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# SOME PROPERTIES OF SEVERAL FUNCTIONS INVOLVING POLYGAMMA FUNCTIONS AND ORIGINATING FROM THE SECTIONAL CURVATURE OF THE BETA MANIFOLD

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# SOME PROPERTIES OF SEVERAL FUNCTIONS INVOLVING POLYGAMMA FUNCTIONS AND ORIGINATING FROM THE SECTIONAL CURVATURE OF THE BETA MANIFOLD

FENG QI

*Dedicated to people facing and fighting COVID-19*

ABSTRACT. In the paper, the author investigates some properties, including analyticity, limits, monotonicity, complete monotonicity, and inequalities, of several functions involving the tri-, tetra-, and penta-gamma functions and originating from computation of the sectional curvature of the beta manifold.

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## 1. MOTIVATIONS AND MAIN RESULTS

Let  $M = \{(x, y) : x, y > 0\}$  denote the first quadrant on  $\mathbb{R}^2$ . Let

$$ds^2 = \psi'(x)dx^2 + \psi'(y)dy^2 - \psi'(x+y)(dx + dy)^2$$

be the Fisher metric equipped with  $M$ , where

$$\psi(x) = [\ln \Gamma(x)]' = \frac{\Gamma'(x)}{\Gamma(x)}$$

and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

In the literature [1, Section 6.4], the function  $\Gamma(z)$  is known as the Euler gamma function, the functions  $\psi(z)$ ,  $\psi'(z)$ ,  $\psi''(z)$ ,  $\psi^{(3)}(z)$ , and  $\psi^{(4)}(z)$  are known as the di-, tri-, tetra-, penta-, and hexa-gamma functions respectively, and, as a whole, all the derivatives  $\psi^{(k)}(z)$  for  $k \geq 0$  are known as the polygamma functions.

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*Key words and phrases.* property; polygamma function; analyticity; monotonicity; complete monotonicity; limit; inequality; trigamma function; tetragamma function; pentagamma function; sectional curvature; beta manifold.

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Proposition 3 in [9, Section 2.4] reads that the sectional curvature  $K(x, y)$  of the Fisher metric  $ds^2$  on the beta manifold  $(M, ds^2)$  is given by

$$K(x, y) = \frac{\psi''(x)\psi''(y)\psi''(x+y) \left[ \frac{\psi'(x)}{\psi''(x)} + \frac{\psi'(y)}{\psi''(y)} - \frac{\psi'(x+y)}{\psi''(x+y)} \right]}{4[\psi'(x)\psi'(x+y) + \psi'(y)\psi'(x+y) - \psi'(x)\psi'(y)]^2}. \quad (1.1)$$

Proposition 4 in [9, Section 2.4] states that the asymptotic behavior of the sectional curvature  $K(x, y)$  is given by

$$\lim_{y \rightarrow 0^+} K(x, y) = \lim_{y \rightarrow 0^+} K(y, x) = \frac{1}{2} \left( \frac{3}{2} - \frac{\psi'(x)\psi^{(3)}(x)}{[\psi''(x)]^2} \right), \quad (1.2)$$

$$\lim_{y \rightarrow \infty} K(x, y) = \lim_{y \rightarrow \infty} K(y, x) = \frac{x\psi''(x) + \psi'(x)}{4[1 - x\psi'(x)]^2}, \quad (1.3)$$

$$\lim_{(x,y) \rightarrow (0^+, 0^+)} K(x, y) = 0, \quad \lim_{(x,y) \rightarrow (\infty, \infty)} K(x, y) = -\frac{1}{2}, \quad (1.4)$$

$$\lim_{(x,y) \rightarrow (0^+, \infty)} K(x, y) = \lim_{(x,y) \rightarrow (\infty, 0^+)} K(x, y) = -\frac{1}{4}. \quad (1.5)$$

Recall from [13, Chapter XIII], [30, Chapter 1], and [31, Chapter IV] that, if a function  $h(t)$  on an interval  $I$  has derivatives of all orders on  $I$  and  $(-1)^n h^{(n)}(t) \geq 0$  for  $t \in I$  and  $n \in \{0\} \cup \mathbb{N}$ , then we call  $h(t)$  a completely monotonic function on  $I$ . Theorem 12b in [31, p. 161] states that a necessary and sufficient condition for  $h(t)$  to be completely monotonic on the infinite interval  $(0, \infty)$  is that

$$h(t) = \int_0^\infty e^{-ts} d\sigma(s), \quad t \in (0, \infty), \quad (1.6)$$

where  $\sigma(s)$  is non-decreasing and the above integral converges for  $t \in (0, \infty)$ . The integral representation (1.6) means that a function  $h(t)$  is completely monotonic on the infinite interval  $(0, \infty)$  if and only if it is a Laplace transform of a non-decreasing measure  $\sigma(s)$  on the infinite interval  $(0, \infty)$ .

