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# Non-controllability of the viscous Burgers equation and a detour into the well-posedness of unbounded entropy solutions to scalar conservation laws

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## Abstract

We contribute an answer to a quantitative variant of the question raised in [Coron, Contemp. Math 2007] concerning the controllability, in the framework of  $L^2$  solutions, of the viscous Burgers equation  $u_t + (u^2/2)_x = u_{xx}$  for initial and terminal data prescribed for  $x \in (0, 1)$ . We investigate the (non)-controllability under the additional *a priori* bound imposed on the (nonlinear) operator that associates the solution to the terminal state.

In contrast to typical techniques on the controllability of the viscous Burgers equation invoking the heat equation, we combine scaling and compensated compactness arguments along with observations on the non-controllability of the inviscid Burgers equation to point out wide sets of terminal states non-attainable from zero initial data by solutions of restricted size. We prove in particular that, given  $L \geq 1$ , for sufficiently large  $|C|$  and  $T < (1 + \Delta)/|C|$  (where  $\Delta > 0$  depends on  $L$ ), the constant terminal state  $u(\cdot, T) := C$  is not attainable at time  $T$ , starting from zero data, by weak solutions of the viscous Burgers equation satisfying a *bounded amplification restriction* of the form  $\|u\|_\infty \leq L|C|$ .

Further, in order to get closer to the original question we develop a basic well-posedness theory of unbounded entropy solutions to the Cauchy problem for multi-dimensional scalar conservation laws with pure  $L^p$  data and polynomial growth up to the critical power  $p$  of the flux function. The case of the Cauchy-Dirichlet problem for the Burgers equation on an interval is also addressed, in the  $L^2$  solution framework which is considerably simpler than the established  $L^1$  theory of renormalized solutions to such problems. Local regularity of unbounded entropy solutions is discussed in the one-dimensional case with convex flux.

Non-controllability results are then extended to solutions of the viscous Burgers equation in the  $L^2$  setting, under the bounded amplification restriction of the form  $\|u\|_2 \leq LT|C|$  and an additional  $L^2 - L^3_{loc}$  regularization assumption.

MSC (2010): 93B03, 35L65, 35D30, 47J35.

KEYWORDS: BURGERS EQUATION, EXACT CONTROLLABILITY, UNBOUNDED ENTROPY SOLUTIONS,  $L^2$  WELL-POSEDNESS, SCALAR CONSERVATION LAWS, CRITICAL GROWTH.

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## 1. Introduction

We are concerned with the controllability of the viscous Burgers equation

$$u_t + (u^2/2)_x = u_{xx} \text{ in } D, \tag{BE}$$

where  $(x, t) \in D = \mathbb{R} \times (0, T)$  (“the strip setting”) or  $(x, t) \in D = (0, 1) \times (0, T)$  (“the box setting”), with a given  $T > 0$ . Our primary motivation comes from [18], where J.-M. Coron asked the following question (Open Problem 4). Let  $T > 0$  and  $C \in \mathbb{R} \setminus \{0\}$ .

Question: *Does there exist  $u \in L^2((0, 1) \times (0, T))$  satisfying (BE) such that for all  $x \in (0, 1)$ ,  $u(\cdot, 0) = 0$  and  $u(\cdot, T) = C$ ?* (Q)

With the method explored in this paper, we focus on non-controllability issues under additional “bounded amplification” assumptions, which essentially mean that the size of the desired solutions is limited relative to the size of the target datum; this can be witnessed through the “amplification factor”  $L$  present in the main statements. The difference with the original question is highlighted in the sequel (see Remark 1.1).

### 1.1. Overview of the results of the paper

The scope of the present article is threefold as it contributes to resolve the following issues:

- (A) We bring a partial negative answer to (Q) (which counterbalances the partial positive answers given in [18, 20, 26], see § 1.2 below) in the  $L^\infty$  setting instead of the original  $L^2$  setting and under an amplification assumption quantified by  $L \geq 1$ .

More precisely, we show that for any given  $L \geq 1$  there exist many pairs  $(C, T)$ , with roughly speaking  $|C|T \leq 1$ , such that the system has no solution  $u \in L^\infty((0, 1) \times (0, T))$  satisfying the bound  $\|u\|_\infty \leq L|C|$ . Actually we point out several families of weakly- $*$  compact sets of states  $u_T \in L^\infty(\Omega)$  not attainable at time  $T$ , starting from zero data, by solutions of (BE), under the *a priori* “amplification assumption”  $\|u\|_\infty \leq L\|u_T\|_\infty$ . This happens for small values of  $T$  and somewhat large (but smaller than  $T^{-1}$ ) values of  $C$ . The details can be found in Section 3 (Corollary 3.5 and more generally, Theorem 3.3). Refinements concerning the non-sharpness of the restriction  $CT \leq 1$  and the case of the strip domain are given in Section 4.

- (B) In order to extend this negative answer – with the ideas developed in this paper and under the adequate amplification assumption – to question (Q) in its original  $L^2$  setting, we need to rely upon a uniqueness theory for *unbounded entropy solutions*. Motivated by the one-dimensional Burgers equation with  $L^2$  data in the strip, we take this opportunity to develop a rather elementary theory of unbounded  $L^\infty(0, T; L^p(\mathbb{R}^d))$  entropy solutions of multi-dimensional scalar conservation laws, under the critical polynomial growth assumption on the flux.

This should be seen as a complement to the well-established pure  $L^1$  theories (kinetic solutions, cf. [42] and the references therein; renormalized solutions, see [13]) and to the deep recent work [45] on  $L^p - L^q$ ,  $p < q \leq \infty$ , regularizing estimates for multi-dimensional Burgers equations. Note that in our setting of unbounded entropy solutions, the subcritical case has already been developed (cf. [46], see also [12]). Also note that although we do not rely on the more sophisticated renormalized solutions theory [13] we follow very similar guidelines in what concerns the existence issue (cf. [4]).

While we are able to develop a complete existence and uniqueness theory for the Cauchy problem with  $L^p$  data (in particular,  $L^2(\mathbb{R})$  data for the Burgers case), the boundary-value

analogue corresponding, for example, to  $D = (0, 1) \times (0, T)$  is more delicate. In the Cauchy-Dirichlet setting, we focus on the basic case of the inviscid Burgers equation to develop existence results for, roughly speaking,  $L^3$  boundary data. We provide a uniqueness result based upon the strong trace properties for “quasi-solutions” ([37]). Boundary data are included in a way which is well-suited for  $L^2$  solutions and compatible with the Bardos-LeRoux-Nédélec interpretation of Dirichlet conditions. The resulting  $L^2$  theory is less general than the  $L^1$  theory of renormalized solutions [3, 43], but it has the advantage of relative technical simplicity.

Finally, for the Burgers case (and more generally, for the strictly convex one-dimensional flux) we give an elementary proof of the  $L_{loc}^\infty$  regularization effect put forward in [45].

We refer the readers interested in these secondary subjects of the paper to Section 5.

- (C) Slightly adapting the techniques of (A) and applying the results of (B), in the setting of  $D = \mathbb{R} \times (0, T)$ , we assess non-controllability of large constant states in  $(0, 1)$  (i.e.,  $u(\cdot, T)|_{(0,1)} = C$  with  $|C| > C_0(L)$ ) for times  $T < 1/|C|$  (or a bit longer times) by solutions of the viscous Burgers equation with  $L^2(\mathbb{R})$  initial data, with  $\|u_0\|_{L^2} \leq L|C|$  taken as the bounded amplification assumption on the initial data. This constitutes a partial, and a quantitative ( $L$ -dependent) negative answer to the question closer to the original setting (Q).

Finally, we also address the setting  $D = (0, 1) \times (0, T)$  where we extend the result of (A) to  $L^2((0, 1) \times (0, T)) \cap L_{loc}^3((0, 1) \times (0, T))$  solutions (see Corollary 6.8), under the  $L^2$  amplification assumption  $\|u\|_{L^2((0,1) \times (0,T))} \leq LT|C|$  on the solutions and an  $L^2 - L_{loc}^3$  regularization assumption. Because of this additional  $L_{loc}^3$  regularity restriction, we do not reach the pure  $L^2((0, 1) \times (0, T))$  setting of (Q), but we get particularly close to it. Indeed, the  $L^2 - L_{loc}^\infty$  regularization can be expected for sufficiently robust notions of solution for the viscous Burgers equation, see Remark 6.4. Details can be found in Section 6.

**Remark 1.1.** *Following the arguments developed in the paper, one can see that they apply as well to the classical heat equation replacing the viscous Burgers equation. Indeed, the key scaling observation, the different bounds on solutions, and the underlying non-attainability results of the inviscid case carry on to this linear setting. Here, one can clearly see that the question we answer negatively is different from the mere question of controllability. Indeed, it is classical that the heat equation on an interval is null controllable at any time by boundary controls starting from any initial datum (cf. [24]), which by the linearity means that all constant states are controllable starting from zero initial datum. It is clear that the “cost” of the controls, which can be quantified by amplification factor  $L$  as in our assumptions, increases as the desired control time decreases. This is why we should interpret the results obtained in this paper as a quantitative version (with limited “costs”) of the original Coron’s question (Q). The issue of (non)-controllability of constant states  $C$  at arbitrarily small times  $T$  for an unbounded cost remains open; the example of the heat equation shows that our method is not suitable for answering negatively this qualitative question.*

## 1.2. The state of the art on controllability of the viscous Burgers equation

Several positive results on exact controllability of constant states for the viscous Burgers equation exist in the literature. One such result is the following:

**Theorem 1.1.** (See [18, 20]) Let  $T > 0$ . There exists  $N = N(T) > 0$  such that for every  $|C| \geq N$ , there exists  $u \in L^2((0, 1) \times (0, T))$  satisfying (BE) and such that  $u(\cdot, 0) = 0$ ,  $u(\cdot, T) = C$  for  $x \in (0, 1)$ .

Another related result in the space  $L^\infty((0, 1) \times (0, T))$  can be found in Glass and Guerrero [26] where the authors consider boundary controls for the viscous Burgers equation with small dissipation. They prove that any nonzero constant state  $C$  can be reached after sufficiently large time. As an immediate consequence of [26], one has the following theorem:

**Theorem 1.2.** (See [26]) There exist  $N > 0$  and  $\beta \geq 1$  such that for every  $|C| > N$ , there exists  $u \in L^\infty((0, 1) \times (0, T))$  satisfying (BE) and such that  $u(\cdot, 0) = 0$ ,  $u(\cdot, T) = C$  for  $x \in (0, 1)$  and all  $T > \beta/|C|$ .

In particular, Glass and Guerrero [26] showed that large constant states can be reached in large time by two boundary controls for viscous Burgers equation with small viscosity coefficient. Later Leautaud in [32] extended this result to scalar viscous conservation laws with more general fluxes. Null-controllability (Marbach, [33]) and small time local controllability (Fursikov and Imanuvilov, [25]) have been achieved with source and one boundary control. It is worth mentioning the result of Guerrero and Imanuvilov [27] where the authors deal with (BE) and two boundary controls and show that exact null controllability indeed fails for small time. Also, they prove a negative result to null exact controllability even for large time. In [23] Fernández-Cara and Guerrero have given an estimate of the time of null controllability depending on the  $L^2$ -norm of the initial data.

On the other hand, the problem has also been investigated under one control and we refer the reader to [25, 23] and references therein.

Regarding the exact controllability for the inviscid Burgers equation (more generally convex conservation laws or even to some particular hyperbolic systems of conservation laws), one can use tools, such as backward characteristics, in order to construct suitable initial and boundary controls. The theory is nevertheless very delicate due to the occurrence of shocks. For more details, we refer the reader to [1, 2, 5, 8, 29, 41].

In Theorems 1.1 and 1.2 stated above, there is clearly a gap in the range of pairs  $(C, T)$  for which the question has not been resolved. Specifically, the range  $T < 1/|C|$  is not covered by these results and we stress the fact that at the level of the inviscid Burgers equation, such states cannot be controlled; see [1] and Propositions 3.1 and 4.2 below. This has been the motivation for the present work in which we provide a partial negative result to question (Q) precisely for such pairs  $(C, T)$ , along with some generalizations directly coming from the techniques we employ.

### 1.3. Outline of the paper, key ideas and techniques

In this paper, we interpret question (Q) as an initial-value problem on  $D = \mathbb{R} \times (0, T)$  or as an underdetermined initial-value problem on  $D = (0, 1) \times (0, T)$  for solutions understood in the appropriate weak sense (see Section 2 for definitions) and under adequate limitations on the size of the solution relative to the size of the target state. We give a series of negative answers for couples  $(C, T)$  satisfying  $T \leq 1/|C|$  (and sometimes  $T < (1 + \Delta)/|C|$  with some  $\Delta > 0$ ), for  $C$  sufficiently large. More generally, such results concern sufficiently large data and the accordingly small times  $(u_T(\cdot), T) \in BV((0, 1)) \times (0, +\infty)$  satisfying properties of the type (NA) (see Proposition 3.1); precise statements are given in Sections 3 and 6. Our method relies on a scaling argument which reduces (BE) to the viscous Burgers equation

$$u_t^\epsilon + \left( \frac{(u^\epsilon)^2}{2} \right)_x = \epsilon u_{xx}^\epsilon,$$

while leaving invariant the product  $Tu_T(\cdot)$  for states  $u_T(\cdot)$  attainable at time  $T$ .

Note that the scaling argument is restricted to the quadratic non-linearity. It is appropriate e.g. to the Navier-Stokes equation and the corresponding inviscid (Euler) equations (see, e.g., [19] for examples of control problems), which are far beyond the scope of this paper. However, the extension to the  $L^2$  setting (Section 6) of the method first developed for  $L^\infty$  solutions (Sections 3,4) is motivated in particular by the fact that the  $L^\infty$  setting, most natural for scalar problems, is not natural for systems.

The conclusion on non-controllability for the viscous Burgers equation follows, upon a careful use of the scaling (Zoom) of solutions of (BE) with  $\epsilon = T$  (see Section 3), i.e. for small times  $T$ , from uniform in  $\epsilon$  bounds ensuring compactness of sequences of solutions  $(u_\epsilon)$ , and from rather elementary non-controllability results for the inviscid Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

The latter is understood in the standard framework of Kruzhkov entropy solutions (Sections 3 and 4) or in the framework of unbounded entropy solutions (Section 6) constructed on purpose in Section 5. An *a priori* bound on solutions is required in our argument which we interpret as an amplification assumption limiting the size of the solutions in terms of the size of the target data. The core arguments are given in Section 3, in the simplest setting; Sections 4 and 6 present more technical extensions of the results, achieved with the same strategy of proof.

We would like to emphasise that, unlike a large number of works on the controllability of the viscous Burgers equation which invoke the heat equation via the Hopf-Cole transformation, our work is not only motivated by, but also based on the inviscid equation through a vanishing viscosity argument. We remark that for the inviscid equation, the most classical solution space is  $L^\infty$  and our main result is posed in this space. We are able to extend the arguments to  $L^2$  solutions by refining the underlying solution concept slightly beyond the classical Kruzhkov setting.

Our investigation of unbounded solutions of critical integrability for the Burgers equation is of some interest on its own right. It is closely related to the theory of renormalized solutions (and to the one of kinetic solutions), the latter ones being more general but also technically and conceptually more involved. Section 5 is devoted to the well-posedness analysis of the Cauchy problem for unbounded entropy solutions to scalar conservation laws far more general than the Burgers equation. The Cauchy-Dirichlet problem in  $L^2((0, 1) \times (0, T))$  for the Burgers flux is also addressed in the context of the *ad hoc* boundary formulation well-adapted to the  $L^2$  integrability of the data. In many aspects of the theory, we heavily rely upon fine regularity properties of the solution and compactness tools put forward by Panov [37, 38, 39], based upon the techniques of parametrized  $H$ -measures as introduced in [35]. We also exploit some of the techniques employed in the past 30 years in the study of renormalized solutions to elliptic and parabolic PDEs (cf. [4, 15, 34]). While the latter were successfully extended to the hyperbolic framework (see in particular [13, 3, 43]), here we avoid the use of truncations found in the renormalization approach, due to the sharp integrability assumption on the flux.

## 2. Precise setting for question (Q)

In addition to the aforementioned interpretations of the original question (Q) raised in [18] (the quantitative “bounded amplification” assumptions, the choice of  $L^\infty$  or  $L^2$  functional framework, the choice  $(0, 1)$  or  $\mathbb{R}$  for the space domain), we also need to make explicit the underlying notion of solution to the viscous Burgers equation. Let us detail the framework(s) we explore.

We discuss the two following situations:

$$\begin{cases} u_t + (u^2/2)_x = u_{xx} & \text{in } (0, 1) \times (0, T), \\ u(\cdot, 0) = 0 \text{ and } u(\cdot, T) = u_T & \text{for } x \in (0, 1), \end{cases} \quad (\mathbf{Pb}_{box})$$

and

$$\begin{cases} u_t + (u^2/2)_x = u_{xx} & \text{in } \mathbb{R} \times (0, T), \\ u(\cdot, 0) = u_0 \text{ with } u_0 = 0 & \text{for } x \in (0, 1), \\ \text{and } u(\cdot, T) = u_T & \text{for } x \in (0, 1). \end{cases} \quad (\mathbf{Pb}_{strip})$$

If one puts aside the assigned terminal conditions for a moment, we recognize in  $(\mathbf{Pb}_{strip})$  a standard Cauchy problem. So the question is a particular instance of control by the initial data (which we will instead refer to as *attainability* in the sequel of the paper). Similarly, one possible interpretation of  $(\mathbf{Pb}_{box})$  would be in terms of boundary control in the Cauchy-Dirichlet (or even Cauchy-Neumann) setting; however, we prefer to consider  $(\mathbf{Pb}_{box})$  as an underdetermined problem with solutions defined locally in  $(0, 1) \times [0, T]$  (attention is paid to the initial and terminal times  $t = 0, T$  but not to the boundaries  $x = 0, 1$ ). Indeed, prescribing boundary traces of the solution  $u$  or of the convection-diffusion flux  $u^2/2 - u_x$  at  $x = 0, 1$  would restrict the generality of problem  $(\mathbf{Pb}_{box})$ .

It is obvious that a solution  $u$  to the problem  $(\mathbf{Pb}_{strip})$  in the strip can be seen as well as a solution to  $(\mathbf{Pb}_{box})$ : it is enough to consider its restriction  $u|_{(0,1) \times (0,T)}$  to the box. Therefore it is more difficult to attain a given state  $u_T$  in the strip setting  $(\mathbf{Pb}_{strip})$  than in the box setting  $(\mathbf{Pb}_{box})$ . Because our focus in this paper is on non-attainability (i.e. on the impossibility to reach the desired states at desired times), we see  $(\mathbf{Q})$  in the strip setting  $(\mathbf{Pb}_{strip})$  as a simpler question than the same question in its box setting  $(\mathbf{Pb}_{box})$ .

Next, although question  $(\mathbf{Q})$  is originally about  $L^2$  solutions of  $(\mathbf{BE})$ , our techniques primarily drive us to replace  $L^2$  by  $L^\infty$  (see Section 3). In order to get closer to the original  $L^2$  setting we need to rely upon a theory of unbounded (more precisely,  $L^2$ ) entropy solutions to the inviscid Burgers equation. Insufficiency of the  $L^2$  uniqueness theory for the Cauchy-Dirichlet problem pushes us to consider also the  $L^3((0, 1) \times (0, T))$  solutions of the viscous Burgers equation.

In our non-attainability results, we will not merely ask that the solutions belong to some  $L^p$  spaces, but also that they obey some uniform bounds that we state in terms of “amplification”. The amplification constants are denoted by  $L$  throughout the paper; their role is to control the size of the solution in terms of the size of the target data  $u_T$ .

Finally, the precise meaning of what a “solution” of  $(\mathbf{BE})$  is in our paper is different from the one found in [18], where solutions are meant in the sense of distributions (usually called “very weak” solutions). Our approach requires that the solution satisfy a local  $L^2$  in time,  $H^1$  in space, energy estimate which means that it should be a weak solution (sometimes called variational solution or finite energy solution) locally in  $D$ ; moreover, the entropy inequalities of parabolic conservation laws are required. Because for non-degenerate parabolic conservation laws weak formulation implies the entropy formulation (see e.g. [17]), in the sequel we use the term *weak solution of the viscous Burgers equation* (supplemented with initial and terminal data) meaning the following.

**Definition 2.1** (Adopted notion of solution for the viscous Burgers equation).

Let  $D = I \times (0, T)$  with  $I = (0, 1)$  or  $I = \mathbb{R}$ . Let  $u_0$  and  $u_T$  belong to  $L^2_{loc}(I)$ . A function  $u \in L^2(D)$  is called a weak solution of  $(\mathbf{BE})$  with initial data  $u_0$  and terminal data  $u_T$  if  $u \in L^2(0, T; H^1_{loc}(I))$



and for all  $\xi \in C_c^\infty(I \times [0, T])$ , there holds

$$\int_0^T \int_0^1 \left( u \xi_t + \frac{u^2}{2} \xi_x - u_x \xi_x \right) dx dt + \int_{\mathbb{R}} u_0(x) \xi(x, 0) dx - \int_{\mathbb{R}} u_T(x) \xi(x, T) dx = 0 \quad (2.1)$$

and, furthermore, for all  $\xi \in C_c^\infty(I \times [0, T])$ ,  $\xi \geq 0$ , for all  $k \in \mathbb{R}$  there holds

$$- \int_0^T \int_0^1 \left( |u - k| \xi_t + |u - k| \frac{u + k}{2} \xi_x - |u - k|_x \xi_x \right) dx dt - \int_{\mathbb{R}} |u_0 - k| \xi(x, 0) dx \leq 0. \quad (2.2)$$

In particular, solutions to  $(\mathbf{Pb}_{\text{box}})$  or  $(\mathbf{Pb}_{\text{strip}})$  are understood in the sense of Definition 2.1 in the sequel; they will be supplemented with additional bounds in  $L^p(D)$  for different choices of  $p$ .

