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# SOME PROPERTIES OF SEVERAL FUNCTIONS INVOLVING POLYGAMMA FUNCTIONS AND ORIGINATING FROM THE SECTIONAL CURVATURE OF THE BETA MANIFOLD

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# Some properties of several functions involving polygamma functions and originating from the sectional curvature of the beta manifold

Feng Qi

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**Abstract** In the paper, the author investigates some properties, including analyticity, limits, monotonicity, complete monotonicity, and inequalities, of several functions involving the tri-, tetra-, and penta-gamma functions and originating from computation of the sectional curvature of the beta manifold.

**Keywords** property · polygamma function · analyticity · monotonicity · complete monotonicity · limit · inequality · trigamma function · tetragamma function · pentagamma function · sectional curvature · beta manifold

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## 1 Motivations and main results

Let  $M = \{(x, y) : x, y > 0\}$  denote the first quadrant on  $\mathbb{R}^2$ . Let

$$ds^2 = \psi'(x)dx^2 + \psi'(y)dy^2 - \psi'(x+y)(dx + dy)^2$$

be the Fisher metric and  $M$  be equipped with  $ds^2$ , where

$$\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{and} \quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

for  $\Re(z) > 0$ . In the literature [1, Section 6.4], the function  $\Gamma(z)$  is known as the Euler gamma function, the functions  $\psi(z)$ ,  $\psi'(z)$ ,  $\psi''(z)$ ,  $\psi'''(z)$ , and  $\psi^{(4)}(z)$  are known as the di-, tri-, tetra-, penta-, and hexa-gamma functions respectively, and, as a whole, all the derivatives  $\psi^{(k)}(z)$  for  $k \geq 0$  are known as the polygamma functions.

Proposition 3 in [8, Section 2.4] and Proposition 13 in [9] read that the sectional curvature  $K(x, y)$  of the Fisher metric  $ds^2$  on the beta manifold  $(M, ds^2)$  is given by

$$K(x, y) = \frac{1}{4} \frac{\psi''(x)\psi''(y)\psi''(x+y) \left[ \frac{\psi'(x)}{\psi''(x)} + \frac{\psi'(y)}{\psi''(y)} - \frac{\psi'(x+y)}{\psi''(x+y)} \right]}{[\psi'(x)\psi'(x+y) + \psi'(y)\psi'(x+y) - \psi'(x)\psi'(y)]^2}. \quad (1.1)$$

Proposition 4 in [8, Section 2.4] and Proposition 14 in [9] state that the asymptotic behavior of the sectional curvature  $K(x, y)$  is given by

$$\lim_{y \rightarrow 0^+} K(x, y) = \lim_{y \rightarrow 0^+} K(y, x) = \frac{1}{2} \left( \frac{3}{2} - \frac{\psi'(x)\psi'''(x)}{[\psi''(x)]^2} \right), \quad (1.2)$$

$$\lim_{y \rightarrow \infty} K(x, y) = \lim_{y \rightarrow \infty} K(y, x) = \frac{\psi'(x) + x\psi''(x)}{4[x\psi'(x) - 1]^2}, \quad (1.3)$$

$$\lim_{(x,y) \rightarrow (0^+, 0^+)} K(x, y) = 0, \quad \lim_{(x,y) \rightarrow (\infty, \infty)} K(x, y) = -\frac{1}{2}, \quad (1.4)$$

$$\lim_{(x,y) \rightarrow (0^+, \infty)} K(x, y) = \lim_{(x,y) \rightarrow (\infty, 0^+)} K(x, y) = -\frac{1}{4}. \quad (1.5)$$

Recall from [12, Chapter XIII], [25, Chapter 1], and [26, Chapter IV] that, if a function  $h(t)$  on an interval  $I$  has derivatives of all orders on  $I$  and  $(-1)^n h^{(n)}(t) \geq 0$  for  $t \in I$  and  $n \in \{0\} \cup \mathbb{N}$ , then we call  $h(t)$  a completely monotonic function on  $I$ . Theorem 12b in [26, p. 161] states that a necessary and sufficient condition for  $h(t)$  to be completely monotonic on the infinite interval  $(0, \infty)$  is that

$$h(t) = \int_0^\infty e^{-ts} d\sigma(s), \quad t \in (0, \infty), \quad (1.6)$$

where  $\sigma(s)$  is non-decreasing and the integral in (1.6) converges for  $t \in (0, \infty)$ . The integral representation (1.6) means that a function  $h(t)$  is completely monotonic

on the infinite interval  $(0, \infty)$  if and only if it is a Laplace transform of a non-decreasing measure  $\sigma(s)$  on the infinite interval  $(0, \infty)$ .

