# Observability for generalized Schrödinger equations and quantum limits on product manifolds 

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#### Abstract

Given a closed product Riemannian manifold $N=M \times M^{\prime}$ equipped with the product Riemannian metric $g=h+h^{\prime}$, we explore the observability properties for the generalized Schrödinger equation $i \partial_{t} u=F\left(\triangle_{g}\right) u$, where $\triangle_{g}$ is the Laplace-Beltrami operator on $N$ and $F:[0,+\infty) \rightarrow[0,+\infty)$ is an increasing function. In this note, we prove observability in finite time on any open subset $\omega$ satisfying the so-called Vertical Geometric Control Condition, stipulating that any vertical geodesic meets $\omega$, under the additional assumption that the spectrum of $F\left(\Delta_{g}\right)$ satisfies a gap condition. A first consequence is that observability on $\omega$ for the Schrödinger equation is a strictly weaker property than the usual Geometric Control Condition on any product of spheres. A second consequence is that the Dirac measure along any geodesic of $N$ is never a quantum limit.


## 1 Introduction and main results

Let $(M, h),\left(M^{\prime}, h^{\prime}\right)$ be closed Riemannian manifolds, and let $\triangle_{h}$ and $\triangle_{h^{\prime}}$ be their respective (nonnegative) Laplace-Beltrami operators. We consider the Riemannian product manifold ( $N, g$ ) defined by $N=M \times M^{\prime}$ and $g=h+h^{\prime}$. Let $\triangle_{g}=\triangle_{h} \otimes \triangle_{h^{\prime}}$ be the corresponding Laplace-Beltrami operator on $N$. Let $F:[0,+\infty) \rightarrow[0,+\infty)$ be an arbitrary increasing function. We consider the generalized Schrödinger equation

$$
\begin{equation*}
i \partial_{t} u=F\left(\triangle_{g}\right) u \tag{1}
\end{equation*}
$$

on $M$, and we are interested in finding characterizations of the observability property for (1) on any open subset $\omega \subset N$.

We denote by $0=\mu_{0} \leqslant \mu_{1} \leqslant \cdots \leqslant \mu_{k} \leqslant \cdots$ (resp., $0=\mu_{0}^{\prime} \leqslant \mu_{1}^{\prime} \leqslant \cdots \leqslant \mu_{k}^{\prime} \leqslant \cdots$ ) the eigenvalues of $\triangle_{h}$ (resp., of $\triangle_{h^{\prime}}$ ), associated with a Hilbert eigenbasis $\left(\phi_{k}\right)_{k \in \mathbb{N}}$ of $L^{2}(M)$

[^0](resp., $\left(\phi_{k}^{\prime}\right)_{k \in \mathbb{N}}$ of $\left.L^{2}\left(M^{\prime}\right)\right)$. We also denote by $0=\lambda_{0}<\lambda_{1}<\cdots<\lambda_{k}<\cdots$ (resp., $\left.0=\lambda_{0}^{\prime}<\lambda_{1}^{\prime}<\cdots<\lambda_{k}^{\prime}<\cdots\right)$ its distinct eigenvalues. There exists an increasing sequence $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ (resp., $\left.\left(\alpha_{k}^{\prime}\right)_{k \in \mathbb{N}}\right)$ such that for $j=\alpha_{k}, \ldots, \alpha_{k+1}-1$ (resp., $j=\alpha_{k}^{\prime}, \ldots, \alpha_{k+1}^{\prime}-1$, $\mu_{j}=\lambda_{k}\left(\right.$ resp.,$\left.\mu_{l}^{\prime}=\lambda_{k}^{\prime}\right)$.

Then, $\left(\phi_{j} \phi_{k}^{\prime}\right)_{j, k \in \mathbb{N}}$ is an orthonormal basis of $L^{2}(N)$ of eigenfunctions of $\triangle_{g}$ associated to the eigenvalues $\mu_{j}+\mu_{k}^{\prime}$. The operator $F\left(\triangle_{g}\right)$ is spectrally defined as the linear operator which, restricted to the eigenspace of $\triangle_{g}$ associated to the eigenvalue $\mu_{j}+\mu_{k}^{\prime}$, is equal to $F\left(\mu_{j}+\mu_{k}^{\prime}\right)$ id. The fact that $F\left(\triangle_{g}\right)$ and $\triangle_{g}$ have the same eigenspaces comes from the fact that $F$ is increasing, so that $F\left(\mu_{j}+\mu_{k}^{\prime}\right)=F\left(\mu_{j^{\prime}}+\mu_{k^{\prime}}^{\prime}\right)$ if and only if $\mu_{j}+\mu_{k}^{\prime}=\mu_{j^{\prime}}+\mu_{k^{\prime}}^{\prime}$.