Let us stress that the more usual, in the context of such definitions,  $L^\infty$  assumption on  $u$  is not needed for the above definition to make sense, indeed, under the  $L_{loc}^2$  assumptions on  $u, u_x$  all terms in (2.1),(2.2) are well defined.

**Remark 2.2.** The  $L_{loc}^2$  regularity of  $u_x$  assumed in Definition 2.1 implies in particular that the solution satisfies variants of classical chain rules in space (following from the Sobolev regularity of  $u$  in space), like  $|u - k|_x = \text{sign}(u - k)u_x$ , and chain rules in time (see e.g. [4, Lemma 2.3]); these chain rules are necessary technical ingredients of the entropy formulation (2.2). In particular, local  $L^1$  estimates on the term  $|u_x|^2$  (we refer to the proof of Lemma 3.2), obtained by formally multiplying the equation by  $u$ , are justified using chain rules.

In this respect, let us recall that classical solutions to the Burgers equation are related to classical solutions of the heat equation by the Hopf-Cole transformation (see, e.g., [22]) which is a nonlinear change of the unknown; the equivalence relies on chain rules for derivatives. While considering very weak solutions to  $(\mathbf{BE})$  as suggested in [18], we do not have any kind of chain rule at our disposal; thus not only the classical regularity cannot be derived from the formal link with the heat equation, but also the entropy formulation cannot be guaranteed. For this reason, we cannot rely upon the notion of merely distributional (very weak) solutions to  $(\mathbf{BE})$ .

As a matter of fact, we could go beyond the weak  $L_{loc}^2$  setting and even the very weak setting, by considering  $L^1$  data and the appropriate notions of solution developed in the literature.

**Remark 2.3.** Recall that the  $L^1$  setting is sharp for inviscid conservation laws provided the solutions are interpreted either in abstract semigroup terms (like in [6]) or in the renormalized setting (like in [13, 43]) or else, in the setting of the kinetic formulation (see [42] and references therein). Because all these solutions can be seen as pointwise limits of Kruzhkov entropy solutions for truncated data (like the unbounded entropy solutions we construct in Section 5), the results of Section 6 can be extended to the  $L^1$  setting as well. Let us remark that, for example, the notion of kinetic solution can be applied in parallel to the viscous and to the inviscid Burgers equations. In general, these solutions are not even solutions in the sense of distributions (very weak solutions) because  $u^2/2$  may fall out of  $L_{loc}^1$ , or at least their  $L_{loc}^1$  regularity is far from being straightforward ([45]). This line of investigation would provide yet another functional and solution framework for interpretation of question  $(\mathbf{Q})$ .

Making precise the notion of solution strongly impacts the results we can prove concerning question  $(\mathbf{Q})$ ; we refer in particular to the final discussion of Remark 6.4.

### 3. Sets of terminal data non-attainable by bounded weak solutions

This section is devoted to partial (negative) answers to question  $(\mathbf{Q})$  in the setting where we assume that the solutions  $u$  are bounded and moreover, the ratio of the  $L^\infty$  norm of  $u$  and the  $L^\infty$

norm of the target data  $u_T$  is controlled by a constant  $L$  given beforehand. Clearly, only  $L \geq 1$  makes sense. For general  $u_T$ , asking  $L = 1$  means roughly speaking that we look for solutions  $u$  with the same amplitude as  $u_T$ ; while letting  $L > 1$  allows for a controlled amplification. For this reason, in the sequel we call  $L$  the *amplification factor*.

Our argument essentially relies upon the scaling

$$\begin{aligned} (t, v) &\mapsto \left(\frac{t}{\epsilon}, \mathcal{S}_\epsilon v\right), \\ \mathcal{S}_\epsilon v &:= \epsilon v(\cdot, \epsilon \cdot), \quad \text{i.e. } t = \epsilon \tau, \mathcal{S}_\epsilon(v)(x, \tau) = \epsilon v(x, \epsilon \tau), \end{aligned} \tag{Zoom}$$

where  $v$  is a function of  $(x, t) \in (0, 1) \times (0, T)$ ; this scaling permits to link (BE) to the viscous Burgers equation with viscosity parameter  $\epsilon > 0$ . The inviscid Burgers equation, under the standard notion of admissibility of solutions, can be seen as the singular limit of the latter as  $\epsilon \rightarrow 0$ . Also note that the inviscid Burgers equation is invariant under the scaling (Zoom).

**Remark 3.1.** *Let us point out that a study analogous to the one we conduct in this section and in Section 6 can be conducted for the problem*

$$u_t + |u|^{p-1}u_x = \left(|u_x|^{p-2}u_x\right)_x$$

for  $p \in (1, \infty)$ ; it possesses a scaling invariance which generalizes (Zoom).

Note that it is well known that weak (energy) solutions of scalar conservation laws regularized with  $p$ -laplacian viscosity  $\epsilon(|u_x|^{p-2}u_x)_x$  converge to entropy solutions of the corresponding inviscid problem. Also note that the theory of the Cauchy problem developed in Section 5 applies to the flux  $F(u) = \text{sign}(u)|u|^p/p$  and  $L^p$  initial data.

We start by constructing a wide family of non-attainable (from initial data  $u_0$  verifying  $u_0 = 0$  in  $(0, 1)$ ) at time  $T = 1$  states in the classical setting of Kruzhkov entropy solutions to the inviscid Burgers equation. The scaling (Zoom), along with the classical vanishing viscosity characterization of the admissible solutions to the inviscid Burgers equation, will permit to transfer the non-attainability result to our target problem (BE). In order to do so, we will need to restrict our attention to  $L^\infty$  solutions of the latter (we relax this restriction in Section 6).

### 3.1. Non-attainable states for the inviscid Burgers equation in the classical entropy solutions setting

The initial value problem addressed in question (Q) is underdetermined (its formulation does not implicitly include boundary data); therefore we first make precise what we mean by solution of the analogous underdetermined inviscid Burgers problem.

**Definition 3.2.** *A function  $u \in L^\infty((0, 1) \times (0, T))$  is a local Kruzhkov entropy solution of the underdetermined problem*

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } D = (0, 1) \times (0, T), \\ u(x, 0) = 0 & \text{on } (0, 1), \end{cases} \tag{Pb}_{box}^0$$

if for all  $k \in \mathbb{R}$ , for all  $\xi \in C_c^\infty((0, 1) \times [0, T])$ ,  $\xi \geq 0$  there holds

$$-\int_0^T \int_0^1 |u - k| \xi_t + |u - k| \frac{u + k}{2} \xi_x \, dx \, dt - \int_0^1 |k| \xi(x, 0) \, dx \leq 0. \tag{3.1}$$

Moreover,  $u_T \in L^\infty((0, 1))$  is the terminal state of a local Kruzhkov entropy solution  $u$  if for all  $\xi \in C_c^\infty((0, 1) \times [0, T])$

$$\int_0^T \int_0^1 u \xi_t + \frac{u^2}{2} \xi_x \, dx \, dt - \int_0^1 u_T(x) \xi(x, T) \, dx = 0.$$

Having in mind a variant of question **(Q)**, we are also interested in the Cauchy problem set on the whole real line:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0, T), \\ u(x, 0) = u_0 \text{ with } u_0(x) = 0 \text{ for } x \in (0, 1). \end{cases} \quad (\mathbf{Pb}_{strip}^0)$$

An analogous definition with  $u_0 \in L^\infty(\mathbb{R})$  is classical [31]; we refer to the corresponding solutions as (global) Kruzhkov entropy solutions.

Further, Kruzhkov entropy solutions can be restricted and they can be glued together:

**Remark 3.3.** *Similarly to Definition 3.2, one defines local Kruzhkov entropy solutions on any open domain  $D \subset \mathbb{R} \times (0, T)$  by localizing the support of the test functions to  $D \cup \left(\overline{D} \cap (\mathbb{R} \times \{0\})\right)$ . Non-zero initial data are easily included (cf. (5.3) in Section 5).*

*It is obvious that a restriction of a local Kruzhkov entropy solution on an open subdomain  $\tilde{D}$  of  $D = (0, 1) \times (0, T)$  is a local Kruzhkov entropy solution in  $\tilde{D}$ . Further, it is easily checked that gluing continuously local Kruzhkov entropy solutions in domains  $\tilde{D}, \hat{D}$  separated by a Lipschitz curve  $\Gamma$  (by continuity we mean coincidence of strong traces from the right and from the left of  $\Gamma$ ), we obtain a Kruzhkov entropy solution in  $\tilde{D} \cup \Gamma \cup \hat{D}$ .*

Note that a terminal state exists for every local Kruzhkov entropy solution; further, every such solution can be seen as the solution of the initial-boundary value problem with appropriately chosen boundary data. In the following Remark, we give precise sense to the initial datum and to the Dirichlet boundary data denoted by  $b_0$  (the Dirichlet datum at  $x = 0^-$ ) and  $b_1$  (the Dirichlet datum at  $x = 1^-$ .) More precisely, we have

**Remark 3.4.** *Local entropy solutions of  $(\mathbf{Pb}_{box}^0)$  possess the following properties:*

(i)  *$u \in C([0, T]; L^1((0, 1)))$ , and in particular, the initial data  $u_0 = 0$  and the terminal data  $u(\cdot, T) = u_T$  can be understood as traces of  $u$ , in the strong  $L^1$  sense, on  $(0, 1) \times \{0\}$  and on  $(0, 1) \times \{T\}$ , respectively (see [36, 16]).*

(ii) *There exist traces (in the strong  $L^1$  sense)  $b_0(\cdot) = u(0^+, \cdot)$  and  $b_1(\cdot) = u(1^-, \cdot)$ ,  $b_0, b_1 \in L^\infty((0, T))$  (see [47, 36]), which can also be seen as the boundary data for the Cauchy-Dirichlet problem understood in the BLN sense (see Bardos, LeRoux and Nédélec [10], see also [9]). One can see  $u$  as the unique solution in the BLN sense corresponding to the initial data  $u_0 = 0$  and boundary data  $u(\cdot, 0^+), u(\cdot, 1^-)$ .*

**Remark 3.5.** *The feature that will be instrumental in Section 5 is the  $L^1$  comparison and contraction property valid (see e.g. [31, 44, 22]) for any two Kruzhkov entropy solutions  $u, \hat{u}$  corresponding to the  $L^\infty(\mathbb{R})$  initial data  $u_0, \hat{u}_0$ , respectively:*

$$\|(u - \hat{u})^\pm\|_{L^\infty(0, T; L^1)} \leq \|(u_0 - \hat{u}_0)^\pm\|_{L^1}, \quad (3.2)$$

where  $z^\pm := \max\{0, \pm z\}$ . Property (3.2) makes sense whenever the right-hand side is finite; it implies in particular that smaller initial data ( $u_0 \leq \hat{u}_0$  on  $\mathbb{R}$ ) give rise to smaller solutions ( $u \leq \hat{u}$  in  $\mathbb{R} \times (0, T)$ ). Note that the comparison principle under the form

$$u_0 \leq \hat{u}_0 \text{ on } (0, 1), b_0 \leq \hat{b}_0, b_1 \leq \hat{b}_1 \text{ on } (0, T) \implies u \leq \hat{u} \text{ in } (0, 1) \times (0, T)$$

is known also for Cauchy-Dirichlet problems (see, e.g., [9]), here in addition to the initial data, boundary data  $b_0, b_1$  (respectively,  $\hat{b}_0, \hat{b}_1$ ) for  $u$  (resp., for  $\hat{u}$ ) are prescribed.

With the above preliminaries at hand, let us introduce some convenient notation. For  $T > 0$ ,

denote by

$$\mathcal{NA}_T^{L^\infty, box} := \left\{ u_T \in L^\infty(0, 1) \mid \begin{array}{l} \#u \text{ solution in the sense of Definition 3.2} \\ \text{to problem } (\mathbf{Pb}_{box}^0) \text{ with } u(\cdot, T) = u_T \end{array} \right\} \quad (3.3)$$

the set of states non-attainable at time  $T$  by local Kruzhkov entropy solutions of the *inviscid* Burgers equation in  $(0, 1) \times (0, T)$  with zero initial data. Consider the scaling (**Zoom**) of the solution of  $(\mathbf{Pb}_{box}^0)$  where we take  $\epsilon = T$  so that the scaled equation is posed in the time interval  $\tau \in (0, 1)$ . It is readily checked that the notion of local Kruzhkov entropy solution is invariant under this scaling. For this reason, we have

$$\mathcal{NA}_T^{L^\infty, box} = \left\{ u_T \in L^\infty(0, 1) \mid Tu_T \in \mathcal{NA}_1^{L^\infty, box} \right\} = T^{-1} \mathcal{NA}_1^{L^\infty, box}, \quad (3.4)$$

i.e. we can fix  $T = 1$  in our study of states non-attainable for the inviscid Burgers equation.

**Remark 3.6.** *It is classical that for  $(\mathbf{Pb}_{box}^0)$ , states that do not belong to  $BV_{loc}((0, 1))$  are not attainable at any time. In the sequel, and having in mind the constant terminal states of question **(Q)**, we will not focus on the BV regularity restrictions but the reader may always suppose that  $u_T$  is at least  $BV_{loc}$  regular.*

Non-attainability for the inviscid equation is naturally studied using the insight from the theory of maximal backward characteristics ([21, 22]), see e.g. [5]. In particular, we have the following key observation.

**Proposition 3.1.** *Let  $u_1 \in BV_{loc}((0, 1))$  verifying*

$$\begin{array}{l} \exists x^* \in [0, 1] \text{ such that} \\ \text{either } 0 < \bar{u}_1(x^*) \leq x^* \text{ ,} \\ \text{or } -(1 - x^*) \leq \underline{u}_1(x^*) < 0 \text{ ,} \end{array} \quad (\mathbf{NA})$$

where  $\underline{u}_1(\cdot)$  (respectively,  $\bar{u}_1(\cdot)$ ) stands for the left-continuous in  $x$  (respectively, right-continuous in  $x$ ) representative of the BV function  $x \mapsto u_1(x)$ . Then there exists no local Kruzhkov entropy solution verifying  $(\mathbf{Pb}_{box}^0)$  and the terminal datum  $u(\cdot, 1) = u_1$ . In other words,  $u_1 \in \mathcal{NA}_1^{L^\infty, box}$  and for all  $T > 0$ ,  $T^{-1}u_1 \in \mathcal{NA}_T^{L^\infty, box}$ . In particular, for all couples  $(C, T) \in (0, +\infty) \times \mathbb{R}_+$  verifying  $|C|T \leq 1$ , there holds  $C \in \mathcal{NA}_T^{L^\infty, box}$ .

*Proof.* We argue by contradiction; let  $u$  be a solution of  $(\mathbf{Pb}_{box}^0)$  corresponding to the terminal data  $u_1$ . We can assume without loss of generality that there exists  $x^* \in (0, 1]$  such that  $u_1(x^*) \leq x^*$ , where  $u_1$  is normalized by the right-continuity in the variable  $x$ ; the case where  $u_1(x^*) \geq -(1 - x^*)$  and the normalization is by the left-continuity is fully analogous.

Let  $u^* = u_1(x^*)$ ; we set  $x_* = x^* - u^* \in [0, 1)$ . We draw from the point  $(1, x^*)$  the maximal backward generalized characteristic ([21, 22]); it crosses the axis  $t = 0$  at the point  $x_*$ , see Figure 1. It follows from the theory of generalized characteristics that  $u(x_* + tu^*, t) = u(x^*, 1) = u^*$  for all  $t \in [0, 1]$ , where we recall that  $u$  is normalized to be right-continuous.

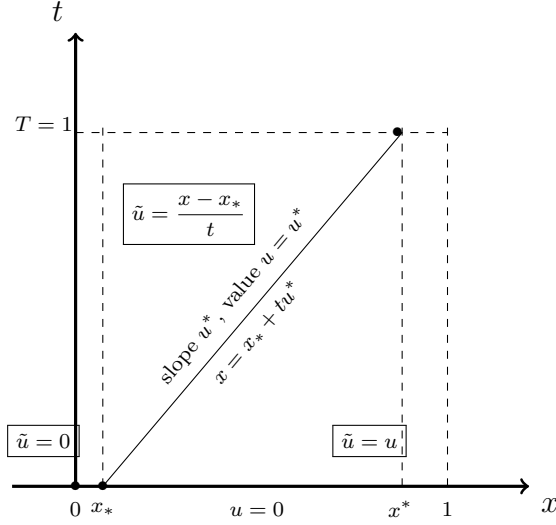


Figure 1: The construction of  $\tilde{u}$

Since  $u(x, 0) = 0$  for  $x \in (0, 1)$ , we reach a contradiction whenever  $x_* > 0$ , which corresponds to the strict inequality  $u^* < x^*$ . In order to include the special case  $x_* = 0$ , and also in order to prepare the ground for different extensions of Proposition 3.1 (see Proposition 4.4 and the results of Section 6), we construct an auxiliary local entropy solution  $\tilde{u}$  of the Burgers equation as follows. We set for  $(x, t) \in [0, 1] \times (0, 1]$

$$\tilde{u}(x, t) := \begin{cases} u(x, t), & x \geq x_* + tu^*, \\ (x - x_*)/t, & x_* \leq x \leq x_* + tu^*, \\ 0, & x \leq x_*. \end{cases} \quad (3.5)$$

In particular,  $\tilde{u}$  is continuous across the lines  $x = x_*$  and  $x = x_* + tu^*$ . Because we glued continuously three patches and each of them is a Kruzhkov entropy solution in the corresponding subdomain (a constant, a rarefaction and our solution  $u$ , from the left to the right), according to Remark 3.3 we find that  $\tilde{u}$  is a local Kruzhkov entropy solution on  $D = (0, 1) \times (0, 1)$ . Moreover,  $\tilde{u}$  assumes zero initial data and zero boundary data on the left boundary (cf. Remark 3.4).

The finite speed of propagation (recall that  $\tilde{u} \in L^\infty$ ) ensures that  $\tilde{u}$  should be zero in some vicinity of the point  $(x_*, 0)$ , which is contradictory because for all  $t \in (0, 1)$  we have  $u(x_* + tu^*, t) = u^* > 0$ . This contradiction proves the non-existence of  $u$  and the non-attainability of  $u_1$  at time  $T = 1$ . The remaining claims follow from the fact that  $u_1 = CT$  satisfies (NA) when  $|C|T \leq 1$ , and from the scaling observation (3.4).  $\square$

To formulate in an optimal way our non-attainability results for the viscous Burgers equations, we will be interested in compact subsets of  $\mathcal{NA}_1^{L^\infty, box}$  with respect to the weak-\* topology of  $L^\infty((0, 1))$ . Below are the main examples we consider.

**Remark 3.7.** *The following subsets of  $\mathcal{NA}_1^{L^\infty, box}$  are weakly-\* compact in  $L^\infty$ :*

(i)  $\mathcal{K}_{\alpha, \beta} := \left\{ u : x \mapsto C \mid \alpha \leq |C| \leq \beta \right\}$ , for any given  $\alpha, \beta$  with  $0 < \alpha \leq \beta \leq 1$ ;

(ii)  $\mathcal{K}_{E, m(\cdot)}^+ := \left\{ u \in L^\infty((0, 1)) \mid \forall x \in E \ m(x) \leq u(x) \leq x \right\}$ ,  
for a given  $E \subset (0, 1)$  of non-zero Lebesgue measure and a given measurable  $m : E \rightarrow (0, 1]$ ;

(iii)  $\mathcal{K}_{E,m(\cdot)}^- := \left\{ u \in L^\infty((0,1)) \mid \forall x \in E - (1-x) \leq u(x) \leq -m(x) \right\}$ , for  $(E, m(\cdot))$  like in (ii).

In this remark, the fact that  $\mathcal{K}_{\alpha,\beta}, \mathcal{K}_{E,m(\cdot)}^\pm \subset \mathcal{NA}_1^{L^\infty, box}$  follows from Proposition 3.1. Their weak-\* precompactness follows from their boundedness; moreover, it is easily seen that they are weakly-\* closed. For example, condition  $m(x) \leq u(x)$  for a.e.  $x \in E$  can be rewritten as

$$\text{for all measurable subsets } F \text{ of } E, \quad \int_F m(x) dx \leq \int_F u(x) dx$$

which is stable with respect to the weak-\* convergence in  $L^\infty$  because the indicator function  $\mathbb{1}_F$  of  $F$  belongs to  $L^1((0,1))$ . Therefore  $\mathcal{K}_{\alpha,\beta}, \mathcal{K}_{E,m(\cdot)}^\pm$  are indeed weakly-\* compact in  $L^\infty((0,1))$ .