In this paper, we will show analyticity of the sectional curvature  $K(x, y)$ , recover the above limits in (1.2), (1.3), (1.4), and (1.5) by alternative approaches, present that the function

$$\mathcal{H}(x) = \frac{1}{2} \left( \frac{3}{2} - \frac{\psi'(x)\psi'''(x)}{[\psi''(x)]^2} \right) \quad (1.7)$$

in (1.2) is decreasing from  $(0, \infty)$  onto  $(-\frac{1}{4}, 0)$ , reveal necessary and sufficient conditions on  $\alpha$  for the function

$$\mathfrak{H}_\alpha(x) = \psi'(x) + x\psi''(x) + \alpha[x\psi'(x) - 1]^2 \quad (1.8)$$

and its negativity to be completely monotonic on  $(0, \infty)$ , and derive a sharp double inequality

$$-2 < \frac{\psi'(x) + x\psi''(x)}{[x\psi'(x) - 1]^2} < -1 \quad (1.9)$$

in the sense that the constants  $-2$  and  $-1$  cannot be replaced by any bigger and smaller ones respectively.

## 2 Analyticity

In this section, we show analyticity of the sectional curvature  $K(x, y)$  on the first quadrant  $M$ .

**Theorem 1** *The sectional curvature  $K(x, y)$  is an analytic function of  $(x, y)$  on the first quadrant  $M$ .*

*Proof* Since

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-zt} dt \quad (2.1)$$

for  $\Re(z) > 0$  and  $n \geq 1$ , see [1, p. 260, 6.4.1], the trigamma function  $\psi'(x)$  is positive, decreasing, and convex on  $(0, \infty)$ , with the limits  $\lim_{x \rightarrow 0^+} \psi'(x) = \infty$  and  $\lim_{x \rightarrow \infty} \psi'(x) = 0$ . Applying the double inequality

$$\frac{n-1}{n} < \frac{[\psi^{(n)}(x)]^2}{\psi^{(n-1)}(x)\psi^{(n+1)}(x)} < \frac{n}{n+1}$$

for  $n \geq 2$ , see [4, Corollary 2.3], [18, Section 3.5], or [27, Eq. (1.4)], leads to

$$\left[ \frac{1}{\psi'(x)} \right]'' = \frac{2[\psi''(x)]^2 - \psi'(x)\psi'''(x)}{[\psi'(x)]^3} > 0.$$

A function  $\varphi(x)$  is said to be sub-additive on an interval  $I$  if  $\varphi(x+y) < \varphi(x) + \varphi(y)$  holds for all  $x, y \in I$  with  $x+y \in I$ . If  $\varphi(x+y) > \varphi(x) + \varphi(y)$ , then the function  $\varphi(x)$  is called super-additive on the interval  $I$ . A function  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is said to be star-shaped if  $\varphi(\nu t) < \nu\varphi(t)$  for  $\nu \in [0, 1]$  and  $t \geq 0$ . Between convex functions, star-shaped functions, and super-additive functions, there are the following relations:

1. if  $\varphi$  is convex on  $[0, \infty)$  with  $\varphi(0) \leq 0$ , then  $\varphi$  is star-shaped;
2. if  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is star-shaped, then  $\varphi$  is super-additive.

For more information on additive and star-shaped functions, please refer to [11, Chapter 16], [13, Section 3.4], the papers [2, 3, 5, 7, 10, 22, 23], and closely related references therein. By these relations, we conclude that the reciprocal  $\frac{1}{\psi'(x)}$  is super-additive. Hence, the factor of the denominator in (1.1), which can be rearranged as

$$\begin{aligned} & \psi'(x)\psi'(x+y) + \psi'(y)\psi'(x+y) - \psi'(x)\psi'(y) \\ &= \psi'(x)\psi'(y)\psi'(x+y) \left[ \frac{1}{\psi'(x)} + \frac{1}{\psi'(y)} - \frac{1}{\psi'(x+y)} \right] > 0, \end{aligned}$$

is positive on  $(0, \infty)$ .

Since  $\psi^{(n)}(z)$  for  $n \geq 0$  is a single valued analytic function over the entire complex plane save at the points  $z = -m$  where it possesses poles of order  $n+1$ , see [1, p. 260, 6.4.1] or the paper [16], the denominator in (1.1) is positive and analytic on  $(0, \infty)$ . The numerator in (1.1) is also analytic on  $(0, \infty)$ . In conclusion, the sectional curvature  $K(x, y)$  is analytic on the first quadrant  $M$ . The proof of Theorem 1 is complete.

