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On the Covering Radius of Codes over \mathbb{Z}_{p^k}

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Abstract: In this correspondence, we investigate the covering radius of various types of repetition codes over \mathbb{Z}_{p^k} ($k \geq 2$) with respect to the Lee distance. We determine the exact covering radius of the various repetition codes, which have been constructed using the zero divisors and units in \mathbb{Z}_{p^k} . We also derive the lower and upper bounds on the covering radius of block repetition codes over \mathbb{Z}_{p^k} .

Keywords: covering radius; codes over rings; repetition codes; Gray map

1. Introduction

Codes over finite fields have been studied since the inception of coding theory. Due to the rich algebraic structure of rings, the codes over rings gained popularity during the seventies [1–3]. In 1994, Hammons et al. [4] obtained the well known non-linear codes as a Gray image of the codes over \mathbb{Z}_4 . After that, working on the codes over rings gained greater attention. What started with the ring \mathbb{Z}_4 , was later generalized to the rings \mathbb{Z}_{2^s} , $\mathbb{Z}_2 + u\mathbb{Z}_2$, $\mathbb{Z}_4 + u\mathbb{Z}_4$, $\mathbb{F}_p + u\mathbb{F}_p$ etc [5–8]. Covering Radius is a widely discussed parameter for the codes with respect to the Hamming weight [9]. A lot of other weights such as Lee weight [4], Homogenous weight [10] and Euclidean weight have been introduced and used in the literature for convenience.

The Covering Radius for the codes with respect to the Lee distance was first investigated for the ring \mathbb{Z}_4 by Aoki [11]. Later, working on the Covering Radius of codes the with respect to the Lee distance gained interest [6,12,13]. We are particularly interested to find the Covering Radius for Repetition Codes, Since the Covering Radius of the Repetition Codes simplifies the process of finding the Covering Radius for many existing codes. For eg., it helps to find the Covering Radius of the well known Simplex and Macdonald Codes, as the generator matrix of Simplex and Macdonald Codes has lot of similarities with the generator matrix of the Repetition Codes. For the Quaternary case, it was discussed in [6].

This motivated us to work on the Covering Radius of Repetition Codes over the ring \mathbb{Z}_{p^k} . The problem of generalising the results for \mathbb{Z}_{p^k} starts with defining a proper Lee weight for \mathbb{Z}_{p^k} and then the extended Gray map defined here is not surjective. Also the zero divisors of different orders are obtained here, which will not be in the case of \mathbb{Z}_4 .

In this correspondence, we have investigated the covering radius of the codes over \mathbb{Z}_{p^k} ($k \geq 2$) with respect to the Lee distance in relation to the codes obtained by the Gray map. In Section 2, we have given some basic preliminaries. We have given several upper and lower bounds on covering radius, including \mathbb{Z}_{p^k} analogue of sphere covering bound, packing bound and Delsarte bound in Section 3. In the next Section, the covering radii of some repetition codes have been discussed, namely repetition codes using the zero divisors in \mathbb{Z}_{p^k} of different orders and the repetition codes from the

units in \mathbb{Z}_{p^k} . We have ended the section with the upper and lower bounds on the covering radius of the block repetition codes $BR_{p^k}^{(p^k-1)n}$ and $BR_{p^k}^{(p^k-2)n}$. Here we have determined the exact value of $r_L(BR_{p^2}^{(p^2-1)n})$. Finally, we have concluded the paper with the future work that can be proceeded with.

2. Preliminaries

A linear code C of length n is an additive subgroup of $\mathbb{Z}_{p^k}^n$. If C is not an additive subgroup of $\mathbb{Z}_{p^k}^n$, then C is simply called a code of length n . Thus, every linear code C is a \mathbb{Z}_{p^k} submodule of $\mathbb{Z}_{p^k}^n$. An element in C is known as a *codeword* of C . A matrix G is said to be a generator matrix of C if C is the row span of G over \mathbb{Z}_{p^k} . Two codes are said to be permutation equivalent if one is obtained from another by permuting the coordinates.