**Remark 3.8.** *It turns out that the sets  $\mathring{\mathcal{K}}_{\alpha,\beta}, \mathring{\mathcal{K}}_{E,m(\cdot)}^\pm$  of  $\mathcal{K}_{\alpha,\beta}, \mathcal{K}_{E,m(\cdot)}^\pm$  defined with strict inequalities “<” in place of “≤” belong not only to the set  $\mathcal{NA}_1^{L^\infty, box}$  – the set of states not attainable by classical (bounded) Kruzhkov entropy solutions – but also to the topological interior of  $\mathcal{NA}_1^{L^\infty, box}$  with respect to the  $L^1$  convergence.*

*This observation will allow us to extend the non-attainability results to unbounded ( $L^2$ ) entropy solutions of the inviscid Burgers equation, see Sections 5 and 6.*

### 3.2. The viscous Burgers equation inherits non-attainability

The key idea of our work is that, in the appropriate regime uncovered via the scaling (Zoom) and under the natural amplification assumptions compatible with this scaling, the viscous Burgers equation inherits the non-attainability of the inviscid one.

We start with the following lemma which is a consequence of Proposition 3.1 and the main technical ingredient of the proof of our main result, Theorem 3.3. The lemma relies on a standard compensated compactness argument.

**Lemma 3.2.** *Let  $\mathcal{K}$  be a subset of  $\mathcal{NA}_1^{L^\infty, box}$  compact in the weak-\* topology of  $L^\infty((0,1))$ . Let  $L \geq 1$ . Then there exists  $\epsilon_0 = \epsilon_0(\mathcal{K}, L) > 0$  such that for all  $\epsilon \in (0, \epsilon_0)$  and all  $u_1 \in \mathcal{K}$ , the small viscosity Burgers equation*

$$u_t + (u^2/2)_x = \epsilon u_{xx} \quad \text{in } D = (0,1) \times (0,1), \quad (\mathbf{Pb}_{box}^\epsilon)$$

*has no weak solution (in the sense of Definition 2.1, with  $\epsilon = 1$  replaced by  $\epsilon > 0$  in the diffusion term) with initial data  $u(\cdot, 0) = 0$  and terminal data  $u(\cdot, 1) = u_1$  within the class of functions verifying  $\|u\|_{L^\infty((0,1) \times (0,1))} \leq L \|u_1\|_{L^\infty((0,1))}$ .*

**Remark 3.9.** *Let us stress that in this and the subsequent results on the viscous Burgers equation, we work with the precise notion of weak solution from Definition 2.1, meaning in particular that  $u \in L^2(0, T; H_{loc}^1((0,1)))$ , and such solutions verify the entropy formulation proper to parabolic conservation laws (cf. [17] for the  $L^\infty$  theory and [4] for extensions that cover, in particular, the  $L_{loc}^2$  case). This entropy formulation is an essential tool in our method, due to its link with the Kruzhkov entropy solutions of the inviscid Burgers equations and to its central role in the compensated compactness argument applied below.*

*While it is obvious that classical solutions of  $(\mathbf{Pb}_{box}^\epsilon)$  are entropy solutions (and classical solutions exist in many situations like the pure Cauchy problem, due to the link between the Burgers equation and the heat equation provided by the Hopf-Cole formula, see e.g. [22]), it is not clear that merely distributional local solutions of the Burgers equation are entropy solutions.*

*Proof of Lemma 3.2.* We argue by contradiction. Assuming that the statement is false, there exists a sequence (which we do not relabel) of values  $\epsilon$  converging to zero and a sequence  $(u_1^\epsilon) \subset \mathcal{K}$  of terminal data such that problem  $(\mathbf{Pb}_{box}^\epsilon)$  has a weak solution  $u^\epsilon$  with zero initial data and the terminal

data  $u_1^\epsilon$  satisfying the desired  $L^\infty$  bound. Due to the assumption of weak-\* compactness of  $\mathcal{K}$  we can find a subsequence (still not relabelled) such that the corresponding terminal data  $u_1^\epsilon$  converge weak-\* in  $L^\infty((0, 1))$  to some  $u_1 \in \mathcal{K}$ . The associated solutions  $u^\epsilon$  fulfill  $\|u^\epsilon\|_{L^\infty((0,1)\times(0,1))} \leq \text{const}$  because  $\mathcal{K}$  is bounded and due to the amplification assumption, therefore up to a further extraction of a subsequence  $u^\epsilon$  converge weak-\* in  $L^\infty((0, 1) \times (0, 1))$  to some function  $u$ .

Using the compensated compactness technique and passing to the limit

- in the local entropy inequalities satisfied by  $u^\epsilon$
  - in the weak formulation of  $(\mathbf{Pb}_{box}^\epsilon)$  including the terminal and the initial data,
- we will show that  $u$  is a local Kruzhkov entropy solution of  $(\mathbf{Pb}_{box}^0)$  with the terminal data  $u_1$ ; this contradicts the non-attainability of  $u_1 \in \mathcal{K} \subset \mathcal{NA}_1^{L^\infty, \text{box}}$ .

First, recall that according to Definition 2.1 weak solutions of the viscous Burgers equation satisfy the associated local entropy inequalities. Moreover, the uniform  $L^\infty$  bound on  $u^\epsilon$  implies the uniform  $L^1_{loc}((0, 1) \times (0, 1))$  bound on  $\epsilon|u_x^\epsilon|^2$ . Indeed, for compact sets of the form  $K_\delta = [\delta, 1 - \delta] \times [\delta, 1 - \delta] \subset (0, 1) \times (0, 1)$  choose a test function  $\xi \in C_c^\infty((0, 1) \times (0, 1))$  such that  $\xi \equiv 1$  on  $K_\delta$  and  $0 \leq \xi \leq 1$ . Recall that arbitrary convex functions can be approximated in the locally uniform sense by linear combinations of  $Id$  and Kruzhkov entropies  $|\cdot - k|$ . Applying this approximation to the convex entropy  $\eta : u \mapsto u^2/2$ , with the associated entropy flux  $q : u \mapsto u^3/3$ , we find that the entropy formulation of the type (2.2) (written for  $\epsilon > 0$  in place of  $\epsilon = 1$ ) implies

$$\epsilon \iint_K |u_x^\epsilon|^2 dx dt \leq \epsilon \int_0^1 \int_0^1 |u_x^\epsilon|^2 \xi dx dt = \int_0^1 \int_0^1 \frac{1}{2} (u^\epsilon)^2 \xi_t + \frac{(u^\epsilon)^3}{3} \xi_x + \frac{1}{2} \epsilon (u^\epsilon)^2 \xi_{xx} dx dt. \quad (3.6)$$

But  $(u^\epsilon)$  is bounded in  $L^\infty((0, 1) \times (0, 1))$  so that  $(\epsilon|u_x^\epsilon|^2)$  is indeed bounded in  $L^1(K_\delta)$ ;  $\delta > 0$  being arbitrary, the claim of  $L^1_{loc}((0, 1) \times (0, 1))$  boundedness is justified.

With these ingredients at hand, standard application of the compensated compactness method (see e.g. [44, Sect. 9.2]) guarantees that  $(u^\epsilon)$  converges to  $u$  a.e. on  $(0, 1) \times (0, 1)$  as  $\epsilon \rightarrow 0$ . Using again the above  $L^1_{loc}((0, 1) \times (0, 1))$  bound on  $\epsilon|u_x^\epsilon|^2$  to make the diffusion term vanish in the limit  $\epsilon \rightarrow 0$ , we find that  $u$  fulfils the local entropy formulation (3.1) of  $(\mathbf{Pb}_{box}^0)$ .

As for the terminal data, for  $\epsilon > 0$ , we write the weak formulation analogous to (2.1) (it is contained in Definition 2.1); we pass to the limit in the latter, using the a.e. convergence in the integrals over  $(0, 1) \times (0, 1)$  and using the weak-\* convergence in the linear in  $u_1^\epsilon$  term accounting for the terminal data. The proof is complete.  $\square$

We are now in a position to state and prove our central result in the setting of uniformly bounded weak solutions to problem  $(\mathbf{Pb}_{box})$ .

**Theorem 3.3.** *Let  $\mathcal{K}$  be a subset of  $\mathcal{NA}_1^{L^\infty, \text{box}}$  compact in the weak-\* topology of  $L^\infty((0, 1))$ . Let  $L \geq 1$ . Then there exists a constant  $\epsilon_0 > 0$  (depending on  $\mathcal{K}$  and  $L$  only) such that for all couples  $(T, u_T) \in (0, \infty) \times L^\infty((0, 1))$  satisfying  $Tu_T \in \mathcal{K}$ ,  $T \leq \epsilon_0$  problem  $(\mathbf{Pb}_{box})$  has no weak solution – in the sense of Definition 2.1 – satisfying the amplification assumption*

$$\|u\|_{L^\infty((0,1)\times(0,T))} \leq L \|u_T\|_{L^\infty((0,1))}. \quad (3.7)$$

The simplest way to interpret this uniform (over states in  $\mathcal{K}$ ) non-attainability result is to particularize it to singletons  $\mathcal{K} = \{u_T\}$ . Then Theorem 3.3 provides information on no-attainability of profiles  $w$  with the shape prescribed by the shape of  $u_T$  (namely,  $w = T^{-1}u_T$ ), the amplitude of  $w$  being entangled with the non-attainability times  $T$ . In particular, we will do so for constant profiles in Corollary 3.5 below.

**Remark 3.10.** Note that given  $\mathcal{K}$ , the non-attainability times  $T$  for  $u_T \in T^{-1}\mathcal{K}$  are small. Let us stress that, due to this fact, the associated non-attainable data in Theorem 3.3 are somewhat large; indeed,  $u_T \in T^{-1}\mathcal{K}$ , being understood that the targets in the weakly-\* compact subset  $\mathcal{K}$  of  $\mathcal{NA}_1^{L^\infty, box}$  satisfy  $\text{dist}(\mathcal{K}, 0) > 0$  due to the fact that  $0 \notin \mathcal{NA}_1^{L^\infty, box}$ .

**Remark 3.11.** Note that in Theorem 3.3, we assume that the solutions are not too large in the  $L^\infty$  norm (compared to the  $L^\infty$  norm of the desired terminal data), and we assume that the solutions are weak (and not merely very weak) solutions. In the sequel of the paper, we will get closer to the pure  $L^2$  setting suggested in [18], however, some a priori bound on the size of the desired solutions (measured via the amplification constant  $L$ ) will always be required.

*Proof of Theorem 3.3.* It is enough to scale a solution of (BE) with terminal data  $u(\cdot, T) = u_T$ ,  $u_T \in T^{-1}\mathcal{K}$ , by (Zoom) with  $\epsilon = T$ ; we need  $T \leq \epsilon_0$  in order to apply Lemma 3.2. It is easily checked that  $u_\epsilon = \mathcal{S}_\epsilon(u)$  solves (Pb $_{box}^\epsilon$ ) (also in the weak sense) on the time interval  $(0, 1)$ . Also note that the amplification assumption (3.7) is invariant under this scaling.  $\square$

Now we concentrate on the case of constant solutions addressed in (Q); to do so, we apply Theorem 3.3 to the sets  $\mathcal{K}_{\alpha, \beta}$  defined in Remark 3.7(i) and we employ the following elementary observation:

**Lemma 3.4.** *If a constant state  $C$  is non-attainable at time  $T$  by weak solutions of (Pb $_{box}$ ) verifying the amplification restriction  $\|u\|_{L^\infty((0,1) \times (0,T))} \leq L|C|$ , then for all  $T' < T$  the state  $C$  remains non-attainable, under the restriction  $\|u\|_{L^\infty((0,1) \times (0,T'))} \leq L|C|$ .*

*Proof.* Arguing by contradiction, one assumes that  $u(\cdot, T') = C$  for some  $T' < T$ . Gluing continuously  $u$  on the time interval  $[0, T']$  and the constant function  $C$  on the time interval  $[T', T]$ , we find that the resulting function is a weak solution to (Pb $_{box}$ ) with terminal state  $C$ ; moreover, the amplification restriction at time  $T$  is inherited from the one that was assumed at time  $T'$ .  $\square$

Combining specific choices of  $\mathcal{K}$  (Remark 3.7(i)) in Theorem 3.3 with Lemma 3.4, we find the following partial negative answer to (the quantitative version of) the question (Q) in the  $L^\infty$  setting. We provide two closely related formulations, the first one focusing on individual states and their guaranteed non-attainability times, the second one highlighting the fact that the result naturally applies to non-attainability of families of target states.

**Corollary 3.5.** *Fix  $L \geq 1$  and consider problem (Pb $_{box}$ ) under the amplification assumption (3.7).*

(i) *(non-attainability of individual constant states)*

*There exists  $C_0 = C_0(L)$  such that whenever  $|C| \geq C_0$ , the state  $C$  is non-attainable at all times  $T \in [0, 1/|C|]$ .*

(ii) *(non-attainability of families of constant states at a given time)*

*Given  $\alpha \in (0, 1]$  there exists  $\epsilon_0 = \epsilon_0(\alpha, L)$  such that for all  $T \leq \epsilon_0$  and all  $C$  with  $\alpha T^{-1} \leq |C| \leq T^{-1}$ ,  $C$  is not attainable at time  $T$  (as well as at any smaller time).*

*Proof.*

(i) Theorem 3.3 yields, for  $\mathcal{K} := \mathcal{K}_{1,1} = \{1\}$ , the existence of  $\epsilon_0 = \epsilon_0(L)$  such that for  $T \leq \epsilon_0$ , the constant state  $C = T^{-1}$  is not attainable at time  $T$ ; in addition, Lemma 3.4 ensures that this state is not attainable at any smaller time. Setting  $C_0 = C_0(L) := 1/\epsilon_0(L)$ , we infer claim (i).

(ii) Theorem 3.3 yields, for  $\mathcal{K} := \mathcal{K}_{\alpha,1}$ , the existence of  $\epsilon_0 = \epsilon_0(\alpha, L)$  such that for  $T \leq \epsilon_0$ , the constant states  $C$  with  $CT \in \mathcal{K}_{\alpha,1}$  – i.e.  $C$  such that  $\alpha \leq |C|T \leq 1$  – are not attainable at time



$T$ . Fixing a value  $T \leq \epsilon_0$  we find non-attainable states  $C$  at this time; then by Lemma 3.4 these states are also not attainable at any time smaller than  $T$ . This proves claim (ii).  $\square$

**Remark 3.12.** *If instead of taking  $\mathcal{K} := \mathcal{K}_{1,1} = \{1\}$  we take  $\mathcal{K} := \mathcal{K}_{\beta,\beta} = \{\beta\}$  for some  $0 < \beta < 1$ , it is not difficult to see that we find a smaller threshold  $C_0$  in Corollary 3.5(i). However, in this case the intervals of non-attainability for the target  $C$  take the form  $[0, \beta/|C|]$  which makes them shorter.*

As stressed in the introduction, assuming an  $L^\infty$  bound on the solutions in Theorem 3.3 and Corollary 3.5, departs from the original question raised in [18]. In order to get closer to the latter, we will develop in Section 5 elements of an  $L^2$  theory of the inviscid Burgers equation, before returning to the non-attainability issues in the last section of the paper.

Prior to this, we examine into more depth the non-attainability by  $L^\infty$  solutions.

#### 4. Some extensions of the results in the $L^\infty$ setting

While retaining the  $L^\infty$  interpretation of (Q) of the previous section, in § 4.1 we address the strip setting ( $\mathbf{Pb}_{strip}$ ); then in § 4.2 we point out the non-optimality of the restriction  $T|C| \leq 1$  in our non-attainability results.

##### 4.1. Non-attainability by bounded solutions in the strip

We start by extending the non-attainability results to the simpler variant of problem (BE), namely for the case  $D = \mathbb{R} \times (0, 1)$ . We introduce the set  $\mathcal{NA}_1^{L^\infty, strip}$  by analogy with  $\mathcal{NA}_1^{L^\infty, box}$ , replacing in (3.3) “solutions to ( $\mathbf{Pb}_{box}^0$ )” by “solutions to ( $\mathbf{Pb}_{strip}^0$ )”. It is obvious that states  $u_T$  on  $(0, 1)$  non attainable at time  $T$  by  $L^\infty((0, 1) \times (0, T))$  weak solutions of the viscous Burgers equation are also non-attainable by  $L^\infty(\mathbb{R} \times (0, T))$  weak solutions, i.e.,

$$\forall T > 0 \quad \mathcal{NA}_T^{L^\infty, box} \subset \mathcal{NA}_T^{L^\infty, strip}; \quad (4.1)$$

also note that the scaling property (3.4) extends to the strip case.

The strip setting ( $\mathbf{Pb}_{strip}$ ) is a pure initial-value problem, therefore it is simpler than ( $\mathbf{Pb}_{box}$ ) in many respects. However, note that Lemma 3.4 does not extend to the strip setting. We state the results analogous to Theorem 3.3 and Corollary 3.5 as a reference point for subsequent refinements (see Corollary 4.3 in the next paragraph and Theorem 6.4 in Section 6). Its proof follows the lines of the proofs in § 3.2.

**Theorem 4.1.** *Let  $\mathcal{K}$  be a subset of  $\mathcal{NA}_1^{L^\infty, strip}$  compact in the weak-\* topology of  $L^\infty((0, 1))$ . Let  $L \geq 1$ . Then there exists a constant  $\epsilon_0 > 0$  (depending on  $\mathcal{K}$  and  $L$  only) such that for all couples  $(T, u_T) \in (0, +\infty) \times L^\infty((0, 1))$  satisfying  $Tu_T \in \mathcal{K}$ ,  $T \leq \epsilon_0$  there exist no initial data satisfying*

$$\|u_0\|_{L^\infty(\mathbb{R})} \leq L\|u_T\|_{L^\infty((0,1))} \quad (4.2)$$

such that the problem ( $\mathbf{Pb}_{strip}$ ) admits a weak solution in the sense of Definition 2.1.

In particular, for given  $\alpha \in (0, 1]$  there exists  $\epsilon_0 = \epsilon_0(\alpha, L)$  such that for all  $T \leq \epsilon_0$  and all  $C$  with  $\alpha T^{-1} \leq |C| \leq T^{-1}$ , the constant state  $C$  is not attainable at time  $T$  for problem ( $\mathbf{Pb}_{strip}$ ) with initial data fulfilling the amplification restriction  $\|u_0\|_{L^\infty(\mathbb{R})} \leq L|C|$ .

The latter conclusion gives a partial negative answer to (the quantitative version of) (Q) in the strip setting. Note that it can also be reformulated as follows: given  $L \geq 1$  and  $\alpha \in (0, 1]$ , setting  $C_0 = C_0(\alpha, L) := 1/\epsilon_0(\alpha, L)$ , there holds the following:

for all  $C$  with  $|C| \geq C_0$ , the constant target state  $u_T(\cdot) = C$  is not attainable

by weak solutions of ( $\mathbf{Pb}_{strip}$ ) under the restriction (4.2) at any  $T \in [\alpha|C|^{-1}, |C|^{-1}]$ .  $\quad (4.3)$

Note that we cannot extend the non-attainability to times smaller than  $\alpha/|C|$  because we don't have the conclusion of Lemma 3.4 in the strip setting.

#### 4.2. Non-attainability for some $T > 1/|C|$

While it may seem from the proof of Proposition 3.1 that the non-attainability at  $T = 1$  argument is limited to constants  $C \leq 1$  (so that the scaling procedure yields the restriction  $T \leq 1/|C|$  in the context of Theorem 3.3, Corollary 3.5, Theorem 4.1 and also in the subsequent results of Section 6 based upon Proposition 3.1), let us point out that this restriction is not sharp. This condition can be weakened due to our introduction of amplification conditions (3.7),(4.2) in the context of problems  $(\mathbf{Pb}_{box}^0), (\mathbf{Pb}_{strip}^0)$ , respectively. Imposing the analogous restrictions in the inviscid setting, in the case  $L = 1$  (no amplification) we can extend the result of non-attainability at time  $T = 1$  in Proposition 3.1 to constants  $C$  with  $0 < |C| < 2$ , in place of  $0 < |C| \leq 1$ . More generally, for the case of constant targets we have the following observation.

**Proposition 4.2.** *Let  $L \geq 1$  be given. For all  $C \neq 0$  with  $|C| < 1 + \Delta$ ,  $\Delta = L^{-2}$ , there exists no local Kruzhkov entropy solution verifying  $(\mathbf{Pb}_{box}^0)$ , the terminal data  $u(\cdot, 1) = C$  and the amplification restriction (3.7).*

*In the case of problem  $(\mathbf{Pb}_{strip}^0)$ , the analogous result (under the amplification restriction (4.2)) holds with the even larger value of  $\Delta$ , namely  $\Delta = (2L - 1)^{-1}$ .*

*Proof.* We give the proof in the box setting. The strip setting is similar and we only sketch the argument. We divide the proof for  $(\mathbf{Pb}_{box}^0)$  into two parts. First, we address the elementary case  $L = 1$  and develop the argument based upon the comparison (see Remark 3.5) with an obvious reference solution. Next, we consider  $L > 1$  where the construction of an adequate reference solution and an analogous comparison argument yields the desired result. It is enough to consider positive constants  $C$ , the case of  $C < 0$  being completely analogous (upon exchanging the role of the two boundaries  $x = 0, x = 1$ ).