### 3 Recoveries of limits

In this section, we recover the limits in (1.2), (1.3), (1.4), and (1.5) by alternative approaches.

**Theorem 2** *The sectional curvature  $K(x, y)$  on the first quadrant  $M$  has the limits*

$$\lim_{x \rightarrow \infty} K(x, y) = \frac{\psi'(y) + y\psi''(y)}{4[1 - y\psi'(y)]^2}, \quad (3.1)$$

$$\lim_{y \rightarrow \infty} K(x, y) = \frac{\psi'(x) + x\psi''(x)}{4[x\psi'(x) - 1]^2}, \quad (3.2)$$

$$\lim_{y \rightarrow 0^+} \lim_{x \rightarrow \infty} K(x, y) = -\frac{1}{4}, \quad (3.3)$$

$$\lim_{y \rightarrow \infty} \lim_{x \rightarrow \infty} K(x, y) = -\frac{1}{2}, \quad (3.4)$$

$$\lim_{x \rightarrow 0^+} \lim_{y \rightarrow \infty} K(x, y) = -\frac{1}{4}, \quad (3.5)$$

$$\lim_{x \rightarrow \infty} \lim_{y \rightarrow \infty} K(x, y) = -\frac{1}{2}, \quad (3.6)$$

$$\lim_{(x, y) \rightarrow (\infty, \infty)} K(x, y) = -\frac{1}{2}. \quad (3.7)$$

*Proof* The expression (1.1) for the sectional curvature  $K(x, y)$  can be rewritten as

$$K(x, y) = \frac{\left( \begin{aligned} & [x^2 \psi''(x)] \psi''(y) [(x+y)^2 \psi''(x+y)] \\ & \times \left[ \frac{\psi'(x)}{\psi''(x)} - \frac{\psi'(x+y)}{\psi''(x+y)} \right] \\ & + [x^2 \psi''(x)] \psi'(y) [(x+y)^2 \psi''(x+y)] \end{aligned} \right)}{4 \left( \begin{aligned} & \psi'(y) x(x+y) [\psi'(x+y) - \psi'(x)] \\ & + [x \psi'(x)] [(x+y) \psi'(x+y)] \end{aligned} \right)^2}. \quad (3.8)$$

Making use of the limit

$$\lim_{x \rightarrow \infty} [x^k \psi^{(k)}(x)] = (-1)^{k-1} (k-1)!, \quad k \geq 1 \quad (3.9)$$

in [6, p. 9896, Eq. (13)], [19, p. 1689, Eq. (3.3)], [20, p. 286, Eq. (2.6)], and [21, p. 81, Eq. (41)], we can obtain

$$\lim_{x \rightarrow \infty} [x^2 \psi''(x)] = \lim_{x \rightarrow \infty} [(x+y)^2 \psi''(x+y)] = -1 \quad (3.10)$$

and

$$\lim_{x \rightarrow \infty} [x \psi'(x)] = \lim_{x \rightarrow \infty} [(x+y) \psi'(x+y)] = 1. \quad (3.11)$$

Direct differentiating and utilizing the limit (3.9) lead to

$$\left[ \frac{\psi'(x)}{\psi''(x)} \right]' = 1 - \frac{[x \psi'(x)] [x^3 \psi'''(x)]}{[x^2 \psi''(x)]^2} \rightarrow 1 - \frac{0!2!}{(1!)^2} = -1$$

as  $x \rightarrow \infty$ . Therefore, by virtue of Lagrange's mean value theorem, we have

$$\lim_{x \rightarrow \infty} \left[ \frac{\psi'(x)}{\psi''(x)} - \frac{\psi'(x+y)}{\psi''(x+y)} \right] = -y \lim_{x \rightarrow \infty} \left[ \frac{\psi'(t)}{\psi''(t)} \right]'_{t=\xi} = y, \quad (3.12)$$

where  $\xi \in (x, x+y)$ .

Employing Lagrange's mean value theorem, simple differentiating, and utilizing the limit (3.9) yield

$$x(x+y) [\psi'(x+y) - \psi'(x)] = x[(x+y) \psi'(x+y) - x \psi'(x)] - y[x \psi'(x)]$$

$$\begin{aligned}
&= x[\psi'(\xi) + \xi\psi''(\xi)] - y[x\psi'(x)], \quad \xi \in (x, x+y) \\
&= \frac{x}{\xi}[\xi\psi'(\xi) + \xi^2\psi''(\xi)] - y[x\psi'(x)], \quad \xi \in (x, x+y) \quad (3.13) \\
&\rightarrow -y, \quad x, \xi \rightarrow \infty.
\end{aligned}$$

Substituting the limits in (3.10), (3.11), (3.12), and (3.13) into (3.8) arrives at the limit (3.1).