The *Hamming weight* $w_H(x)$ of a vector $x \in \mathbb{Z}_{p^k}^n$ is the number of non-zero coordinates in x . The *Lee weight* of $x \in \mathbb{Z}_{p^k}$ in the sense of [14] is given by

$$w_L(x) = \begin{cases} x & \text{for } 0 \leq x \leq p^{k-1} - 1 \\ p^{k-1} & \text{for } p^{k-1} \leq x \leq p^k - p^{k-1} \\ p^k - x & \text{for } p^k - p^{k-1} + 1 \leq x \leq p^k - 1 \end{cases}$$

Note that this weight coincides with the classical Lee weight when $p = 2$, and is different when $p > 2$. The *Lee weight* of $x \in \mathbb{Z}_{p^k}^n$ is the sum of the Lee weight of its coordinates. The Hamming (Lee) distance $d_H(x, y)$ ($d_L(x, y)$) between two vectors x and y is $w_H(x - y)$ ($w_L(x - y)$). The minimum Hamming (Lee) distance is the smallest Hamming (Lee) weight among all non-zero codewords of C . A code of length n , size M , minimum Hamming distance d_H , minimum Lee distance d_L over \mathbb{Z}_{p^k} is a (n, M, d_H, d_L) code.

The dual code C^\perp of C is defined as $C^\perp = \{x \in \mathbb{Z}_{p^k}^n \mid x_1 \cdot y_1 + x_2 \cdot y_2 + \cdots + x_n \cdot y_n = 0 \pmod{p^k} \text{ for all } y \in C\}$. As various distances are possible for the codes over \mathbb{Z}_{p^k} , we have given a definition of the covering radius for a general distance. Let d be the general distance out of the various possible distances such as Hamming and Lee. The Covering radius $r_d(C)$ of a code C over \mathbb{Z}_{p^k} with respect to the general distance is given by

$$r_d(C) = \max_{x \in \mathbb{Z}_{p^k}^n} \left\{ \min_{c \in C} d(x, c) \right\}$$

and hence $\mathbb{Z}_{p^k}^n = \bigcup_{c \in C} S_{r_d}(c)$, where $r_d = r_d(C)$

and $S_{r_d}(c) = \{v \in \mathbb{Z}_{p^k}^n \mid d(c, v) \leq r_d\}$. To alleviate notation, we have written r_L (resp. r_H) for r_{d_L} (resp. r_{d_H}).

In [14], a distance preserving Gray map $\phi_L : (\mathbb{Z}_{p^k}, d_L) \rightarrow (\mathbb{Z}_p^{p^{k-1}}, d_H)$ was defined as follows, for, $1 \leq j \leq p - 1$,

$$(j-1)p^{k-1} + i \longrightarrow (\overbrace{jj \cdots j}^i \overbrace{j-1 \cdots j-1}^{p^{k-1}-i}), \text{ for } 0 \leq i \leq p^{k-1} - 1$$

and then we can extend the map ϕ_L to $\phi : \mathbb{Z}_{p^k}^n \rightarrow \mathbb{Z}_p^{np^{k-1}}$ by the coordinate wise extension of the Gray map. Let C be a code of length n with M codewords and minimum Lee distance d over \mathbb{Z}_{p^k} . Then by the above Gray map, the image $\phi(C)$ is a code of length np^{k-1} with M codewords and minimum Hamming distance d . We have summed up the idea below without proof.

Proposition 1. [11] If C is a linear code over \mathbb{Z}_{p^k} of length n , size M and minimum Lee distance d , then the Gray image $\phi(C)$ is a code over \mathbb{Z}_p of length np^{k-1} , size M and minimum Hamming distance d and also $r_L(C) \leq r_H(\phi(C))$.

Note that since the Gray map is injective but not surjective in general, the covering radius of a code C for the Lee metric is at most that of $\phi(C)$ for the Hamming metric, but could be different.

3. Covering Radius of Codes

We have discussed several bounds on covering radius of codes in this section, including the \mathbb{Z}_{p^k} analogue of the packing bound, the sphere covering bound and the Delsarte bound. The following bound is called the packing bound, which is similar to the bound given for \mathbb{Z}_4 in [11].