By the Stone theorem, $\left(e^{i t F\left(\Delta_{g}\right)}\right)_{t \geqslant 0}$ is a unitary strongly continuous semigroup on $L^{2}(N)$. Given any $y \in L^{2}(N)$, there exists a unique solution $u \in C^{0}\left([0,+\infty), L^{2}(N)\right) \cap$ $C^{1}\left((0,+\infty), H^{-2}(N)\right)$ of (1) such that $u(0)=y$, given by $u(t)=e^{i t F\left(\Delta_{g}\right)} y$.

If $F(s)=s$ then (1) is the usual Schrödinger equation, and if $F(s)=\sqrt{s}$ then (1) is the half-wave equation.

We denote by $d x_{g}$ the Riemannian volume form on $N$. Given any $T>0$ and any measurable subset $\omega$ of $N$, we define the observability constant $C_{T}(\omega) \geqslant 0$ as the largest constant $C \geqslant 0$ such that

$$
\begin{equation*}
\int_{0}^{T} \int_{\omega}\left|e^{i t F\left(\Delta_{g}\right)} y\right|^{2} \geqslant C\|y\|_{L^{2}(N)} \quad \forall y \in L^{2}(N) \tag{2}
\end{equation*}
$$

## obs

(observability inequality), i.e.,

$$
\begin{aligned}
C_{T}(\omega) & =\inf \left\{\int_{0}^{T} \int_{\omega}\left|e^{i t F\left(\Delta_{g}\right)} y\right|^{2} d x_{g} d t \mid y \in L^{2}(N),\|y\|_{L^{2}(N)}=1\right\} \\
& =\inf \left\{\left.\int_{0}^{T} \int_{\omega}\left|\sum_{l, m} b_{j k} e^{i t F\left(\mu_{j}+\mu_{k}\right)} \phi_{j} \phi_{k}^{\prime}\right|^{2} d x_{g} d t\left|b_{j k} \in \ell^{2}(\mathbb{C}), \sum_{j, k=0}^{+\infty}\right| b_{j k}\right|^{2}=1\right\}
\end{aligned}
$$

We say that the observability property is satisfied for (1) on $(\omega, T)$ if $C_{T}(\omega)>0$.
Definition 1. A vertical (resp., horizontal) geodesic of $N$ is a geodesic of the form $t \rightarrow$ $(x, \gamma(t))$ (resp., $(\gamma(t), x)$ ) for some $x \in M$ (resp., for some $x \in M^{\prime}$ ) and some geodesic $\gamma$ of $M^{\prime}$ (resp., of $M$ ).

Definition 2. Let $\omega \subset N$ and let $T>0$. We say that $(\omega, T)$ satisfies the Vertical Geometric Control Condition (in short, VGCC) if all vertical geodesics meet $\omega$ within time $T$, i.e., $\gamma([0, T]) \cap \omega \neq \emptyset$.

Definition 3. We say that a family $\left(a_{k}\right)_{k \in \mathbb{N}}$ of real numbers satisfies the gap condition if there exists a constant $C>0$ such that for all $j, k \in \mathbb{N}$, we have either $a_{j}=a_{k}$ or $\left|a_{k}-a_{l}\right| \geqslant C$, i.e., if all distinct elements are at a distance of at least $C$ one from each other.
${ }^{\langle m a i n\rangle}$ Theorem 1. Let $T>0$ and $\omega$ be an open subset of N. If $(\omega, T)$ satisfies VGCC and if the family $\left(F\left(\lambda_{j}+\lambda_{k}^{\prime}\right)\right)_{j, k \in \mathbb{N}}$ satisfies the gap condition, then the observability property is satisfied for (1) on ( $\omega, T$ ).

