



HAL
open science

Observability for generalized Schrödinger equations and quantum limits on product manifolds

Emmanuel Humbert, Yannick Privat, Emmanuel Trélat

► **To cite this version:**

Emmanuel Humbert, Yannick Privat, Emmanuel Trélat. Observability for generalized Schrödinger equations and quantum limits on product manifolds. 2020. hal-02496644

HAL Id: hal-02496644

<https://hal.science/hal-02496644v1>

Preprint submitted on 3 Mar 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Observability for generalized Schrödinger equations and quantum limits on product manifolds

Emmanuel Humbert* Yannick Privat[†] Emmanuel Trélat[‡]

March 3, 2020

Abstract

Given a closed product Riemannian manifold $N = M \times M'$ equipped with the product Riemannian metric $g = h + h'$, we explore the observability properties for the generalized Schrödinger equation $i\partial_t u = F(\Delta_g)u$, where Δ_g is the Laplace-Beltrami operator on N and $F : [0, +\infty) \rightarrow [0, +\infty)$ is an increasing function. In this note, we prove observability in finite time on any open subset ω satisfying the so-called Vertical Geometric Control Condition, stipulating that any vertical geodesic meets ω , under the additional assumption that the spectrum of $F(\Delta_g)$ satisfies a gap condition. A first consequence is that observability on ω for the Schrödinger equation is a strictly weaker property than the usual Geometric Control Condition on any product of spheres. A second consequence is that the Dirac measure along any geodesic of N is never a quantum limit.

1 Introduction and main results

Let (M, h) , (M', h') be closed Riemannian manifolds, and let Δ_h and $\Delta_{h'}$ be their respective (nonnegative) Laplace-Beltrami operators. We consider the Riemannian product manifold (N, g) defined by $N = M \times M'$ and $g = h + h'$. Let $\Delta_g = \Delta_h \otimes \Delta_{h'}$ be the corresponding Laplace-Beltrami operator on N . Let $F : [0, +\infty) \rightarrow [0, +\infty)$ be an arbitrary increasing function. We consider the generalized Schrödinger equation

$$i\partial_t u = F(\Delta_g)u \tag{1} \boxed{\text{schrod}}$$

on M , and we are interested in finding characterizations of the observability property for (1) on any open subset $\omega \subset N$.

We denote by $0 = \mu_0 \leq \mu_1 \leq \dots \leq \mu_k \leq \dots$ (resp., $0 = \mu'_0 \leq \mu'_1 \leq \dots \leq \mu'_k \leq \dots$) the eigenvalues of Δ_h (resp., of $\Delta_{h'}$), associated with a Hilbert eigenbasis $(\phi_k)_{k \in \mathbb{N}}$ of $L^2(M)$

*Institut Denis Poisson, UFR Sciences et Technologie, Faculté François Rabelais, Parc de Grandmont, 37200 Tours, France (emmanuel.humbert@lmpt.univ-tours.fr).

[†]IRMA, Université de Strasbourg, CNRS UMR 7501, 7 rue René Descartes, 67084 Strasbourg, France (yannick.privat@unistra.fr).

[‡]Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions (LJLL), F-75005 Paris, France (emmanuel.trelat@sorbonne-universite.fr).

(resp., $(\phi'_k)_{k \in \mathbb{N}}$ of $L^2(M')$). We also denote by $0 = \lambda_0 < \lambda_1 < \dots < \lambda_k < \dots$ (resp., $0 = \lambda'_0 < \lambda'_1 < \dots < \lambda'_k < \dots$) its distinct eigenvalues. There exists an increasing sequence $(\alpha_k)_{k \in \mathbb{N}}$ (resp., $(\alpha'_k)_{k \in \mathbb{N}}$) such that for $j = \alpha_k, \dots, \alpha_{k+1} - 1$ (resp., $j = \alpha'_k, \dots, \alpha'_{k+1} - 1$), $\mu_j = \lambda_k$ (resp., $\mu'_j = \lambda'_k$).

Then, $(\phi_j \phi'_k)_{j, k \in \mathbb{N}}$ is an orthonormal basis of $L^2(N)$ of eigenfunctions of Δ_g associated to the eigenvalues $\mu_j + \mu'_k$. The operator $F(\Delta_g)$ is spectrally defined as the linear operator which, restricted to the eigenspace of Δ_g associated to the eigenvalue $\mu_j + \mu'_k$, is equal to $F(\mu_j + \mu'_k)$ id. The fact that $F(\Delta_g)$ and Δ_g have the same eigenspaces comes from the fact that F is increasing, so that $F(\mu_j + \mu'_k) = F(\mu_{j'} + \mu'_{k'})$ if and only if $\mu_j + \mu'_k = \mu_{j'} + \mu'_{k'}$.

