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Observability for generalized Schrödinger equations and quantum limits on product manifolds

Emmanuel Humbert* Yannick Privat[†] Emmanuel Trélat[‡]

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Abstract

Given a closed product Riemannian manifold $N = M \times M'$ equipped with the product Riemannian metric $g = h + h'$, we explore the observability properties for the generalized Schrödinger equation $i\partial_t u = F(\Delta_g)u$, where Δ_g is the Laplace-Beltrami operator on N and $F : [0, +\infty) \rightarrow [0, +\infty)$ is an increasing function. In this note, we prove observability in finite time on any open subset ω satisfying the so-called Vertical Geometric Control Condition, stipulating that any vertical geodesic meets ω , under the additional assumption that the spectrum of $F(\Delta_g)$ satisfies a gap condition. A first consequence is that observability on ω for the Schrödinger equation is a strictly weaker property than the usual Geometric Control Condition on any product of spheres. A second consequence is that the Dirac measure along any geodesic of N is never a quantum limit.

1 Introduction and main results

Let (M, h) , (M', h') be closed Riemannian manifolds, and let Δ_h and $\Delta_{h'}$ be their respective (nonnegative) Laplace-Beltrami operators. We consider the Riemannian product manifold (N, g) defined by $N = M \times M'$ and $g = h + h'$. Let $\Delta_g = \Delta_h \otimes \Delta_{h'}$ be the corresponding Laplace-Beltrami operator on N . Let $F : [0, +\infty) \rightarrow [0, +\infty)$ be an arbitrary increasing function. We consider the generalized Schrödinger equation

$$i\partial_t u = F(\Delta_g)u \tag{1} \boxed{\text{schrod}}$$

on M , and we are interested in finding characterizations of the observability property for (1) on any open subset $\omega \subset N$.

We denote by $0 = \mu_0 \leq \mu_1 \leq \dots \leq \mu_k \leq \dots$ (resp., $0 = \mu'_0 \leq \mu'_1 \leq \dots \leq \mu'_k \leq \dots$) the eigenvalues of Δ_h (resp., of $\Delta_{h'}$), associated with a Hilbert eigenbasis $(\phi_k)_{k \in \mathbb{N}}$ of $L^2(M)$

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(resp., $(\phi'_k)_{k \in \mathbb{N}}$ of $L^2(M')$). We also denote by $0 = \lambda_0 < \lambda_1 < \dots < \lambda_k < \dots$ (resp., $0 = \lambda'_0 < \lambda'_1 < \dots < \lambda'_k < \dots$) its distinct eigenvalues. There exists an increasing sequence $(\alpha_k)_{k \in \mathbb{N}}$ (resp., $(\alpha'_k)_{k \in \mathbb{N}}$) such that for $j = \alpha_k, \dots, \alpha_{k+1} - 1$ (resp., $j = \alpha'_k, \dots, \alpha'_{k+1} - 1$), $\mu_j = \lambda_k$ (resp., $\mu'_j = \lambda'_k$).

Then, $(\phi_j \phi'_k)_{j,k \in \mathbb{N}}$ is an orthonormal basis of $L^2(N)$ of eigenfunctions of Δ_g associated to the eigenvalues $\mu_j + \mu'_k$. The operator $F(\Delta_g)$ is spectrally defined as the linear operator which, restricted to the eigenspace of Δ_g associated to the eigenvalue $\mu_j + \mu'_k$, is equal to $F(\mu_j + \mu'_k) \text{id}$. The fact that $F(\Delta_g)$ and Δ_g have the same eigenspaces comes from the fact that F is increasing, so that $F(\mu_j + \mu'_k) = F(\mu_{j'} + \mu'_{k'})$ if and only if $\mu_j + \mu'_k = \mu_{j'} + \mu'_{k'}$.