Theorem 1. Let C be a (n, M, d_H, d) code over \mathbb{Z}_{p^k} . Then $r_L(C) \geq \frac{d}{2}$.

Proof. Let x, y in C be with $x \neq y$. Choose $x_0 \in \mathbb{Z}_{p^k}^n$ such that $w_L(x_0) = \lfloor \frac{d}{2} \rfloor$. Consider,

$$\begin{aligned} w_L(x - y) &= d_L(x, y) \leq d_L(x, y + x_0) + d_L(y + x_0, y) \\ d_L(x, y + x_0) &\geq d_L(x, y) - w_L(x_0) \geq d - \left\lfloor \frac{d}{2} \right\rfloor \\ &\geq d - \frac{d}{2} = \frac{d}{2} \end{aligned}$$

$d_L(x, x_0 + y) \geq \frac{d}{2}$ for any codeword x of C . It implies that $r_L(C) \geq \frac{d}{2}$. \square

The proof of the following Proposition 2 and 3, is similar to but distinct from the case of \mathbb{Z}_4 [11]. Note that the covering radius of a code C for the Lee metric is at most that of $\phi(C)$ for the Hamming metric, but could be different.

Proposition 2. For any code C of length n over \mathbb{Z}_{p^k} .

$$\frac{p^{np^{k-1}}}{|C|} \leq \sum_{i=0}^{r_L(C)} \binom{np^{k-1}}{i} (p-1)^i.$$

This bound is known as the Sphere Covering Bound.

Proof. Because the map ϕ is an isometry the image of a ball of radius r centered in x is a ball of radius r centered in $\phi(x)$ of the same cardinality. The result follows by the usual sphere covering argument. \square

Let C be a code over \mathbb{Z}_{p^k} and let $s(C^\perp) = \#\{i | A_i(C^\perp) \neq 0, i \neq 0\}$ where $A_i(C^\perp)$ denotes the number of codewords of the Lee weight i in C^\perp . Then we have the Delsarte bound

Proposition 3. For any $C \subseteq \mathbb{Z}_{p^k}^n$, $r_L(C) \leq s(C^\perp)$.

Proof. As noted above the covering radius $r_L(C)$ of a code C for the Lee metric is at most that of $\phi(C)$ for the Hamming metric. Denote by $r_H(\phi(C))$ that latter quantity. Since ϕ is an isometry, it can be shown by using the duality of complete weight enumerators and specialization of variables, so that the Hamming weight enumerators of $\phi(C)$ and $\phi(C^\perp)$ are MacWilliams duals of each other, a fact already noted in [4] for $k = 2$. This implies that the number of Gray weights of C^\perp , that is the number of weights of $\phi(C^\perp)$ equals the number of Hamming weights of $\phi(C)^\perp$. By the Delsarte

bound in the Hamming metric we conclude that $r_H(\phi(C)) \leq s(C^\perp)$. The result follows upon writing $r_L(C) \leq r_H(\phi(C))$. \square

If C is a code of length n over a group $(G, +)$, then the covering radius of the code C is defined by, $r(C) = \max_{x \in G^n} d(x, C) = \max_{x \in G^n} \{\min_{c \in C} d(x, c)\} = \max_{x \in G^n} \{\min_{c \in C} wt(x - c)\}$. Hence the following result, which is a generalization of codes over finite rings from finite field by G.D Cohen et al. [9].

Theorem 2. Let C be the Cartesian Product of two Codes C_1 and C_2 , then the covering radius of C is $r(C) = r(C_1) + r(C_2)$ with respect to all distances.

