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# Observability for generalized Schrödinger equations and quantum limits on product manifolds

Emmanuel Humbert\*      Yannick Privat<sup>†</sup>      Emmanuel Trélat<sup>‡</sup>

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## Abstract

Given a closed product Riemannian manifold  $N = M \times M'$  equipped with the product Riemannian metric  $g = h + h'$ , we explore the observability properties for the generalized Schrödinger equation  $i\partial_t u = F(\Delta_g)u$ , where  $\Delta_g$  is the Laplace-Beltrami operator on  $N$  and  $F : [0, +\infty) \rightarrow [0, +\infty)$  is an increasing function. In this note, we prove observability in finite time on any open subset  $\omega$  satisfying the so-called Vertical Geometric Control Condition, stipulating that any vertical geodesic meets  $\omega$ , under the additional assumption that the spectrum of  $F(\Delta_g)$  satisfies a gap condition. A first consequence is that observability on  $\omega$  for the Schrödinger equation is a strictly weaker property than the usual Geometric Control Condition on any product of spheres. A second consequence is that the Dirac measure along any geodesic of  $N$  is never a quantum limit.

## 1 Introduction and main results

Let  $(M, h)$ ,  $(M', h')$  be closed Riemannian manifolds, and let  $\Delta_h$  and  $\Delta_{h'}$  be their respective (nonnegative) Laplace-Beltrami operators. We consider the Riemannian product manifold  $(N, g)$  defined by  $N = M \times M'$  and  $g = h + h'$ . Let  $\Delta_g = \Delta_h \otimes \Delta_{h'}$  be the corresponding Laplace-Beltrami operator on  $N$ . Let  $F : [0, +\infty) \rightarrow [0, +\infty)$  be an arbitrary increasing function. We consider the generalized Schrödinger equation

$$i\partial_t u = F(\Delta_g)u \tag{1} \boxed{\text{schrod}}$$

on  $M$ , and we are interested in finding characterizations of the observability property for (1) on any open subset  $\omega \subset N$ .

We denote by  $0 = \mu_0 \leq \mu_1 \leq \dots \leq \mu_k \leq \dots$  (resp.,  $0 = \mu'_0 \leq \mu'_1 \leq \dots \leq \mu'_k \leq \dots$ ) the eigenvalues of  $\Delta_h$  (resp., of  $\Delta_{h'}$ ), associated with a Hilbert eigenbasis  $(\phi_k)_{k \in \mathbb{N}}$  of  $L^2(M)$

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(resp.,  $(\phi'_k)_{k \in \mathbb{N}}$  of  $L^2(M')$ ). We also denote by  $0 = \lambda_0 < \lambda_1 < \dots < \lambda_k < \dots$  (resp.,  $0 = \lambda'_0 < \lambda'_1 < \dots < \lambda'_k < \dots$ ) its distinct eigenvalues. There exists an increasing sequence  $(\alpha_k)_{k \in \mathbb{N}}$  (resp.,  $(\alpha'_k)_{k \in \mathbb{N}}$ ) such that for  $j = \alpha_k, \dots, \alpha_{k+1} - 1$  (resp.,  $j = \alpha'_k, \dots, \alpha'_{k+1} - 1$ ),  $\mu_j = \lambda_k$  (resp.,  $\mu'_j = \lambda'_k$ ).

Then,  $(\phi_j \phi'_k)_{j,k \in \mathbb{N}}$  is an orthonormal basis of  $L^2(N)$  of eigenfunctions of  $\Delta_g$  associated to the eigenvalues  $\mu_j + \mu'_k$ . The operator  $F(\Delta_g)$  is spectrally defined as the linear operator which, restricted to the eigenspace of  $\Delta_g$  associated to the eigenvalue  $\mu_j + \mu'_k$ , is equal to  $F(\mu_j + \mu'_k) \text{id}$ . The fact that  $F(\Delta_g)$  and  $\Delta_g$  have the same eigenspaces comes from the fact that  $F$  is increasing, so that  $F(\mu_j + \mu'_k) = F(\mu_{j'} + \mu'_{k'})$  if and only if  $\mu_j + \mu'_k = \mu_{j'} + \mu'_{k'}$ .