In this paper, we will show analyticity of the sectional curvature  $K(x, y)$ , recover the above limits in (1.2), (1.3), (1.4), and (1.5) by alternative approaches, present that the function

$$\frac{1}{2} \left( \frac{3}{2} - \frac{\psi'(x)\psi^{(3)}(x)}{[\psi''(x)]^2} \right) \quad (1.7)$$

in (1.2) is decreasing from  $(0, \infty)$  onto  $(-\frac{1}{4}, 0)$ , reveal that the function

$$H(x) = \psi'(x) + x\psi''(x) + 2[1 - x\psi'(x)]^2 \quad (1.8)$$

is completely monotonic on  $(0, \infty)$ , and derive a sharp double inequality

$$-\frac{1}{2} < \frac{x\psi''(x) + \psi'(x)}{4[1 - x\psi'(x)]^2} < 0 \quad (1.9)$$

in the sense that the constant  $-\frac{1}{2}$  and 0 can not be replaced by bigger and smaller ones respectively.

## 2. ANALYTICITY

In this section, we show analyticity of the sectional curvature  $K(x, y)$  on the first quadrant  $M$ .

**Theorem 2.1.** *The sectional curvature  $K(x, y)$  is an analytic function of  $(x, y)$  on the first quadrant  $M$ .*

*Proof.* Since

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-zt} dt \quad (2.1)$$

for  $\Re(z) > 0$  and  $n \geq 1$ , see [1, p. 260, 6.4.1], the trigamma function  $\psi'(x)$  is positive, decreasing, and convex on  $(0, \infty)$ , with the limits  $\lim_{x \rightarrow 0^+} \psi'(x) = \infty$  and  $\lim_{x \rightarrow \infty} \psi'(x) = 0$ . Applying the double inequality

$$\frac{n-1}{n} < \frac{[\psi^{(n)}(x)]^2}{\psi^{(n-1)}(x)\psi^{(n+1)}(x)} < \frac{n}{n+1}$$

for  $n \geq 2$ , see [4, Corollary 2.3], [18, Section 3.5], or [32, Eq. (1.4)], leads to

$$\left[ \frac{1}{\psi'(x)} \right]'' = \frac{2[\psi''(x)]^2 - \psi'(x)\psi^{(3)}(x)}{[\psi'(x)]^3} > 0.$$

A function  $\varphi(x)$  is said to be sub-additive on an interval  $I$  if  $\varphi(x+y) \leq \varphi(x) + \varphi(y)$  holds for all  $x, y \in I$  with  $x+y \in I$ . If  $\varphi(x+y) \geq \varphi(x) + \varphi(y)$ , then the function  $\varphi(x)$  is called super-additive on the interval  $I$ . A function  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is said to be star-shaped if  $\varphi(\nu t) \leq \nu\varphi(t)$  for  $\nu \in [0, 1]$  and  $t \geq 0$ . Between convex functions, star-shaped functions, and super-additive functions, there are the following relations:

- (1) if  $\varphi$  is convex on  $[0, \infty)$  with  $\varphi(0) \leq 0$ , then  $\varphi$  is star-shaped;
- (2) if  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is star-shaped, then  $\varphi$  is super-additive.

For more information on additive and star-shaped functions, please refer to [12, Chapter 16], [14, Section 3.4], the papers [2, 3, 5, 7, 8, 10, 11, 16, 21, 23, 24, 25, 26, 28, 29], and closely related references therein. By these relations, we conclude that the reciprocal  $\frac{1}{\psi'(x)}$  is super-additive. Hence, the factor of the denominator in (1.1), which can be rearranged as

$$\begin{aligned} & \psi'(x)\psi'(x+y) + \psi'(y)\psi'(x+y) - \psi'(x)\psi'(y) \\ &= \psi'(x)\psi'(y)\psi'(x+y) \left[ \frac{1}{\psi'(x)} + \frac{1}{\psi'(y)} - \frac{1}{\psi'(x+y)} \right] > 0, \end{aligned}$$

is positive on  $(0, \infty)$ .