Let us comment on this theorem and on VGCC.
Recall that $(\omega, T)$ satisfies the usual Geometric Control Condition (GCC, see $[2,8,11])$ whenever every geodesic (not necessarily vertical) meets $\omega$ within time $T$. Let $\omega$ be an open subset of $N$ and $T>0$. If $(\omega, T)$ satisfies GCC then it also satisfies VGCC. There exist examples where $(\omega, T)$ satisfies VGCC but not GCC: for every $x \in M$, we define $\omega_{x}:=\left(\{x\} \times M^{\prime}\right) \cap \omega$. Then, $(\omega, T)$ satisfies VGCC if and only if $\left(\omega_{x}, T\right)$ satisfies GCC on $M^{\prime}$ for every $x \in M$. In particular, we obtain the following examples:

- Let $\left(U_{i}\right)_{i \in I}$ be an open covering of $M$, and let $\left(\omega_{i}\right)_{i \in I}$ be a family of open subsets of $M^{\prime}$ satisfying GCC within time $T$. Then, setting $\omega=\cup_{i \in I} U_{i} \times \omega_{i},(\omega, T)$ satisfies VGCC. In particular, if $\omega^{\prime}$ is an open subset of $M^{\prime}$ satisfying GCC, then $M \times \omega^{\prime}$ satisfies VGCC.
- Let $\gamma$ be a non-vertical geodesic. Given any $\varepsilon>0$, we consider the closed $\varepsilon$ neighborhood of the support $\Gamma$ of $\gamma$ defined by $U_{\varepsilon}=\left\{x \in N \quad \mid \quad d_{g}(x, \Gamma) \leqslant \varepsilon\right\}$, where $d_{g}$ is the Riemannian distance on $N$. We set $\omega_{\varepsilon}=N \backslash U_{\varepsilon}$. Then, for any $T>0$ and any $\varepsilon>0$ small enough, $\left(\omega_{\varepsilon}, T\right)$ satisfies VGCC.
For instance, if $\gamma$ is horizontal, we can choose $\varepsilon<\frac{T}{2}$. For the general case, note that for every $x \in M,\left(\omega_{\varepsilon}\right)_{x}$ (with the notations above) is contained in the complement of a small ball in $M^{\prime}$.

Let us now recall some existing results. It is well known that, when $\omega$ is open, GCC is a sufficient condition for observability of the Schrödinger equation (see [8]). It is also well known that, except for Zoll manifolds, i.e., manifolds whose all geodesics are periodic (see [9]), GCC is not a necessary assumption. An example where the Schrödinger is observable on $(\omega, T)$ but where $(\omega, T)$ does not satisfy GCC is given in [6]: in the flat 2 D torus, any non empty open set gives observability in any time $T$. This example has been extended to high dimensions in [7]. We also refer to [1] for another example, in the Dirichlet disk.

Remark 1. The spectrum of $\triangle_{g}^{1 / 2}$ can never satisfy the gap condition on the product manifold $N$.

Application to the Schrödinger equation. We assume that $F(s)=s$ so that (1) is now the usual Schrödinger equation. Theorem 1 can be applied as soon as the spectrum of $\triangle_{g}$ satisfies the gap condition. This is true for instance when $M$ and $M^{\prime}$ have an integer spectrum, in particular when $M$ and $M^{\prime}$ are a finite product of standard spheres.
? 〈cor1〉? Corollary 1. Assume that the spectrum of $\triangle_{g}$ satisfies the gap condition. Let $T>0$ and $\omega$ be an open subset of $N$, such that $(\omega, T)$ satisfies $V G C C$ but not $G C C$. Then the Schrödinger equation is observable on $(\omega, T)$, while $G C C$ is not satisfied.

This result provides new examples of configurations where one has observability but not GCC.

Quantum limits on a product manifold. The definition of a quantum limit is recalled in Appendix A.1.
${ }^{\langle c o r 3\rangle}$ Corollary 2. The support of any quantum limit of $N$ must contain at least an horizontal and a vertical geodesic. In particular, the Dirac measure along any periodic geodesic of $N$ is not a quantum limit.