In the case  $L = 1$ , the function  $u_{ref}(x, t) = C$  for  $0 \leq x < Ct/2$ ,  $u = 0$  otherwise, is an obvious solution to  $(\mathbf{Pb}_{box}^0)$  and it attains the terminal data  $u_T(\cdot, T) = C$  in  $(0, 1)$  if and only if  $CT \geq 2$ . For any smaller time  $T$ , there holds  $u_{ref}(x, T) = 0 < C$  for  $x \in (CT/2, 1)$ . Now, observe that  $u_{ref}$  solves the Cauchy-Dirichlet problem in  $(0, 1) \times (0, T)$  with initial data  $u_0 = 0$  and boundary data  $b_0 = C$  (the Dirichlet datum at  $x = 0^+$ ),  $b_1 = C$  (the Dirichlet datum at  $x = 1^-$ ), the boundary data being assumed in the Bardos-LeRoux-Nédélec [10] sense, see Remark 3.4. In the sequel, we rescale this solution to fit our reference setting  $T = 1$ ; this ensures that for  $C < 2$ ,  $u_{ref}(\cdot, 1) = 0 < C$  in  $(C/2, 1)$ .

Now, fix  $0 < C < 2$  and take any local Kruzhkov entropy solution  $u$  of (2.2) attaining the terminal data  $C$  at time  $T$ ; according to Remark 3.4, it corresponds to some boundary data  $b_0, b_1$  which are  $[-C, C]$ -valued due to the restriction (3.7) and our assumption  $L = 1$ . The comparison principle (Remark 3.5) for Cauchy-Dirichlet problems yields  $C = u(\cdot, 1) \leq u_{ref}(\cdot, 1)$  which is a contradiction on the interval  $(C/2, 1)$ . This proves the claim for  $L = 1$ .

Now, we address the case  $L > 1$ . Let us indicate the reference solution which achieves the final constant state  $C$  precisely at the critical time  $T = 1$ ; it corresponds to the critical value  $C = 1 + L^{-2}$  and takes the following form. Introduce  $\delta = 1 - 1/C = 1/(1 + L^2)$  and define the following curves in  $(0, 1) \times (0, 1)$ :

$$\begin{aligned} \Gamma_1 &:= \{(x, t) \mid \delta \leq t \leq 1, x = C(t - \delta)\}, & \Gamma_2 &:= \{(x, t) \mid \delta \leq t \leq 2\delta, x = LC(t - \delta)\}, \\ \Gamma_3 &:= \{(x, t) \mid 0 \leq t \leq 2\delta, x = LCt/2\}, & \Gamma_4 &:= \{(x, t) \mid 2\delta \leq t \leq 1, x = LC(\delta(t - \delta))^{\frac{1}{2}}\}. \end{aligned}$$

Note that the choices  $C = 1 + L^{-2}$ ,  $\delta = 1 - 1/C$  ensure that  $\Gamma_1$  meets  $\Gamma_4$  at the point  $(x, t) = (1, 1)$ .

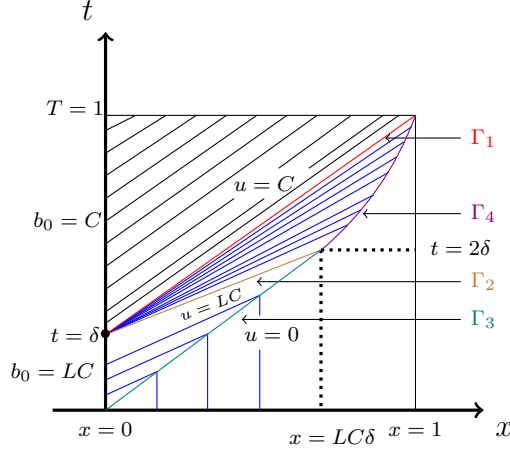


Figure 2: Solution  $u_{ref}$  to  $(\mathbf{Pb}_{strip}^0)$  in the critical case  $C = 1 + L^{-2}$ ,  $L > 1$ .

Then (see Figure 2) we set  $u_{ref} = C$  above  $\Gamma_1$ ,  $u_{ref}(x, t) = x/(t - \delta)$  between  $\Gamma_1$  and  $\Gamma_2 \cup \Gamma_4$ ,  $u_{ref} = LC$  between  $\Gamma_2$  and  $\Gamma_3$ , and  $u_{ref} = 0$  below  $\Gamma_3 \cup \Gamma_4$ . It is easily checked that  $u_{ref}$  is a local Kruzhkov entropy solution to  $(\mathbf{Pb}_{box}^0)$ , in particular, the Rankine-Hugoniot and the entropy admissibility conditions on  $\Gamma_3 \cup \Gamma_4$  hold true.

It is also easy to verify that for any  $C < 1 + L^{-2}$  the solution constructed in the same way (somewhat abusively, we will keep the notation  $u_{ref}$  for this solution) exhibits a crossing of  $\Gamma_1$  and  $\Gamma_4$  before  $T = 1$ , and therefore it attains some state  $u_{ref}(\cdot, 1)$  which takes zero values in a vicinity of  $x = 1$ . It also assumes the boundary condition  $b_1 = LC$  at  $x = 1^-$  in the BLN sense.

Now by applying the maximum principle, we conclude that any solution  $u$  to  $(\mathbf{Pb}_{box}^0)$  with  $u(\cdot, T) = C$  actually lies below  $u_{ref}$ . Recalling Remark 3.4(ii), let  $b_0, b_1$  be boundary data that lead to a local Kruzhkov entropy solution to  $(\mathbf{Pb}_{box}^0)$  with terminal data  $C$  at  $T = 1$ , then  $b_0(t) = C$  on  $(1 - 1/C, 1)$  (this follows by the backward characteristics construction [21]) and  $b_0 \leq LC$  on  $(0, 1 - 1/C)$ ,  $b_1 \leq LC$  on  $(0, 1)$  due to assumption (3.7). Thus  $u_{ref}$  corresponds exactly to the largest possible boundary data; yet in a vicinity of  $x = 1$ , its terminal state lies strictly below the target state, so also  $u(\cdot, 1)$  cannot achieve the target state  $C$ . This proves that states  $C < 1 + L^{-2}$  are not attainable for  $(\mathbf{Pb}_{box}^0)$ .

As for problem  $(\mathbf{Pb}_{strip}^0)$ , the initial data leading to the reference solution  $u_{ref}$  are given by  $u_{0,ref} = 0$  for  $x \in (0, 1)$ ,  $u_{0,ref} = C$  for  $x \in (-\infty, 1 - C)$ ,  $u_{0,ref} = LC$  for  $x \in (1 - C, 0) \cup (1, +\infty)$ . With the choice  $C = 1 + (2L - 1)^{-1}$ , the shock starting from the point  $(0, 0)$  encounters the rarefaction starting from the point  $(1 - C, 0)$  at  $(C - 1, 2(C - 1)/(LC))$ , crosses the rarefaction and gets out of the rarefaction precisely at the point  $(1, 1)$ , quite similarly to what happens in Figure 2. For any smaller value of  $C$ , the shock crosses the rarefaction before  $T = 1$  and therefore leads to a reference solution with  $u_{ref}(x, 1) = 0$  in some vicinity of  $x = 1$ .

To conclude using the maximum principle as above, we have to remark first that, if we know that a Kruzhkov entropy solution  $u$  assumes the target datum  $C$  for  $x \in (0, 1)$  and  $T = 1$ , then the values of  $u_0$  for  $x < -C$  do not influence the values of  $u$  in the domain  $\{(x, t) \mid x > C(t - 1)\}$ , because the boundary of this domain is a maximal backward generalized characteristic for the solution. For  $x > -C$ , the initial data  $u_{0,ref}$  taken to generate  $u_{ref}$  are the largest ones compatible with the reconstruction of  $u_0$  in  $(-C, 1 - C)$  by backward characteristics, with the requirement  $u(\cdot, 0) = 0$  in  $(0, 1)$  and with the amplification constraint (4.2).  $\square$

Using Proposition 4.2 in place of Proposition 3.1, following the same strategies of proof as in § 3.2, we can improve the result of Corollary 3.5(i) by extending the interval of non-attainability times  $T$  by the factor  $(1 + \Delta)$ ,  $\Delta = L^{-2} \in (0, 1)$ :

**Corollary 4.3.** *Let  $L \geq 1$  and restrict attention to weak solutions of  $(\mathbf{Pb}_{box})$  that verify the amplification restriction  $\|u\|_{L^\infty((0,1) \times (0,T))} \leq L|C|$ . There exists  $C_0 = C_0(L)$  such that whenever  $|C| \geq C_0$ , the state  $C$  is non-attainable at all times  $T \in [0, (1 + \Delta)/|C|]$  with  $\Delta = L^{-2}$ .*

Similarly, the last conclusion of Theorem 4.1 for problem  $(\mathbf{Pb}_{strip})$  holds for constants  $C$  satisfying the weaker restriction  $\alpha T^{-1} \leq |C| < (1 + \Delta)T^{-1}$ ,  $\Delta = (2L - 1)^{-1}$ , while the non-attainability of large individual constants  $C$  in (4.3) can be extended to  $T \in [\alpha|C|^{-1}, (1 + \Delta)T^{-1}]$ .

For more general target data, we have the following variant of Proposition 3.1. For simplicity, we formulate it for the inviscid Burgers problem in the strip and only for half of the cases covered by assumption (NA).

**Proposition 4.4.** *Let  $L, M \geq 1$  and  $m > 0$  be given. Consider target states  $u_1 \in BV_{loc}((0, 1))$ , normalized by right-continuity, verifying*

$$\begin{aligned} &\exists x^* \in (0, 1] \text{ such that } m \leq u_1(x^*) \leq (1 + \Delta)x^* \\ &\text{and moreover, } \|u_1\|_\infty \leq Mu_1(x^*). \end{aligned}$$

Assume that  $\Delta < m(2LM(LM + 1))^{-1}$ . Then there exists no Kruzhkov entropy solution verifying  $(\mathbf{Pb}_{strip}^0)$ , the terminal data  $u(\cdot, 1) = u_1$  in  $(0, 1)$  and the bound  $\|u\|_\infty \leq L\|u_1\|_\infty$ .

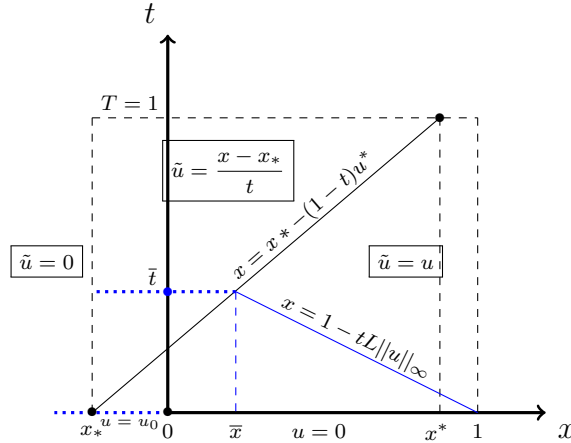


Figure 3: Construction of the point  $(\bar{x}, \bar{t})$  in the proof of Proposition 4.4.

*Proof.* We write  $u^* = u_1(x^*)$  and conduct the construction of the proof of Proposition 3.1, being understood that this time,  $x_* = 1 - u^*$  can be negative. In the sequel, we assume that  $x_*$  is negative, since otherwise the contradiction is readily given by the argument of Proposition 3.1. We define  $\tilde{u}$  by (3.5), but this time for all  $x \in \mathbb{R}$ . Define  $\bar{t}$  by the relation  $x_* + u^*\bar{t} = 1 - LMu^*\bar{t}$ ; this yields

$$\bar{t} = \frac{1 - x_*}{LMu^* + u^*} = \frac{1}{LM + 1},$$

keeping in mind that  $1 - x_* = u^*$ . Set  $\bar{x} = x_* + u^*\bar{t}$ ; we refer to Figure 3 for the geometric interpretation of the point  $(\bar{x}, \bar{t})$ .

In view of the bound  $\|\tilde{u}\|_\infty \leq \|u\|_\infty \leq LMu^*$ , the classical Kruzhkov propagation estimates [31] imply in particular that

$$\int_{-\infty}^{\bar{x}} |\tilde{u}(x, \bar{t})| dx \leq \int_{-\infty}^1 |\tilde{u}(x, 0)| dx.$$

The expression (3.5) of  $\tilde{u}$  being explicit for  $t = \bar{t}$ ,  $x \in (-\infty, \bar{x})$ , the calculation of the left-hand side of the above inequality, bearing in mind the bound  $|\tilde{u}_0(x)| = |u(x)| \leq L\|u_1\|_\infty$  for  $x \in (x_*, 0)$ , yields

$$\frac{(u^*)^2}{2(LM+1)} = \frac{(u^*)^2\bar{t}}{2} = \int_{x_*}^{x_*+u^*\bar{t}} \frac{x-x_*}{\bar{t}} dx \leq \int_{x_*}^0 |\tilde{u}(x,0)| \leq LMu^*|x_*| = LMu^*(u^*-1) \leq LMu^*\Delta.$$

Because  $u^* \geq 0$ , this leads to a contradiction as soon as  $2LM(LM+1)\Delta < m$ .  $\square$

Note that a qualitatively analogous to Proposition 4.4 result can be formulated for the problem ( $\mathbf{Pb}_{box}^0$ ); but the explicit bound for  $\Delta$  in terms of  $L, M, m$  is more delicate to compute because the control of the  $L^1$  norm of the solution in terms of the  $L^1$  norm of the boundary data makes the Lipschitz constant of  $f : u \mapsto u^2/2$  on  $[-\|u\|_\infty, \|u\|_\infty]$  to appear (cf. the stability estimate in [43]). We will not pursue this further.

Using Proposition 4.4 in place of Proposition 3.1, following the same strategy of proof as in § 3.2, one can improve the results of Theorem 4.1 by requiring that

$$T(1+\Delta)u_T \in \mathcal{K}, \quad \Delta < m(2LM(LM+1))^{-1},$$

provided  $\mathcal{K}$  consists of states verifying (4.4).

**Remark 4.1.** *While we do not pursue the goal of giving optimal statements in this and related situations, let us stress that the case  $|C| = T^{-1}$  (that appeared as critical in the non-attainability statements of § 3.2) is actually situated in the interior of the non-attainable (under amplification restrictions!) set, and not on its boundary (cf. Remark 3.8); the same is true at least for terminal states  $u_T \in \mathcal{NA}_T^{L^\infty, box}$  having the shape (4.4) and the amplitude  $T^{-1}$ . This fact becomes important in Section 6.*

## 5. Entropy solutions of scalar conservation laws in critical $L^p$ setting

Question (Q) raised in the survey [18] concerns  $L^2$  solutions of the one-dimensional viscous Burgers equation. Adaptation of our strategy of analysis to this case requires an extension of the theory of the inviscid Burgers equation to  $L^2$  solutions; this is the purpose of the present section, where we put forward a basic theory covering the desired case. Let us point out that for the (multidimensional) Burgers equation, a pure  $L^p$  theory for the Cauchy problem was constructed in the recent work [45] of Serre and Silvestre. Well-posedness and instantaneous regularizing effect (from  $L^p$  to  $L^\infty$ ) are shown, with the help of dispersion estimates.

The goal of the present section is two-fold:

On the one hand, we provide an elementary theory of the Cauchy problem for scalar multidimensional conservation laws for general flux functions exhibiting polynomial (of degree  $p$ ) growth in the critical case of  $L^p$  initial data. The criticality is to be understood in comparison with the results of Szepessy [46] and Bendahmane, Karlsen [12], where for a flux of degree  $(p-\alpha)$  polynomial growth,  $\alpha > 0$ , an  $L^1 \cap L^p$  theory is constructed. It should be stressed that the critical case requires neither adaptation of the notion of solution (cf. [13] and [42] for the pure  $L^1$  theories of, respectively, renormalized and kinetic solutions), nor the fine regularization estimates of the super-critical case [45]. Compared to the subcritical case of [46, 12], it requires a careful control of approximations in the existence proof. The precise well-posedness result for the Cauchy problem is given in Theorem 5.1 below. Further, we also investigate the analogous Cauchy-Dirichlet problem in the one-dimensional case, limiting our attention to the Burgers equation. We provide sufficient conditions on boundary data for existence of  $L^2$  entropy solutions, and give the uniqueness result

based upon the automatic boundary regularity of such solutions. Concerning the Cauchy-Dirichlet version of our problem, let us point out the related work [6] of Ancona and Marson where semi-group solutions with initial and boundary data of appropriate integrability are considered, though not fully characterized by entropy inequalities, and the works [43] of Porretta and Vovelle, [3] of Ammar, Carrillo and Wittbold where the problem is treated in the renormalized solutions setting, i.e. for a more involved truncated entropy formulation.

On the other hand, we provide an elementary argument for the  $L^2 - L^\infty$  regularization effect observed in [45] in the case of the one-dimensional Burgers equation. In addition to being elementary, it applies also to the viscous Burgers equation (cf. Remark 6.4). The precise instantaneous regularization result is given in Propositions 5.6 and 5.8 below. We stress that this result permits us to apply the tools of generalized characteristics, in arguments like Propositions 3.1 and 4.4, to  $L^2$  solutions of the inviscid Burgers equation.

### 5.1. Well-posedness of the Cauchy problem in the critical integrability setting

We consider the problem

$$\begin{cases} u_t + \operatorname{div} F(u) = 0 & \text{in } \mathbb{R}^d \times (0, T), \\ u(0, \cdot) = u_0 & \text{on } \mathbb{R}^d, \end{cases} \quad (5.1)$$

for  $u_0 \in L^p(\mathbb{R}^d)$ ,  $1 \leq p < \infty$ , and the nonlinearity  $F \in C(\mathbb{R}; \mathbb{R}^d)$  verifying for some  $C > 0$  the critical growth assumption

$$\forall u \in \mathbb{R} \quad |F(u)| \leq C(1 + |u|^p). \quad (5.2)$$

Up to the issue of integrability, the notion of solution for problem (5.1) is the standard notion in the entropy framework of Kruzhkov [31]:

**Definition 5.1.** *A function  $u \in L^\infty(0, T; L^p(\mathbb{R}^d))$  is said to be an entropy solution of (5.1) with  $F$  satisfying (5.2) and  $u_0 \in L^p(\mathbb{R}^d)$  if for all  $\xi \in C_c^\infty(\mathbb{R}^d \times [0, T])$ ,  $\xi \geq 0$  and all  $k \in \mathbb{R}$  there holds*

$$- \int_0^T \int_{\mathbb{R}^d} |u - k| \xi_t + \operatorname{sign}(u - k)(F(u) - F(k)) \cdot \nabla \xi \, dx \, dt - \int_{\mathbb{R}^d} |u_0 - k| \xi(x, 0) \, dx \leq 0. \quad (5.3)$$

We remark that the above extension of the Kruzhkov [31] definition can be compared to the approach and the techniques of renormalized solutions of conservation laws (Bénilan, Carrillo and Wittbold [13]). In our setting, there is no need to rely upon truncations of  $u$  to define solutions. This is due to (5.2) and the integrability assumption on  $u$ , making all terms in the above formulation well defined. However, like in the renormalized solution theory, it is not immediately clear from the entropy inequalities (5.3) (even letting  $k \rightarrow \pm\infty$ ) that an entropy solution is also a weak one; we therefore include this point into the well-posedness result below. Like in the renormalized solutions theory, we can link entropy solutions in the sense of Definition 5.1 to their approximation by classical  $L^\infty \cap L^1$  Kruzhkov entropy solutions. Actually, this approximation - in a monotone way, following Ammar and Wittbold [4] - is in the heart of the existence proof; it also permits to recover easily the structural properties highlighted in Proposition 5.6 and Corollary 5.7 below.

**Theorem 5.1.** *Under assumption (5.2), for all  $u_0 \in L^p(\mathbb{R}^d)$  there exists a unique entropy solution  $u$  to problem (5.1). The solutions for different data verify the  $L^1$  contraction and comparison property (3.2) (with possibly infinite right-hand side).*

*Moreover, the solution  $u$  is the pointwise (a.e.) limit, as  $M \rightarrow +\infty$ , of the sequence  $(u^M)_M$  of Kruzhkov entropy solutions of the Cauchy problem for the same conservation law with the truncated initial data  $u_0^M = \operatorname{sign}(u_0) \min\{|u_0|, M\}$ .*

Further, the solution  $u$  satisfies the weak formulation of problem (5.1), it is time-continuous, i.e.  $u \in C([0, T]; L^1_{loc}(\mathbb{R}^d))$ , and  $\sup_{t>0} \|u - u^M\|_{L^1(\mathbb{R}^d)} \rightarrow 0$  as  $M \rightarrow +\infty$ . In particular,  $(u^M)$  converges to  $u$  in  $C([0, T]; L^1_{loc}(\mathbb{R}^d))$ .

*Proof.* The proof follows [31] and assembles several techniques developed, for different purposes in related literature.

**Uniqueness and comparison property**<sup>1</sup>. Let  $u, \hat{u}$  be solutions of (5.1) in the sense of Definition 5.1 with the same initial data  $u_0 = \hat{u}_0$ .

The first step is to obtain from the entropy inequalities (5.3) the so-called Kato inequality: for all  $\xi \in C_c^\infty(\mathbb{R}^d \times [0, T])$ ,  $\xi \geq 0$  there holds

$$- \int_0^T \int_{\mathbb{R}^d} |u - \hat{u}| \xi_t + q(u, \hat{u}) \cdot \nabla \xi \, dx \, dt \leq 0, \quad (5.4)$$

where

$$q(z, r) := \text{sign}(z - r)(F(z) - F(r)).$$

This is done by the classical doubling of variables argument [31]. We stress that the properties  $u, \hat{u} \in L^1_{loc}(\mathbb{R}^d \times [0, T])$ ,  $F(u), F(\hat{u}) \in L^1_{loc}(\mathbb{R}^d \times [0, T])$  are precisely the ones that are required in this argument (one can use the technique based upon the upper semicontinuity of the  $L^1$  bracket<sup>2</sup> in [7, Lemma 6, Prop. 3]; see also [11, Remark 4]). By the standard density argument (in the weak-\* topology of  $W^{1, \infty}$ ), it is immediate to extend (5.4) to Lipschitz continuous test functions with compact support in  $\mathbb{R}^d \times [0, T]$ .