4. Repetition Codes

Let $\mathbb{F}_q = \{\alpha_0 = 0, \alpha_1 = 1, \dots, \alpha_{q-1}\}$ be a finite field. A q -ary repetition code $C = \{\bar{\alpha} | \alpha \in \mathbb{F}_q\}$ is a (n, q, n) over \mathbb{F}_q , where $\bar{\alpha} = (\alpha, \dots, \alpha) \in \mathbb{F}_q^n$. The covering radius of the repetition code C over \mathbb{F}_q is given by $\lceil \frac{n(q-1)}{q} \rceil$ [15]. Here we have introduced three types of repetition codes over \mathbb{Z}_{p^k} ,

4.1. Zero Divisor Repetition Codes

Let z be a zero divisor of \mathbb{Z}_{p^k} . The code generated by the generator matrix $\begin{bmatrix} \overbrace{zz \dots z}^n \end{bmatrix}$, is called a zero divisor repetition code. The $p^{k-1} - 1$ zero divisors of \mathbb{Z}_{p^k} are given by $\alpha_1 p^{k-1} + \alpha_2 p^{k-2} + \dots + \alpha_{k-2} p^2 + \alpha_{k-1} p$, $\alpha_i \in \{0, 1, \dots, p-1\}$, $1 \leq i \leq k-1$, but not all α'_i s are zero. The additive order of the zero divisors is p, p^2, \dots, p^{k-1} .

First, we have considered the zero divisors of order p . There are $p-1$ zero divisors of order p , namely $\alpha_1 p^{k-1}$, $\alpha_1 \in \{1, \dots, p-1\}$. Let C_p be the code generated by the generator matrix $G_p = \begin{bmatrix} \overbrace{p^{k-1} p^{k-1} \dots p^{k-1}}^n \end{bmatrix}$. Then

$$C_p = \{0, p^{k-1}, 2p^{k-1}, \dots, (p-1)p^{k-1}\} \subset \mathbb{Z}_{p^k}^n, \text{ where } \mathbf{a} = (a, \dots, a) \in \mathbb{Z}_{p^k}^n$$

C_p is a (n, p, n, np^{k-1}) code over \mathbb{Z}_{p^k} . $\phi(C_p) = \{0, 1, 2, \dots, p-1\}$ is a repetition code of length np^{k-1} over the field \mathbb{Z}_p . Then the covering radius $r_L(C_p)$ is given by

$$r_L(C_p) \leq r_H(\phi(C_p)) = \left\lceil \frac{np^{k-1}(p-1)}{p} \right\rceil = np^{k-2}(p-1). \text{ For the reverse inequality, let}$$

$$x = \overbrace{0 \dots 0}^l \overbrace{1 \dots 1}^l \overbrace{2 \dots 2}^l \dots \overbrace{(p^k-1) \dots (p^k-1)}^{n-(p^{k-1}-1)l} \in \mathbb{Z}_{p^k}^n, \text{ where } l = \left\lceil \frac{n}{p^k} \right\rceil.$$

Consider

$$\begin{aligned} d_L(x, \mathbf{0}) &= n + p^k(p^{k-1} - p^{k-2} - 1)l \\ &\geq n(p^{k-1} - p^{k-2}) \text{ since } l \geq \frac{n}{p^k} \\ d_L(x, p^{k-1}) &= (p^{k-1} + 1)n + p^k(p^{k-1} - p^{k-2} - p^{k-1} - 1)l \\ &\geq n(p^{k-1} - p^{k-2}) \text{ since } l \geq \frac{n}{p^k} \\ &\vdots \\ d_L(x, (p-1)p^{k-1}) &= (p^k - p^{k-1} + 1)n + p^k(2p^{k-1} - p^k - p^{k-1} + 1)l \\ &\geq n(p^{k-1} - p^{k-2}) \text{ since } l \geq \frac{n}{p^k} \end{aligned}$$

$$\begin{aligned}
d_L(x, \mathbf{i}) &\geq n(p^{k-1} - p^{k-2}) \\
&= np^{k-2}(p-1) \\
r_L(C_p) &\geq d_L(x, C_p) \geq np^{k-2}(p-1). \text{ Hence we sum up,}
\end{aligned}$$

Theorem 3. $r_L(C_p) = np^{k-2}(p-1)$

Note that this is a short alternate proof of [Theorem 3.1, [12,13]]. The $p^2 - p$ zero divisors of order p^2 in \mathbb{Z}_{p^k} are given by $\alpha_i p^{k-1} + \alpha_j p^{k-2}$ for all $\alpha_i \in \{0, 1, \dots, p-1\}, \alpha_j \in \{1, 2, \dots, p-1\}$. C_{p^2} is a code generated by the generator matrix $G_{p^2} = \left[\overbrace{p^{k-2} p^{k-2} \dots p^{k-2}}^n \right]$. Clearly C_{p^2} is a (n, p^2, n, np^{k-2}) code over \mathbb{Z}_{p^k} .