## 2 Proofs

### 2.1 Proof of Theorem 1

Let $\omega$ be an open subset of $N$. For any $x \in M$, we set $\omega_{x}=\omega \cap\left(\{x\} \times M^{\prime}\right)$. Theorem 1 follows from the following lemmas, which are in order.
? (lemma_main)? Lemma 1. Assume that there exists $c, T>0$ such that for all complex numbers $\left(a_{k, m}\right)_{k, m \in \mathbb{N}}$ and every $x \in M$,

$$
\int_{0}^{T} \int_{\omega_{x}}\left|\sum_{k, m} a_{k, m} \phi_{m}^{\prime} e^{i F\left(\lambda_{k}+\mu_{m}^{\prime}\right) t}\right|^{2} d x_{h^{\prime}} d t \geqslant c \sum_{k, m}\left|a_{k, m}\right|^{2}
$$

(3) main_estimate
then (1) is observable on $(\omega, T)$.
Proof. The objective is to prove (2). Writing $y=\sum_{l, m \geqslant 0} b_{l, m} \phi_{l} \phi_{m}^{\prime}$, we have $e^{i t F\left(\Delta_{g}\right)} y=$ $\sum_{k, m} b_{k, m} \phi_{k} \phi_{m}^{\prime} e^{i F\left(\mu_{k}+\mu_{m}^{\prime}\right) t}$. We denote by $G_{x}: C^{\infty}(N) \rightarrow C^{\infty}\left(\{x\} \times M^{\prime}\right)$ the mapping $\left(G_{x} f\right)\left(q, q^{\prime}\right)=f\left(x, q^{\prime}\right)$. Setting $a_{k, m}(x)=\sum_{l=\alpha_{k}}^{\alpha_{k+1}-1} b_{l, m} \phi_{l}(x)$, using (3) and the definition of $\alpha_{k}$, there exists $T, c>0$ such that

$$
\begin{aligned}
& \int_{0}^{T} \int_{\omega}\left|e^{i t F\left(\Delta_{g}\right)} y\right|^{2} d x_{g} d t=\int_{0}^{T} \int_{M} \int_{\omega_{x}}\left|G_{x} e^{i t F\left(\Delta_{g}\right)} y\right|^{2} d x_{h^{\prime}} d x_{h}(x) d t \\
& \quad=\int_{0}^{T} \int_{M} \int_{\omega_{x}}\left|\sum_{l, m \geqslant 0} b_{l, m} \phi_{l}(x) \phi_{m}^{\prime}\left(x^{\prime}\right) e^{i F\left(\mu_{k}+\mu_{m}^{\prime}\right) t t}\right|^{2} d x_{h^{\prime}}\left(x^{\prime}\right) d x_{h}(x) d t \\
& \quad=\int_{0}^{T} \int_{M} \int_{\omega_{x}}\left|\sum_{k, m} a_{k, m}(x) \phi_{m}^{\prime}\left(x^{\prime}\right) e^{i F\left(\lambda_{k}+\mu_{m}^{\prime}\right) t}\right|^{2} d x_{h^{\prime}}\left(x^{\prime}\right) d x_{h}(x) d t \\
& \geqslant c \int_{0}^{T} \int_{M} \sum_{k, m}\left|a_{k, m}(x)\right|^{2} d x_{h}(x) d t=c T \sum_{k, m} \int_{M}\left|\sum_{l=\alpha_{k}}^{\alpha_{k+1}-1} b_{l, m} \phi_{l}\right|^{2} d x_{h} \\
& =c T \sum_{k, m} \int_{M} \sum_{\alpha_{k} \leqslant l, l^{\prime} \leqslant \alpha_{k+1}-1} b_{l, m} b_{l^{\prime}, m} \phi_{l} \phi_{l}^{\prime} d x_{h}=c T \sum_{k, m} \int_{M} \sum_{l=\alpha_{k}}^{\alpha_{k+1}-1} b_{l, m}^{2} \phi_{l}^{2} d x_{h} \\
& \quad=c T \sum_{k, m} b_{k, m}^{2} \int_{M} \phi_{k}^{2} d x_{h}=c T \sum_{k, m} b_{k, m}^{2}=c T\|y\|_{L^{2}(N)}^{2}
\end{aligned}
$$

This proves observability in time $T$.