Since  $\psi^{(n)}(z)$  for  $n \geq 0$  is a single valued analytic function over the entire complex plane save at the points  $z = -m$  where it possesses poles of order  $n+1$ , see [1, p. 260, 6.4.1] or the paper [17], the denominator in (1.1) is positive and analytic on  $(0, \infty)$ . The numerator in (1.1) is also analytic on  $(0, \infty)$ . In conclusion, the sectional curvature  $K(x, y)$  is analytic on the first quadrant  $M$ . The proof of Theorem 2.1 is complete.  $\square$

### 3. RECOVERIES OF LIMITS

In this section, we recover the limits in (1.2), (1.3), (1.4), and (1.5) by alternative approaches.

**Theorem 3.1.** *The sectional curvature  $K(x, y)$  on the first quadrant  $M$  has the limits*

$$\lim_{x \rightarrow \infty} K(x, y) = \frac{\psi'(y) + y\psi''(y)}{4[1 - y\psi'(y)]^2}, \quad (3.1)$$

$$\lim_{y \rightarrow \infty} K(x, y) = \frac{\psi'(x) + x\psi''(x)}{4[1 - x\psi'(x)]^2}, \quad (3.2)$$

$$\lim_{y \rightarrow 0^+} \lim_{x \rightarrow \infty} K(x, y) = -\frac{1}{4}, \quad (3.3)$$

$$\lim_{x \rightarrow 0^+} \lim_{y \rightarrow \infty} K(x, y) = -\frac{1}{4}, \quad (3.4)$$

$$\lim_{x \rightarrow \infty} \lim_{y \rightarrow \infty} K(x, y) = -\frac{1}{2}, \quad (3.5)$$

$$\lim_{y \rightarrow \infty} \lim_{x \rightarrow \infty} K(x, y) = -\frac{1}{2}, \quad (3.6)$$

$$\lim_{(x, y) \rightarrow (\infty, \infty)} K(x, y) = -\frac{1}{2}. \quad (3.7)$$

*Proof.* The expression (1.1) for the sectional curvature  $K(x, y)$  can be rewritten as

$$K(x, y) = \frac{\left( \begin{aligned} & [x^2\psi''(x)]\psi''(y)[(x+y)^2\psi''(x+y)] \\ & \times \left[ \frac{\psi'(x)}{\psi''(x)} - \frac{\psi'(x+y)}{\psi''(x+y)} \right] \\ & + [x^2\psi''(x)]\psi'(y)[(x+y)^2\psi''(x+y)] \end{aligned} \right)}{4 \left( \begin{aligned} & \psi'(y)x(x+y)[\psi'(x+y) - \psi'(x)] \\ & + [x\psi'(x)][(x+y)\psi'(x+y)] \end{aligned} \right)^2}. \quad (3.8)$$

Making use of the limit

$$\lim_{x \rightarrow \infty} [x^k \psi^{(k)}(x)] = (-1)^{k-1} (k-1)!, \quad k \geq 1 \quad (3.9)$$

in [6, p. 9896, Eq. (13)], [19, p. 1689, Eq. (3.3)], [20, p. 286, Eq. (2.6)], and [22, p. 81, Eq. (41)], we can obtain

$$\lim_{x \rightarrow \infty} [x^2 \psi''(x)] = \lim_{x \rightarrow \infty} [(x+y)^2 \psi''(x+y)] = -1 \quad (3.10)$$

and

$$\lim_{x \rightarrow \infty} [x\psi'(x)] = \lim_{x \rightarrow \infty} [(x+y)\psi'(x+y)] = 1. \quad (3.11)$$

Direct differentiating and utilizing the limit (3.9) lead to

$$\left[ \frac{\psi'(x)}{\psi''(x)} \right]' = 1 - \frac{[x\psi'(x)][x^3\psi^{(3)}(x)]}{[x^2\psi''(x)]^2} \rightarrow 1 - \frac{0!2!}{(1!)^2} = -1$$

as  $x \rightarrow \infty$ . Therefore, by virtue of Lagrange's mean value theorem, we have

$$\lim_{x \rightarrow \infty} \left[ \frac{\psi'(x)}{\psi''(x)} - \frac{\psi'(x+y)}{\psi''(x+y)} \right] = -y \lim_{x \rightarrow \infty} \left[ \frac{\psi'(t)}{\psi''(t)} \right]'_{t=\xi} = y, \quad (3.12)$$

where  $\xi \in (x, x+y)$ .