By the Stone theorem, $(e^{itF(\Delta_g)})_{t \geq 0}$ is a unitary strongly continuous semigroup on $L^2(N)$. Given any $y \in L^2(N)$, there exists a unique solution $u \in C^0([0, +\infty), L^2(N)) \cap C^1((0, +\infty), H^{-2}(N))$ of (1) such that $u(0) = y$, given by $u(t) = e^{itF(\Delta_g)}y$.

If $F(s) = s$ then (1) is the usual Schrödinger equation, and if $F(s) = \sqrt{s}$ then (1) is the half-wave equation.

We denote by dx_g the Riemannian volume form on N . Given any $T > 0$ and any measurable subset ω of N , we define the observability constant $C_T(\omega) \geq 0$ as the largest constant $C \geq 0$ such that

$$\int_0^T \int_\omega \left| e^{itF(\Delta_g)} y \right|^2 \geq C \|y\|_{L^2(N)}^2 \quad \forall y \in L^2(N) \quad (2) \quad \boxed{\text{obs}}$$

(observability inequality), i.e.,

$$\begin{aligned} C_T(\omega) &= \inf \left\{ \int_0^T \int_\omega \left| e^{itF(\Delta_g)} y \right|^2 dx_g dt \mid y \in L^2(N), \|y\|_{L^2(N)} = 1 \right\} \\ &= \inf \left\{ \int_0^T \int_\omega \left| \sum_{l,m} b_{jk} e^{itF(\mu_j + \mu'_k)} \phi_j \phi'_k \right|^2 dx_g dt \mid b_{jk} \in \ell^2(\mathbb{C}), \sum_{j,k=0}^{+\infty} |b_{jk}|^2 = 1 \right\} \end{aligned}$$

We say that the *observability property* is satisfied for (1) on (ω, T) if $C_T(\omega) > 0$.

Definition 1. A vertical (resp., horizontal) geodesic of N is a geodesic of the form $t \rightarrow (x, \gamma(t))$ (resp., $(\gamma(t), x)$) for some $x \in M$ (resp., for some $x \in M'$) and some geodesic γ of M' (resp., of M).

Definition 2. Let $\omega \subset N$ and let $T > 0$. We say that (ω, T) satisfies the Vertical Geometric Control Condition (in short, VGCC) if all vertical geodesics meet ω within time T , i.e., $\gamma([0, T]) \cap \omega \neq \emptyset$.

Definition 3. We say that a family $(a_k)_{k \in \mathbb{N}}$ of real numbers satisfies the gap condition if there exists a constant $C > 0$ such that for all $j, k \in \mathbb{N}$, we have either $a_j = a_k$ or $|a_k - a_l| \geq C$, i.e., if all distinct elements are at a distance of at least C one from each other.

^(main) **Theorem 1.** Let $T > 0$ and ω be an open subset of N . If (ω, T) satisfies VGCC and if the family $(F(\lambda_j + \lambda'_k))_{j,k \in \mathbb{N}}$ satisfies the gap condition, then the observability property is satisfied for (1) on (ω, T) .

Let us comment on this theorem and on VGCC.

Recall that (ω, T) satisfies the usual *Geometric Control Condition* (GCC, see [2, 8, 11]) whenever every geodesic (not necessarily vertical) meets ω within time T . Let ω be an open subset of N and $T > 0$. If (ω, T) satisfies GCC then it also satisfies VGCC. There exist examples where (ω, T) satisfies VGCC but not GCC: for every $x \in M$, we define $\omega_x := (\{x\} \times M') \cap \omega$. Then, (ω, T) satisfies VGCC if and only if (ω_x, T) satisfies GCC on M' for every $x \in M$. In particular, we obtain the following examples:

- Let $(U_i)_{i \in I}$ be an open covering of M , and let $(\omega_i)_{i \in I}$ be a family of open subsets of M' satisfying GCC within time T . Then, setting $\omega = \cup_{i \in I} U_i \times \omega_i$, (ω, T) satisfies VGCC. In particular, if ω' is an open subset of M' satisfying GCC, then $M \times \omega'$ satisfies VGCC.
- Let γ be a non-vertical geodesic. Given any $\varepsilon > 0$, we consider the closed ε -neighborhood of the support Γ of γ defined by $U_\varepsilon = \{x \in N \mid d_g(x, \Gamma) \leq \varepsilon\}$, where d_g is the Riemannian distance on N . We set $\omega_\varepsilon = N \setminus U_\varepsilon$. Then, for any $T > 0$ and any $\varepsilon > 0$ small enough, (ω_ε, T) satisfies VGCC.