The second step is to pass to the limit with a well-chosen sequence of test functions  $\xi_k$  verifying  $\xi_k(x, t) \rightarrow (e^{-t} - e^{-T})$  as  $k \rightarrow +\infty$ . Note that we cannot rely on the Kruzhkov choice of localized in space and time test functions, because in our setting the conservation law exhibits infinite speed of propagation. We pick the simple technique suitable in the case of infinite speed of propagation and integrable flux (cf. [14]). According to the preceding density remark, one can choose

$$\xi_R(x, t) = \min\{1, (|x| - R)^+\}(e^{-t} - e^{-T}) \quad (5.5)$$

in the Kato inequality (5.4). Then  $|\nabla \xi|$  is bounded by 1 and supported in  $\{R \leq |x| \leq R + 1\}$ ; therefore the fact that

$$q(u(x, t), \hat{u}(x, t)) \in L^\infty(0, T; L^1(\mathbb{R}^d))$$

(due to  $u, \hat{u} \in L^\infty(0, T; L^p(\mathbb{R}^d))$  and the growth assumption (5.2)) permits to pass to the limit  $R \rightarrow \infty$  in (5.4) and infer  $\int_0^T \int_{\mathbb{R}^d} e^{-t} |u(x, t) - \hat{u}(x, t)| \, dx \, dt \leq 0$ . This proves uniqueness.

The contraction and comparison properties (3.2) can be proved similarly, taking  $(u_0 - \hat{u}_0)^+ \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$  and using the half-entropies  $(\cdot - k)^\pm$  ([44, 17]) instead of the Kruzhkov entropies and weights  $e^{-ct}$ ,  $c \rightarrow 0^+$ , instead of  $e^{-t}$ .

**Existence.** Let us introduce for  $M, N > 0$  the truncation functions

$$T^{M, N} : \mathbb{R} \rightarrow \mathbb{R}, \quad T^{M, N}(z) = \max\{\min\{z, M\}, -N\}. \quad (5.6)$$

The existence argument is based on the established theory of (bounded) Kruzhkov entropy solutions

<sup>1</sup>Note that the uniqueness proof in [12], written for  $L^1 \cap L^p$  data and subcritical (polynomial of degree  $p - \alpha$ ,  $\alpha > 0$ ) growth of  $F$ , actually extends to our case of the critical growth and pure  $L^p$  data. Here we point out simpler techniques for dealing with the Kato inequality, because we are concerned with a less general family of PDEs.

<sup>2</sup>For  $u, v \in L^1(\Omega)$ , their bracket in  $L^1(\Omega)$  is  $[u, v]_{L^1} := \int_{\Omega} \text{sign}(u)v \, dy + \int_{\{y \in \Omega \mid u(y)=0\}} |v| \, dy$ , see, e.g., [11].

(with the comparison property (3.2)) and on the use of bi-monotone sequences of approximations of initial data put forward by Ammar and Wittbold in [4]. Let  $u_0^{M,N} = T^{M,N}(u_0)$ . It is clear that for all  $M, N > 0$  we have  $u_0^{M,N} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  with the uniform in  $M, N$  bounds

$$\|u_0^{M,N}\|_{L^p(\mathbb{R}^d)} \leq \|u_0\|_{L^p(\mathbb{R}^d)}. \quad (5.7)$$

The corresponding Kruzhkov entropy solutions  $u^{M,N}$  exist and the classical Kruzhkov theory also yields the bound  $\|u^{M,N}\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} \leq \|u_0\|_{L^p(\mathbb{R}^d)}$ , in view of the above bounds on the initial data and because  $\eta : u \mapsto |u|^p$  is an entropy. By construction, we have the pointwise convergence

$$\forall N > 0 \quad u_0^{M,N} \uparrow u_0^{\infty,N} := \max\{u_0, -N\} \quad \text{as } M \rightarrow +\infty.$$

The comparison property contained in (3.2) ensures that  $(u^{M,N})_M$  is a non-decreasing sequence uniformly bounded in  $L^\infty(0, T; L^p(\mathbb{R}^d))$ . Therefore it admits a pointwise limit (which is  $\mathbb{R} \cup \{+\infty\}$ -valued) that we denote  $u^{\infty,N}$ , and the monotone convergence theorem ensures that  $u^{\infty,N} \in L^\infty(0, T; L^p(\mathbb{R}^d))$ , in particular  $u^{\infty,N}$  is a.e. finite. Note that the monotonicity implies the uniform in  $M$  domination  $|u^{M,N}| \leq \max\{(u^{1,N})^-, (u^{\infty,N})^+\} \in L^\infty(0, T; L^p(\mathbb{R}^d))$ . Moreover, it follows by assumption (5.2) that

$$|F(u^{M,N})| \leq C(1 + \max\{(u^{1,N})^-, (u^{\infty,N})^+\}^p) \in L^1(\mathbb{R}^d \times (0, T)). \quad (5.8)$$

This makes it possible to apply the dominated convergence theorem, as  $M \rightarrow +\infty$ , in entropy inequalities (5.3) written for  $u^{M,N}$ . We deduce that  $u^{\infty,N}$  is an entropy solution in the sense of Definition 5.1 for the initial datum  $u^{\infty,N}$ . By construction,

$$u_0^{\infty,N} \downarrow u_0 \quad \text{as } N \rightarrow +\infty.$$

The same monotonicity argument ensures that the resulting sequence  $(u^{\infty,N})_N$  converges pointwise to a limit  $\bar{u} \in L^\infty(0, T; L^p(\mathbb{R}^d))$ , which implies the domination analogous to (5.8). Consequently, the entropy inequalities (5.3) satisfied by  $u^{\infty,N}$  are inherited by  $\bar{u}$ . This concludes the existence proof.

In addition, exchanging the roles of  $M$  and  $N$  in the above argument, we construct  $\underline{u} = \lim_{M \rightarrow +\infty} \lim_{N \rightarrow +\infty} u^{M,N}$ . We have by construction

$$u^{M,\infty} \leq u^{M,N} \leq u^{\infty,N},$$

$$u^{M,\infty} \rightarrow \underline{u} \quad \text{as } M \rightarrow +\infty \quad \text{and} \quad u^{\infty,N} \rightarrow \bar{u} \quad \text{as } N \rightarrow +\infty.$$

Since  $\underline{u}, \bar{u}$  are entropy solutions of (5.1) corresponding to the same  $L^p$  datum  $u_0$ , they coincide according to the above uniqueness analysis. Therefore  $u^{M,N}$  converges to the unique entropy solution  $\underline{u} \equiv \bar{u}$  of (5.1) as  $M, N \rightarrow +\infty$ , and in particular the sequence  $(u^{M,M})_M$  of Kruzhkov entropy solutions corresponding to initial data  $T^{M,M}(u_0)$  converge pointwise to  $\underline{u} \equiv \bar{u}$  as  $M \rightarrow +\infty$ .

**Weak formulation and time continuity.** It is classical that Kruzhkov entropy solutions in the standard  $L^\infty$  setting satisfy the weak formulation of the problem. It is enough to point out that the weak formulation passes to the limit as  $u = \bar{u}$  is approximated by  $u^{\infty,N}$  which is in turn approximated by  $u^{M,N}$ , due to the dominated convergence argument of the above the existence proof.

As for the time continuity of  $u$ , we apply the  $L^1$  contraction principle of the uniqueness claim to solutions  $u, u^M$ , being understood that the  $L^1 \cap L^\infty$  solutions  $u^M$  fulfill as well the formulation



of Definition 5.1. Combining Hölder and Chebyshev inequalities we find

$$\begin{aligned} \sup_{t>0} \|u(\cdot, t) - u^M(\cdot, t)\|_{L^1(\mathbb{R}^d)} &\leq \|u_0 - u_0^M\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (|u_0| - M)^+ dx \\ &\leq \|(|u_0| - M)^+\|_{L^2(\mathbb{R}^d)} \left( \text{meas}(\{x \mid |u_0(x)| > M\}) \right)^{1/2} \leq \|u_0\|_{L^2(\mathbb{R}^d)} \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{M} \rightarrow 0 \end{aligned}$$

as  $M \rightarrow \infty$ . This uniform-in-time convergence ensures that the time continuity of functions  $u^M$  with values in  $L^1_{loc}(\mathbb{R}^d)$  (see Remark 3.4) is inherited by  $u$ ; moreover, the difference  $(u - u^M)$ , as a function of  $t$ , vanishes in  $L^1(\mathbb{R}^d)$  uniformly in time, as  $M \rightarrow +\infty$ .  $\square$

**Remark 5.2.** *While proving Theorem 6.4 below, we sketch the analogous theory in  $L^\infty(0, T; L^2(\mathbb{R}))$  for the viscous Burgers equation, with the additional requirement that  $u \in L^2(0, T; H^1(\mathbb{R}))$  needed to ensure uniqueness.*

*Let us point out that one can address such theory of unbounded entropy solutions in the more general degenerate anisotropic diffusion case. For the subcritical case, well-posedness is established by Bendahmane and Karlsen in [12]; in order to extend their work to the critical case, one needs to insert the bi-monotonicity construction of [4] within the argumentation and employ sharp  $L^1$  arguments for the derivation of the Kato inequalities, similarly to the proof of Theorem 5.1.*

### 5.2. On $L^2$ solutions of the Cauchy-Dirichlet problem for the inviscid Burgers equation

Here, our goal is to sketch an  $L^2$  theory of entropy solutions to the inviscid Burgers equations in  $(0, 1) \times (0, T)$ , with assigned initial and boundary data. This issue appeared as a byproduct of question (Q), and the resulting theory can be seen as a simpler but less general alternative to the setting of renormalized solutions to Cauchy-Dirichlet problems, as developed by Ammar, Carrillo and Wittbold [3] and Porretta and Vovelle [43].

We set the following definition. The notation  $z^\pm$  refers to the decomposition of  $z \in \mathbb{R}$  into the difference  $z = z^+ - z^-$  of two positive numbers, whereas  $\text{sign}^\pm(z)$  stands for  $\pm \text{sign}(z^\pm)$ . We also write  $z \top w$  for  $\max\{z, w\}$ .

**Definition 5.3.** *Assume  $u_0 \in L^2((0, 1))$  and let  $b_1, b_2$  be measurable functions such that the positive part  $b_0^+$  of  $b_0$  and the negative part  $b_1^-$  of  $b_1$  belong to  $L^2((0, T))$ . The function  $u \in L^2((0, 1) \times (0, T))$  is an unbounded entropy solution of the Cauchy-Dirichlet problem*

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } D = (0, 1) \times (0, T), \\ u(x, 0) = u_0 & \text{on } (0, 1), \\ u(0, t) = b_0 & \text{on } (0, T), \\ u(1, t) = b_1 & \text{on } (0, T), \end{cases} \quad (5.9)$$

if for all  $k \in \mathbb{R}$ , for all  $\xi \in C_c^\infty([0, 1] \times [0, T])$ ,  $\xi \geq 0$  there holds

$$\begin{aligned} & - \int_0^T \int_0^1 (u - k)^\pm \xi_t + \text{sign}^\pm(u - k) \frac{u^2 - k^2}{2} \xi_x dx dt - \int_0^1 (u_0 - k)^\pm \xi(x, 0) dx \\ & \leq \int_0^T \frac{1}{2} \left( (k^- \top b_0^+)^2 - k^2 \right)^\pm \xi(0, t) dt + \int_0^T \frac{1}{2} \left( (k^+ \top b_1^-)^2 - k^2 \right)^\mp \xi(1, t) dt. \end{aligned} \quad (5.10)$$

**Remark 5.4.** *The unusual form of the boundary terms in the above definition is motivated by the formalism of the work [9] of Sbihi and the first author, the precise form of these expressions being adapted to our setting of unbounded solutions. In the expression  $k \mapsto \frac{1}{2}(k^- \top b_0^+)^2$  (respectively*

$k \mapsto \frac{1}{2}(k^+ \mp b_1^-)^2$ ), one can recognize the projected graph in the sense of [9] corresponding to the Dirichlet data  $b_0$  imposed at  $x = 0$  (respectively, to the Dirichlet data  $b_1$  imposed at  $x = 1$ ).

We first discuss several technical issues related to the above definition. To start with, we note that all terms in (5.10) make sense, due to the  $L^2$  assumptions on  $u$  and on  $(b_0)^+, (b_1)^-$ . In the sequel we are able to establish existence of  $L^2$  solutions to (5.9) in the sense of Definition 5.3 only under the  $L^3$  assumption on  $(b_0)^+, (b_1)^-$ , which comes quite naturally from the *a priori* estimates. Because in our application in § 6.3 we will need to go slightly beyond the setting of  $L^3$  boundary data (see Remark 5.7), we will link our notion of solution to the well-established but far more technical (in what concerns both the definitions and the proofs) notion of renormalized solution [3, 43]. In this relation, let us also mention the early work [6] of Ancona and Marson, where semigroup type solutions with  $L^1$  initial data and  $L^2$  boundary data (for the case of the left boundary alone) are constructed. We remark that by the theory of [6] these solutions satisfy a representation of the Lax-Oleinik type and they have the  $L_{loc}^\infty$  regularity; these two features are the essential ingredients of our analysis in § 6.3. Unfortunately, it is not clear whether the solutions of [6] can be fully characterized by entropy inequalities like (3.1); whereas in the context of § 6.3, we rely solely upon such local entropy characterization. Therefore the theory of [6] is not instrumental for the application we have in mind.

We stress that the boundary conditions at  $x = 0, 1$  are interpreted in a relaxed sense, in the spirit of Bardos, LeRoux and Nédélec [10]. Note that (5.10) is a variant of the expression of the BLN conditions put forward in [9, Proposition 3.3(iv)], and here we adapt its expression to the Burgers flux and to the  $L^2$  setting. To show that this adaptation is consistent with the established theory of [10, 47], we prove the existence of strong boundary traces for unbounded entropy solutions of (5.9).

For convenience of application of the results of [37, 39], we introduce a notation closely related to (5.6): for  $a, b \in \mathbb{R}$ ,  $a < b$ , the  $[a, b]$ -valued truncation of the identity map is

$$s_{a,b} : r \mapsto \min\{b, \max\{r, a\}\} \equiv T^{b,-a}(r). \quad (5.11)$$

**Lemma 5.2.** *For  $a, b \in \mathbb{R}$ ,  $a < b$ , there exist nonnegative locally bounded in  $[0, 1] \times [0, T]$  measures  $\gamma_b^+, \gamma_a^+$  such that there holds in  $\mathcal{D}'((0, 1) \times (0, T))$ :*

$$s_{a,b}(u)_t + \frac{1}{2}(s_{a,b}(u)^2)_x = \gamma_b^+ - \gamma_a^+. \quad (5.12)$$

*In other words, the solutions verifying (5.10) are also quasi-solutions of the inviscid Burgers equation in the sense of Panov [37, 39].*

*Proof.* Writing  $s_{a,b}(\cdot) = a + (\cdot - a)^+ - (\cdot - b)^+$  and using the representation theorem for nonnegative distributions, from (5.10) we infer (5.12) where for  $k \in \{a, b\}$  and  $\xi \in \mathcal{D}((0, 1) \times (0, T))$ ,  $\int_0^1 \int_0^T \xi d\gamma_k^+$  stands for the difference between the right-hand side and the left-hand side of (5.10) (for the choice of  $\cdot^+$  in  $\cdot^\pm$ ). We only have to estimate the total variation of  $\gamma_a^+, \gamma_b^+$  up to the boundaries. To do so, for a compact set  $K \subset [0, 1] \times [0, T]$  we fix some  $\xi \in C_c^\infty([0, 1] \times [0, T])$  with  $\xi \geq 0$ ,  $\xi = 1$  on  $K$ . Plugging it into the definition of  $\gamma_k^+$ , we find

$$\begin{aligned} |\gamma_k^+|(K) &\leq \int_0^1 \int_0^T \xi d\gamma_k^+ = \int_0^1 \int_0^T \left( (u - k)^+ \xi_t + q^+(u, k) \xi_x \right) dx dt + \int_0^1 (u_0 - k)^+ \xi(\cdot, 0) dx \\ &\quad + \int_0^T \frac{1}{2} \left( (k^- \mp b_0^+)^2 - k^2 \right)^+ \xi(0, t) dt + \int_0^T \frac{1}{2} \left( (k^+ \mp b_1^-)^2 - k^2 \right)^- \xi(1, t) dt \end{aligned} \quad (5.13)$$

where  $q^+(\cdot, k) = \frac{1}{2}\text{sign}^+(u - k)(u^2 - k^2)$ . For a fixed  $k \in \{a, b\}$  we find that  $\gamma_k^+(K)$  is finite, due to the assumption that the solution  $u$  and the data  $u_0, b_0, b_1$  are  $L^2$  functions.  $\square$

Based on Lemma 5.2, we can apply the result of [37] to give pointwise values to  $u$  on the lateral boundaries of the box  $(0, 1) \times (0, T)$  (an analogous claim is true at  $t = 0+$ , cf. [36] and Remark 3.4, but we do not directly exploit this feature).

**Corollary 5.5.** *An unbounded entropy solution of (5.9) in the sense of Definition 5.3 admits strong boundary traces in the  $L^1((0, T))$  sense at  $x = 0^+$  and  $x = 1^-$ .*

Now we are ready to connect the boundary terms of inequalities (5.10) to the usual pointwise BLN interpretation of the Dirichlet boundary conditions.

**Proposition 5.3.** *If  $u \in L^2((0, 1) \times (0, T))$  solves (5.9) in the sense of Definition 5.3 then its strong boundary traces  $u(0^+, \cdot)$  and  $u(1^-, \cdot)$  verify the so-called BLN conditions for a.e.  $t \in (0, T)$ :*

*if  $b_0(t) \leq 0$  then  $u(0^+, t) \leq 0$  and if  $b_0(t) > 0$  then either  $u(0^+, t) = b_0(t)$  or  $u(0^+, t) \leq -b_0(t)$ ,  
if  $b_1(t) \geq 0$  then  $u(1^-, t) \geq 0$  and if  $b_1(t) < 0$  then either  $u(1^-, t) = b_1(t)$  or  $u(1^-, t) \geq -b_1(t)$ .*

*Proof.* The existence of traces is guaranteed by Corollary 5.5. Pointwise characterizations of  $u(0^+, \cdot)$  and  $u(1^-, \cdot)$  can be shown using test functions  $\xi$  concentrating on the boundary like in [9]; a systematic case study of restrictions imposed by (5.10) for different values of  $k$  yields the claimed conclusion.

Note that another way to infer the claim of the Proposition is to replace, in the proof of [9, Proposition 3.3(iv)], equation [9, (3.5)] by our equation (5.10) adapted specifically to the Burgers flux and  $L^2$  data.  $\square$

Finally, we point out that our notion of solution is compatible with the renormalized solution framework; this will be needed in Proposition 5.8 (see however Remark 5.7), used in § 6.3 to achieve the optimal non-controllability result.

**Proposition 5.4.** *If  $u \in L^2((0, 1) \times (0, T))$  solves (5.9) in the sense of Definition 5.3, then it is also a renormalized solution of (5.9) in the sense of [43].*

*Proof.* Denote by  $\delta_0$ , respectively  $\delta_1$ , the Dirac delta functions of  $x$  charging the left, respectively the right, boundary of  $(0, 1) \times (0, T)$ . Write  $u \perp \ell$  for  $\min\{u, \ell\}$ ,  $u \top (-\ell)$  for  $\max\{u, (-\ell)\}$  and  $q^\pm(\cdot, k)$  for  $\text{sign}^\pm(\cdot - k)(\cdot^2 - k^2)/2$ . In view of [43, Definition 1], we need to check that for all  $\ell > 0$  the truncated solution  $u \perp \ell$  fulfills, for  $|k| < \ell$ ,

$$(u \perp \ell - k)_t^+ + \frac{1}{2}q^+(u \perp \ell, k)_x \leq \mu_\ell^+ + \ell \sum_{i=0,1} (b_i \perp \ell - k)^+ \delta_i, \quad (5.14)$$

for some locally bounded in  $[0, 1] \times [0, T)$  measure  $\mu_\ell^+$  with vanishing total variation<sup>3</sup> as  $\ell \rightarrow +\infty$ . An analogous property should be shown for  $(u \top (-\ell) - k)^-$  with some measure  $\mu_{-\ell}^-$  of vanishing total variation, as  $-\ell \rightarrow -\infty$ . We concentrate on (5.14).