We define

$$
g_{1}^{V}(\omega)=\inf _{x, \phi^{\prime}} \int_{\omega_{x}} \phi^{\prime 2}
$$

where the infimum is taken over the set of all possible $x \in M$ and all possible eigenfunctions $\phi^{\prime}$ of $\Delta_{h^{\prime}}$ such that $\left\|\phi^{\prime}\right\|_{L^{2}(M)}=1$.
?(case2)? Lemma 2. Assume that the family $\left(F\left(\lambda_{k}+\lambda_{m}^{\prime}\right)\right)_{k, m \in \mathbb{N}}$ satisfies the gap condition. Then (3) is satisfied with $c=g_{1}^{V}(\omega) / 2$ for $T$ large enough.

Proof. Define $\Lambda_{k, m}=F\left(\lambda_{k}+\mu_{m}^{\prime}\right)$. By assumption, there exists $C_{0}>0$ such that if $\Lambda_{k, m} \neq \Lambda_{k^{\prime}, m^{\prime}}$, then

$$
\begin{equation*}
\left|\Lambda_{k, m}-\Lambda_{k^{\prime}, m^{\prime}}\right| \geqslant C_{0} . \tag{4}
\end{equation*}
$$

Let $T>0$ and $\psi_{T}$ the characteristic function of the interval $[0,2 T]$. Its Fourier transform $\hat{\psi}_{T}$ is equal to $\hat{\psi}_{T}(\xi)=\frac{e^{i T \xi}-1}{T \xi}$. Noting that $\hat{\psi}_{T}(0)=1$, we have

$$
\begin{align*}
& \int_{0}^{2 T} \int_{\omega_{x}}\left|\sum_{k, m} a_{k, m} \phi_{m}^{\prime} e^{i F\left(\lambda_{k}+\mu_{m}^{\prime}\right) t}\right|^{2} d x_{h^{\prime}} d t \\
&=\sum_{k, m, k^{\prime}, m^{\prime}} a_{k, m} \overline{a_{k^{\prime}, m^{\prime}}} \hat{\psi}_{T}\left(\Lambda_{k, m}-\Lambda_{k^{\prime}, m^{\prime}}\right) \int_{\omega_{x}} \phi_{m}^{\prime} \phi_{m^{\prime}}^{\prime}=A+B \tag{5}
\end{align*}
$$

with

$$
A=\sum_{k, m}\left|a_{k, m}\right|^{2} \int_{\omega_{x}}\left|\phi_{m}^{\prime}\right|^{2}, \quad B=\sum_{(k, m) \neq\left(k^{\prime}, m^{\prime}\right)} a_{k, m} \overline{a_{k^{\prime}, m^{\prime}}} \hat{\psi}_{T}\left(\Lambda_{k, m}-\Lambda_{k^{\prime}, m^{\prime}}\right) \int_{\omega_{x}} \phi_{m}^{\prime} \phi_{m^{\prime}}^{\prime} .
$$

Using the gap condition (4), it follows from Montgomery-Vaughan inequality (see [10]) that $|B| \leqslant \frac{2}{T C_{0}} A$. Hence, we obtain from (5) that

$$
\int_{0}^{2 T} \int_{\omega_{x}}\left|\sum_{k, m} a_{k, m} \phi_{m}^{\prime} e^{i F\left(\lambda_{k}+\mu_{m}^{\prime}\right) t}\right|^{2} d x_{h^{\prime}} d t \geqslant\left(1-\frac{2}{T C}\right) A \geqslant \frac{1}{2} A
$$