By the Stone theorem,  $(e^{itF(\Delta_g)})_{t \geq 0}$  is a unitary strongly continuous semigroup on  $L^2(N)$ . Given any  $y \in L^2(N)$ , there exists a unique solution  $u \in C^0([0, +\infty), L^2(N)) \cap C^1((0, +\infty), H^{-2}(N))$  of (1) such that  $u(0) = y$ , given by  $u(t) = e^{itF(\Delta_g)}y$ .

If  $F(s) = s$  then (1) is the usual Schrödinger equation, and if  $F(s) = \sqrt{s}$  then (1) is the half-wave equation.

We denote by  $dx_g$  the Riemannian volume form on  $N$ . Given any  $T > 0$  and any measurable subset  $\omega$  of  $N$ , we define the observability constant  $C_T(\omega) \geq 0$  as the largest constant  $C \geq 0$  such that

$$\int_0^T \int_\omega \left| e^{itF(\Delta_g)} y \right|^2 \geq C \|y\|_{L^2(N)}^2 \quad \forall y \in L^2(N) \quad (2) \quad \boxed{\text{obs}}$$

(observability inequality), i.e.,

$$\begin{aligned} C_T(\omega) &= \inf \left\{ \int_0^T \int_\omega \left| e^{itF(\Delta_g)} y \right|^2 dx_g dt \mid y \in L^2(N), \|y\|_{L^2(N)} = 1 \right\} \\ &= \inf \left\{ \int_0^T \int_\omega \left| \sum_{l,m} b_{jk} e^{itF(\mu_j + \mu'_k)} \phi_j \phi'_k \right|^2 dx_g dt \mid b_{jk} \in \ell^2(\mathbb{C}), \sum_{j,k=0}^{+\infty} |b_{jk}|^2 = 1 \right\} \end{aligned}$$

We say that the *observability property* is satisfied for (1) on  $(\omega, T)$  if  $C_T(\omega) > 0$ .

**Definition 1.** A vertical (resp., horizontal) geodesic of  $N$  is a geodesic of the form  $t \rightarrow (x, \gamma(t))$  (resp.,  $(\gamma(t), x)$ ) for some  $x \in M$  (resp., for some  $x \in M'$ ) and some geodesic  $\gamma$  of  $M'$  (resp., of  $M$ ).

**Definition 2.** Let  $\omega \subset N$  and let  $T > 0$ . We say that  $(\omega, T)$  satisfies the Vertical Geometric Control Condition (in short, VGCC) if all vertical geodesics meet  $\omega$  within time  $T$ , i.e.,  $\gamma([0, T]) \cap \omega \neq \emptyset$ .

**Definition 3.** We say that a family  $(a_k)_{k \in \mathbb{N}}$  of real numbers satisfies the gap condition if there exists a constant  $C > 0$  such that for all  $j, k \in \mathbb{N}$ , we have either  $a_j = a_k$  or  $|a_k - a_l| \geq C$ , i.e., if all distinct elements are at a distance of at least  $C$  one from each other.

<sup>(main)</sup> **Theorem 1.** Let  $T > 0$  and  $\omega$  be an open subset of  $N$ . If  $(\omega, T)$  satisfies VGCC and if the family  $(F(\lambda_j + \lambda'_k))_{j,k \in \mathbb{N}}$  satisfies the gap condition, then the observability property is satisfied for (1) on  $(\omega, T)$ .

Let us comment on this theorem and on VGCC.

Recall that  $(\omega, T)$  satisfies the usual *Geometric Control Condition* (GCC, see [2, 8, 11]) whenever every geodesic (not necessarily vertical) meets  $\omega$  within time  $T$ . Let  $\omega$  be an open subset of  $N$  and  $T > 0$ . If  $(\omega, T)$  satisfies GCC then it also satisfies VGCC. There exist examples where  $(\omega, T)$  satisfies VGCC but not GCC: for every  $x \in M$ , we define  $\omega_x := (\{x\} \times M') \cap \omega$ . Then,  $(\omega, T)$  satisfies VGCC if and only if  $(\omega_x, T)$  satisfies GCC on  $M'$  for every  $x \in M$ . In particular, we obtain the following examples:

- Let  $(U_i)_{i \in I}$  be an open covering of  $M$ , and let  $(\omega_i)_{i \in I}$  be a family of open subsets of  $M'$  satisfying GCC within time  $T$ . Then, setting  $\omega = \cup_{i \in I} U_i \times \omega_i$ ,  $(\omega, T)$  satisfies VGCC. In particular, if  $\omega'$  is an open subset of  $M'$  satisfying GCC, then  $M \times \omega'$  satisfies VGCC.
- Let  $\gamma$  be a non-vertical geodesic. Given any  $\varepsilon > 0$ , we consider the closed  $\varepsilon$ -neighborhood of the support  $\Gamma$  of  $\gamma$  defined by  $U_\varepsilon = \{x \in N \mid d_g(x, \Gamma) \leq \varepsilon\}$ , where  $d_g$  is the Riemannian distance on  $N$ . We set  $\omega_\varepsilon = N \setminus U_\varepsilon$ . Then, for any  $T > 0$  and any  $\varepsilon > 0$  small enough,  $(\omega_\varepsilon, T)$  satisfies VGCC.

For instance, if  $\gamma$  is horizontal, we can choose  $\varepsilon < \frac{T}{2}$ . For the general case, note that for every  $x \in M$ ,  $(\omega_\varepsilon)_x$  (with the notations above) is contained in the complement of a small ball in  $M'$ .

Let us now recall some existing results. It is well known that, when  $\omega$  is open, GCC is a sufficient condition for observability of the Schrödinger equation (see [8]). It is also well known that, except for Zoll manifolds, i.e., manifolds whose all geodesics are periodic (see [9]), GCC is not a necessary assumption. An example where the Schrödinger is observable on  $(\omega, T)$  but where  $(\omega, T)$  does not satisfy GCC is given in [6]: in the flat 2D torus, any non empty open set gives observability in any time  $T$ . This example has been extended to high dimensions in [7]. We also refer to [1] for another example, in the Dirichlet disk.

**Remark 1.** The spectrum of  $\Delta_g^{1/2}$  can never satisfy the gap condition on the product manifold  $N$ .

**Application to the Schrödinger equation.** We assume that  $F(s) = s$  so that (1) is now the usual Schrödinger equation. Theorem 1 can be applied as soon as the spectrum of  $\Delta_g$  satisfies the gap condition. This is true for instance when  $M$  and  $M'$  have an integer spectrum, in particular when  $M$  and  $M'$  are a finite product of standard spheres.

?<cor1>? **Corollary 1.** Assume that the spectrum of  $\Delta_g$  satisfies the gap condition. Let  $T > 0$  and  $\omega$  be an open subset of  $N$ , such that  $(\omega, T)$  satisfies VGCC but not GCC. Then the Schrödinger equation is observable on  $(\omega, T)$ , while GCC is not satisfied.

This result provides new examples of configurations where one has observability but not GCC.

**Quantum limits on a product manifold.** The definition of a quantum limit is recalled in Appendix A.1.

<sup>(cor3)</sup> **Corollary 2.** *The support of any quantum limit of  $N$  must contain at least an horizontal and a vertical geodesic. In particular, the Dirac measure along any periodic geodesic of  $N$  is not a quantum limit.*

## 2 Proofs

### 2.1 Proof of Theorem 1

Let  $\omega$  be an open subset of  $N$ . For any  $x \in M$ , we set  $\omega_x = \omega \cap (\{x\} \times M')$ . Theorem 1 follows from the following lemmas, which are in order.

<sup>(lemma\_main)?</sup> **Lemma 1.** *Assume that there exists  $c, T > 0$  such that for all complex numbers  $(a_{k,m})_{k,m \in \mathbb{N}}$  and every  $x \in M$ ,*

$$\int_0^T \int_{\omega_x} \left| \sum_{k,m} a_{k,m} \phi'_m e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'} dt \geq c \sum_{k,m} |a_{k,m}|^2 \quad (3) \quad \boxed{\text{main\_estimate}}$$

then (1) is observable on  $(\omega, T)$ .

*Proof.* The objective is to prove (2). Writing  $y = \sum_{l,m \geq 0} b_{l,m} \phi_l \phi'_m$ , we have  $e^{itF(\Delta_g)} y = \sum_{k,m} b_{k,m} \phi_k \phi'_m e^{iF(\mu_k + \mu'_m)t}$ . We denote by  $G_x : C^\infty(N) \rightarrow C^\infty(\{x\} \times M')$  the mapping  $(G_x f)(q, q') = f(x, q')$ . Setting  $a_{k,m}(x) = \sum_{l=\alpha_k}^{\alpha_{k+1}-1} b_{l,m} \phi_l(x)$ , using (3) and the definition of  $\alpha_k$ , there exists  $T, c > 0$  such that