For the proof, we first observe that  $(u \perp \ell - k)^+ = (u - k)^+ - (u - \ell)^+$  because  $k < \ell$ . With the arguments and the notation of Lemma 5.2, we readily write the following analogue of (5.14):

$$(u \perp \ell - k)_t^+ + \frac{1}{2}q^+(u \perp \ell, k)_x \leq \gamma_\ell^+ + \frac{1}{2} \sum_{i=0,1} R_i \delta_i,$$

---

<sup>3</sup>While the authors of [43] require the vanishing total variation on  $\bar{Q} := [0, 1] \times [0, T]$ , here we work locally on  $[0, 1] \times [0, T)$ , i.e., we fix some time horizon  $\tau < T$ . This is enough to apply the well-posedness results of [43].

with

$$R_0 = ((k^- \mp b_0^+)^2 - k^2)^+ - ((\ell^- \mp b_0^+)^2 - \ell^2)^+, \quad R_1 = ((k^+ \mp b_1^-)^2 - k^2)^- - ((\ell^+ \mp b_1^-)^2 - \ell^2)^-. \quad (5.15)$$

Therefore we will prove (5.14) with  $\mu_\ell^+ := \gamma_\ell^+$ . To check the vanishing, as  $\ell \rightarrow +\infty$ , of this measure's total variation we examine closer the expressions of measures  $\gamma_k^+$  obtained in the proof of Lemma 5.2. We fix  $\tau < T$  and take  $K = [0, 1] \times [0, \tau]$ . Because  $u$  and  $(b_0)^+$ ,  $(b_1)^-$  are assumed to belong to  $L^2$  in their domains of definition, (5.13) and the absolute continuity of the Lebesgue integral ensure the vanishing property  $|\gamma_\ell^+|([0, 1] \times [0, \tau]) \rightarrow 0$ , as  $\ell \rightarrow +\infty$ . From this we get the formulation of the form (5.14) in any time domain  $(0, \tau)$ ,  $\tau < T$ , but with  $(b_i \perp \ell - k)^+$  replaced with  $R_i$  in (5.15). A case study shows that  $R_0 \leq 2\ell(b_0 \perp \ell - k)^+$  and that  $R_1 \leq 0$ , thus concluding the proof.  $\square$

After these preliminaries, we turn to the well-posedness analysis of (5.9) in the setting of  $L^2$  entropy solutions with  $L^2$  or  $L^3$  boundary data.

**Theorem 5.5.** *Assume that  $u_0 \in L^2((0, 1))$  and let  $b_0, b_1$  be measurable functions with  $b_0^+, b_1^- \in L^2((0, T))$ . Then problem (5.9) admits at most one solution in the sense of Definition 5.3. Moreover, if we reinforce the assumption on the boundary data to  $b_0^+, b_1^- \in L^3((0, T))$ , then there exists a unique solution to (5.9). Moreover, the solution  $u$  is the pointwise (a.e.) limit, as  $M \rightarrow +\infty$ , of the sequence of  $(u^M)_M$  of Kruzhkov entropy solutions of the Cauchy-Dirichlet problem for the same conservation law with the truncated initial data  $u_0^M = \text{sign}(u_0) \min\{|u_0|, M\}$  and boundary data  $b_{0,1}^M = \text{sign}(b_{0,1}) \min\{|b_{0,1}|, M\}$ .*

*Further, the solution  $u$  belongs to  $L^\infty(0, T; L^2((0, 1))) \cap C(0, T; L^1((0, 1)))$ , it satisfies the weak formulation of problem (5.1) in  $\mathcal{D}'((0, 1) \times [0, T])$ , and it assumes the boundary data in the pointwise BLN sense of Proposition 5.3.*

One can also infer from the below proof that solutions for different data verify the comparison property of Remark 3.5. The  $L^1$  contraction property was explored by Porretta and Vovelle in [43] in the related setting of renormalized solutions; here we do not extend on this as it is not needed for the application we have in mind in Section 6. We note the fact that, in view of Proposition 5.4, the uniqueness claim could be deduced from the one of [43]. Also existence could be justified along the guidelines of [43], we would need however to guarantee the  $L^2$  integrability of the solutions and pass to the limit  $\ell \rightarrow \infty$  in (5.14). We will instead follow a more direct and much simpler than [43] approach since, due to Proposition 5.3 and the  $L^1$  integrability of the flux term  $u^2/2$ , we can avoid truncations.

*Proof of Theorem 5.5.* The arguments follow those of the proof of Theorem 5.1.

**Uniqueness.** Let  $u, \hat{u}$  be solutions of (5.1) in the sense of Definition 5.3 with the same initial and boundary data. Precisely as in the proof of Theorem 5.1, from the entropy formulation we obtain the Kato inequality (5.4) with  $C_c^\infty((0, 1) \times [0, T])$  test functions (so that the domain of integration in space is  $(0, 1)$  in place of  $\mathbb{R}^d$  in (5.4)). Taking instead of (5.5) smooth approximations of the Lipschitz test functions

$$\xi_h(x, t) = \frac{1}{h} \min\{h, x, 1 - x\}(e^{-t} - e^{-T})$$

and using Corollary 5.5, we find with the notation  $q(u, k) = \text{sign}(u - k)(u^2 - k^2)/2$ ,

$$\int_0^1 e^{-t} |u - \hat{u}|(x, t) dx \leq - \int_0^t (e^{-\tau} - e^{-T}) \left( q(u(0^+, \tau), \hat{u}(0^+, \tau)) + q(u(1^-, \tau), \hat{u}(1^-, \tau)) \right) d\tau. \quad (5.16)$$

It is well known (see e.g. [9] for an interpretation in terms of monotone graphs) that the BLN conditions stated in Proposition 5.3 ensure that the right-hand side of (5.16) is non-positive, thus

ensuring that  $u = \hat{u}$  a.e. in  $(0, 1) \times (0, T)$ . Note that the comparison principle is shown in the same way, using the  $(\cdot - k)^\pm$  variants of the entropy and the Kato inequalities.

**Existence.** Recalling (5.6), we take truncated data  $u_0^{M,N} = T^{M,N}(u_0)$ ,  $b_{0,1}^{M,N} = T^{M,N}(b_{0,1})$  and consider the inviscid Burgers equation in the usual BLN setting for these truncated initial and boundary data. We denote  $u^{M,N}$  the unique solution of this problem and recall that the comparison principle stated in Remark 3.5 holds true (we can also apply the comparison result claimed above). This permits to mimick the reasoning of the existence proof in Theorem 5.1, but we have to ensure that the construction leads to  $u^{M,N}$  satisfying a uniform in  $M, N$  bound in  $L^2((0, 1) \times (0, T))$ ; at this point, the  $L^3$  assumption on  $b_0^+, b_1^-$  appears from the estimate. To be precise, because  $u^{M,N}$  is a bounded entropy solution we can apply the classical entropy inequalities with the entropy  $\eta : z \mapsto z^2/2$  and the associated flux  $q : z \mapsto z^3/3$ . Taking  $\xi$  approaching the indicator function of the time interval  $[0, t]$ , bearing in mind the time continuity of  $u^{M,N}$  (Remark 3.4) we infer that for all  $t \in (0, T)$

$$\int_{(0,1)} |u^{M,N}(t, x)|^2 dx \leq \int_{(0,1)} |u_0^{M,N}(x)|^2 dx + \frac{2}{3} \int_0^t \left( (u^{M,N}(0^+, \tau))^3 - (u^{M,N}(1^-, \tau))^3 \right) d\tau.$$

Bearing in mind the BLN interpretation (see Proposition 5.3) of the boundary data  $b_{0,1}^{M,N}$ , we further find

$$\begin{aligned} \|u^{M,N}\|_{L^\infty(0,T;L^2((0,1)))}^2 &\leq \|u_0^{M,N}\|_{L^2((0,1))}^2 + \frac{2}{3} \int_0^t \left( (b_0(\tau)^+)^3 - (b_1(\tau)^-)^3 \right) d\tau \\ &\leq \|u_0\|_{L^2((0,1))}^2 + \frac{2}{3} (\|b_0^+\|_{L^3((0,T))}^3 + \|b_1^-\|_{L^3((0,T))}^3). \end{aligned}$$

This bound ensures that the limit  $\underline{u} = \lim_{M \rightarrow +\infty} \lim_{N \rightarrow +\infty} u^{M,N}$  belongs to  $L^2$  in the box – actually to  $L^\infty(0, T; L^2(0, 1))$  – and permits the passage to the limit in the half-entropy inequalities satisfied by  $u^{M,N}$ . We stress that the boundary conditions can be included into the formulation under the form (5.10), due to their relation to the pointwise BLN conditions (cf. [9, Proposition 3.3] for the converse statement of Proposition 5.3). Therefore,  $\underline{u}$  is an unbounded entropy solution of (5.9) in the sense of Definition 5.3. Like in the proof of Theorem 5.1, we introduce analogously  $\bar{u} = \lim_{N \rightarrow +\infty} \lim_{M \rightarrow +\infty} u^{M,N}$ , use the already proved uniqueness claim for (5.9) in our  $L^2$  setting to prove that  $\underline{u} = \bar{u}$ , and conclude that also  $(u^M)$  (as defined in the statement of the theorem) converges to  $u$  as  $M \rightarrow +\infty$ . This concludes the proof of existence.

**Weak formulation, time continuity and boundary data.** Going back to the above existence proof, writing the weak formulation for  $u^{M,N}$ , we infer the one for  $u^{\infty,N}$  and then the one for  $\bar{u} = u$  with the help of the dominated convergence arguments, like for the entropy formulation. Recall that we also inferred that  $u \in L^\infty(0, T; L^2((0, 1)))$  from the uniform bound on  $u^{M,N}$  in this space. Further, the boundary formulation in the pointwise BLN sense is contained in Proposition 5.3, which includes the existence of strong boundary traces of the unbounded entropy solution  $u$  due to Corollary 5.5. Here we point out that analogously to this corollary, existence of strong traces  $u(\cdot, t^\pm)$  in the  $L^1_{loc}((0, 1))$  sense, for all  $t \in [0, T)$  (with one-sided trace at  $t = 0^+$ ) follows from the quasi-solution formulation of Lemma 5.2 ([36, 37]). Formally extending the boundary data by 0 on  $(T, T + \delta)$ , we extend the construction of the solution beyond the time horizon  $T$ , so the argument actually applies up to  $t = T^-$ . For  $t \in (0, T)$ , the existence of  $u(\cdot, t^\pm)$  and the already justified weak formulation of (5.9) guarantee that  $u \in C([0, T]; L^1_{loc}((0, 1)))$ . Then the  $L^\infty(0, T; L^2((0, 1)))$

integrability of  $u$  yields the uniform in  $t$  estimate

$$\int_{(0,\delta)\cup(1-\delta,1)} |u(x,t)| dx \leq \sqrt{2\delta} \|u\|_{L^\infty(0,T;L^2((0,1)))}.$$

In particular,  $u \in C([0, T]; L^1((0, 1)))$ .  $\square$

### 5.3. One-dimensional instantaneous regularization effect

In this paragraph we point out that in the case of dimension  $N = 1$  and strictly convex flux, the solutions constructed in the previous paragraph have the same qualitative properties as the classical Kruzhkov entropy solutions as soon as we regard them locally inside  $I \times (0, T]$ ,  $I = \mathbb{R}$  or  $I = (0, 1)$ .

**Proposition 5.6.** *Let  $p \in (1, +\infty)$ ,  $p' = p/(p-1)$  and  $f \in C^2(\mathbb{R}; \mathbb{R})$  a strictly convex function satisfying the growth restrictions*

$$\exists A > 0 \forall u \in \mathbb{R} \quad |f(u)| \leq A(1 + |u|^p), \quad |f'(u)| \leq A(1 + |u|^{p-1}).$$

*Then the unique entropy solution  $u$  of the problem  $u_t + (f(u))_x = 0$  in  $\mathbb{R} \times (0, T)$ ,  $u(0, \cdot) = u_0 \in L^p(\mathbb{R})$  admits the representation*

$$u(x, t) = (f')^{-1}\left(\frac{x - y(x, t)}{t}\right) \text{ for a.e. } (x, t) \in \mathbb{R} \times (0, +\infty) \quad (5.17)$$

where for a.e.  $t > 0$ ,  $y(\cdot, t)$  is a non-decreasing function.

*In particular, it follows that for all  $t_0 > 0$  there holds  $u \in L^\infty(\mathbb{R} \times (t_0, T))$ , more precisely, there exists a constant  $B$  that depends on  $t_0, T, \|u_0\|_{L^p(\mathbb{R})}$  and  $f$  such that  $\|u\|_{L^\infty(\mathbb{R} \times (t_0, T))} \leq B$ .*

*Proof.* First, note that the Lax-Oleinik representation (5.17) is inherited by  $u = \lim_{M \rightarrow +\infty} u^M$  (the limit is taken in the pointwise sense, see Theorem 5.1) because the  $L^\infty$  entropy solutions  $u^M$  satisfy the Lax-Oleinik representation with  $y^M(x, t) = x - tf'(u^M(x, t))$  and  $f'$  is a continuous function. Indeed, it is enough to define

$$y(x, t) := x - tf'(u(x, t))$$

and observe that  $y = \lim_{M \rightarrow +\infty} y^M$  is monotone in  $x$ , for a.e.  $t$ .

Observe that the information from (5.7) along with our assumptions on  $f$  imply that  $f'(u) \in L^\infty(0, T; L^p_{loc}(\mathbb{R}))$ , with the bound depending on the  $f$ -specific constant  $A$  and on  $\|u_0\|_{L^p(\mathbb{R})}$ . Hence, Hölder's inequality implies the property

$$\exists C > 0 \int_{(x, x+1)} |f'(u(z, t))| dz \leq \int_{(x, x+1)} |f'(u(z, t))|^{p'} dz \leq C \quad (5.18)$$

uniformly with respect to  $t \in (0, T)$  and  $x \in \mathbb{R}$ , with  $C$  that depends on  $f, \|u_0\|_{L^p(\mathbb{R})}$ .

Now, fix  $R > 0$  and restrict our attention to  $x \in [-R, R]$ . Bearing in mind the expression (5.17) of  $u$  given  $y$ , in order to bound  $u$  in  $L^\infty([-R, R] \times [t_0, T])$  it is enough to show that an  $L^\infty$  bound by a constant depending on  $f, T$  and  $\|u_0\|_{L^p(\mathbb{R})}$  is valid for  $y$ ; the claimed  $L^\infty_{loc}$  result for  $u$ , with the additional dependence on the minimal allowed time  $t_0$ , then follows. We are intended to prove an upper bound on  $y(x, t)$ ; the lower bound is proved analogously. The functions  $y(\cdot, t)$  are nondecreasing and in particular,

$$y(z, t) \geq y(x, t) \text{ for all } z \in [x, x+1].$$

Then, by integrating and using the fact that  $y(z, t) = z - tf'(u(z, t))$ , we infer that

$$y(x, t) \leq \int_x^{x+1} y(z, t) dz \leq t \|f'(u(\cdot, t))\|_{L^1([x, x+1])} + \int_x^{x+1} z dz \leq TC + R + 1 \quad (5.19)$$

due to (5.18). This yields the desired bound, whence we conclude that  $\|u\|_{L^\infty([-R,R]\times[t_0,T])} \leq B$  with the constant  $B$  depending on  $f$ ,  $t_0$ ,  $T$ ,  $\|u_0\|_{L^p(\mathbb{R})}$  and the chosen fixed value  $R > 0$ . It remains to extend this bound from  $x \in [-R, R]$  to  $x \in \mathbb{R}$ . To this end, let us consider  $v$  defined, for some  $r \in \mathbb{R}$ , by

$$v(x, t) := u(x - r, t) \text{ for } (x, t) \in \mathbb{R} \times (0, T). \quad (5.20)$$

Note that  $v$  solves the same conservation law with  $v_0(\cdot) = u(\cdot - r)$  and thus  $\|u_0\|_{L^p(\mathbb{R})} = \|u_0\|_{L^p(\mathbb{R})}$ . Therefore there holds  $\|v\|_{L^\infty([-R,R]\times[t_0,T])} \leq B$ . Making  $r \in \mathbb{R}$  vary, we transfer the above proved  $L^\infty_{loc}$  bound on  $u$  from  $x \in [-R, R]$  to  $x \in [-R + r, R + r]$ ; in view of the arbitrariness of  $r$ , this yields the required bound  $\|u\|_{L^\infty(\mathbb{R}\times[t_0,T])} \leq B$ .  $\square$

**Remark 5.6.** *Note that combining the  $L^2 - L^\infty$  regularization with the classical  $L^\infty - BV$  regularization known for solutions of one-dimensional conservation laws with uniformly convex flux, we observe the instantaneous  $L^2 - BV$  regularization.*

The main property that we need from solutions, in the sense of Definition 5.1, of the Burgers equation with  $L^2$  data is the following.

**Corollary 5.7.** *Let  $u$  be the entropy solution of the problem  $u_t + (u^2/2)_x = 0$  in  $\mathbb{R} \times (0, T)$ ,  $u(0, \cdot) = u_0 \in L^2(\mathbb{R})$ . Then for all  $(x, t) \in \mathbb{R} \times (0, T]$  there exist minimal and maximal backward generalized characteristics in the sense of Dafermos [21, 22].*

*Proof.* The result comes from the fact that  $u$  is a classical Kruzhkov entropy solution to the Burgers equation for  $t \geq t_0$ , for all  $t_0$ . The backward characteristics (Dafermos [21], [22, Chapters 10, 11.1]) are then defined first for  $t \in [t_0, T]$  and then extended to  $t \in (0, T] = \cup_{t_0 > 0} [t_0, T]$  by the exhausting argument.  $\square$

Note that the analogous result holds for the Cauchy-Dirichlet problem for Burgers equation, as addressed in Theorem 5.5.

**Proposition 5.8.** *Let  $\theta > 0$  and  $u$  be the entropy solution of  $u_t + (\theta u^2/2)_x = 0$  in  $(0, 1) \times (0, T)$ ,  $u(0, \cdot) = u_0 \in L^2(\mathbb{R})$ ,  $u(0, \cdot) = b_0$  and  $u(1, \cdot) = b_1$  on  $(0, T)$  constructed in Theorem 5.5. Then  $u$  admits the representation*

$$u(t, x) = \frac{1}{\theta} \frac{x - y(x, t)}{t} \text{ for a.e. } (x, t) \in (0, 1) \times (0, +\infty) \quad (5.21)$$

where for a.e.  $t > 0$ ,  $y(\cdot, t)$  is a non-decreasing function. In particular, for all  $t_0 > 0$ ,  $0 < \delta < 1/2$  it holds that  $u \in L^\infty((\delta, 1 - \delta) \times (t_0, T))$ .

*Proof.* We treat the Burgers case  $\theta = 1$  which is our main objective, the general case playing only an auxiliary role. We have  $u \in L^2((0, 1) \times (0, T))$  that we regard here as  $L^2(0, 1; L^2((0, T)))$ . Exploiting the notion of Lebesgue points (see, e.g., [40] for details), one finds that for a.e.  $\delta \in (0, 1/2)$  the function  $u$  admits one-sided traces (in the sense of a.e. convergence)  $b_0 \in L^2((0, T))$  at  $x = \delta^+$  and  $b_1 \in L^2((0, T))$  at  $x = (1 - \delta)^-$  (we denote them  $b_{0,1}$  in view of the rescaling that will be operated from now on). In what follows we argue on the restriction  $u|_{(\delta, 1 - \delta) \times (0, T)}$  of  $u$ . Upon changing  $x$  into  $(x - \delta)/(1 - 2\delta)$  and rescaling the Burgers flux by the factor  $(1 - 2\delta)$ , we see that the rescaled function that we denote by  $u_\delta$  is an unbounded entropy solution of (5.9) where the Burgers flux is multiplied by a constant (which changes nothing to the theory developed in Theorem 5.5). By Proposition 5.4,  $u_\delta$  is also a renormalized solution of the same problem. Combining the uniqueness claim and the solution construction of [43] we know that  $u$  is a pointwise limit of  $L^\infty$  (standard Kruzhkov) entropy solutions  $u_\delta^M$  to Cauchy-Dirichlet problems with truncated initial and boundary

data  $u_0^M, b_{0,1}^M$ . According to Joseph and Gowda [30], each of  $u^M$  admits a representation of the form (5.21) with the appropriate value  $\theta = \theta_\delta$ . We have already seen that such representations are stable under the pointwise convergence, therefore  $u_\delta$  inherits such representation. As  $\delta \rightarrow 0$  we have  $\theta_\delta \rightarrow 1$  and  $u_\delta \rightarrow u$ , so that  $u$  inherits the representation (5.21) with  $\theta = 1$ .

With (5.21) at hand, the proof of the  $L^\infty(0, T; L^2((0, 1))) - L_{loc}^\infty((0, 1) \times (0, T])$  regularization follows the lines of the proof of Proposition 5.6, upon replacing  $[x_j, x_{j+1}]$  by the interval  $[x_j, x_j + \delta] \subset (0, 1)$ . As a concluding observation in relation to Remark 5.7, let us point out that under the additional assumption  $u \in L_{loc}^3((0, 1) \times [0, T])$  and using the same arguments, we can reduce the question to the setting of Theorem 5.5 and avoid the arguments referring to renormalized solutions.  $\square$

**Remark 5.7.** *The proof of Proposition 5.8 is the only point in the paper where we have to go beyond the unbounded solutions setting of § 5.2 and refer to fine properties of renormalized solutions in the sense of [43]. We can bypass this ingredient if we assume that  $u \in L_{loc}^3((0, 1) \times [0, T])$ . Here, by  $L_{loc}^3$  on  $(0, 1) \times [0, T]$  we mean the space of functions that are in  $L^3$  on any rectangle of the form  $(\delta, 1 - \delta) \times (0, T)$ ,  $\delta > 0$ .*

*Because in our application to non-controllability (Theorem 6.7, Corollary 6.8) we have to require the  $L_{loc}^3((0, 1) \times (0, T))$  integrability of solutions, our unbounded entropy solutions theory for (5.9) is very close to be enough (but not exactly enough: between  $L_{loc}^3((0, 1) \times [0, T])$  and  $L_{loc}^3((0, 1) \times (0, T))$  there is a gap) in order to attain the optimal result in § 6.3.*

Similarly to Corollary 5.7, we derive the following claim that will be instrumental in § 6.3. We stress that it concerns *local* unbounded entropy solutions of the inviscid Burgers equation (without any prescribed boundary behavior). Such solutions appear in our analysis of question (Q).