By the Stone theorem, $(e^{itF(\Delta_g)})_{t \geq 0}$ is a unitary strongly continuous semigroup on $L^2(N)$. Given any $y \in L^2(N)$, there exists a unique solution $u \in C^0([0, +\infty), L^2(N)) \cap C^1((0, +\infty), H^{-2}(N))$ of (1) such that $u(0) = y$, given by $u(t) = e^{itF(\Delta_g)}y$.

If $F(s) = s$ then (1) is the usual Schrödinger equation, and if $F(s) = \sqrt{s}$ then (1) is the half-wave equation.

We denote by dx_g the Riemannian volume form on N . Given any $T > 0$ and any measurable subset ω of N , we define the observability constant $C_T(\omega) \geq 0$ as the largest constant $C \geq 0$ such that

$$\int_0^T \int_{\omega} \left| e^{itF(\Delta_g)} y \right|^2 \geq C \|y\|_{L^2(N)}^2 \quad \forall y \in L^2(N) \quad (2) \quad \boxed{\text{obs}}$$

(observability inequality), i.e.,

$$\begin{aligned} C_T(\omega) &= \inf \left\{ \int_0^T \int_{\omega} \left| e^{itF(\Delta_g)} y \right|^2 dx_g dt \mid y \in L^2(N), \|y\|_{L^2(N)} = 1 \right\} \\ &= \inf \left\{ \int_0^T \int_{\omega} \left| \sum_{l, m} b_{jk} e^{itF(\mu_j + \mu'_k)} \phi_j \phi'_k \right|^2 dx_g dt \mid b_{jk} \in \ell^2(\mathbb{C}), \sum_{j, k=0}^{+\infty} |b_{jk}|^2 = 1 \right\} \end{aligned}$$

We say that the *observability property* is satisfied for (1) on (ω, T) if $C_T(\omega) > 0$.

Definition 1. A vertical (resp., horizontal) geodesic of N is a geodesic of the form $t \rightarrow (x, \gamma(t))$ (resp., $(\gamma(t), x)$) for some $x \in M$ (resp., for some $x \in M'$) and some geodesic γ of M' (resp., of M).

Definition 2. Let $\omega \subset N$ and let $T > 0$. We say that (ω, T) satisfies the Vertical Geometric Control Condition (in short, VGCC) if all vertical geodesics meet ω within time T , i.e., $\gamma([0, T]) \cap \omega \neq \emptyset$.

Definition 3. We say that a family $(a_k)_{k \in \mathbb{N}}$ of real numbers satisfies the gap condition if there exists a constant $C > 0$ such that for all $j, k \in \mathbb{N}$, we have either $a_j = a_k$ or $|a_k - a_l| \geq C$, i.e., if all distinct elements are at a distance of at least C one from each other.

^(main) **Theorem 1.** Let $T > 0$ and ω be an open subset of N . If (ω, T) satisfies VGCC and if the family $(F(\lambda_j + \lambda'_k))_{j, k \in \mathbb{N}}$ satisfies the gap condition, then the observability property is satisfied for (1) on (ω, T) .

Let us comment on this theorem and on VGCC.

Recall that (ω, T) satisfies the usual *Geometric Control Condition* (GCC, see [2, 8, 11]) whenever every geodesic (not necessarily vertical) meets ω within time T . Let ω be an open subset of N and $T > 0$. If (ω, T) satisfies GCC then it also satisfies VGCC. There exist examples where (ω, T) satisfies VGCC but not GCC: for every $x \in M$, we define $\omega_x := (\{x\} \times M') \cap \omega$. Then, (ω, T) satisfies VGCC if and only if (ω_x, T) satisfies GCC on M' for every $x \in M$. In particular, we obtain the following examples:

- Let $(U_i)_{i \in I}$ be an open covering of M , and let $(\omega_i)_{i \in I}$ be a family of open subsets of M' satisfying GCC within time T . Then, setting $\omega = \cup_{i \in I} U_i \times \omega_i$, (ω, T) satisfies VGCC. In particular, if ω' is an open subset of M' satisfying GCC, then $M \times \omega'$ satisfies VGCC.
- Let γ be a non-vertical geodesic. Given any $\varepsilon > 0$, we consider the closed ε -neighborhood of the support Γ of γ defined by $U_\varepsilon = \{x \in N \mid d_g(x, \Gamma) \leq \varepsilon\}$, where d_g is the Riemannian distance on N . We set $\omega_\varepsilon = N \setminus U_\varepsilon$. Then, for any $T > 0$ and any $\varepsilon > 0$ small enough, (ω_ε, T) satisfies VGCC.