$$\begin{aligned} \int_0^T \int_{\omega} |e^{itF(\Delta_g)} y|^2 dx_g dt &= \int_0^T \int_M \int_{\omega_x} |G_x e^{itF(\Delta_g)} y|^2 dx_{h'} dx_h(x) dt \\ &= \int_0^T \int_M \int_{\omega_x} \left| \sum_{l,m \geq 0} b_{l,m} \phi_l(x) \phi'_m(x') e^{iF(\mu_k + \mu'_m)t} \right|^2 dx_{h'}(x') dx_h(x) dt \\ &= \int_0^T \int_M \int_{\omega_x} \left| \sum_{k,m} a_{k,m}(x) \phi'_m(x') e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'}(x') dx_h(x) dt \\ &\geq c \int_0^T \int_M \sum_{k,m} |a_{k,m}(x)|^2 dx_h(x) dt = cT \sum_{k,m} \int_M \left| \sum_{l=\alpha_k}^{\alpha_{k+1}-1} b_{l,m} \phi_l \right|^2 dx_h \\ &= cT \sum_{k,m} \int_M \sum_{\alpha_k \leq l, l' \leq \alpha_{k+1}-1} b_{l,m} b_{l',m} \phi_l \phi_{l'} dx_h = cT \sum_{k,m} \int_M \sum_{l=\alpha_k}^{\alpha_{k+1}-1} b_{l,m}^2 \phi_l^2 dx_h \\ &= cT \sum_{k,m} b_{k,m}^2 \int_M \phi_k^2 dx_h = cT \sum_{k,m} b_{k,m}^2 = cT \|y\|_{L^2(N)}^2. \end{aligned}$$

This proves observability in time  $T$ . □

We define

$$g_1^V(\omega) = \inf_{x, \phi'} \int_{\omega_x} \phi'^2$$

where the infimum is taken over the set of all possible  $x \in M$  and all possible eigenfunctions  $\phi'$  of  $\Delta_{h'}$  such that  $\|\phi'\|_{L^2(M)} = 1$ .

**Lemma 2.** *Assume that the family  $(F(\lambda_k + \lambda'_m))_{k,m \in \mathbb{N}}$  satisfies the gap condition. Then (3) is satisfied with  $c = g_1^V(\omega)/2$  for  $T$  large enough.*

*Proof.* Define  $\Lambda_{k,m} = F(\lambda_k + \mu'_m)$ . By assumption, there exists  $C_0 > 0$  such that if  $\Lambda_{k,m} \neq \Lambda_{k',m'}$ , then

$$|\Lambda_{k,m} - \Lambda_{k',m'}| \geq C_0. \quad (4) \quad \boxed{\text{gap}}$$

Let  $T > 0$  and  $\psi_T$  the characteristic function of the interval  $[0, 2T]$ . Its Fourier transform  $\hat{\psi}_T$  is equal to  $\hat{\psi}_T(\xi) = \frac{e^{iT\xi} - 1}{T\xi}$ . Noting that  $\hat{\psi}_T(0) = 1$ , we have

$$\begin{aligned} \int_0^{2T} \int_{\omega_x} \left| \sum_{k,m} a_{k,m} \phi'_m e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'} dt \\ = \sum_{k,m,k',m'} a_{k,m} \overline{a_{k',m'}} \hat{\psi}_T(\Lambda_{k,m} - \Lambda_{k',m'}) \int_{\omega_x} \phi'_m \phi'_{m'} = A + B \end{aligned} \quad (5) \quad \boxed{\text{sum}}$$

with

$$A = \sum_{k,m} |a_{k,m}|^2 \int_{\omega_x} |\phi'_m|^2, \quad B = \sum_{(k,m) \neq (k',m')} a_{k,m} \overline{a_{k',m'}} \hat{\psi}_T(\Lambda_{k,m} - \Lambda_{k',m'}) \int_{\omega_x} \phi'_m \phi'_{m'}.$$

Using the gap condition (4), it follows from Montgomery-Vaughan inequality (see [10]) that  $|B| \leq \frac{2}{TC_0} A$ . Hence, we obtain from (5) that

$$\int_0^{2T} \int_{\omega_x} \left| \sum_{k,m} a_{k,m} \phi'_m e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'} dt \geq \left(1 - \frac{2}{TC}\right) A \geq \frac{1}{2} A$$

when  $T$  is large enough. Noting that  $A \geq \sum_{k,m} |a_{k,m}|^2 g_1^V(\omega)$ , the inequality (3) follows with  $c = g_1^V(\omega)/2$ .  $\square$