For  $\Re(z) > 0$  and  $k \geq 1$ , we have

$$\psi^{(k-1)}(z+1) = \psi^{(k-1)}(z) + (-1)^{k-1} \frac{(k-1)!}{z^k}. \quad (3.14)$$

See [1, p. 260, 6.4.6]. From this, it follows that

$$\begin{aligned}
\lim_{x \rightarrow 0^+} [x^k \psi^{(k-1)}(x)] &= \lim_{x \rightarrow 0^+} \left\{ x^k \left[ \psi^{(k-1)}(x+1) - (-1)^{k-1} \frac{(k-1)!}{x^k} \right] \right\} \\
&= (-1)^k (k-1)! \quad (3.15)
\end{aligned}$$

for  $k \geq 1$ . See [15, p. 260, (2.2)] or [27, p. 769]. From the limits (3.1) and (3.15), it follows that

$$\lim_{y \rightarrow 0^+} \lim_{x \rightarrow \infty} K(x, y) = \lim_{y \rightarrow 0^+} \frac{y^2 \psi'(y) + y^3 \psi''(y)}{4[y - y^2 \psi'(y)]^2} = \frac{1! - 2!}{4(0-1)!} = -\frac{1}{4}.$$

The limit (3.3) is thus proved.

In [1, p. 260, 6.4.11], it was given that, for  $|\arg z| < \pi$ , as  $z \rightarrow \infty$ ,

$$\psi^{(n)}(z) \sim (-1)^{n-1} \left[ \frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k+n-1)!}{(2k)!z^{2k+n}} \right], \quad (3.16)$$

see also [24], where  $B_{2k}$  for  $n \geq 0$  are known as the Bernoulli numbers which can be generated [14] by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

Accordingly, we can derive

$$\begin{aligned}
\lim_{y \rightarrow \infty} (y^2 [\psi'(y) + y\psi''(y)]) &= \lim_{y \rightarrow \infty} \left\{ y^2 \left[ \frac{1}{y} + \frac{1}{2y^2} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k)!}{(2k)!y^{2k+1}} \right. \right. \\
&\quad \left. \left. - y \left( \frac{1}{y^2} + \frac{1}{y^3} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k+1)!}{(2k)!y^{2k+2}} \right) \right] \right\} \\
&= \lim_{y \rightarrow \infty} \left[ -\frac{1}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k)! - (2k+1)!}{(2k)!y^{2k-1}} \right]
\end{aligned}$$

$$= -\frac{1}{2}$$

and

$$\begin{aligned} \lim_{y \rightarrow \infty} (y[1 - y\psi'(y)]) &= \lim_{y \rightarrow \infty} \left\{ y \left[ 1 - y \left( \frac{1}{y} + \frac{1}{2y^2} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k)!}{(2k)!y^{2k+1}} \right) \right] \right\} \\ &= \lim_{y \rightarrow \infty} \left[ -\frac{1}{2} - \sum_{k=1}^{\infty} B_{2k} \frac{(2k)!}{(2k)!y^{2k-1}} \right] \\ &= -\frac{1}{2}. \end{aligned}$$

Consequently, it follows that

$$\begin{aligned} \lim_{y \rightarrow \infty} \lim_{x \rightarrow \infty} K(x, y) &= \frac{1}{4} \lim_{y \rightarrow \infty} \frac{\psi'(y) + y\psi''(y)}{[1 - y\psi'(y)]^2} \\ &= \frac{1}{4} \lim_{y \rightarrow \infty} \frac{y^2[\psi'(y) + y\psi''(y)]}{y^2[1 - y\psi'(y)]^2} \\ &= \frac{1}{4} \frac{\lim_{y \rightarrow \infty} (y^2[\psi'(y) + y\psi''(y)])}{\lim_{y \rightarrow \infty} (y[1 - y\psi'(y)])^2} \\ &= \frac{1}{4} \frac{-1/2}{(-1/2)^2} \\ &= -\frac{1}{2}. \end{aligned}$$

The limit (3.4) is thus proved.

Since  $K(x, y) = K(y, x)$ , repeating the above arguments acquires the limits (3.2), (3.5), and (3.6).

Since the sectional curvature  $K(x, y)$  is analytic on the first quadrant  $M$ , see Theorem 1 in this paper, we have

$$\lim_{(x, y) \rightarrow (\infty, \infty)} K(x, y) = \lim_{x \rightarrow \infty} \lim_{y \rightarrow \infty} K(x, y) = \lim_{y \rightarrow \infty} \lim_{x \rightarrow \infty} K(x, y) = -\frac{1}{2}.$$

The limit in (3.7) is thus proved. The proof of Theorem 2 is complete.