Employing Lagrange's mean value theorem, simple differentiating, and utilizing the limit (3.9) yield

$$\begin{aligned} x(x+y)[\psi'(x+y) - \psi'(x)] &= x[(x+y)\psi'(x+y) - x\psi'(x)] - y[x\psi'(x)] \\ &= x[\psi'(\xi) + \xi\psi''(\xi)] - y[x\psi'(x)], \quad \xi \in (x, x+y) \\ &= \frac{x}{\xi} [\xi\psi'(\xi) + \xi^2\psi''(\xi)] - y[x\psi'(x)], \quad \xi \in (x, x+y) \\ &\rightarrow -y, \quad x, \xi \rightarrow \infty. \end{aligned} \quad (3.13)$$

Substituting the limits in (3.10), (3.11), (3.12), and (3.13) into (3.8) arrives at the limit (3.1).

For  $\Re(z) > 0$  and  $k \geq 1$ , we have

$$\psi^{(k-1)}(z+1) = \psi^{(k-1)}(z) + (-1)^{k-1} \frac{(k-1)!}{z^k}. \quad (3.14)$$

See [1, p. 260, 6.4.6]. From this, it follows that

$$\begin{aligned} \lim_{x \rightarrow 0^+} [x^k \psi^{(k-1)}(x)] &= \lim_{x \rightarrow 0^+} \left\{ x^k \left[ \psi^{(k-1)}(x+1) - (-1)^{k-1} \frac{(k-1)!}{x^k} \right] \right\} \\ &= (-1)^k (k-1)! \end{aligned} \quad (3.15)$$

for  $k \geq 1$ . From the limits (3.1) and (3.15), it follows that

$$\lim_{y \rightarrow 0^+} \lim_{x \rightarrow \infty} K(x, y) = \lim_{y \rightarrow 0^+} \frac{y^2 \psi'(y) + y^3 \psi''(y)}{4[y - y^2 \psi'(y)]^2} = \frac{1! - 2!}{4(0-1)!} = -\frac{1}{4}.$$

The limit (3.3) is thus proved.

In [1, p. 260, 6.4.11], it was given that, for  $|\arg z| < \pi$ , as  $z \rightarrow \infty$ ,

$$\psi^{(n)}(z) \sim (-1)^{n-1} \left[ \frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k+n-1)!}{(2k)!z^{2k+n}} \right],$$

see also [27], where  $B_{2k}$  for  $n \geq 0$  are known as the Bernoulli numbers which can be generated [15] by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

Accordingly, we can derive

$$\begin{aligned} \lim_{y \rightarrow \infty} (y^2 [\psi'(y) + y\psi''(y)]) &= \lim_{y \rightarrow \infty} \left\{ y^2 \left[ \frac{1}{y} + \frac{1}{2y^2} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k)!}{(2k)!y^{2k+1}} \right. \right. \\ &\quad \left. \left. - y \left( \frac{1}{y^2} + \frac{1}{y^3} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k+1)!}{(2k)!y^{2k+2}} \right) \right] \right\} \\ &= \lim_{y \rightarrow \infty} \left[ -\frac{1}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k)! - (2k+1)!}{(2k)!y^{2k-1}} \right] = -\frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \lim_{y \rightarrow \infty} (y[1 - y\psi'(y)]) &= \lim_{y \rightarrow \infty} \left\{ y \left[ 1 - y \left( \frac{1}{y} + \frac{1}{2y^2} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k)!}{(2k)!y^{2k+1}} \right) \right] \right\} \\ &= \lim_{y \rightarrow \infty} \left[ -\frac{1}{2} - \sum_{k=1}^{\infty} B_{2k} \frac{(2k)!}{(2k)!y^{2k-1}} \right] = -\frac{1}{2}. \end{aligned}$$

Consequently, it follows that

$$\begin{aligned} \lim_{y \rightarrow \infty} \lim_{x \rightarrow \infty} K(x, y) &= \frac{1}{4} \lim_{y \rightarrow \infty} \frac{\psi'(y) + y\psi''(y)}{[1 - y\psi'(y)]^2} = \frac{1}{4} \lim_{y \rightarrow \infty} \frac{y^2 [\psi'(y) + y\psi''(y)]}{y^2 [1 - y\psi'(y)]^2} \\ &= \frac{1}{4} \frac{\lim_{y \rightarrow \infty} (y^2 [\psi'(y) + y\psi''(y)])}{\lim_{y \rightarrow \infty} (y[1 - y\psi'(y)])^2} = \frac{1}{4} \frac{-1/2}{(-1/2)^2} = -\frac{1}{2}. \end{aligned}$$

The limit (3.6) is thus proved.