Theorem 4. $r_L(C_{p^2}) = np^{k-2}(p-1)$.

Proof. The proof is the same as the proof of the Theorem 5, which is the more general. \square

For each $i, 1 \leq i \leq k-1$ the number of zero divisors of order p^i is $p^{i-1}(p-1)$ which are given by $\alpha_1 p^{k-1} + \alpha_2 p^{k-2} + \dots + \alpha_i p^{k-i}, \alpha_j \in \{0, 1, \dots, p-1\}, 1 \leq j \leq i-1$ and $\alpha_i \in \{1, 2, \dots, p-1\}$.

C_{p^i} is a code generated by the generator matrix $G_{p^i} = \left[\overbrace{p^{k-i} p^{k-i} \dots p^{k-i}}^n \right]$.

Hence C_{p^i} is an (n, p^i, n, np^{k-i}) code. As we sum up the above ideas we get

Theorem 5. For $1 \leq i \leq k-1$, $r_L(C_{p^i}) = np^{k-2}(p-1)$.

Proof. Let $x \in \mathbb{Z}_{p^k}^n$ and let w_i be the number of i coordinates in x for $0 \leq i \leq p^k - 1$. Then $\sum_{i=0}^{p^k-1} w_i = n$. Consider,

$$\begin{aligned}
d_L(x, \mathbf{0}) &= (w_1 + w_{p^{k-1}}) + 2(w_2 + w_{p^{k-2}}) + \dots + (p^{k-1} - 1) \\
&\quad (w_{p^{k-p^{k-1}+1}} + w_{p^{k-1}-1}) + p^{k-1}(w_{p^{k-1}} + \dots + w_{p^{k-p^{k-1}}}) \\
d_L(x, \mathbf{p}^{k-i}) &= (w_{p^{k-i}+1} + w_{p^{k-i}-1}) + 2(w_{p^{k-i}+2} + w_{p^{k-i}-2}) + \dots + \\
&\quad (p^{k-1} - 1)(w_{p^{k-i-p^{k-1}+1}} + w_{p^{k-i}+p^{k-1}-1}) \\
&\quad + p^{k-1}(w_{p^{k-1}+p^{k-i}} + \dots + w_{p^{k-p^{k-1}+p^{k-i}}}) \\
&\quad \vdots \\
d_L(x, (p-1)p^{k-1} + (p-1)p^{k-2} + \dots + (p-1)p^{k-i}) &= d_L(x, \mathbf{p}^k - \mathbf{p}^{k-i}) \\
&= (w_{p^{k-p^{k-i}+1}} + w_{p^{k-p^{k-i}-1}}) + 2(w_{p^{k-p^{k-i}+2}} + w_{p^{k-p^{k-i}-2}}) \\
&\quad + \dots + (p^{k-1} - 1)(w_{p^{k-p^{k-i}-p^{k-1}+1}} + w_{p^{k-p^{k-i}+p^{k-1}-1}}) \\
&\quad + p^{k-1}(w_{p^{k-1}+p^{k-i}} + \dots + w_{p^{k-p^{k-1}+p^{k-i}}})
\end{aligned}$$

We know that the minimum is always less than the average. So we get,

$$\begin{aligned}
 r_L(C_{p^i}) &\leq \frac{\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} d_L(x, (p-1)p^{k-1} + (p-1)p^{k-2} + \dots + (p-1)p^{k-i})}{p^2} \\
 &= \frac{((p^i - 2p^{i-1} + 1)p^{k-1} + 2p^{k-i}(1 + 2 + \dots + (p^{i-1} - 1))) \sum_{i=0}^{p^k-1} w_i}{p^i} \\
 &= \frac{n((p^i - 2p^{i-1} + 1)p^{k-1} + p^{k-i}(p^{i-1} - 1)p^{i-1})}{p^i} \\
 &= np^{k-2}(p-1)
 \end{aligned}$$