when $T$ is large enough. Noting that $A \geqslant \sum_{k, m}\left|a_{k, m}\right|^{2} g_{1}^{V}(\omega)$, the inequality (3) follows with $c=g_{1}^{V}(\omega) / 2$.
? $\langle g 1 \mathrm{vgcc}\rangle$ ? Lemma 3. If $(\omega, T)$ satisfies $V G C C$ then $g_{1}^{V}(\omega)>0$.
Proof. Assume that $\omega$ satisfies VGCC. By contradiction, let us assume that $g_{1}^{V}(\omega)=0$. This means that for every $\varepsilon>0$, there exists $x_{\varepsilon} \in M$ and an eigenfunction $\phi_{\varepsilon}^{\prime}$ of $\Delta_{h^{\prime}}$ such that $\left\|\phi_{\varepsilon}^{\prime}\right\|_{L^{2}\left(M^{\prime}\right)}=1$ and such that $\int_{\omega_{x_{\varepsilon}}} \phi_{\varepsilon}^{2} d x_{g} \leqslant \varepsilon$, where we recall that $\omega_{x_{\varepsilon}}=$ $\omega \cap\left(\left\{x_{\varepsilon}\right\} \times M^{\prime}\right)$. By compactness, we assume that $x_{\varepsilon} \rightarrow x_{0} \in M$ and that $\left(\phi_{\varepsilon}^{\prime}\right)^{2} \rightarrow \mu$ weakly, where $\mu$ is a quantum limit of $M^{\prime}$. Let $U_{k}$ be an increasing sequence of open sets such that $\overline{U_{k}} \subset U_{k+1}$ and such that $\cup_{k} U_{k}=\omega_{0}=\omega \cap\left(\left\{x_{0}\right\} \times M^{\prime}\right)$. Since $\omega$ is open, for all $k \in \mathbb{N}$ and $\varepsilon>0$ small enough, we have $U_{k} \subset \omega_{x_{\varepsilon}}$. This implies that $\int_{U_{k}}\left(\phi_{\varepsilon}^{\prime}\right)^{2} d x_{g} \leqslant \varepsilon$. We infer from the Portmanteau theorem (see Appendix A.2) that $\mu\left(U_{k}\right)=0$, and thus
$\mu\left(\omega_{0}\right)=0$. This implies that GCC does not hold for $\omega_{0}$ in any time. Indeed, by the Egorov theorem (see [4, 12]), $\mu$ is invariant under the geodesic flow, as a measure on $S^{*} M^{\prime}$. By the Krein-Milman theorem, $\mu$ can be approximated by a sequence $\left(\mu_{k}\right)_{k \in \mathbb{N}}$ of convex combinations of Dirac measures along periodic geodesics. Since $\mu_{k}\left(\omega_{0}\right) \rightarrow 0$, there exists a sequence of periodic geodesics $\gamma_{k}$ such that, if $\delta_{k}$ is the Dirac measure along $\gamma_{k}$, we have $\delta_{k}\left(\mu_{0}\right) \rightarrow 0$. This means that the time spent by $\gamma_{k}$ (actually, by its projection onto $M^{\prime}$ ) in $\omega_{0}$ tends to 0 . By compactness of geodesics, $\gamma_{k}$ converges to some geodesic $\gamma$. Again by the Portmanteau theorem, $\gamma$ does not meet $\omega$, hence GCC on $M^{\prime}$ fails for $\omega_{0}$ and this contradicts that VGCC is satisfied for $\omega$.

### 2.2 Proof of Corollary 2

We prove the vertical case, the horizontal case being symmetric. We rearrange the set $\left\{\lambda_{j}+\lambda_{k}^{\prime} \mid j, k \in \mathbb{N}\right\}=\left\{d_{k} \mid k \in \mathbb{N}\right\}$ with an increasing sequence $\left(d_{k}\right)_{k \in \mathbb{N}}$. Let $F$ be an increasing function such that $F\left(d_{k}\right)=k$ for every $k \in \mathbb{N}$. By construction, the set $\left\{F\left(\lambda_{j}+\lambda_{k}^{\prime}\right) \mid j, k \in \mathbb{N}\right\}$ satisfies the gap condition. Let $\Gamma$ be the support of a quantum limit $\mu$ on $M \times M^{\prime}$. Since $F\left(\triangle_{g}\right)$ and $\triangle_{g}$ have the same eigenfunctions, $\mu$ is also the weak limit of a sequence of $\psi_{j}^{2} d x_{g} d_{\xi}$ where $\psi_{j}$ are eigenfunctions of $F\left(\triangle_{g}\right)$ satisfying $\left\|\psi_{j}\right\|_{L^{2}}=1$. We set $\omega_{\varepsilon}=\left\{x \in N \mid d_{g}(x, \Gamma)>\varepsilon\right\}$, for $\varepsilon>0$ small enough. For every $T>0,\left(\omega_{\varepsilon}, T\right)$ is not observable for (1) because $y=\psi_{j}$ provides a sequence of test functions which, at the limit, lie on $\Gamma$. Hence, by Theorem $1,\left(\omega_{\varepsilon}, T\right)$ does not satisfy VGCC. Remark ?? implies that $\Gamma$ must contain a vertical geodesic.