**Corollary 5.9.** *Let  $u \in L^2((0, 1) \times (0, T))$  be a local unbounded entropy solution (in the sense of (5.10) but with test functions that vanish on the lateral boundaries of  $(0, 1) \times (0, T)$ ) of the problem  $u_t + (u^2/2)_x = 0$  in  $(0, 1) \times (0, T)$ ,  $u(\cdot, 0) = u_0 \in L^2((0, 1))$ . Then for all  $(x, t) \in (0, 1) \times (0, T]$  there exist minimal and maximal backward generalized characteristics in the sense of Dafermos [21, 22].*

## 6. Non-attainability in the setting of unbounded entropy solutions

In this section we focus on problems  $(\mathbf{Pb}_{strip})$  and  $(\mathbf{Pb}_{box})$  in the setting of  $L^2$  solutions. As in all the results of this paper, we rely upon non-attainability results for the inviscid Burgers equation. Therefore, non-attainable states for  $(\mathbf{Pb}_{strip})$  belong to

$$\mathcal{NA}_T^{L^2, strip} := \left\{ u_T \in L^2(0, 1) \mid \nexists u \text{ solution in the sense of Definition 5.1} \right. \\ \left. \text{to problem } (\mathbf{Pb}_{strip}^0) \text{ with } u(\cdot, T) = u_T \text{ for } x \in (0, 1) \right\}, \quad (6.1)$$

while non-attainable states for  $(\mathbf{Pb}_{box})$  belong to the analogously defined set  $\mathcal{NA}_T^{L^2, box}$ . The invariance of the invoked solution notions under scaling (Zoom) leads to analogues of the rescaling property (3.4) for non-attainable sets of the inviscid Burgers case; this permits us to fix the reference time  $T = 1$  in the definition of non-attainability sets like (6.1).

The general part of the non-attainability statement of Theorems 3.3 and 4.1 is extended to non-attainability, by  $L^2$  solutions of  $(\mathbf{Pb}_{strip})$  or  $(\mathbf{Pb}_{box})$ , of weakly compact subsets of  $\mathcal{NA}_1^{L^2, box}$  or  $\mathcal{NA}_1^{L^2, strip}$ , respectively, where an appropriate amplification assumption uncovered through the scaling argument is mandatory. Note however that the box setting requires, at this stage, the extra  $L_{loc}^3$  regularization assumption on the solutions. We give the details in the two next paragraphs.



In order to conclude, from the above indicated non-attainability claims, that constants  $C \neq 0$ ,  $|C| \geq C_0$ , are not attainable by  $L^2$  weak solutions of  $(\mathbf{Pb}_{strip})$  or  $(\mathbf{Pb}_{box})$  for times  $T \leq 1/|C|$  (or even  $T < (1 + \Delta)/|C|$  for some  $\Delta = \Delta(L) > 0$ , cf. Proposition 4.2) one would need to show that constant states  $C \neq 0$  with  $|C| \leq 1$  (respectively,  $|C| < 1 + \Delta$ ) do belong to the sets  $\mathcal{NA}_1^{L^2, box}$ ,  $\mathcal{NA}_1^{L^2, strip}$ . We address this question in two distinct ways.

First, we invoke the property

$$\text{Int}\left(\mathcal{NA}_1^{L^\infty, strip}\right) \subset \mathcal{NA}_1^{L^2, strip} \quad (6.2)$$

where  $\text{Int}$  denotes, from now on, the topological interior with respect to the  $L^1$  convergence. The analogous statement holds in the box setting. Observation (6.2) (proved in Lemma 6.3 below) follows from the limit connection between bounded (Kruzhkov) entropy solutions and their unbounded analogues pointed out in the construction of Section 5. Note that both the approximation and the uniqueness claims from Theorem 5.1 are required for this argument. With (6.2) at hand, it is enough to notice that constant states  $C$  with  $\alpha < |C| < \beta$  belong to the interior of  $\mathcal{NA}_1^{L^\infty, strip}$ ,  $\mathcal{NA}_1^{L^\infty, box}$  in order to negatively answer the strip (respectively, the box) interpretation of (the quantitative version of)  $(\mathbf{Q})$ . This is done in § 6.1, along with generalizations (see Remarks 3.8 and 4.1).

Moreover, in § 6.3 we directly address the generalization of Proposition 3.1 claiming that the states of the form  $(\mathbf{NA})$  (see Proposition 3.1) with  $0 < x^* < 1$  (including constants  $C$ ,  $0 < |C| < 1$ ) fall not only within  $\mathcal{NA}_1^{L^\infty, box}$  but also within  $\mathcal{NA}_1^{L^2, box}$  (and consequently, within  $\mathcal{NA}_1^{L^2, strip}$  as well). To this end, the regularity claims of Proposition 5.8 and Corollary 5.9 stated in Section 5 are instrumental.

### 6.1. Robustness of non-attainability for the inviscid Burgers equation

The set  $\mathcal{NA}_1^{L^\infty, box}$  has a large interior with respect to the  $L^1$  (or to the a.e.) convergence, so that one can say that non-attainability for the inviscid Burgers equation is a robust property allowing, for instance, the extension to unbounded entropy solutions.

Specifically, we point out the following observation already mentioned in Remark 3.8.

**Proposition 6.1.** *The following subsets of  $\mathcal{NA}_1^{L^\infty, box}$  belong to the interior of  $\mathcal{NA}_1^{L^\infty, box}$  with respect to the  $L^1((0, 1))$  topology:*

- (i)  $\mathring{\mathcal{K}}_{\alpha, \beta} := \left\{ u : x \mapsto C \mid \alpha < |C| < \beta \right\}$ , for any given  $\alpha, \beta$  with  $0 < \alpha < \beta \leq 1$ ;
- (ii)  $\mathring{\mathcal{K}}_{E, m(\cdot)}^+ := \left\{ u \in L^\infty((0, 1)) \mid \forall x \in E \ m(x) < u(x) < x \right\}$ ,  
for a given  $E \subset (0, 1)$  of non-zero Lebesgue measure and a given measurable  $m : E \rightarrow (0, 1]$ ;
- (iii)  $\mathring{\mathcal{K}}_{E, m(\cdot)}^- := \left\{ u \in L^\infty((0, 1)) \mid \forall x \in E \ - (1 - x) < u(x) < -m(x) \right\}$ , for  $(E, m(\cdot))$  like in (ii).

Note that the elementary inclusion (4.1) readily yields the analogous statement in the strip:

**Corollary 6.2.** *The sets  $\mathring{\mathcal{K}}_{\alpha, \beta}$ ,  $\mathring{\mathcal{K}}_{E, m(\cdot)}^\pm$  defined in Proposition 6.1 belong to the interior of  $\mathcal{NA}_1^{L^\infty, strip}$  with respect to the  $L^1((0, 1))$  topology.*

*Proof of Proposition 6.1.* We give the proof for  $\mathring{\mathcal{K}}_{E, m(\cdot)}^+$ , the other cases being entirely similar.

Let  $\mu > 0$  denote the Lebesgue measure  $\text{meas}(E)$  of the set  $E$ . Upon setting for  $\gamma > 0$ ,

$$E^\gamma := \left\{ x \in E \mid m(x) + \gamma < u(x) < x - \gamma \right\},$$

the sets  $E^\gamma$  are nested and their union over all  $\gamma > 0$  gives the whole set  $E$ . It follows by the continuity of the Lebesgue measure that  $\text{meas}(E^\gamma) \rightarrow \text{meas}(E)$  as  $\gamma \rightarrow 0$ , so that  $\text{meas}(E^{\bar{\gamma}}) \geq \mu/2$  for some  $\bar{\gamma} > 0$  which we fix.

Let  $v \in L^1((0, 1))$  be such that  $\|v - u\|_{L^1((0,1))} \leq \bar{\gamma}\mu/4$ . The Chebyshev inequality yields

$$\text{meas}(F_v^{\bar{\gamma}}) \leq \mu/4, \quad F_v^{\bar{\gamma}} := \{x \in (0, 1) \mid |v(x) - u(x)| \geq \bar{\gamma}\}$$

which implies that  $m(x) < v(x) < x$  at least on  $E_v := E^{\bar{\gamma}} \setminus F_v^{\bar{\gamma}}$ , where  $\text{meas}(E_v) \geq \mu/2 - \mu/4 = \mu/4 > 0$ . Therefore we can conclude that  $v \in \mathcal{K}_{E_v, m(\cdot)}^+ \subset \mathcal{NA}_1^{L^\infty, \text{box}}$ , i.e. the ball of  $L^1((0, 1))$  centered at  $u$ , of radius  $\bar{\gamma}\mu/4$ , lies in  $\mathcal{NA}_1^{L^\infty, \text{box}}$ . This completes the proof.  $\square$

**Lemma 6.3.** *Property (6.2) holds true, that is*

$$\text{Int}\left(\mathcal{NA}_1^{L^\infty, \text{strip}}\right) \subset \mathcal{NA}_1^{L^2, \text{strip}}.$$

We remark that the uniqueness claim of Theorem 5.1 is exploited in the proof of the above lemma. As a matter of fact, we need that unbounded  $L^2$  solutions to the Burgers equation enter some framework where uniqueness is guaranteed in order to connect their properties to classical properties of Kruzhkov entropy solutions. A similar situation occurred in Proposition 5.8 which is crucial for the results of § 6.3.

*Proof of Lemma 6.3.* Recall that for general sets  $A, B$  in a topological space,  $\text{Int}(A) \subset B$  is equivalent to  $B^c \subset \overline{A^c}$ . Therefore it is enough to justify that states  $u_1$  attainable by  $L^2$  unbounded entropy solutions of  $(\mathbf{Pb}_{\text{strip}}^0)$  are limits, in the  $L^1$  topology, of states attainable by bounded (Kruzhkov) entropy solutions of  $(\mathbf{Pb}_{\text{strip}}^0)$ . This follows from the fact that *any* such unbounded solution  $u$  is the  $C(0, T; L_{loc}^1(\mathbb{R}^d))$  limit of bounded solutions  $u^M$  corresponding to truncated data, see the last claim of Theorem 5.1.  $\square$

## 6.2. Non-attainability: extension to $L^2$ setting in the strip

We start by the following – straightforward but quite technical – extension of the claim of Theorem 3.3 to the  $L^2$  setting.

**Theorem 6.4.** *Let  $\mathcal{K}$  be a subset of  $\mathcal{NA}_1^{L^2, \text{strip}}$  compact in the weak topology of  $L^2((0, 1))$ . Let  $L \geq 1$ . Then there exists a constant  $\epsilon_0 > 0$  (depending on  $\mathcal{K}$  and  $L$  only) such that for all couples  $(T, u_T) \in (0, \infty) \times L^2((0, 1))$  satisfying  $Tu_T \in \mathcal{K}$ ,  $T \leq \epsilon_0$  problem  $(\mathbf{Pb}_{\text{strip}})$  has no weak solution  $u \in L^2(\mathbb{R} \times (0, T)) \cap L^2(0, T; H^1(\mathbb{R}))$  with initial data  $u_0$  satisfying the amplification assumption*

$$\|u_0\|_{L^2(\mathbb{R})} \leq L\|u_T\|_{L^2((0,1))}. \quad (6.3)$$

**Remark 6.1.** *In the above statement we limit our framework to the  $L^2(\mathbb{R} \times (0, T)) \cap L^2(0, T; H^1(\mathbb{R}))$  weak solutions of the Burgers equation in the strip corresponding to  $L^2(\mathbb{R})$  initial data; the proof indicates that we could as well ask that  $u$  belongs to  $L^\infty(0, T; L^2(\mathbb{R}))$  rather than to  $L^2(\mathbb{R} \times (0, T))$ , since the bounds (6.4) below are automatically satisfied in our framework.*

*Proof of Theorem 6.4.* The proof follows the lines of the proof of Theorem 3.3, the delicate point being to justify the analogue of Lemma 3.2 (recall that it is stated for fixed time  $T = 1$  and variable viscosity parameter  $\epsilon$ ). Arguing by contradiction, we assume targets  $u_1^\epsilon \in \mathcal{K}$  are attainable at time  $T = 1$  by solutions of

$$u_t + (u^2/2)_x = \epsilon u_{xx} \quad \text{in } D = \mathbb{R} \times (0, 1) \quad (\mathbf{Pb}_{\text{strip}}^\epsilon)$$

starting from some initial data  $u_0^\epsilon \in L^2(\mathbb{R})$ ,  $u_0^\epsilon = 0$  in  $(0, 1)$ . We use the boundedness of  $\mathcal{K}$  and the bound (6.3) to extract a (not relabelled) weakly convergent in  $L^2((0, 1))$  sequence  $(u_1^\epsilon)$

corresponding to the weakly converging in  $L^2(\mathbb{R})$  sequence of initial data  $(u_0^\epsilon)$ ; denote by  $u_1 \in \mathcal{K}$ ,  $u_0 \in \mathcal{K}$  their respective weak limits.

We also need to extract a convergent subsequence from  $(u^\epsilon)$  in the strip  $\mathbb{R} \times (0, 1)$ . At this point, it is essential to observe that initial data  $u_0^\epsilon \in L^2(\mathbb{R})$  give rise to a unique  $L^2(\mathbb{R} \times (0, 1))$  weak solution  $u^\epsilon$  of the viscous Burgers problem  $(\mathbf{Pb}_{strip}^\epsilon)$  satisfying in addition

$$\|u^\epsilon\|_{L^\infty(0,1;L^2(\mathbb{R}))} \leq \|u_0^\epsilon\|_{L^2(\mathbb{R})} \leq \text{const}, \quad \|\epsilon|u_x^\epsilon|^2\|_{L^1(\mathbb{R} \times (0,1))} \leq \|u_0^\epsilon\|_{L^2(\mathbb{R})} \leq \text{const}. \quad (6.4)$$

The above bounds follow from the combination of two arguments, quite similarly to what is done in the proof of Theorem 5.1 for the inviscid case.

- We rely upon the construction of solutions via approximation by truncated data, being understood that estimates (6.4) are standard for the resulting  $L^2 \cap L^\infty$  solutions and they are uniform with respect to the approximation (truncation) parameters. Indeed, starting from (3.6), having in mind the uniform in  $t$  bounds on solutions in both  $L^\infty(\mathbb{R})$  and  $L^1(\mathbb{R})$ , we can pass to the limit and get the analogue of (3.6) with space-independent test functions (using e.g. (5.5) like in the proof of Theorem 5.1). To sum up, the convective term in  $(\mathbf{Pb}_{strip}^\epsilon)$  – as soon as a meaning can be given to it – does not contribute to the energy estimates. Let us stress that this feature is closely related to the ideas of Boccardo, Murat et al. that eventually led to the development of the theory of renormalized solutions to elliptic and parabolic PDEs, see in particular [15, 34].
- We invoke the uniqueness of  $L^2(\mathbb{R} \times (0, 1)) \cap L^2(0, T; H^1(\mathbb{R}))$  weak solutions to  $(\mathbf{Pb}_{strip}^\epsilon)$  for given  $u_0^\epsilon \in L^2$ . The uniqueness claim is proved following the argument borrowed from [7]. It is based upon the Kato inequality derived with the method of Carrillo [17] from the entropy formulation of the viscous Burgers equation (inequalities (2.2) of Definition 2.1). To conclude the argument one picks the specific choice (5.5) of the test functions in the Kato inequality (see the proof of Theorem 5.1); at this point, the assumption  $u \in L^2(0, 1; H^1(\mathbb{R}))$  contained in the statement of the theorem is used while letting  $R \rightarrow \infty$ .

In view of the amplification assumption (6.3) (the latter being invariant under the scaling (Zoom)), the estimates (6.4) yield uniform  $L^\infty(0, 1; L^2(\mathbb{R})) \cap L^2(0, 1; H^1(\mathbb{R}))$  estimates on the sequence  $(u^\epsilon)$ . Moreover, the global  $L^1(\mathbb{R} \times (0, 1))$  estimate of  $\epsilon|u_x^\epsilon|^2$  replaces the localized estimate (3.6) of the proof of Lemma 3.2. Using the embedding of  $H^1(\mathbb{R})$  into  $L^\infty(\mathbb{R})$  and the interpolation of  $L^\infty(0, T; L^2(\mathbb{R}))$  and  $L^2(0, T; L^\infty(\mathbb{R}))$ , we find that

$$(u^\epsilon) \text{ is bounded in } L^4(0, T; \mathbb{R}). \quad (6.5)$$

To conclude the proof, we replace the compensated compactness argument of Lemma 3.2 by a more general compactness argument suitable for uniformly bounded in  $L^p$ ,  $p > 2$ , solutions (in place of uniformly bounded in  $L^\infty$  solutions). We choose to exploit the argument based on quasi-solutions, following Panov [39, Remark 3] as detailed in [38, Theorem 6]. To this end, we exploit the  $[a, b]$ -valued truncation of the identity map  $s_{a,b}$ , as introduced in (5.11). For fixed  $a < b$ , we apply  $s_{a,b}$  to the sequence  $(u^\epsilon)$  and write the resulting weak formulation. To do so, first we combine the weak and the entropy formulations of Definition 2.1 for  $C_c^\infty(\mathbb{R} \times (0, 1))$  test functions to get the local “half-entropy” inequalities (cf. [44, 17, 43, 9]). Using the fact that nonnegative distributions are locally finite measures, we find that for all  $k \in \mathbb{R}$  there exist locally finite nonnegative measures  $\gamma_k^{\epsilon,\pm}$  on  $\mathbb{R} \times (0, 1)$  such that

$$(u^\epsilon - k)^\pm_t + \frac{1}{2}(\text{sign}^\pm(u^\epsilon - k)((u^\epsilon)^2 - k^2))_x = \epsilon(u^\epsilon - k)^\pm_{xx} - \gamma_k^{\epsilon,\pm} \text{ in } \mathcal{D}'(\mathbb{R} \times (0, 1)). \quad (6.6)$$

Then, writing  $s_{a,b}(\cdot) = a + (\cdot - a)^+ - (\cdot - b)^+$ , we find that  $u^\epsilon$  is a quasi-solution of the inviscid

Burgers equation in the sense introduced by Panov [37, 39]: for all  $a < b$ , there holds

$$s_{a,b}(u^\epsilon)_t + \frac{1}{2}(s_{a,b}(u^\epsilon)^2 = \gamma_b^{\epsilon,+} - \gamma_a^{\epsilon,+} - \sqrt{\epsilon}(\sqrt{\epsilon}s_{a,b}(u^\epsilon)_x)_x =: r_{a,b}^\epsilon \quad (6.7)$$

in the sense of distributions. We intend to apply [38, Theorem 6(a)] (see also [39, Lemma 5, Remark 3]) to ensure that  $u^\epsilon$  converge to a measurable function  $u$  a.e. on  $\mathbb{R} \times (0, 1)$  and also (similarly to the statement of [38, Theorem 6(b)]) in  $L^2(\mathbb{R} \times (0, 1))$ .