For instance, if γ is horizontal, we can choose $\varepsilon < \frac{T}{2}$. For the general case, note that for every $x \in M$, $(\omega_\varepsilon)_x$ (with the notations above) is contained in the complement of a small ball in M' .

Let us now recall some existing results. It is well known that, when ω is open, GCC is a sufficient condition for observability of the Schrödinger equation (see [8]). It is also well known that, except for Zoll manifolds, i.e., manifolds whose all geodesics are periodic (see [9]), GCC is not a necessary assumption. An example where the Schrödinger is observable on (ω, T) but where (ω, T) does not satisfy GCC is given in [6]: in the flat 2D torus, any non empty open set gives observability in any time T . This example has been extended to high dimensions in [7]. We also refer to [1] for another example, in the Dirichlet disk.

Remark 1. The spectrum of $\Delta_g^{1/2}$ can never satisfy the gap condition on the product manifold N .

Application to the Schrödinger equation. We assume that $F(s) = s$ so that (1) is now the usual Schrödinger equation. Theorem 1 can be applied as soon as the spectrum of Δ_g satisfies the gap condition. This is true for instance when M and M' have an integer spectrum, in particular when M and M' are a finite product of standard spheres.

?<cor1>? **Corollary 1.** Assume that the spectrum of Δ_g satisfies the gap condition. Let $T > 0$ and ω be an open subset of N , such that (ω, T) satisfies VGCC but not GCC. Then the Schrödinger equation is observable on (ω, T) , while GCC is not satisfied.

This result provides new examples of configurations where one has observability but not GCC.

Quantum limits on a product manifold. The definition of a quantum limit is recalled in Appendix A.1.

^(cor3) **Corollary 2.** *The support of any quantum limit of N must contain at least an horizontal and a vertical geodesic. In particular, the Dirac measure along any periodic geodesic of N is not a quantum limit.*

2 Proofs

2.1 Proof of Theorem 1

Let ω be an open subset of N . For any $x \in M$, we set $\omega_x = \omega \cap (\{x\} \times M')$. Theorem 1 follows from the following lemmas, which are in order.

^{(lemma_main)?} **Lemma 1.** *Assume that there exists $c, T > 0$ such that for all complex numbers $(a_{k,m})_{k,m \in \mathbb{N}}$ and every $x \in M$,*

$$\int_0^T \int_{\omega_x} \left| \sum_{k,m} a_{k,m} \phi'_m e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'} dt \geq c \sum_{k,m} |a_{k,m}|^2 \quad (3) \quad \boxed{\text{main_estimate}}$$

then (1) is observable on (ω, T) .

Proof. The objective is to prove (2). Writing $y = \sum_{l,m \geq 0} b_{l,m} \phi_l \phi'_m$, we have $e^{itF(\Delta_g)} y = \sum_{k,m} b_{k,m} \phi_k \phi'_m e^{iF(\mu_k + \mu'_m)t}$. We denote by $G_x : C^\infty(N) \rightarrow C^\infty(\{x\} \times M')$ the mapping $(G_x f)(q, q') = f(x, q')$. Setting $a_{k,m}(x) = \sum_{l=\alpha_k}^{\alpha_{k+1}-1} b_{l,m} \phi_l(x)$, using (3) and the definition of α_k , there exists $T, c > 0$ such that