## A Appendix

## A. 1 Quantum limits

$\left.{ }^{\langle q 1}\right\rangle$ We recall that a quantum limit (QL in short) $\mu$, also called semi-classical measure, is a probability Radon (i.e., probability Borel regular) measure on $S^{*} M$ that is a closure point (weak limit), as $\lambda \rightarrow+\infty$, of the family of Radon measures $\mu_{\lambda}(a)=\left\langle\operatorname{Op}(a) \phi_{\lambda}, \phi_{\lambda}\right\rangle$ (which are asymptotically positive by the Gårding inequality), where $\phi_{\lambda}$ denotes an eigenfunction of norm 1 associated with the eigenvalue $\lambda$ of $\sqrt{\triangle}$. Here, Op is any quantization. We speak of a $Q L$ on $M$ to refer to a closure point (for the weak topology) of the sequence of probability Radon measures $\phi_{\lambda}^{2} d x_{g}$ on $M$ as $\lambda \rightarrow+\infty$. Note that QLs do not depend on the choice of a quantization. We denote by $\mathcal{Q}\left(S^{*} M\right.$ ) (resp., $\mathcal{Q}(M)$ ) the set of QLs (resp., the set of QLs on $M$ ). Both are compact sets.

Given any $\mu \in \mathcal{Q}\left(S^{*} M\right)$, the Radon measure $\pi_{*} \mu$, image of $\mu$ under the canonical projection $\pi: S^{*} M \rightarrow M$, is a probability Radon measure on $M$. It is defined, equivalently, by $\left(\pi_{*} \mu\right)(f)=\mu\left(\pi^{*} f\right)=\mu(f \circ \pi)$ for every $f \in C^{0}(M)$ (note that, in local coordinates $(x, \xi)$ in $S^{*} M$, the function $f \circ \pi$ is a function depending only on $x$ ), or by $\left(\pi_{*} \mu\right)(\omega)=\mu\left(\pi^{-1}(\omega)\right)$ for every $\omega \subset M$ Borel measurable (or Lebesgue measurable, by regularity). It is easy to
see that ${ }^{1}$

$$
\pi_{*} \mathcal{Q}\left(S^{*} M\right)=\mathcal{Q}(M)
$$

In other words, QLs on $M$ are exactly the image measures under $\pi$ of QLs.

## A. 2 Portmanteau theorem

${ }^{\langle c i n t r e\rangle}$ Let us recall the Portmanteau theorem (see, e.g., [3]). Let $X$ be a topological space, endowed with its Borel $\sigma$-algebra. Let $\mu$ and $\mu_{n}, n \in \mathbb{N}^{*}$, be finite Borel measures on $X$. Then the following items are equivalent:

- $\mu_{n} \rightarrow \mu$ for the narrow topology, i.e., $\int f d \mu_{n} \rightarrow \int f d \mu$ for every bounded continuous function $f$ on $X$;
- $\int f d \mu_{n} \rightarrow \int f d \mu$ for every Borel bounded function $f$ on $X$ such that $\mu\left(\Delta_{f}\right)=0$, where $\Delta_{f}$ is the set of points at which $f$ is not continuous;
- $\mu_{n}(B) \rightarrow \mu(B)$ for every Borel subset $B$ of $X$ such that $\mu(\partial B)=0$;
- $\mu(F) \geqslant \lim \sup \mu_{n}(F)$ for every closed subset $F$ of $X$, and $\mu_{n}(X) \rightarrow \mu(X)$;
- $\mu(O) \leqslant \lim \inf \mu_{n}(O)$ for every open subset $O$ of $X$, and $\mu_{n}(X) \rightarrow \mu(X)$.

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$$
\begin{aligned}
& \hline{ }^{1} \text { Indeed, given any } f \in C^{0}(M) \text { and any } \lambda \in \operatorname{Spec}(\sqrt{\triangle}) \text {, we have } \\
& \qquad\left(\pi_{*} \mu_{\lambda}\right)(f)=\mu_{\lambda}\left(\pi^{*} f\right)=\left\langle\operatorname{Op}\left(\pi^{*} f\right) \phi_{\lambda}, \phi_{\lambda}\right\rangle=\int_{M} f \phi_{\lambda}^{2} d x_{g},
\end{aligned}
$$

because $\operatorname{Op}\left(\pi^{*} f\right) \phi_{\lambda}=f \phi_{\lambda}$. The equality then easily follows by weak compactness of probability Radon measures.

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