\sqrt{\alpha'(t)\gamma'(t) - \beta'(t)^2}} (\beta'(t)A(t, z) + \gamma'(t)B(t, z))$ for $t > 0$

and $\mathcal{E}(-t, z) \triangleq \overline{\mathcal{E}(t, \bar{z})}$

Theorem (De Branges spaces)

Let $\sigma(t) = \alpha'(t) - \frac{\beta'(t)^2}{\gamma'(t)}$ and $\sigma(-t) \triangleq \sigma(t), t > 0$. Then:

- 1 For each fixed $\tau > 0$ the set $\mathcal{H}\{\mathcal{E}(\tau, z)\}$ of functions

$$\widehat{f}_\tau(z) = \frac{1}{2\pi} \int_{-\tau}^{\tau} f(t) \mathcal{E}(t, z) \sigma(t) dt, \quad (2)$$

where $f \in L_2([-\tau, \tau], \sigma(t)dt)$, is a RKHS of entire functions

- 2 $\omega \in \mathbb{R}, \widehat{f}(\omega) = \lim_{\tau \rightarrow \infty} \widehat{f}_\tau(\omega) \in L_2(d\mu)$ for some μ and $\mathcal{H}\{\mathcal{E}(a, z)\} \subset L_2(d\mu)$.

De Branges decomposition

De Branges's theory constructs a map:

$$\mathfrak{m}(t) \mapsto \mathcal{H}_{\mathfrak{m}} = \mathcal{H}\{\mathcal{E}\}$$

Next, we give an example of construction of de Branges's matrix $\mathfrak{m}_y(t)$ from a given signal $y(t)$:

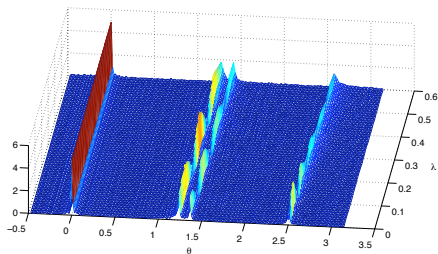
$$y \mapsto \mathfrak{m}_y(t) \mapsto \mathcal{H}_y = \mathcal{H}\{\mathcal{E}\}$$

De Branges Chirp-transform

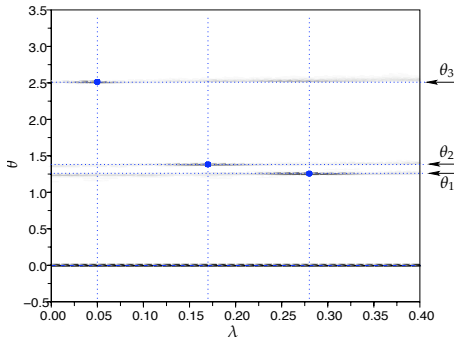
- $m(t) = t\mathbf{I} \Rightarrow \mathcal{E}(t, \omega) = e^{-i\omega t}, \sigma(t) = 1, t > 0 \rightsquigarrow$ classical Fourier analysis.
- $m(t) = \psi(t)\mathbf{I} \Rightarrow \mathcal{E}(t, \omega) = e^{-i\omega\psi(t)}, \sigma(t) = \psi'(|t|), t > 0.$
- $y(t) = \psi(t) = t^2 \mapsto m_y(t) = t^2\mathbf{I} \rightsquigarrow$ natural decomposition for chirp signals.

$$x(t) = \sum_{n=1}^N \alpha_n e^{-i\omega_n \psi_n(t)} + \text{noise}, \quad \omega_n \psi_n(t) = \theta_n (t - \lambda_n)^2 + \eta_n$$

Chirp-transform: Spectrogram



a) 3D view



b) 2D projection

Data-driven transform: a proof of concept

- Let $y_n = u_n + iv_n, n = 0, 1, \dots$ be a discrete complex signal (t_n : sampling instants).
- (For real signals, set $u_n = y_n$ and consider the analytic signal $\tilde{y}_n = u_n + iv_n$).
- Consider the piecewise constant 2×2 -matrix function $\mathbf{m}_y(t)$ defined s.t.

$$\mathbf{m}'_y(t) = \frac{d\mathbf{m}_y(t)}{dt} = \begin{pmatrix} u_n^2 & u_n v_n \\ v_n u_n & v_n^2 \end{pmatrix} \delta(t - t_n), \quad n = 0, 1, 2, \dots$$

\rightsquigarrow finite dimensional De Branges space

This realizes the map $\{y_n\} \mapsto \mathbf{m}_y(t)$ (or $\{\tilde{y}_n\} \mapsto \mathbf{m}_y(t)$)

... Data-driven transform: a proof of concept

Set $X_n(z) = X(t_n, z) = \begin{pmatrix} A_n(z) \\ B_n(z) \end{pmatrix}$. Then with m_y , we have:

$$X_{n+1}(z) = \begin{pmatrix} 1 - zu_n v_n & -zv_n^2 \\ zu_n^2 & 1 + zu_n v_n \end{pmatrix} X_n(z) \quad \text{with} \quad X_0(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Define

$$S_n(z) \triangleq u_n A_n(z) + v_n B_n(z)$$

Theorem (De Branges, Alpay & Dym)

If $u_n \neq 0$ and $v_n \neq 0$, $n = 0, 1, \dots, N$, then $\{S_n(z)\}_{n=0}^N$ is an orthonormal basis of $\mathcal{H}\{\mathcal{E}(t_N, z)\}$.

... Data-driven transform: a proof of concept

Given two causal signals $y_n = u_n + iv_n$ and $x_n = p_n + iq_n$, $n = 0, 1, \dots, N$, we define the y -transform of x by

$$\widehat{x}_y(z) = \frac{1}{\pi} \sum_{n=0}^N (p_n u_n + q_n v_n) S_n(z)$$

Some Remarks:

- For all n , $S_n(0) = u_n$ ($A_n(0) = 1, B_n(0) = 0$)

$$\widehat{x}_y(0) = \frac{1}{\pi} \sum_{n=0}^N (p_n u_n^2 + q_n u_n v_n)$$

↪ 3rd order cross moments

- $\|\widehat{x}_y(z)\|^2 = \frac{1}{\pi} \sum_{n=0}^N (p_n u_n + q_n v_n)^2$

↪ 4th order cross moments

Thank you !