**Theorem 3** *The sectional curvature  $K(x, y)$  on the first quadrant  $M$  has the limits*

$$\lim_{x \rightarrow 0^+} K(x, y) = \frac{1}{2} \left( \frac{3}{2} - \frac{\psi'(y)\psi'''(y)}{[\psi''(y)]^2} \right), \quad (3.17)$$

$$\lim_{y \rightarrow 0^+} K(x, y) = \frac{1}{2} \left( \frac{3}{2} - \frac{\psi'(x)\psi'''(x)}{[\psi''(x)]^2} \right), \quad (3.18)$$

$$\lim_{y \rightarrow \infty} \lim_{x \rightarrow 0^+} K(x, y) = -\frac{1}{4}, \quad (3.19)$$

$$\lim_{y \rightarrow 0^+} \lim_{x \rightarrow 0^+} K(x, y) = 0, \quad (3.20)$$

$$\lim_{x \rightarrow \infty} \lim_{y \rightarrow 0^+} K(x, y) = -\frac{1}{4}, \quad (3.21)$$

$$\lim_{x \rightarrow 0^+} \lim_{y \rightarrow 0^+} K(x, y) = 0, \quad (3.22)$$

$$\lim_{(x, y) \rightarrow (0^+, 0^+)} K(x, y) = 0, \quad (3.23)$$

$$\lim_{(x, y) \rightarrow (0^+, \infty)} K(x, y) = -\frac{1}{4}, \quad (3.24)$$

$$\lim_{(x, y) \rightarrow (\infty, 0^+)} K(x, y) = -\frac{1}{4}. \quad (3.25)$$

*Proof* By virtue of the L'Hospital rule, we have

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\frac{\psi''(x+y)}{\psi'(x+y)} - \frac{\psi''(y)}{\psi'(y)}}{x} &= \lim_{x \rightarrow 0^+} \frac{d}{dx} \left[ \frac{\psi''(x+y)}{\psi'(x+y)} \right] \\ &= \lim_{x \rightarrow 0^+} \frac{\psi'(x+y)\psi'''(x+y) - [\psi''(x+y)]^2}{[\psi'(x+y)]^2} \\ &= \frac{\psi'(y)\psi'''(y) - [\psi''(y)]^2}{[\psi'(y)]^2} \end{aligned} \quad (3.26)$$

and

$$\lim_{x \rightarrow 0^+} \frac{\psi'(x+y) - \psi'(y)}{x} = \lim_{x \rightarrow 0^+} \psi''(x+y) = \psi''(y). \quad (3.27)$$

Using the expression in (1.1) and the limits (3.15), (3.26), and (3.27) yields

$$\begin{aligned} K(x, y) &= \frac{\left( [x^3\psi''(x)]\psi'(x+y)\psi'(y)\frac{\frac{\psi''(x+y)}{\psi'(x+y)} - \frac{\psi''(y)}{\psi'(y)}}{x} \right)}{4([x^2\psi'(x)]\frac{\psi'(x+y) - \psi'(y)}{x} + x\psi'(x+y)\psi'(y))^2} \\ &\rightarrow \frac{-2![\psi'(y)]^2\frac{\psi'(y)\psi'''(y) - [\psi''(y)]^2}{[\psi'(y)]^2} + 1![\psi''(y)]^2}{4[1!\psi''(y) + 0]^2} \\ &= \frac{3[\psi''(y)]^2 - 2\psi'(y)\psi'''(y)}{4[\psi''(y)]^2} \end{aligned}$$

as  $x \rightarrow 0^+$ . The limit (3.17) is thus proved.

From the limits (3.9) and (3.17), it follows that

$$\lim_{y \rightarrow \infty} \lim_{x \rightarrow 0^+} K(x, y) = \frac{3}{4} - \frac{1}{2} \lim_{y \rightarrow \infty} \frac{y\psi'(y)y^3\psi'''(y)}{[y^2\psi''(y)]^2} = \frac{3}{4} - \frac{1}{2} \frac{0!2!}{(1!)^2} = -\frac{1}{4}.$$

The limit (3.19) is thus proved.