Since  $K(x, y) = K(y, x)$ , repeating the above arguments acquires the limits (3.2), (3.4), and (3.5).

Since the sectional curvature  $K(x, y)$  is analytic on the first quadrant  $M$ , see Theorem 2.1 in this paper, we have

$$\lim_{(x,y) \rightarrow (\infty, \infty)} K(x, y) = \lim_{x \rightarrow \infty} \lim_{y \rightarrow \infty} K(x, y) = \lim_{y \rightarrow \infty} \lim_{x \rightarrow \infty} K(x, y) = -\frac{1}{2}.$$

The limit in (3.7) is thus proved. The proof of Theorem 3.1 is complete.  $\square$

**Theorem 3.2.** *The sectional curvature  $K(x, y)$  on the first quadrant  $M$  has the limits*

$$\lim_{x \rightarrow 0^+} K(x, y) = \frac{1}{2} \left( \frac{3}{2} - \frac{\psi'(y)\psi^{(3)}(y)}{[\psi''(y)]^2} \right), \quad (3.16)$$

$$\lim_{y \rightarrow 0^+} K(x, y) = \frac{1}{2} \left( \frac{3}{2} - \frac{\psi'(x)\psi^{(3)}(x)}{[\psi''(x)]^2} \right), \quad (3.17)$$

$$\lim_{y \rightarrow \infty} \lim_{x \rightarrow 0^+} K(x, y) = -\frac{1}{4}, \quad (3.18)$$

$$\lim_{x \rightarrow \infty} \lim_{y \rightarrow 0^+} K(x, y) = -\frac{1}{4}, \quad (3.19)$$

$$\lim_{y \rightarrow 0^+} \lim_{x \rightarrow 0^+} K(x, y) = 0, \quad (3.20)$$

$$\lim_{x \rightarrow 0^+} \lim_{y \rightarrow 0^+} K(x, y) = 0, \quad (3.21)$$

$$\lim_{(x,y) \rightarrow (0^+, 0^+)} K(x, y) = 0, \quad (3.22)$$

$$\lim_{(x,y) \rightarrow (0^+, \infty)} K(x, y) = -\frac{1}{4}, \quad (3.23)$$

$$\lim_{(x,y) \rightarrow (\infty, 0^+)} K(x, y) = -\frac{1}{4}. \quad (3.24)$$

*Proof.* By virtue of the L'Hôpital rule, we have

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\frac{\psi''(x+y)}{\psi'(x+y)} - \frac{\psi''(y)}{\psi'(y)}}{x} &= \lim_{x \rightarrow 0^+} \frac{d}{dx} \left[ \frac{\psi''(x+y)}{\psi'(x+y)} \right] \\ &= \lim_{x \rightarrow 0^+} \frac{\psi'(x+y)\psi^{(3)}(x+y) - [\psi''(x+y)]^2}{[\psi'(x+y)]^2} \\ &= \frac{\psi'(y)\psi^{(3)}(y) - [\psi''(y)]^2}{[\psi'(y)]^2} \end{aligned} \quad (3.25)$$

and

$$\lim_{x \rightarrow 0^+} \frac{\psi'(x+y) - \psi'(y)}{x} = \lim_{0 \rightarrow 0^+} \psi''(x+y) = \psi''(y). \quad (3.26)$$