It shows that, $r_L(C_{p^i}) \leq np^{k-2}(p-1)$. For the reverse inequality, let

$$x = \underbrace{0 \dots 0}_l \underbrace{1 \dots 1}_l \underbrace{2 \dots 2}_l \dots \underbrace{(p^k-1) \dots (p^k-1)}_{n-(p^k-1)l} \in \mathbb{Z}_{p^k}^n, \text{ where } l = \lceil \frac{n}{p^k} \rceil.$$

Consider

$$\begin{aligned}
 d_L(x, \mathbf{0}) &= n + p^k(p^{k-1} - p^{k-2} - 1)l \\
 d_L(x, p^{k-i}) &= (p^{k-i} + 1)n + p^k(p^{k-1} - p^{k-2} - (p^{k-i} + 1))l \\
 &\vdots \\
 d_L(x, (p-1)p^{k-1} + (p-1)p^{k-2} + \dots + (p-1)p^{k-i}) &= d_L(x, p^k - p^{k-i}) \\
 &= (p^k - p^{k-i} + 1)n + p^k(p^{k-1} - p^{k-2} - (p^k - p^{k-i} + 1))l
 \end{aligned}$$

Therefore, for all $\mathbf{i} \in C_{p^i}$

$$\begin{aligned}
 d_L(x, \mathbf{i}) &\geq n(p^{k-1} - p^{k-2}), \text{ since } l \geq \frac{n}{p^k}. \\
 &= np^{k-2}(p-1) \\
 r_L(C_{p^i}) &\geq d_L(x, C_{p^i}) \geq np^{k-2}(p-1)
 \end{aligned}$$

Hence, $r_L(C_{p^i}) = np^{k-2}(p-1)$. \square

4.2. Unit Repetition Codes

Let u be a unit in \mathbb{Z}_{p^k} . Then code C_u generated by the matrix $G_u = [\overbrace{uu \dots u}^n]$ is called unit repetition code. Clearly C_u is a (n, p^k, n, n) code over \mathbb{Z}_{p^k} .

Theorem 6. Let u be a unit in \mathbb{Z}_{p^k} , then $r_L(C_u) = np^{k-2}(p-1)$

Proof. Let $x \in \mathbb{Z}_{p^k}^n$ and let w_i be the number of i coordinates in x for $0 \leq i \leq p^k - 1$. Consider,

$$\begin{aligned} d_L(x, \mathbf{0}) &= w_0(0) + (w_1 + w_{p^{k-1}}) + \cdots + (p^{k-1} - 1)(w_{p^k - p^{k-1} + 1} + w_{p^{k-1} - 1}) \\ &\quad + p^{k-1}(w_{p^{k-1}} + \cdots + w_{p^k - p^{k-1}}) \\ d_L(x, \mathbf{1}) &= w_1(0) + (w_0 + w_2) + \cdots + (p^{k-1} - 1)(w_{p^k - p^{k-1} + 2} + w_{p^{k-1}}) \\ &\quad + p^{k-1}(w_{p^{k-1} + 1} + \cdots + w_{p^k - p^{k-1} + 1}) \\ &\vdots \\ d_L(x, \mathbf{p^k - 1}) &= w_{p^{k-1}}(0) + (w_0 + w_{p^{k-2}}) + \cdots + (p^{k-1} - 1)(w_{p^k + p^{k-1} - 2} + w_{p^k - p^{k-1}}) \\ &\quad + p^{k-1}(w_{p^k + p^{k-1} - 1} + \cdots + w_{2p^k - p^{k-1} - 1}) \end{aligned}$$