For instance, if γ is horizontal, we can choose $\varepsilon < \frac{T}{2}$. For the general case, note that for every $x \in M$, $(\omega_\varepsilon)_x$ (with the notations above) is contained in the complement of a small ball in M' .

Let us now recall some existing results. It is well known that, when ω is open, GCC is a sufficient condition for observability of the Schrödinger equation (see [8]). It is also well known that, except for Zoll manifolds, i.e., manifolds whose all geodesics are periodic (see [9]), GCC is not a necessary assumption. An example where the Schrödinger is observable on (ω, T) but where (ω, T) does not satisfy GCC is given in [6]: in the flat 2D torus, any non empty open set gives observability in any time T . This example has been extended to high dimensions in [7]. We also refer to [1] for another example, in the Dirichlet disk.

Remark 1. The spectrum of $\Delta_g^{1/2}$ can never satisfy the gap condition on the product manifold N .

Application to the Schrödinger equation. We assume that $F(s) = s$ so that (1) is now the usual Schrödinger equation. Theorem 1 can be applied as soon as the spectrum of Δ_g satisfies the gap condition. This is true for instance when M and M' have an integer spectrum, in particular when M and M' are a finite product of standard spheres.

?<cor1>? **Corollary 1.** *Assume that the spectrum of Δ_g satisfies the gap condition. Let $T > 0$ and ω be an open subset of N , such that (ω, T) satisfies VGCC but not GCC. Then the Schrödinger equation is observable on (ω, T) , while GCC is not satisfied.*

This result provides new examples of configurations where one has observability but not GCC.

Quantum limits on a product manifold. The definition of a quantum limit is recalled in Appendix A.1.

(cor3) **Corollary 2.** *The support of any quantum limit of N must contain at least an horizontal and a vertical geodesic. In particular, the Dirac measure along any periodic geodesic of N is not a quantum limit.*

2 Proofs

2.1 Proof of Theorem 1

Let ω be an open subset of N . For any $x \in M$, we set $\omega_x = \omega \cap (\{x\} \times M')$. Theorem 1 follows from the following lemmas, which are in order.

(lemma_main)? **Lemma 1.** *Assume that there exists $c, T > 0$ such that for all complex numbers $(a_{k,m})_{k,m \in \mathbb{N}}$ and every $x \in M$,*

$$\int_0^T \int_{\omega_x} \left| \sum_{k,m} a_{k,m} \phi'_m e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'} dt \geq c \sum_{k,m} |a_{k,m}|^2 \quad (3) \quad \boxed{\text{main_estimate}}$$

then (1) is observable on (ω, T) .

Proof. The objective is to prove (2). Writing $y = \sum_{l,m \geq 0} b_{l,m} \phi_l \phi'_m$, we have $e^{itF(\Delta_g)} y = \sum_{k,m} b_{k,m} \phi_k \phi'_m e^{iF(\mu_k + \mu'_m)t}$. We denote by $G_x : C^\infty(N) \rightarrow C^\infty(\{x\} \times M')$ the mapping $(G_x f)(q, q') = f(x, q')$. Setting $a_{k,m}(x) = \sum_{l=\alpha_k}^{\alpha_{k+1}-1} b_{l,m} \phi_l(x)$, using (3) and the definition of α_k , there exists $T, c > 0$ such that