$$\begin{aligned} \int_0^T \int_{\omega} |e^{itF(\Delta_g)} y|^2 dx_g dt &= \int_0^T \int_M \int_{\omega_x} |G_x e^{itF(\Delta_g)} y|^2 dx_{h'} dx_h(x) dt \\ &= \int_0^T \int_M \int_{\omega_x} \left| \sum_{l,m \geq 0} b_{l,m} \phi_l(x) \phi'_m(x') e^{iF(\mu_k + \mu'_m)t} \right|^2 dx_{h'}(x') dx_h(x) dt \\ &= \int_0^T \int_M \int_{\omega_x} \left| \sum_{k,m} a_{k,m}(x) \phi'_m(x') e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'}(x') dx_h(x) dt \\ &\geq c \int_0^T \int_M \sum_{k,m} |a_{k,m}(x)|^2 dx_h(x) dt = cT \sum_{k,m} \int_M \left| \sum_{l=\alpha_k}^{\alpha_{k+1}-1} b_{l,m} \phi_l \right|^2 dx_h \\ &= cT \sum_{k,m} \int_M \sum_{\alpha_k \leq l, l' \leq \alpha_{k+1}-1} b_{l,m} b_{l',m} \phi_l \phi_{l'} dx_h = cT \sum_{k,m} \int_M \sum_{l=\alpha_k}^{\alpha_{k+1}-1} b_{l,m}^2 \phi_l^2 dx_h \\ &= cT \sum_{k,m} b_{k,m}^2 \int_M \phi_k^2 dx_h = cT \sum_{k,m} b_{k,m}^2 = cT \|y\|_{L^2(N)}^2. \end{aligned}$$

This proves observability in time T . □

We define

$$g_1^V(\omega) = \inf_{x, \phi'} \int_{\omega_x} \phi'^2$$

where the infimum is taken over the set of all possible $x \in M$ and all possible eigenfunctions ϕ' of $\Delta_{h'}$ such that $\|\phi'\|_{L^2(M)} = 1$.

Lemma 2. *Assume that the family $(F(\lambda_k + \lambda'_m))_{k,m \in \mathbb{N}}$ satisfies the gap condition. Then (3) is satisfied with $c = g_1^V(\omega)/2$ for T large enough.*

Proof. Define $\Lambda_{k,m} = F(\lambda_k + \mu'_m)$. By assumption, there exists $C_0 > 0$ such that if $\Lambda_{k,m} \neq \Lambda_{k',m'}$, then

$$|\Lambda_{k,m} - \Lambda_{k',m'}| \geq C_0. \quad (4) \quad \boxed{\text{gap}}$$

Let $T > 0$ and ψ_T the characteristic function of the interval $[0, 2T]$. Its Fourier transform $\hat{\psi}_T$ is equal to $\hat{\psi}_T(\xi) = \frac{e^{iT\xi} - 1}{T\xi}$. Noting that $\hat{\psi}_T(0) = 1$, we have

$$\begin{aligned} \int_0^{2T} \int_{\omega_x} \left| \sum_{k,m} a_{k,m} \phi'_m e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'} dt \\ = \sum_{k,m,k',m'} a_{k,m} \overline{a_{k',m'}} \hat{\psi}_T(\Lambda_{k,m} - \Lambda_{k',m'}) \int_{\omega_x} \phi'_m \phi'_{m'} = A + B \end{aligned} \quad (5) \quad \boxed{\text{sum}}$$

with

$$A = \sum_{k,m} |a_{k,m}|^2 \int_{\omega_x} |\phi'_m|^2, \quad B = \sum_{(k,m) \neq (k',m')} a_{k,m} \overline{a_{k',m'}} \hat{\psi}_T(\Lambda_{k,m} - \Lambda_{k',m'}) \int_{\omega_x} \phi'_m \phi'_{m'}.$$