We know that the minimum is always less than the average. So we get,

$$\begin{aligned} r_L(C_u) &\leq \frac{\sum_{t=0}^{p^k-1} d_L(x, \mathbf{t})}{p^k} \\ &= \frac{((p^k - 2p^{k-1} + 1)p^{k-1} + 2(1 + 2 + \cdots + (p^{k-1} - 1))) \sum_{i=0}^{p^k-1} w_i}{p^k} \\ &= \frac{n((p^{2k-1} - 2p^{2k-2} + p^{k-1} + (p^{k-1} - 1)p^{k-1}))}{p^k} \\ &= np^{k-2}(p - 1) \end{aligned}$$

Thus, $r_L(C_u) \leq np^{k-2}(p - 1)$.

Let $x = (\underbrace{00 \cdots 0}_l \underbrace{11 \cdots 1}_l \underbrace{22 \cdots 2}_l \cdots \underbrace{(p^k - 1)(p^k - 1) \cdots (p^k - 1)}_{n - (p^k - 1)l}) \in \mathbb{Z}_{p^k}^n$ where $l = \lceil \frac{n}{p^k} \rceil$. Then

$$\begin{aligned} d_L(x, \mathbf{0}) &= n + p^k(p^{k-1} - p^{k-2} - 1)l \\ d_L(x, \mathbf{1}) &= 2n + p^k(p^{k-1} - p^{k-2} - 2)l \\ &\vdots \\ d_L(x, \mathbf{p^k - 1}) &= p^{2k-2}(p - 1)l \end{aligned}$$

Thus the covering radius $r_L(C_u)$ is given by

$$\begin{aligned} r_L(C_u) &\geq d_L(x, C_u) \\ &= \min_{0 \leq i \leq p^k-1} \{d_L(x, \mathbf{i})\} \\ &= \min_{0 \leq i \leq p^k-2} \{d_L(x, \mathbf{i}), d_L(x, \mathbf{p^k - 1})\} \\ &\geq p^{2k-2}(p - 1)l \end{aligned}$$

Since $l \geq \frac{n}{p^k}$, $r_L(C_u) \geq np^{k-2}(p - 1)$. Finally, we have $r_L(C_u) = np^{k-2}(p - 1)$. \square

4.3. Block Repetition Codes of \mathbb{Z}_{p^k}

We have defined a few block repetition codes over \mathbb{Z}_{p^k} and found their covering radius. Let

$$G = \left[\overbrace{11 \dots 1}^n \overbrace{22 \dots 2}^n \dots \overbrace{(p^k - 1)(p^k - 1) \dots (p^k - 1)}^n \right]$$

be a matrix over \mathbb{Z}_{p^k} . Then the code generated by G is a $(n(p^k - 1), p^k)$ code. This code is called a block repetition code over \mathbb{Z}_{p^k} and is denoted by $BR_{p^k}^{(p^k-1)n}$. The covering radius of the code generated by the above matrix is $\left\lceil \frac{n(p-1)^2}{p} \right\rceil$ [15].

The following theorem gives the upper and lower bounds of this code with respect to the Lee distance,

Theorem 7. $n(p^{2k-1} - p^{2k-2}) \leq r_L(BR_{p^k}^{(p^k-1)n}) \leq n(p^{2k-1} - p^k)$.

Proof. Let $x = 00 \dots 0 \in \mathbb{Z}_{p^k}^{(p^k-1)n}$, then we get $d_L(x, BR_{p^k}^{(p^k-1)n}) = n(p^{2k-1} - p^{2k-2})$.

This implies, $r_L(BR_{p^k}^{(p^k-1)n}) \geq n(p^{2k-1} - p^{2k-2})$.