$$\begin{aligned} \int_0^T \int_{\omega} |e^{itF(\Delta_g)} y|^2 dx_g dt &= \int_0^T \int_M \int_{\omega_x} |G_x e^{itF(\Delta_g)} y|^2 dx_{h'} dx_h(x) dt \\ &= \int_0^T \int_M \int_{\omega_x} \left| \sum_{l,m \geq 0} b_{l,m} \phi_l(x) \phi'_m(x') e^{iF(\mu_k + \mu'_m)t} \right|^2 dx_{h'}(x') dx_h(x) dt \\ &= \int_0^T \int_M \int_{\omega_x} \left| \sum_{k,m} a_{k,m}(x) \phi'_m(x') e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'}(x') dx_h(x) dt \\ &\geq c \int_0^T \int_M \sum_{k,m} |a_{k,m}(x)|^2 dx_h(x) dt = cT \sum_{k,m} \int_M \left| \sum_{l=\alpha_k}^{\alpha_{k+1}-1} b_{l,m} \phi_l \right|^2 dx_h \\ &= cT \sum_{k,m} \int_M \sum_{\alpha_k \leq l, l' \leq \alpha_{k+1}-1} b_{l,m} b_{l',m} \phi_l \phi_{l'} dx_h = cT \sum_{k,m} \int_M \sum_{l=\alpha_k}^{\alpha_{k+1}-1} b_{l,m}^2 \phi_l^2 dx_h \\ &= cT \sum_{k,m} b_{k,m}^2 \int_M \phi_k^2 dx_h = cT \sum_{k,m} b_{k,m}^2 = cT \|y\|_{L^2(N)}^2. \end{aligned}$$

This proves observability in time T . □

We define

$$g_1^V(\omega) = \inf_{x, \phi'} \int_{\omega_x} \phi'^2$$

where the infimum is taken over the set of all possible $x \in M$ and all possible eigenfunctions ϕ' of $\Delta_{h'}$ such that $\|\phi'\|_{L^2(M)} = 1$.

?(case2)? **Lemma 2.** *Assume that the family $(F(\lambda_k + \lambda'_m))_{k, m \in \mathbb{N}}$ satisfies the gap condition. Then (3) is satisfied with $c = g_1^V(\omega)/2$ for T large enough.*

Proof. Define $\Lambda_{k, m} = F(\lambda_k + \mu'_m)$. By assumption, there exists $C_0 > 0$ such that if $\Lambda_{k, m} \neq \Lambda_{k', m'}$, then

$$|\Lambda_{k, m} - \Lambda_{k', m'}| \geq C_0. \quad (4) \quad \boxed{\text{gap}}$$

Let $T > 0$ and ψ_T the characteristic function of the interval $[0, 2T]$. Its Fourier transform $\hat{\psi}_T$ is equal to $\hat{\psi}_T(\xi) = \frac{e^{iT\xi} - 1}{T\xi}$. Noting that $\hat{\psi}_T(0) = 1$, we have

$$\begin{aligned} & \int_0^{2T} \int_{\omega_x} \left| \sum_{k, m} a_{k, m} \phi'_m e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'} dt \\ &= \sum_{k, m, k', m'} a_{k, m} \overline{a_{k', m'}} \hat{\psi}_T(\Lambda_{k, m} - \Lambda_{k', m'}) \int_{\omega_x} \phi'_m \phi'_{m'} = A + B \quad (5) \quad \boxed{\text{sum}} \end{aligned}$$

with

$$A = \sum_{k, m} |a_{k, m}|^2 \int_{\omega_x} |\phi'_m|^2, \quad B = \sum_{(k, m) \neq (k', m')} a_{k, m} \overline{a_{k', m'}} \hat{\psi}_T(\Lambda_{k, m} - \Lambda_{k', m'}) \int_{\omega_x} \phi'_m \phi'_{m'}.$$

Using the gap condition (4), it follows from Montgomery-Vaughan inequality (see [10]) that $|B| \leq \frac{2}{TC_0} A$. Hence, we obtain from (5) that

$$\int_0^{2T} \int_{\omega_x} \left| \sum_{k, m} a_{k, m} \phi'_m e^{iF(\lambda_k + \mu'_m)t} \right|^2 dx_{h'} dt \geq \left(1 - \frac{2}{TC}\right) A \geq \frac{1}{2} A$$

when T is large enough. Noting that $A \geq \sum_{k, m} |a_{k, m}|^2 g_1^V(\omega)$, the inequality (3) follows with $c = g_1^V(\omega)/2$. \square