Utilizing the limit (3.15) gives

$$\frac{3[\psi''(y)]^2 - 2\psi'(y)\psi'''(y)}{[\psi''(y)]^2} = \frac{3[y^3\psi''(y)]^2 - 2[y^2\psi'(y)][y^4\psi'''(y)]}{[y^3\psi''(y)]^2}$$

$$\begin{aligned} &\rightarrow \frac{3(-2!)^2 - 2 \times 1! \times 3!}{(-2!)^2} \\ &= 0 \end{aligned}$$

as  $y \rightarrow 0^+$ . This means that the limit (3.20) is valid.

Since  $K(x, y) = K(y, x)$ , repeating the above proofs of the limits (3.17), (3.19), and (3.20), we can obtain the limits (3.18), (3.21), and (3.22) immediately.

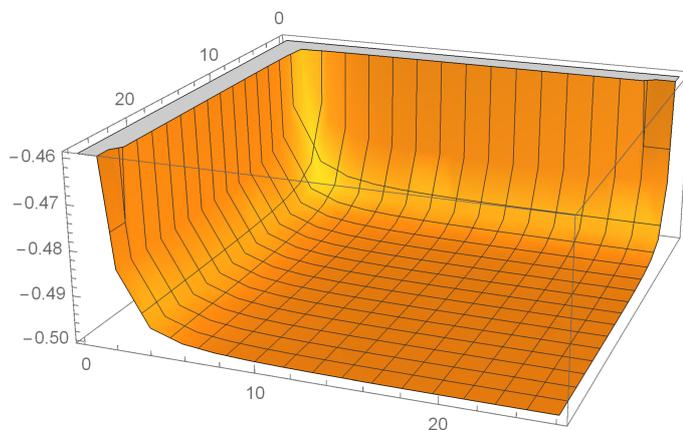
Combining analyticity of the sectional curvature  $K(x, y)$  on  $M$ , see Theorem 1 in this paper, with the limits (3.20) and (3.22) leads to the limit (3.23).

Combining analyticity of the sectional curvature  $K(x, y)$  on the first quadrant  $M$  with the limits (3.3), (3.5), (3.19), and (3.21) arrives at the limits (3.24) and (3.25). The proof of Theorem 3 is complete.

#### 4 Boundness of the sectional curvature

As did in [8, Proposition 5] and its proof, basing on analyticity and limits recovered in the above two sections, we can deduce that the sectional curvature  $K(x, y)$  is negative and bounded on the first quadrant  $M$ .

Judging from the graph of  $K(x, y)$  in Figure 1 and basing on analyticity in



**Fig. 1** The lower bound of  $K(x, y)$  is conjectured to be  $-\frac{1}{2}$

Theorem 1 and the limits in Theorems 2 and 3, the authors of the papers [8, 9] conjectured that  $K(x, y) > -\frac{1}{2}$  on  $M$ . Till 9 September 2020, this conjecture has not been verified and is still open.

## 5 Complete monotonicity and a double inequality

In the is section, we demonstrate decreasing monotonicity of the function  $\mathcal{H}(x)$  in (1.7), find necessary and sufficient conditions for the function  $\pm\mathfrak{H}_\alpha(x)$  in (1.8) to be completely monotonic on  $(0, \infty)$ , and derive the double inequality (1.9).

**Theorem 4** *The function  $\mathcal{H}(x)$  in (1.7) is decreasing from  $(0, \infty)$  onto  $(-\frac{1}{4}, 0)$ .*

*The function  $\mathfrak{H}_\alpha(x)$  in (1.8) is completely monotonic on  $(0, \infty)$  if and only if  $\alpha \geq 2$ , while the function  $-\mathfrak{H}_\alpha(x)$  is completely monotonic on  $(0, \infty)$  if and only if  $\alpha \leq 1$ .*

*The double inequality (1.9) is valid and sharp in the sense that the constants  $-2$  and  $-1$  cannot be replaced by any bigger and smaller ones respectively.*

*Proof* The decreasing monotonicity of the function  $\mathcal{H}(x)$  in (1.7) follows from the fact that the ratio  $\frac{[\psi''(x)]^2}{\psi'(x)\psi'''(x)}$  is decreasing from  $(0, \infty)$  onto  $(\frac{1}{2}, \frac{2}{3})$ . This fact is a special case of [27, Theorem 2] which states that the function  $\frac{[\psi^{(n+1)}(x)]^2}{\psi^{(n)}(x)\psi^{(n+2)}(x)}$  for  $n \geq 1$  is decreasing from  $(0, \infty)$  onto  $(\frac{n}{n+1}, \frac{n+1}{n+2})$ .