The flux  $u \mapsto u^2/2$  of the Burgers equation being non-degenerately nonlinear, in order to apply Panov's precompactness result it is enough to check that given  $a < b$ , the family  $(r_{a,b}^\epsilon)$  is precompact in  $W_{loc}^{-1,d}(\mathbb{R} \times (0, 1))$  for some  $d > 1$ . To do so, we observe that the uniform  $L^2$  bound on  $(u^\epsilon)$  and the argument of [39, Lemma 5] yield a uniform in  $\epsilon$  bound on the total variation of the measures  $\gamma_b^+$ ,  $\gamma_a^+$  on every compact subset  $K$  of  $\mathbb{R} \times (0, 1)$ . This is analogous to the argument in Lemma 5.2. Indeed, fixing  $\xi \in C_c^\infty(\mathbb{R} \times (0, 1))$ ,  $\xi \geq 0$  with  $\xi = 1$  on  $K$  and  $\text{supp}(\xi) = F_\xi$ , from (6.6) we have

$$\begin{aligned} |\gamma_b^{\epsilon,+}|(K) &\leq \int_{\mathbb{R}} \int_0^1 \xi \, d\gamma_b^{\epsilon,+} = \int_{\mathbb{R}} \int_0^1 (u^\epsilon - b)^+ \xi_t + \frac{\text{sign}^+(u^\epsilon - b)((u^\epsilon)^2 - b^2)}{2} \xi_x + \epsilon(u^\epsilon - b)^+ \xi_{xx} \, dx \, dt \\ &\leq \max\{\|\xi_t\|_\infty, \|\xi_x\|_\infty, \|\xi_{xx}\|_\infty\} \left( (1+\epsilon)\|u^\epsilon\|_{L^1(F_\xi)} + \frac{1}{2}\|u^\epsilon\|_{L^2(F_\xi)}^2 + ((1+\epsilon)b + b^2/2)\text{meas}(F_\xi) \right), \end{aligned} \quad (6.8)$$

which is bounded uniformly in  $\epsilon$ . An analogous estimate holds for  $|\gamma_a^{\epsilon,+}|(K)$ . Writing  $\mathcal{M}_{loc}(\mathbb{R} \times (0, 1))$  for the space of bounded Radon measures, we use the compact embedding  $\mathcal{M}_{loc}(\mathbb{R} \times (0, 1)) \subset W_{loc}^{-1,d}(\mathbb{R} \times (0, 1))$  for  $1 \leq d < 2$  as in the proof of [38, Theorem 1]. In order to justify that  $(r_{a,b}^\epsilon)$  is relatively compact, it remains to recall that from (6.4) and the obvious bound  $|s_{a,b}(u^\epsilon)_x| \leq |u_x^\epsilon|$ , we find that  $(\sqrt{\epsilon}s_{a,b}(u^\epsilon)_x)$  is bounded in  $L^2(\mathbb{R} \times (0, 1)) \subset L_{loc}^d(\mathbb{R} \times (0, 1))$  so that  $\sqrt{\epsilon}(\sqrt{\epsilon}s_{a,b}(u^\epsilon)_x)_x \rightarrow 0$  in  $W_{loc}^{-1,d}(\mathbb{R} \times (0, 1))$ . At this point, we know that for all  $a < b$ ,  $a, b \in \mathbb{Q}$ ,  $s_{a,b}(u^\epsilon)$  converge in  $L_{loc}^1$  and a.e. on  $\mathbb{R} \times (0, 1)$  up to a subsequence. Following a diagonal argument, and based on the fact that for  $a \leq a' < b' \leq b$  there holds  $s_{a',b'}(r) = s_{a',b'}(s_{a,b}(r))$ , we deduce the a.e. convergence of (a non relabelled subsequence of)  $u^\epsilon$  to some measurable function  $u$  on  $\mathbb{R} \times (0, 1)$ .

In view of the bound (6.5), the de la Vallée Poussin criterion of equi-integrability and the Vitali convergence theorem imply, like in [38, Theorem 6(b)], that  $(u^\epsilon)$  converges to  $u$  in  $L^p(\mathbb{R} \times (0, 1))$  for all  $p < 4$ , and in particular for  $p = 2$ . Now, we can pass to the limit in the entropy inequalities (2.2) of Definition 2.1, where we have to limit our attention to test functions vanishing at  $t = 0$ . Using once more the  $L^1$  bound on  $\epsilon|u_x^\epsilon|^2$ , we infer that  $u$  is an (unbounded) entropy solution of the inviscid Burgers equation in the open domain  $\mathbb{R} \times (0, 1)$ . Passing similarly to the limit in the weak formulation (2.1) with test functions localized to  $(x, t) \in (0, 1) \times [0, 1]$ , we find that  $u$  assumes in the weak sense the initial data  $u_0 = 0$  and the terminal data  $u_1$  (obtained as the weak limit of  $u_1^\epsilon$ ) for  $x \in (0, 1)$ . The delicate point now is to prove that  $u$  is the unbounded entropy solution of the inviscid Burgers equation corresponding to the initial data  $u_0$  (obtained as the weak limit of  $u_0^\epsilon$ ) in the entropy sense, on the whole of  $\mathbb{R}$ . We refer to the example [16, pp. 53-54] to highlight the importance of this issue. Indeed, the initial data – particularly for unbounded solutions – should be imposed in the entropy sense (or in the equivalent  $\text{esslim}_{t \rightarrow 0^+}$  sense, like in the original setting of [31]) in order to guarantee the uniqueness statement of Theorem 5.1. To this end, passing to the limit in (2.1) of Definition 2.1 with  $\xi \in C_c^\infty(\mathbb{R} \times [0, 1])$  we find that the initial data  $u_0$  are assumed in the weak sense. Now, we exploit the strong traces result of Panov [37] for quasi-solutions. Going back to (6.7), we can pass to the limit in  $\mathcal{D}'$  using the uniform local bounds like (6.8), and infer

that

$$s_{a,b}(u)_t + \frac{1}{2}(s_{a,b}(u)^2)_x = \gamma_b^+ - \gamma_a^+$$

with some locally finite nonnegative measures  $\gamma_b^+, \gamma_a^+$ . In order to apply the result of [37] and infer the existence of a strong (in the  $L^1_{loc}(\mathbb{R})$  sense) trace  $\tau_{a,b}u$  of  $s_{a,b}(u)$ , it remains to refine (6.8) by letting  $\xi$  be non-zero at  $t = 0$ . In this case the additional term  $(u_0^\epsilon - b)^+\xi(\cdot, 0)$  appears in the right-hand side of (6.8); given  $b$  and  $\xi$  (i.e. given the compact  $K \subset \mathbb{R} \times [0, 1)$  on which the variation of  $\gamma_b^+$  is estimated) this term is bounded uniformly in  $\epsilon$ . Thus  $\gamma_b^+$  is locally bounded up to  $t = 0$ , as required by the assumptions of [37], and the same argument applies to  $\gamma_a^+$ . Starting from the traces  $\tau_{a,b}u$ , using once more the diagonal construction while passing to the limit as  $a, b \in \mathbb{Q}$  tend to  $-\infty, +\infty$ , respectively, we infer the existence of a trace  $\tau u$  at  $t = 0$  in the sense of the a.e. convergence. Bearing in mind that  $u \in L^\infty(0, 1; L^2(\mathbb{R}))$  in view of (6.4), we get a uniform in  $t \in (0, 1)$  bound in  $L^2(\mathbb{R})$  on the family  $(u(t, \cdot))$ , which ensures that the trace  $\tau u$  is also assumed in the  $L^1_{loc}$  sense, like in [37]. It remains to notice that  $\tau u = u_0$  by identification from the weak formulation satisfied by  $u$ . This concludes the passage to the limit in (2.2) of Definition 2.1 with  $\xi \in C_c^\infty(\mathbb{R} \times [0, 1))$ . Also note that  $u_0 = 0$  in  $(0, 1)$  because  $u_0^\epsilon = 0$  in  $(0, 1)$  for all  $\epsilon > 0$ .

Therefore,  $u$  is an unbounded entropy solution of the inviscid Burgers equation for the initial data  $u_0 \in L^2(\mathbb{R})$  satisfying  $u_0 = 0$  in  $(0, 1)$ , its terminal state at  $T = 1$ , for  $x \in (0, 1)$ , being  $u(\cdot, 1) = u_1 \in \mathcal{NA}_1^{L^2, strip}$ . This contradicts the definition of the non-attainability set  $\mathcal{NA}_1^{L^2, strip}$ , which concludes the justification of the required analogue of Lemma 3.2 and the proof of Theorem 6.4.  $\square$

**Remark 6.2.** *Note that in view of estimate (6.5), we could address the passage to the limit in  $u^\epsilon$  via the results of Szepessy [46]. Indeed, the  $L^4$  bound ensures that we are in the subcritical setting (cf. the discussion of the above Theorem 5.1). The strong convergence would then come as a consequence of reduction of a measure-valued solution  $\nu_{(x,t)}$  corresponding to some initial data  $u_0$  (which is a trivial Young measure, i.e. a function in the usual sense). The obstacle in doing so is that we need to know, beforehand, that  $(u_0^\epsilon)$  converges to some limit  $u_0$  in the  $L^1_{loc}$  sense. Unfortunately, in the setting of measure-valued solutions employed in [46], strong trace results exploited in the above proof are not available. Therefore the line of argumentation from [46] does not seem to be applicable.*

Combining Theorem 6.4, Lemma 6.3, the result of Corollary 6.2 and the argument of the proof of Corollary 3.5, we get the following negative answer to (the quantitative version of) the question (Q), in its strip interpretation.

**Corollary 6.5.** *For given  $0 < \alpha \leq \beta < 1$  there exists  $\epsilon_0 = \epsilon_0(\alpha, \beta, L)$  such that for all  $T \leq \epsilon_0$  and all  $C$  with  $\alpha T^{-1} \leq |C| \leq \beta T^{-1}$ , the constant state  $C$  is not attainable at time  $T$  for problem (Pb<sub>strip</sub>) in the weak  $L^2$  setting, with initial data fulfilling the amplification restriction  $\|u_0\|_{L^2(\mathbb{R})} \leq L|C|$ .*

**Remark 6.3.** *In the spirit of Remark 4.1, in the statement of Corollary 6.5 one can actually go up to  $\beta = 1$  and even slightly beyond. To this end, one has to replace the  $L^\infty$ -based estimates*

$$|\tilde{u}_0(x)| = |u(x)| \leq L\|u_T\|_\infty, \quad \int_{x_*}^0 |\tilde{u}_0(x)| \, dx \leq L\|u_T\|_\infty |x_*|$$

*in the proof of Proposition 4.4 by the  $L^2$ -based analogue*

$$\int_{x_*}^0 |\tilde{u}_0(x)| \, dx \leq \sqrt{|x_*|} \left( \int_{x_*}^0 |\tilde{u}_0(x)|^2 \, dx \right)^{1/2} \leq L\|u_T\|_{L^2((0,1))} \sqrt{|x_*|},$$

and follow the guidelines of the proof of Proposition 4.4. One ends up with the restriction  $C < 1 + \Delta$ , for some  $\Delta = \Delta(L) > 0$  in the case of a constant target  $u_1 = C$  (thus, with  $m = C$  and  $M = 1$  in the notation of Proposition 4.4).

More general targets of the type (NA) can be dealt with in this way. We do not pursue this line of investigation because we can also address this issue directly, by adapting the proof of Proposition 3.1 to unbounded entropy solutions of  $(\mathbf{Pb}_{strip}^0)$ . We will do so in the more general box setting in § 6.3 below.

### 6.3. Non-attainability by $L^2 \cap L^3_{loc}$ solutions in the box

With the help of Corollary 5.9, we can revisit the non-attainability result of Proposition 3.1, i.e. for the inviscid Burgers equation in the  $L^2$  setting, as follows.

**Proposition 6.6.** *Assume that  $x^*$  in (NA) fulfils  $0 < x^* < 1$ . Then the claim of Proposition 3.1 remains valid if one replaces local Kruzhkov entropy solutions by unbounded entropy solutions in  $L^2((0, 1) \times (0, 1))$ , i.e. one can replace  $\mathcal{NA}_1^{L^\infty, box}$ ,  $\mathcal{NA}_T^{L^\infty, box}$  by  $\mathcal{NA}_1^{L^2, box}$ ,  $\mathcal{NA}_T^{L^2, box}$ , respectively, in the conclusion of Proposition 3.1.*

*Proof.* The assumption  $x^* < 1$  is used in order to ensure, via the application of Corollary 5.9, that the maximal backward generalized characteristics can be utilized; indeed, recall that we can define them in sets  $[\delta, 1 - \delta] \times [\delta, 1]$  where our unbounded entropy solution is actually bounded, and then let  $\delta \rightarrow 0$ .

Then we pursue the construction of the auxiliary solution  $\tilde{u}$ . The definition of  $\tilde{u}$  in the proof of Proposition 3.1 does not lead to a contradiction in our new setting of unbounded entropy solutions; indeed, we cannot rely any more upon the property of finite speed of propagation, specifically as  $t$  approaches  $0^+$  or as  $x$  approaches  $1^-$  (we stress the local character of the regularization in Proposition 5.6). Therefore we deeper analyze and modify the construction of  $\tilde{u}$ .

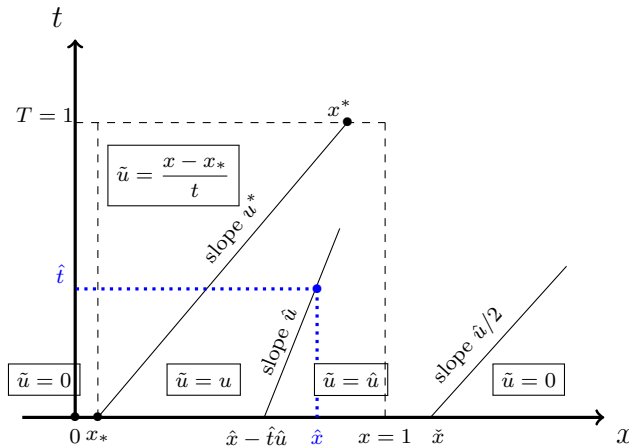


Figure 4: Modification of the auxiliary function  $\tilde{u}$  (cf. Fig. 1): construction based upon the existence of a point  $(\hat{x}, \hat{t}) \in (0, 1) \times (0, 1)$  with  $\hat{x} > x^* - (1 - t)u^*$  such that  $\hat{u} := u(\hat{x}, \hat{t}) > 0$ ; the point  $\tilde{x} > \hat{x}$  is arbitrary.

To this end, first observe that we can assume that  $u$  in the region below the characteristics  $x = x_* + tu^*$ , normalized by the left continuity (which is possible due to the  $BV_{loc}$  regularity, see Remark 5.6) takes a nonnegative value  $\hat{u}$  at least at some point  $(\hat{x}, \hat{t})$ ,  $0 < \hat{t} < 1$ ,  $x_* + \hat{t}u^* < \hat{x} < 1$ . Indeed, in the contrary case  $\tilde{u}$  undergoes a jump across the line  $x = \gamma(t) := x_* + tu^*$  from the value  $u^* > 0$  to some values  $u_*(t) := \lim_{x \rightarrow x_* + tu^*} u(x, t)$ ,  $u_*(t) \leq 0$ . This is prohibited by the

Rankine-Hugoniot conditions (recall that  $\tilde{u}$  is a classical entropy solution locally in the box, due to Proposition 5.6) because we have  $\gamma'(t) = u^* > (u^* + u_*(t))/2$ .

Now we modify  $\tilde{u}$  to the right from the minimal backward characteristic  $x = \hat{x} - (\hat{t} - t)\hat{u}$ , by setting it equal to the constant  $\hat{u}$  for all  $(x, t)$  to the left of the line  $x = \tilde{x} + t\hat{u}/2$  where  $\tilde{x} \in (\hat{x}, +\infty)$  is arbitrary (see Figure 4). The construction is completed by setting  $\tilde{u} = 0$  to the right of the line  $x = \tilde{x} + t\hat{u}/2$ , leading to an entropy-admissible jump across this line.

The resulting function  $\tilde{u}$  is glued continuously from pieces of (bounded or unbounded) entropy solutions, therefore it is an (unbounded)  $L^\infty(0, \hat{t}; L^2(\mathbb{R}))$  entropy solution to the Burgers equation. Moreover, it assumes the  $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  initial data  $u_0 = \hat{u}$  for  $x \in (\hat{x} - \hat{t}\hat{u}, \tilde{x})$ , while  $u_0 = 0$  otherwise. According to the uniqueness claim of Theorem 5.1,  $\tilde{u}$  should coincide with the classical solution of the Burgers equation with these initial data. However, unlike  $\tilde{u}$ , this classical solution lacks the rarefaction centered at  $(x_*, 0)$ . This contradiction completes the proof.  $\square$

Now, we return to the viscous problem ( $\mathbf{Pb}_{box}$ ). We have the following analogue of Theorems 3.3, 4.1, 6.4.

**Theorem 6.7.** *Let  $\mathcal{K}$  be a subset of  $\mathcal{NA}_1^{L^2, box}$  compact in the weak topology of  $L^2((0, 1))$ . Let  $L \geq 1$ . Then there exists a constant  $\epsilon_0 > 0$  (depending on  $\mathcal{K}$  and  $L$  only) such that for all couples  $(T, u_T) \in (0, \infty) \times L^2((0, 1))$  satisfying  $Tu_T \in \mathcal{K}$ ,  $T \leq \epsilon_0$  problem ( $\mathbf{Pb}_{box}$ ) has no weak solution*

$$u \in L^2((0, 1) \times (0, T)) \cap L_{loc}^3((0, 1) \times (0, T)) \quad (6.9)$$

satisfying the  $L^2$  amplification assumption

$$\|u\|_{L^2((0,1) \times (0,T))} \leq LT \|u_T\|_{L^2((0,1))}. \quad (6.10)$$

along with the following  $L_{loc}^3$  regularization assumption:

$$\forall \delta \in (0, 1/2) \exists M_\delta \text{ such that } \|u\|_{L^3((\delta, 1-\delta) \times (\delta, T-\delta))} \leq M_\delta \|u\|_{L^2((0,1) \times (0,T))}. \quad (6.11)$$

*Proof.* We only sketch the proof, pointing out the changes with respect to those of Theorems 3.3 and 6.4. As a matter of fact, we only need to check the application of the variant of Lemma 3.2 in the  $L^2$  box setting, which is close to the proof of Theorem 6.4. The bounds, including the scaling-compatible amplification assumption (6.10), permit to extract  $L^2$  weakly convergent subsequences from  $(u_1^\epsilon)$  and  $(u^\epsilon)$ , while we recall that  $u_0^\epsilon = 0$  converge to  $u_0 = 0$  strongly on  $(0, 1)$ . In order to apply the compactness arguments of [38, 39] and pass to the weak limit in the entropy formulation (recall that weak solutions of ( $\mathbf{Pb}_{box}$ ) satisfy (2.2) of Definition 2.1) we need to show the uniform  $L_{loc}^1$  bound on  $\epsilon|u_x^\epsilon|^2$  and to provide a replacement for the ‘‘higher integrability’’ bound (6.5). Both issues rely upon the  $L_{loc}^3$  bound on the sequence  $(u^\epsilon)$ . Indeed, first, a bound in  $L_{loc}^p((0, 1) \times (0, 1))$ ,  $p > 2$  is enough to replace the global  $L^4$  bound (6.5) in the Vitali argument of the proof of Theorem 6.4. Such a bound, with  $p = 3$ , is ensured by our hypotheses (6.10), (6.11) (we stress that both of them are scaling-invariant). Second, the estimation of  $\epsilon|u_x^\epsilon|^2$  can be done through (3.6) where the uniform  $L_{loc}^3$  bound on the sequence  $(u^\epsilon)$  is mandatory. The remaining proof is simpler than the one in Theorem 6.4 because the initial condition  $u_0 = 0$  is enforced in the strong sense by passage to the limit from entropy inequalities (2.2).  $\square$

We note that in order to apply the above statement to concrete situations, such as constant targets, we need to point out some subsets of  $\mathcal{NA}_1^{L^2, box}$ , in particular, the constant states within it. This could be done via Proposition 6.1 and the box version of property (6.2). However, in our final result, we prefer to exploit Proposition 6.6 to provide an alternative, more direct argument.

**Corollary 6.8.** *For given  $0 < \alpha \leq \beta < 1$  there exists  $\epsilon_0 = \epsilon_0(\alpha, \beta, L)$  such that for all  $T \leq \epsilon_0$  and all  $C$  with  $\alpha T^{-1} \leq |C| \leq \beta T^{-1}$ , the constant state  $C$  is not attainable at time  $T$  for problem  $(\mathbf{Pb}_{\text{box}})$  in the weak  $L^2((0, 1) \times (0, T)) \cap L^3_{\text{loc}}((0, 1) \times (0, T))$  setting, with solutions fulfilling the amplification restriction  $\|u\|_{L^2((0,1)\times(0,T))} \leq LT|C|$  and the regularization assumption (6.11).*

*Proof.* It is enough to apply Proposition 6.6 to establish that  $\mathring{\mathcal{K}}_{\alpha, \beta} \subset \mathcal{NA}_1^{L^2, \text{box}}$ , and then apply Theorem 6.7.  $\square$

We do not pursue the goal of stating further and finer results with *ad hoc* choices of  $\mathcal{K}$  in the setting of Theorem 6.7. Interested readers can infer such results from Proposition 6.6, and even go further using the ideas and auxiliary results collected in Proposition 4.4, Remarks 4.1 and 6.3.

We conclude with the following remark that relates the most general negative answer (for the above indicated couples  $(C, T)$  and under the *ad hoc* amplification bounds) to question (Q) of Coron [18] to the precise choice of the underlying notion of solution.

**Remark 6.4.** *Assumption (6.11) should be seen as an  $L^2 - L^3$  regularizing effect in the interior of  $(0, 1) \times (0, T)$ . Let us underline that this property holds true for sufficiently regular weak solutions of  $(\mathbf{Pb}_{\text{box}})$ , with  $(\delta, 1 - \delta) \times (\delta, T - \delta)$  replaced by  $(\delta, 1 - \delta) \times (\delta, T)$ . We have in mind the following point. First, like for the inviscid Burgers equation (cf. Theorems 5.1, 5.5), the  $L^2((0, 1) \times (0, T))$  solutions to the viscous Burgers equation that correspond to some prescribed boundary controls can be expected to belong to  $L^\infty(0, T; L^2(0, 1))$ . Second, it is known since the founding work of Hopf [28] that classical solutions of the viscous Burgers equation satisfy the Lax-Oleinik representation (5.21). If we assume additionally that our weak solutions are pointwise a.e. limits of such regular solutions, then representation (5.21) is inherited (see Proposition 5.8, cf. the proof of Proposition 5.6). At least in this setting, the  $L^2 - L^\infty$  local regularization result of Proposition 5.8 – result solely based upon the  $L^\infty(0, T, L^2((0, 1)))$  regularity and the representation (5.21) with monotone functions  $y(\cdot, t)$  – extends to solutions of the viscous Burgers equation. To sum up, assumption (6.11) in Theorem 6.7 and Corollary 6.8 may be automatically satisfied in practical control situations.*

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