Using the expression in (1.1) and the limits (3.15), (3.25), and (3.26) yields

$$K(x, y) = \frac{\left( \begin{aligned} &[x^3\psi''(x)]\psi'(x+y)\psi'(y) \frac{\frac{\psi''(x+y)}{\psi'(x+y)} - \frac{\psi''(y)}{\psi'(y)}}{x} \\ &+ [x^2\psi'(x)]\psi''(y)\psi''(x+y) \end{aligned} \right)}{4 \left( [x^2\psi'(x)] \frac{\psi'(x+y) - \psi'(y)}{x} + x\psi'(x+y)\psi'(y) \right)^2}$$

$$\begin{aligned} & \rightarrow \frac{-2![\psi'(y)]^2 \frac{\psi'(y)\psi^{(3)}(y) - [\psi''(y)]^2}{[\psi'(y)]^2} + 1![\psi''(y)]^2}{4[1!\psi''(y) + 0]^2} \\ & = \frac{3[\psi''(y)]^2 - 2\psi'(y)\psi^{(3)}(y)}{4[\psi''(y)]^2} \end{aligned}$$

as  $x \rightarrow 0^+$ . The limit (3.16) is thus proved.

From the limits (3.9) and (3.16), it follows that

$$\lim_{y \rightarrow \infty} \lim_{x \rightarrow 0^+} K(x, y) = \frac{3}{4} - \frac{1}{2} \lim_{y \rightarrow \infty} \frac{y\psi'(y)y^3\psi^{(3)}(y)}{[y^2\psi''(y)]^2} = \frac{3}{4} - \frac{1}{2} \frac{0!2!}{(1!)^2} = -\frac{1}{4}.$$

The limit (3.18) is thus proved.

Utilizing the limit (3.15) gives

$$\begin{aligned} \frac{3[\psi''(y)]^2 - 2\psi'(y)\psi^{(3)}(y)}{[\psi''(y)]^2} & = \frac{3[y^3\psi''(y)]^2 - 2[y^2\psi'(y)][y^4\psi^{(3)}(y)]}{[y^3\psi''(y)]^2} \\ & \rightarrow \frac{3(-2!)^2 - 2 \times 1! \times 3!}{(-2!)^2} = 0 \end{aligned}$$

as  $y \rightarrow 0^+$ . This means that the limit (3.20) is valid.

Since  $K(x, y) = K(y, x)$ , repeating the above proofs of the limits (3.16), (3.18), and (3.20), we can obtain the limits (3.17), (3.19), and (3.21) immediately.

Combining analyticity of the sectional curvature  $K(x, y)$  on  $M$ , see Theorem 2.1 in this paper, with the limits (3.20) and (3.21) leads to the limit (3.22).

Combining analyticity of the sectional curvature  $K(x, y)$  on the first quadrant  $M$  with the limits (3.3), (3.4), (3.18), and (3.19) arrives at the limits (3.23) and (3.24). The proof of Theorem 3.2 is complete.  $\square$

#### 4. COMPLETE MONOTONICITY AND A DOUBLE INEQUALITY

In this section, we demonstrate decreasing monotonicity of the function in (1.7), complete monotonicity of the function in (1.8), and the double inequality (1.9).

**Theorem 4.1.** *The function defined in (1.7) is decreasing from  $(0, \infty)$  onto  $(-\frac{1}{4}, 0)$ . The function defined in (1.8) is completely monotonic on  $(0, \infty)$ . The double inequality (1.9) is valid and sharp in the sense that the constant  $-\frac{1}{2}$  and 0 can not be replaced by bigger and smaller ones respectively.*

*Proof.* The decreasing monotonicity of the function in (1.7) follows from the fact that the ratio  $\frac{[\psi''(x)]^2}{\psi'(x)\psi^{(3)}(x)}$  is decreasing from  $(0, \infty)$  onto  $(\frac{1}{2}, \frac{2}{3})$ . This fact is a special case of [32, Theorem 2] which states that the function  $\frac{[\psi^{(n+1)}(x)]^2}{\psi^{(n)}(x)\psi^{(n+2)}(x)}$  for  $n \geq 1$  is decreasing from  $(0, \infty)$  onto  $(\frac{n}{n+1}, \frac{n+1}{n+2})$ .