Making use of the integral representation (2.1) and integrating by parts yield

$$\begin{aligned} x\psi'(x) &= x \int_0^\infty \frac{t}{1-e^{-t}} e^{-xt} dt \\ &= - \int_0^\infty \frac{t}{1-e^{-t}} \frac{de^{-xt}}{dt} dt \\ &= - \left[ \frac{te^{-xt}}{1-e^{-t}} \Big|_{t \rightarrow 0^+}^{t \rightarrow \infty} - \int_0^\infty \left( \frac{t}{1-e^{-t}} \right)' e^{-xt} dt \right] \\ &= \int_0^\infty \frac{e^t(e^t-1-t)}{(e^t-1)^2} e^{-xt} dt + 1 \\ &> 1, \end{aligned} \tag{5.1}$$

$$\begin{aligned} x[x\psi'(x) - 1] &= x \int_0^\infty \frac{e^t(e^t-1-t)}{(e^t-1)^2} e^{-xt} dt \\ &= - \int_0^\infty \frac{e^t(e^t-1-t)}{(e^t-1)^2} \frac{de^{-xt}}{dt} dt \\ &= - \left[ \frac{e^t(e^t-1-t)e^{-xt}}{(e^t-1)^2} \Big|_{t \rightarrow 0^+}^{t \rightarrow \infty} - \int_0^\infty \left[ \frac{e^t(e^t-1-t)}{(e^t-1)^2} \right]' e^{-xt} dt \right] \\ &= \frac{1}{2} + \int_0^\infty \frac{e^t[e^t(t-2) + t + 2]}{(e^t-1)^3} e^{-xt} dt \\ &\rightarrow \frac{1}{2}, \quad x \rightarrow \infty, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
 x\psi''(x) &= -x \int_0^\infty \frac{t^2}{1-e^{-t}} e^{-xt} dt \\
 &= \int_0^\infty \frac{t^2}{1-e^{-t}} \frac{de^{-xt}}{dt} dt \\
 &= \left[ \frac{t^2 e^{-xt}}{1-e^{-t}} \Big|_{t \rightarrow 0^+} - \int_0^\infty \left( \frac{t^2}{1-e^{-t}} \right)' e^{-xt} dt \right] \\
 &= - \int_0^\infty \frac{e^t (2e^t - t - 2)t}{(e^t - 1)^2} e^{-xt} dt.
 \end{aligned} \tag{5.3}$$

Therefore, applying (2.1) for  $n = 1$  and (5.3) gives

$$\psi'(x) + x\psi''(x) = - \int_0^\infty \frac{t(e^t - 1 - t)e^t}{(e^t - 1)^2} e^{-xt} dt < 0, \quad x \in (0, \infty). \tag{5.4}$$

If the function  $\mathfrak{H}_\alpha(x)$  is completely monotonic on  $(0, \infty)$ , then its first derivative

$$\mathfrak{H}'_\alpha(x) = 2\alpha[x\psi'(x) - 1][\psi'(x) + x\psi''(x)] + 2\psi''(x) + x\psi'''(x) \leq 0$$

which is equivalent to

$$\begin{aligned}
 \alpha &\geq - \frac{2\psi''(x) + x\psi'''(x)}{2[x\psi'(x) - 1][\psi'(x) + x\psi''(x)]} \\
 &= - \frac{1}{2} \frac{x^3 [2\psi''(x) + x\psi'''(x)]}{(x[x\psi'(x) - 1])(x^2[\psi'(x) + x\psi''(x)])} \\
 &\rightarrow - \frac{1}{2} \frac{\lim_{x \rightarrow \infty} (x^3 [2\psi''(x) + x\psi'''(x)])}{\lim_{x \rightarrow \infty} (x[x\psi'(x) - 1]) \lim_{x \rightarrow \infty} (x^2[\psi'(x) + x\psi''(x)])}, \quad x \rightarrow \infty \\
 &= - \frac{1}{2} \frac{1}{\frac{1}{2} \times (-\frac{1}{2})} \\
 &= 2,
 \end{aligned}$$

where we used (5.1), (5.2), (5.4), and the asymptotic expansion (3.16). Similarly, if the function  $-H_\alpha(x)$  is completely monotonic on  $(0, \infty)$ , then

$$\begin{aligned}
 \alpha &\leq - \frac{2\psi''(x) + x\psi'''(x)}{2[x\psi'(x) - 1][\psi'(x) + x\psi''(x)]} \\
 &= - \frac{1}{2} \frac{x^3 [2\psi''(x) + x\psi'''(x)]}{(x[x\psi'(x) - 1])(x^2[\psi'(x) + x\psi''(x)])} \\
 &\rightarrow - \frac{1}{2} \frac{\lim_{x \rightarrow 0^+} (x^3 [2\psi''(x) + x\psi'''(x)])}{\lim_{x \rightarrow 0^+} (x[x\psi'(x) - 1]) \lim_{x \rightarrow 0^+} (x^2[\psi'(x) + x\psi''(x)])}, \quad x \rightarrow 0^+ \\
 &= - \frac{1}{2} \frac{2}{1 \times (-1)} \\
 &= 1,
 \end{aligned}$$

where we used (5.1), (5.4), and the limit (3.15). In a word, the necessary condition for  $H_\alpha(x)$  to be completely monotonic on  $(0, \infty)$  is  $\alpha \geq 2$ , while the necessary condition for  $-H_\alpha(x)$  to be completely monotonic on  $(0, \infty)$  is  $\alpha \leq 1$ .