Using the gap condition (4), it follows from Montgomery-Vaughan inequality (see [10]) that $|B| \leq \frac{2}{TC_0} A$. Hence, we obtain from (5) that

$$\int_0^{2T} \int_{\omega_x} \left| \sum_{k,m} a_{k,m} \phi'_m e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'} dt \geq \left(1 - \frac{2}{TC}\right) A \geq \frac{1}{2} A$$

when T is large enough. Noting that $A \geq \sum_{k,m} |a_{k,m}|^2 g_1^V(\omega)$, the inequality (3) follows with $c = g_1^V(\omega)/2$. $\square$

Lemma 3. *If (ω, T) satisfies VGCC then $g_1^V(\omega) > 0$.*

Proof. Assume that ω satisfies VGCC. By contradiction, let us assume that $g_1^V(\omega) = 0$. This means that for every $\varepsilon > 0$, there exists $x_\varepsilon \in M$ and an eigenfunction ϕ'_ε of $\Delta_{h'}$ such that $\|\phi'_\varepsilon\|_{L^2(M')} = 1$ and such that $\int_{\omega_{x_\varepsilon}} \phi_\varepsilon^2 dx_g \leq \varepsilon$, where we recall that $\omega_{x_\varepsilon} = \omega \cap (\{x_\varepsilon\} \times M')$. By compactness, we assume that $x_\varepsilon \rightarrow x_0 \in M$ and that $(\phi'_\varepsilon)^2 \rightarrow \mu$ weakly, where μ is a quantum limit of M' . Let U_k be an increasing sequence of open sets such that $\overline{U_k} \subset U_{k+1}$ and such that $\cup_k U_k = \omega_0 = \omega \cap (\{x_0\} \times M')$. Since ω is open, for all $k \in \mathbb{N}$ and $\varepsilon > 0$ small enough, we have $U_k \subset \omega_{x_\varepsilon}$. This implies that $\int_{U_k} (\phi'_\varepsilon)^2 dx_g \leq \varepsilon$. We infer from the Portmanteau theorem (see Appendix A.2) that $\mu(U_k) = 0$, and thus

$\mu(\omega_0) = 0$. This implies that GCC does not hold for ω_0 in any time. Indeed, by the Egorov theorem (see [4, 12]), μ is invariant under the geodesic flow, as a measure on S^*M' . By the Krein-Milman theorem, μ can be approximated by a sequence $(\mu_k)_{k \in \mathbb{N}}$ of convex combinations of Dirac measures along periodic geodesics. Since $\mu_k(\omega_0) \rightarrow 0$, there exists a sequence of periodic geodesics γ_k such that, if δ_k is the Dirac measure along γ_k , we have $\delta_k(\mu_0) \rightarrow 0$. This means that the time spent by γ_k (actually, by its projection onto M') in ω_0 tends to 0. By compactness of geodesics, γ_k converges to some geodesic γ . Again by the Portmanteau theorem, γ does not meet ω , hence GCC on M' fails for ω_0 and this contradicts that VGCC is satisfied for ω . $\square$

2.2 Proof of Corollary 2

We prove the vertical case, the horizontal case being symmetric. We rearrange the set $\{\lambda_j + \lambda'_k \mid j, k \in \mathbb{N}\} = \{d_k \mid k \in \mathbb{N}\}$ with an increasing sequence $(d_k)_{k \in \mathbb{N}}$. Let F be an increasing function such that $F(d_k) = k$ for every $k \in \mathbb{N}$. By construction, the set $\{F(\lambda_j + \lambda'_k) \mid j, k \in \mathbb{N}\}$ satisfies the gap condition. Let Γ be the support of a quantum limit μ on $M \times M'$. Since $F(\Delta_g)$ and Δ_g have the same eigenfunctions, μ is also the weak limit of a sequence of $\psi_j^2 dx_g d\xi$ where ψ_j are eigenfunctions of $F(\Delta_g)$ satisfying $\|\psi_j\|_{L^2} = 1$. We set $\omega_\varepsilon = \{x \in N \mid d_g(x, \Gamma) > \varepsilon\}$, for $\varepsilon > 0$ small enough. For every $T > 0$, (ω_ε, T) is not observable for (1) because $y = \psi_j$ provides a sequence of test functions which, at the limit, lie on Γ . Hence, by Theorem 1, (ω_ε, T) does not satisfy VGCC. Remark ?? implies that Γ must contain a vertical geodesic.