On the other hand, the gray image $\phi(BR_{p^k}^{(p^k-1)n})$ contains a codeword

$$y = (\overbrace{11 \dots 1}^{p^k n} \overbrace{22 \dots 2}^{p^k n} \dots \overbrace{(p-1)(p-1) \dots (p-1)}^{p^k n} \overbrace{00 \dots 0}^{n(p^{2k-1} - p^{k-1} - p^{k+1} + p^k)})$$

Let C'_1 be the code generated by y . Let C'_2 be the code generated by the matrix

$$G_2 = \left[\overbrace{11 \dots 1}^{p^k n} \overbrace{22 \dots 2}^{p^k n} \dots \overbrace{(p-1)(p-1) \dots (p-1)}^{p^k n} \right]$$

Then, C'_2 is equivalent to the repetition code $((p-1)p^k n, p, (p-1)p^k n)$.

$$r_H(C'_2) = \left\lceil \frac{(p-1)p^k n(p-1)}{p} \right\rceil = np^{k-1}(p-1)^2 = n(p^{k+1} + p^{k-1} - 2p^k)$$

Let $C'_3 = \{\overbrace{00 \dots 0}^{n(p^{2k-1} - p^{k-1} - p^{k+1} + p^k)}\}$, then we get $r_H(C'_3) = n(p^{2k-1} - p^{k-1} - p^{k+1} + p^k)$.

Note that C'_1 is a cartesian product of C'_2 and C'_3 . Then, by Theorem 2,

$$\begin{aligned} r_H(C'_2 \times C'_3) &= r_H(C'_2) + r_H(C'_3) \\ &= n(p^{k+1} + p^{k-1} - 2p^k) + n(p^{2k-1} - p^{k-1} - p^{k+1} + p^k) \\ r_H(C'_2 \times C'_3) &= n(p^{2k-1} - p^k) \end{aligned}$$

Since $C'_1 \subset \phi(BR_{p^k}^{(p^k-1)n})$, we get

$$\begin{aligned} r_H(\phi(BR_{p^k}^{(p^k-1)n})) &\leq r_H(C'_1) \\ &= r_H(C'_2 \times C'_3) \\ &= n(p^{2k-1} - p^k) \end{aligned}$$

Hence $n(p^{2k-1} - p^{2k-2}) \leq r_L(BR_{p^k}^{(p^k-1)n}) \leq n(p^{2k-1} - p^k) \quad \square$

On Substituting $k = 2$ in Theorem 7, it results in $r_L(BR_{p^2}^{(p^2-1)n}) = n(p^3 - p^2)$. This gives an exact value of $r_L(BR_{p^2}^{(p^2-1)n})$, which is better than the known bound in (Theorem 3.4, [12]). Now, we have

defined a new matrix G' , which is obtained by removing $\overbrace{(p^k - 1)(p^k - 1) \cdots (p^k - 1)}^n$ from G . Let,

$$G' = \left[\overbrace{11 \cdots 1}^n \overbrace{22 \cdots 2}^n \cdots \overbrace{(p^k - 2)(p^k - 2) \cdots (p^k - 2)}^n \right]$$

This matrix G' generates a new block repetition code over \mathbb{Z}_{p^k} and is denoted by $BR_{p^k}^{(p^k-2)n}$. The following theorem gives the upper and lower bounds on the covering radius of $BR_{p^k}^{(p^k-2)n}$,

Theorem 8. $n(p^{2k-1} - p^{2k-2} - 1) \leq r_L(BR_{p^k}^{(p^k-2)n}) \leq n(p^{2k-1} - p^{k-1} - p^k)$.

Proof. The proof is the same as the proof of the Theorem 7. \square

5. Conclusions

We have discussed some well known bounds such as the sphere covering bound, the Delsarte bound and the packing bound with respect to the Lee distance for the codes over \mathbb{Z}_{p^k} . We have determined the exact value of the covering radius of the zero divisor (unit) repetition codes. We have obtained the lower and upper bounds on the covering radius of the block repetition codes over \mathbb{Z}_{p^k} . The results obtained in this article is definitely helpful, if we are able to obtain the similarities between the generator matrix of existing codes over \mathbb{Z}_{p^k} with the generator matrix of the repetition codes over \mathbb{Z}_{p^k} , then we will able to apply all the existing results on the covering radius. And also it would be an interesting task to discuss the covering radius for the more generalized ring \mathbb{Z}_n . We can also obtain the weight enumeration of these codes in Lee distance and compare it with the Hamming distance.

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