**Lemma 3.** *If  $(\omega, T)$  satisfies VGCC then  $g_1^V(\omega) > 0$ .*

*Proof.* Assume that  $\omega$  satisfies VGCC. By contradiction, let us assume that  $g_1^V(\omega) = 0$ . This means that for every  $\varepsilon > 0$ , there exists  $x_\varepsilon \in M$  and an eigenfunction  $\phi'_\varepsilon$  of  $\Delta_{h'}$  such that  $\|\phi'_\varepsilon\|_{L^2(M')} = 1$  and such that  $\int_{\omega_{x_\varepsilon}} \phi_\varepsilon^2 dx_g \leq \varepsilon$ , where we recall that  $\omega_{x_\varepsilon} = \omega \cap (\{x_\varepsilon\} \times M')$ . By compactness, we assume that  $x_\varepsilon \rightarrow x_0 \in M$  and that  $(\phi'_\varepsilon)^2 \rightarrow \mu$  weakly, where  $\mu$  is a quantum limit of  $M'$ . Let  $U_k$  be an increasing sequence of open sets such that  $\overline{U_k} \subset U_{k+1}$  and such that  $\cup_k U_k = \omega_0 = \omega \cap (\{x_0\} \times M')$ . Since  $\omega$  is open, for all  $k \in \mathbb{N}$  and  $\varepsilon > 0$  small enough, we have  $U_k \subset \omega_{x_\varepsilon}$ . This implies that  $\int_{U_k} (\phi'_\varepsilon)^2 dx_g \leq \varepsilon$ . We infer from the Portmanteau theorem (see Appendix A.2) that  $\mu(U_k) = 0$ , and thus

$\mu(\omega_0) = 0$ . This implies that GCC does not hold for  $\omega_0$  in any time. Indeed, by the Egorov theorem (see [4, 12]),  $\mu$  is invariant under the geodesic flow, as a measure on  $S^*M'$ . By the Krein-Milman theorem,  $\mu$  can be approximated by a sequence  $(\mu_k)_{k \in \mathbb{N}}$  of convex combinations of Dirac measures along periodic geodesics. Since  $\mu_k(\omega_0) \rightarrow 0$ , there exists a sequence of periodic geodesics  $\gamma_k$  such that, if  $\delta_k$  is the Dirac measure along  $\gamma_k$ , we have  $\delta_k(\mu_0) \rightarrow 0$ . This means that the time spent by  $\gamma_k$  (actually, by its projection onto  $M'$ ) in  $\omega_0$  tends to 0. By compactness of geodesics,  $\gamma_k$  converges to some geodesic  $\gamma$ . Again by the Portmanteau theorem,  $\gamma$  does not meet  $\omega$ , hence GCC on  $M'$  fails for  $\omega_0$  and this contradicts that VGCC is satisfied for  $\omega$ .  $\square$

## 2.2 Proof of Corollary 2

We prove the vertical case, the horizontal case being symmetric. We rearrange the set  $\{\lambda_j + \lambda'_k \mid j, k \in \mathbb{N}\} = \{d_k \mid k \in \mathbb{N}\}$  with an increasing sequence  $(d_k)_{k \in \mathbb{N}}$ . Let  $F$  be an increasing function such that  $F(d_k) = k$  for every  $k \in \mathbb{N}$ . By construction, the set  $\{F(\lambda_j + \lambda'_k) \mid j, k \in \mathbb{N}\}$  satisfies the gap condition. Let  $\Gamma$  be the support of a quantum limit  $\mu$  on  $M \times M'$ . Since  $F(\Delta_g)$  and  $\Delta_g$  have the same eigenfunctions,  $\mu$  is also the weak limit of a sequence of  $\psi_j^2 dx_g d\xi$  where  $\psi_j$  are eigenfunctions of  $F(\Delta_g)$  satisfying  $\|\psi_j\|_{L^2} = 1$ . We set  $\omega_\varepsilon = \{x \in N \mid d_g(x, \Gamma) > \varepsilon\}$ , for  $\varepsilon > 0$  small enough. For every  $T > 0$ ,  $(\omega_\varepsilon, T)$  is not observable for (1) because  $y = \psi_j$  provides a sequence of test functions which, at the limit, lie on  $\Gamma$ . Hence, by Theorem 1,  $(\omega_\varepsilon, T)$  does not satisfy VGCC. Remark ?? implies that  $\Gamma$  must contain a vertical geodesic.