A Appendix

A.1 Quantum limits

^(q1) We recall that a *quantum limit* (QL in short) μ , also called *semi-classical measure*, is a probability Radon (i.e., probability Borel regular) measure on S^*M that is a closure point (weak limit), as $\lambda \rightarrow +\infty$, of the family of Radon measures $\mu_\lambda(a) = \langle \text{Op}(a)\phi_\lambda, \phi_\lambda \rangle$ (which are asymptotically positive by the Gårding inequality), where ϕ_λ denotes an eigenfunction of norm 1 associated with the eigenvalue λ of $\sqrt{\Delta}$. Here, Op is any quantization. We speak of a *QL on M* to refer to a closure point (for the weak topology) of the sequence of probability Radon measures $\phi_\lambda^2 dx_g$ on M as $\lambda \rightarrow +\infty$. Note that QLs do not depend on the choice of a quantization. We denote by $\mathcal{Q}(S^*M)$ (resp., $\mathcal{Q}(M)$) the set of QLs (resp., the set of QLs on M). Both are compact sets.

Given any $\mu \in \mathcal{Q}(S^*M)$, the Radon measure $\pi_*\mu$, image of μ under the canonical projection $\pi : S^*M \rightarrow M$, is a probability Radon measure on M . It is defined, equivalently, by $(\pi_*\mu)(f) = \mu(\pi^*f) = \mu(f \circ \pi)$ for every $f \in C^0(M)$ (note that, in local coordinates (x, ξ) in S^*M , the function $f \circ \pi$ is a function depending only on x), or by $(\pi_*\mu)(\omega) = \mu(\pi^{-1}(\omega))$ for every $\omega \subset M$ Borel measurable (or Lebesgue measurable, by regularity). It is easy to

see that¹

$$\pi_* \mathcal{Q}(S^*M) = \mathcal{Q}(M).$$

In other words, QLs on M are exactly the image measures under π of QLs.

A.2 Portmanteau theorem

^(centre) Let us recall the Portmanteau theorem (see, e.g., [3]). Let X be a topological space, endowed with its Borel σ -algebra. Let μ and μ_n , $n \in \mathbb{N}^*$, be finite Borel measures on X . Then the following items are equivalent:

- $\mu_n \rightarrow \mu$ for the narrow topology, i.e., $\int f d\mu_n \rightarrow \int f d\mu$ for every bounded continuous function f on X ;
- $\int f d\mu_n \rightarrow \int f d\mu$ for every Borel bounded function f on X such that $\mu(\Delta_f) = 0$, where Δ_f is the set of points at which f is not continuous;
- $\mu_n(B) \rightarrow \mu(B)$ for every Borel subset B of X such that $\mu(\partial B) = 0$;
- $\mu(F) \geq \limsup \mu_n(F)$ for every closed subset F of X , and $\mu_n(X) \rightarrow \mu(X)$;
- $\mu(O) \leq \liminf \mu_n(O)$ for every open subset O of X , and $\mu_n(X) \rightarrow \mu(X)$.

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¹Indeed, given any $f \in C^0(M)$ and any $\lambda \in \text{Spec}(\sqrt{\Delta})$, we have

$$(\pi_* \mu_\lambda)(f) = \mu_\lambda(\pi^* f) = \langle \text{Op}(\pi^* f) \phi_\lambda, \phi_\lambda \rangle = \int_M f \phi_\lambda^2 dx_g,$$

because $\text{Op}(\pi^* f) \phi_\lambda = f \phi_\lambda$. The equality then easily follows by weak compactness of probability Radon measures.

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