Making use of the integral representation (2.1) and integrating by parts yield

$$\begin{aligned} x\psi'(x) & = x \int_0^\infty \frac{t}{1-e^{-t}} e^{-xt} dt = - \int_0^\infty \frac{t}{1-e^{-t}} \frac{de^{-xt}}{dt} dt \\ & = - \left[ \frac{te^{-xt}}{1-e^{-t}} \Big|_{t \rightarrow 0^+}^{t \rightarrow \infty} - \int_0^\infty \left( \frac{t}{1-e^{-t}} \right)' e^{-xt} dt \right] \\ & = \int_0^\infty \frac{e^t(e^t - 1 - t)}{(e^t - 1)^2} e^{-xt} dt + 1 > 1 \end{aligned} \tag{4.1}$$



and

$$\begin{aligned}
x\psi''(x) &= -x \int_0^\infty \frac{t^2}{1-e^{-t}} e^{-xt} dt = \int_0^\infty \frac{t^2}{1-e^{-t}} \frac{de^{-xt}}{dt} dt \\
&= \left[ \frac{t^2 e^{-xt}}{1-e^{-t}} \Big|_{t \rightarrow 0^+} - \int_0^\infty \left( \frac{t^2}{1-e^{-t}} \right)' e^{-xt} dt \right] \\
&= - \int_0^\infty \frac{e^t (2e^t - t - 2)t}{(e^t - 1)^2} e^{-xt} dt.
\end{aligned} \tag{4.2}$$

Therefore, applying (2.1) for  $n = 1$  and (4.2) gives

$$\psi'(x) + x\psi''(x) = - \int_0^\infty \frac{t(e^t - 1 - t)e^t}{(e^t - 1)^2} e^{-xt} dt < 0$$

on  $(0, \infty)$ . The right hand side of the double inequality (1.9) is thus proved.

Using the recurrent formula (3.14) and straightforward computing result in

$$\begin{aligned}
H_1(x) &\triangleq H(x) - H(x+1) = 4 \left( \frac{1}{x^2} + \frac{2}{x} + 2 \right) \psi'(x) \\
&\quad - \frac{4x^3 + 7x^2 + 6x + 2}{x^4} - \psi''(x) - 2(2x+1)[\psi'(x)]^2, \\
H_2(x) &\triangleq H_1(x) - H_1(x+1) = 4[\psi'(x)]^2 \\
&\quad + \frac{\left[ 4x^6 + 20x^5 + 41x^4 + 48x^3 + 37x^2 + 16x + 4 \right] - 4(x+1)^2(2x^3 + 5x^2 + 4x + 2)x^2\psi'(x)}{x^4(x+1)^4}, \\
H_2(x) - H_2(x+1) &= \frac{4 \left[ \psi'(x) - \frac{2x^4 + 13x^3 + 29x^2 + 27x + 8}{2x(x+1)^2(x+2)^2} \right]}{(x+1)^2(x+2)^2} \\
&\triangleq \frac{4H_3(x)}{(x+1)^2(x+2)^2},
\end{aligned}$$

where

$$\begin{aligned}
H_3(x) &= \psi'(x) - \left[ \frac{1}{x} - \frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{2(x+2)} + \frac{1}{2(x+2)^2} \right] \\
&= \int_0^\infty \left[ \frac{t}{1-e^{-t}} - \left( 1 - \frac{e^{-t}}{2} + \frac{te^{-t}}{2} + \frac{e^{-2t}}{2} + \frac{te^{-2t}}{2} \right) \right] e^{-xt} dt \\
&= \frac{1}{2} \int_0^\infty \frac{2(t-1)e^{3t} - (t-3)e^{2t} - 2e^t + t + 1}{e^t - 1} e^{-(x+2)t} dt \\
&= \frac{1}{2} \int_0^\infty \left[ \sum_{k=3}^\infty \frac{2(3 \times 2^{k-2} - 1) + (k-3)(2 \times 3^{k-1} - 2^{k-1})}{k!} \right] \frac{e^{-(x+2)t}}{e^t - 1} dt.
\end{aligned}$$

Consequently, the function  $H_3(x)$ , and then the difference  $H_2(x) - H_2(x+1)$ , is a completely monotonic function on  $(0, \infty)$ . Hence, we have

$$(-1)^n [H_2(x) - H_2(x+1)]^{(n)} = (-1)^n [H_2(x)]^{(n)} - (-1)^n [H_2(x+1)]^{(n)} \geq 0$$

for  $n \geq 0$ . By induction, it follows that

$$(-1)^n [H_2(x)]^{(n)} \geq (-1)^n [H_2(x+1)]^{(n)} \geq (-1)^n [H_2(x+2)]^{(n)} \geq \dots$$

$$\geq (-1)^n [H_2(x+m)]^{(n)} \geq \dots \geq \lim_{m \rightarrow \infty} (-1)^n [H_2(x+m)]^{(n)}.$$