# A Appendix

## A.1 Quantum limits

<sup>(q1)</sup> We recall that a *quantum limit* (QL in short)  $\mu$ , also called *semi-classical measure*, is a probability Radon (i.e., probability Borel regular) measure on  $S^*M$  that is a closure point (weak limit), as  $\lambda \rightarrow +\infty$ , of the family of Radon measures  $\mu_\lambda(a) = \langle \text{Op}(a)\phi_\lambda, \phi_\lambda \rangle$  (which are asymptotically positive by the Gårding inequality), where  $\phi_\lambda$  denotes an eigenfunction of norm 1 associated with the eigenvalue  $\lambda$  of  $\sqrt{\Delta}$ . Here,  $\text{Op}$  is any quantization. We speak of a *QL on  $M$*  to refer to a closure point (for the weak topology) of the sequence of probability Radon measures  $\phi_\lambda^2 dx_g$  on  $M$  as  $\lambda \rightarrow +\infty$ . Note that QLs do not depend on the choice of a quantization. We denote by  $\mathcal{Q}(S^*M)$  (resp.,  $\mathcal{Q}(M)$ ) the set of QLs (resp., the set of QLs on  $M$ ). Both are compact sets.

Given any  $\mu \in \mathcal{Q}(S^*M)$ , the Radon measure  $\pi_*\mu$ , image of  $\mu$  under the canonical projection  $\pi : S^*M \rightarrow M$ , is a probability Radon measure on  $M$ . It is defined, equivalently, by  $(\pi_*\mu)(f) = \mu(\pi^*f) = \mu(f \circ \pi)$  for every  $f \in C^0(M)$  (note that, in local coordinates  $(x, \xi)$  in  $S^*M$ , the function  $f \circ \pi$  is a function depending only on  $x$ ), or by  $(\pi_*\mu)(\omega) = \mu(\pi^{-1}(\omega))$  for every  $\omega \subset M$  Borel measurable (or Lebesgue measurable, by regularity). It is easy to

see that<sup>1</sup>

$$\pi_* \mathcal{Q}(S^*M) = \mathcal{Q}(M).$$

In other words, QLs on  $M$  are exactly the image measures under  $\pi$  of QLs.

## A.2 Portmanteau theorem

<sup>(centre)</sup> Let us recall the Portmanteau theorem (see, e.g., [3]). Let  $X$  be a topological space, endowed with its Borel  $\sigma$ -algebra. Let  $\mu$  and  $\mu_n$ ,  $n \in \mathbb{N}^*$ , be finite Borel measures on  $X$ . Then the following items are equivalent:

- $\mu_n \rightarrow \mu$  for the narrow topology, i.e.,  $\int f d\mu_n \rightarrow \int f d\mu$  for every bounded continuous function  $f$  on  $X$ ;
- $\int f d\mu_n \rightarrow \int f d\mu$  for every Borel bounded function  $f$  on  $X$  such that  $\mu(\Delta_f) = 0$ , where  $\Delta_f$  is the set of points at which  $f$  is not continuous;
- $\mu_n(B) \rightarrow \mu(B)$  for every Borel subset  $B$  of  $X$  such that  $\mu(\partial B) = 0$ ;
- $\mu(F) \geq \limsup \mu_n(F)$  for every closed subset  $F$  of  $X$ , and  $\mu_n(X) \rightarrow \mu(X)$ ;
- $\mu(O) \leq \liminf \mu_n(O)$  for every open subset  $O$  of  $X$ , and  $\mu_n(X) \rightarrow \mu(X)$ .

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<sup>1</sup>Indeed, given any  $f \in C^0(M)$  and any  $\lambda \in \text{Spec}(\sqrt{\Delta})$ , we have

$$(\pi_* \mu_\lambda)(f) = \mu_\lambda(\pi^* f) = \langle \text{Op}(\pi^* f) \phi_\lambda, \phi_\lambda \rangle = \int_M f \phi_\lambda^2 dx_g,$$

because  $\text{Op}(\pi^* f) \phi_\lambda = f \phi_\lambda$ . The equality then easily follows by weak compactness of probability Radon measures.



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