?(g1vgcc)? **Lemma 3.** *If (ω, T) satisfies VGCC then $g_1^V(\omega) > 0$.*

Proof. Assume that ω satisfies VGCC. By contradiction, let us assume that $g_1^V(\omega) = 0$. This means that for every $\varepsilon > 0$, there exists $x_\varepsilon \in M$ and an eigenfunction ϕ'_ε of $\Delta_{h'}$ such that $\|\phi'_\varepsilon\|_{L^2(M')} = 1$ and such that $\int_{\omega_{x_\varepsilon}} \phi_\varepsilon^2 dx_g \leq \varepsilon$, where we recall that $\omega_{x_\varepsilon} = \omega \cap (\{x_\varepsilon\} \times M')$. By compactness, we assume that $x_\varepsilon \rightarrow x_0 \in M$ and that $(\phi'_\varepsilon)^2 \rightarrow \mu$ weakly, where μ is a quantum limit of M' . Let U_k be an increasing sequence of open sets such that $\overline{U_k} \subset U_{k+1}$ and such that $\cup_k U_k = \omega_0 = \omega \cap (\{x_0\} \times M')$. Since ω is open, for all $k \in \mathbb{N}$ and $\varepsilon > 0$ small enough, we have $U_k \subset \omega_{x_\varepsilon}$. This implies that $\int_{U_k} (\phi'_\varepsilon)^2 dx_g \leq \varepsilon$. We infer from the Portmanteau theorem (see Appendix A.2) that $\mu(U_k) = 0$, and thus

$\mu(\omega_0) = 0$. This implies that GCC does not hold for ω_0 in any time. Indeed, by the Egorov theorem (see [4, 12]), μ is invariant under the geodesic flow, as a measure on S^*M' . By the Krein-Milman theorem, μ can be approximated by a sequence $(\mu_k)_{k \in \mathbb{N}}$ of convex combinations of Dirac measures along periodic geodesics. Since $\mu_k(\omega_0) \rightarrow 0$, there exists a sequence of periodic geodesics γ_k such that, if δ_k is the Dirac measure along γ_k , we have $\delta_k(\mu_0) \rightarrow 0$. This means that the time spent by γ_k (actually, by its projection onto M') in ω_0 tends to 0. By compactness of geodesics, γ_k converges to some geodesic γ . Again by the Portmanteau theorem, γ does not meet ω , hence GCC on M' fails for ω_0 and this contradicts that VGCC is satisfied for ω . \square

2.2 Proof of Corollary 2

We prove the vertical case, the horizontal case being symmetric. We rearrange the set $\{\lambda_j + \lambda'_k \mid j, k \in \mathbb{N}\} = \{d_k \mid k \in \mathbb{N}\}$ with an increasing sequence $(d_k)_{k \in \mathbb{N}}$. Let F be an increasing function such that $F(d_k) = k$ for every $k \in \mathbb{N}$. By construction, the set $\{F(\lambda_j + \lambda'_k) \mid j, k \in \mathbb{N}\}$ satisfies the gap condition. Let Γ be the support of a quantum limit μ on $M \times M'$. Since $F(\Delta_g)$ and Δ_g have the same eigenfunctions, μ is also the weak limit of a sequence of $\psi_j^2 dx_g d\xi$ where ψ_j are eigenfunctions of $F(\Delta_g)$ satisfying $\|\psi_j\|_{L^2} = 1$. We set $\omega_\varepsilon = \{x \in N \mid d_g(x, \Gamma) > \varepsilon\}$, for $\varepsilon > 0$ small enough. For every $T > 0$, (ω_ε, T) is not observable for (1) because $y = \psi_j$ provides a sequence of test functions which, at the limit, lie on Γ . Hence, by Theorem 1, (ω_ε, T) does not satisfy VGCC. Remark ?? implies that Γ must contain a vertical geodesic.

A Appendix

A.1 Quantum limits

^(q1) We recall that a *quantum limit* (QL in short) μ , also called *semi-classical measure*, is a probability Radon (i.e., probability Borel regular) measure on S^*M that is a closure point (weak limit), as $\lambda \rightarrow +\infty$, of the family of Radon measures $\mu_\lambda(a) = \langle \text{Op}(a)\phi_\lambda, \phi_\lambda \rangle$ (which are asymptotically positive by the Gårding inequality), where ϕ_λ denotes an eigenfunction of norm 1 associated with the eigenvalue λ of $\sqrt{\Delta}$. Here, Op is any quantization. We speak of a *QL on M* to refer to a closure point (for the weak topology) of the sequence of probability Radon measures $\phi_\lambda^2 dx_g$ on M as $\lambda \rightarrow +\infty$. Note that QLs do not depend on the choice of a quantization. We denote by $\mathcal{Q}(S^*M)$ (resp., $\mathcal{Q}(M)$) the set of QLs (resp., the set of QLs on M). Both are compact sets.