It is not difficult to see that

$$\begin{aligned} [H_2(x)]^{(n)} &= 4([\psi'(x)]^2)^{(n)} - \left[ \frac{4(2x^3 + 5x^2 + 4x + 2)}{x^2(x+1)^2} \psi'(x) \right]^{(n)} \\ &\quad + \left[ \frac{4x^6 + 20x^5 + 41x^4 + 48x^3 + 37x^2 + 16x + 4}{x^4(x+1)^4} \right]^{(n)} \rightarrow 0 \end{aligned}$$

as  $x \rightarrow \infty$  for all  $n \geq 0$ . Accordingly, we obtain  $(-1)^n [H_2(x)]^{(n)} \geq 0$  on  $(0, \infty)$  for all  $n \geq 0$ , that is, the function  $H_2(x)$  is completely monotonic on  $(0, \infty)$ . Hence, we have

$$(-1)^n [H_1(x) - H_1(x+1)]^{(n)} = (-1)^n [H_1(x)] - (-1)^n [H_1(x+1)]^{(n)} \geq 0$$

for  $n \geq 0$ . By induction, it follows that

$$\begin{aligned} (-1)^n [H_1(x)]^{(n)} &\geq (-1)^n [H_1(x+1)]^{(n)} \geq (-1)^n [H_1(x+2)]^{(n)} \geq \dots \\ &\geq (-1)^n [H_1(x+m)]^{(n)} \geq \dots \geq \lim_{m \rightarrow \infty} (-1)^n [H_1(x+m)]^{(n)}. \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} [H_1(x)]^{(n)} &= 4 \left[ \left( \frac{1}{x^2} + \frac{2}{x} + 2 \right) \psi'(x) \right]^{(n)} - \left( \frac{4x^3 + 7x^2 + 6x + 2}{x^4} \right)^{(n)} \\ -[\psi''(x)]^{(n)} &- \begin{cases} 2(2x+1)[\psi'(x)]^2 \rightarrow 0, & n=0 \\ 2(2x+1)([\psi'(x)]^2)^{(n)} + 4n([\psi'(x)]^2)^{(n-1)} \rightarrow 0, & n \geq 1 \end{cases} \end{aligned}$$

as  $x \rightarrow \infty$ . This means that  $(-1)^n [H_1(x)]^{(n)} \geq 0$  on  $(0, \infty)$  for all  $n \geq 0$ . In other words,

$$(-1)^n [H(x) - H(x+1)]^{(n)} = (-1)^n [H(x)]^{(n)} - (-1)^n [H(x+1)]^{(n)} \geq 0$$

which inductively reduces to

$$\begin{aligned} (-1)^n [H(x)]^{(n)} &\geq (-1)^n [H(x+1)]^{(n)} \geq (-1)^n [H(x+2)]^{(n)} \geq \dots \\ &\geq (-1)^n [H(x+m)]^{(n)} \geq \dots \geq \lim_{m \rightarrow \infty} (-1)^n [H(x+m)]^{(n)} \end{aligned}$$

for all  $n \geq 0$ . From the integral representation (2.1) and the formulas (4.1) and (4.2), it is immediate that

$$\begin{aligned} [H(x)]^{(n)} &= \psi^{(n+1)}(x) - (-1)^n \int_0^\infty \frac{e^t(2e^t - t - 2)t^{n+1}}{(e^t - 1)^2} e^{-xt} dt \\ &\quad + 2(-1)^n \sum_{k=0}^n \binom{n}{k} \left[ \int_0^\infty \frac{e^t(e^t - 1 - t)t^k}{(e^t - 1)^2} e^{-xt} dt \right] \\ &\quad \times \left[ \int_0^\infty \frac{e^t(e^t - 1 - t)t^{n-k}}{(e^t - 1)^2} e^{-xt} dt \right] \rightarrow 0 \end{aligned}$$

as  $x \rightarrow \infty$  for all  $n \geq 0$ . This means that  $(-1)^n [H(x)]^{(n)} \geq 0$  on  $(0, \infty)$  for all  $n \geq 0$ . In other words, the function  $H(x)$  is completely monotonic on  $(0, \infty)$ .

The complete monotonicity of  $H(x)$  implies  $H(x) > 0$  which is equivalent to the left hand side of the double inequality (1.9) on  $(0, \infty)$ . The proof of Theorem 4.1 is complete.  $\square$

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