Given any $\mu \in \mathcal{Q}(S^*M)$, the Radon measure $\pi_*\mu$, image of μ under the canonical projection $\pi : S^*M \rightarrow M$, is a probability Radon measure on M . It is defined, equivalently, by $(\pi_*\mu)(f) = \mu(\pi^*f) = \mu(f \circ \pi)$ for every $f \in C^0(M)$ (note that, in local coordinates (x, ξ) in S^*M , the function $f \circ \pi$ is a function depending only on x), or by $(\pi_*\mu)(\omega) = \mu(\pi^{-1}(\omega))$ for every $\omega \subset M$ Borel measurable (or Lebesgue measurable, by regularity). It is easy to

see that¹

$$\pi_*\mathcal{Q}(S^*M) = \mathcal{Q}(M).$$

In other words, QLs on M are exactly the image measures under π of QLs.

A.2 Portmanteau theorem

(cintre) Let us recall the Portmanteau theorem (see, e.g., [3]). Let X be a topological space, endowed with its Borel σ -algebra. Let μ and μ_n , $n \in \mathbb{N}^*$, be finite Borel measures on X . Then the following items are equivalent:

- $\mu_n \rightarrow \mu$ for the narrow topology, i.e., $\int f d\mu_n \rightarrow \int f d\mu$ for every bounded continuous function f on X ;
- $\int f d\mu_n \rightarrow \int f d\mu$ for every Borel bounded function f on X such that $\mu(\Delta_f) = 0$, where Δ_f is the set of points at which f is not continuous;
- $\mu_n(B) \rightarrow \mu(B)$ for every Borel subset B of X such that $\mu(\partial B) = 0$;
- $\mu(F) \geq \limsup \mu_n(F)$ for every closed subset F of X , and $\mu_n(X) \rightarrow \mu(X)$;
- $\mu(O) \leq \liminf \mu_n(O)$ for every open subset O of X , and $\mu_n(X) \rightarrow \mu(X)$.

Acknowledgment. The first author is supported by the project THESPEGE (APR IA), Région Centre-Val de Loire, France, 2018-2020.

References

ananth [1] N. Anantharaman, M. Léautaud, F. Macià, *Wigner measures and observability for the Schrödinger equation on the disk*, *Invent. Math.* **206** (2016), no. 2, 485–599.

BardosLebeauRauch [2] C. Bardos, G. Lebeau, J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, *SIAM J. Control Optim.* **30** (1992), no. 5, 1024–1065.

Billingsley [3] P. Billingsley, *Convergence of Probability Measures*, 2nd ed., Wiley, 1999.

Egorov [4] Y. Egorov, *The canonical transformations of a pseudo-differential operator*, *Uspehi. Mat. Nauk.* **24** (1969), 235–236.

?hpt? [5] E. Humbert, Y. Privat, E. Trélat, *Observability properties of the homogeneous wave equation on a closed manifold*, *Comm. Partial Differential Equations* **44** (2019), no. 9, 749–772.

jaffard [6] S. Jaffard, *Contrôle interne exact des vibrations d'une plaque rectangulaire*, *Portugal. Math.* **47** (1990), 423–429.

¹Indeed, given any $f \in C^0(M)$ and any $\lambda \in \text{Spec}(\sqrt{\Delta})$, we have

$$(\pi_*\mu_\lambda)(f) = \mu_\lambda(\pi^*f) = \langle \text{Op}(\pi^*f)\phi_\lambda, \phi_\lambda \rangle = \int_M f\phi_\lambda^2 dx_g,$$

because $\text{Op}(\pi^*f)\phi_\lambda = f\phi_\lambda$. The equality then easily follows by weak compactness of probability Radon measures.

- Komornik** [7] V. Komornik, P. Loreti, *Fourier Series in Control Theory*, Springer-Verlag, New York, 2005.
- lebeau** [8] G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. **71** (1992), 267–291.
- Macia_2011** [9] F. Macià, *The Schrödinger flow in a compact manifold: high-frequency dynamics and dispersion*, in: Modern Aspects of the Theory of Partial Differential Equations, volume 216 of Operator Theory: Advances and Applications, pp. 275–289. Birkhäuser/Springer Basel AG, Basel (2011).
- MV** [10] H. L. Montgomery, R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. **2** (1974), no.8, 73–82.
- rauch-taylor** [11] J. Rauch, M. Taylor, *Exponential decay of solutions to hyperbolic equations in bounded domains*, Indiana Univ. Math. J. **24** (1974), 79–86.
- zworski** [12] M. Zworski. *Semiclassical Analysis*, Graduate Studies in Mathematics, Vol. 38, AMS, 2012.