Using the recurrent formula (3.14) and straightforward computing result in

$$\begin{aligned} \mathfrak{H}_2(x) - \mathfrak{H}_2(x+1) &= 4\left(\frac{1}{x^2} + \frac{2}{x} + 2\right)\psi'(x) - \frac{4x^3 + 7x^2 + 6x + 2}{x^4} \\ &\quad - \psi''(x) - 2(2x+1)[\psi'(x)]^2. \end{aligned}$$

Let  $H_1(x) = \mathfrak{H}_2(x) - \mathfrak{H}_2(x+1)$ . Then

$$H_1(x) - H_1(x+1) = 4[\psi'(x)]^2 + \frac{\begin{bmatrix} 4x^6 + 20x^5 + 41x^4 + 48x^3 + 37x^2 + 16x + 4 \\ -4(x+1)^2(2x^3 + 5x^2 + 4x + 2)x^2\psi'(x) \end{bmatrix}}{x^4(x+1)^4}.$$

Let  $H_2(x) = H_1(x) - H_1(x+1)$ . Then

$$H_2(x) - H_2(x+1) = \frac{4\left[\psi'(x) - \frac{2x^4 + 13x^3 + 29x^2 + 27x + 8}{2x(x+1)^2(x+2)^2}\right]}{(x+1)^2(x+2)^2}.$$

Let  $H_3(x) = \frac{(x+1)^2(x+2)^2}{4}[H_2(x) - H_2(x+1)]$ . Then, by virtue of (2.1) for  $n = 1$  and

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt \quad (5.5)$$

in [1, p. 255, 6.1.1], we obtain

$$\begin{aligned} H_3(x) &= \psi'(x) - \left[ \frac{1}{x} - \frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{2(x+2)} + \frac{1}{2(x+2)^2} \right] \\ &= \int_0^\infty \left[ \frac{t}{1-e^{-t}} - \left( 1 - \frac{e^{-t}}{2} + \frac{te^{-t}}{2} + \frac{e^{-2t}}{2} + \frac{te^{-2t}}{2} \right) \right] e^{-xt} dt \\ &= \frac{1}{2} \int_0^\infty \frac{2(t-1)e^{3t} - (t-3)e^{2t} - 2e^t + t + 1}{e^t - 1} e^{-(x+2)t} dt \\ &= \frac{1}{2} \int_0^\infty \left[ \sum_{k=3}^\infty \frac{2(3 \times 2^{k-2} - 1) + (k-3)(2 \times 3^{k-1} - 2^{k-1})}{k!} t^k \right] \frac{e^{-(x+2)t}}{e^t - 1} dt. \end{aligned}$$

Consequently, the function  $H_3(x)$ , and then the difference  $H_2(x) - H_2(x+1)$ , is a completely monotonic function on  $(0, \infty)$ . Hence, we have

$$(-1)^n [H_2(x) - H_2(x+1)]^{(n)} = (-1)^n [H_2(x)]^{(n)} - (-1)^n [H_2(x+1)]^{(n)} \geq 0$$

for  $n \geq 0$ . By induction, it follows that

$$(-1)^n [H_2(x)]^{(n)} \geq (-1)^n [H_2(x+1)]^{(n)} \geq (-1)^n [H_2(x+2)]^{(n)} \geq \dots$$

$$\geq (-1)^n [H_2(x+m)]^{(n)} \geq \dots \geq \lim_{m \rightarrow \infty} (-1)^n [H_2(x+m)]^{(n)}.$$

It is not difficult to see that

$$\begin{aligned} [H_2(x)]^{(n)} &= 4([\psi'(x)]^2)^{(n)} - \left[ \frac{4(2x^3 + 5x^2 + 4x + 2)}{x^2(x+1)^2} \psi'(x) \right]^{(n)} \\ &\quad + \left[ \frac{4x^6 + 20x^5 + 41x^4 + 48x^3 + 37x^2 + 16x + 4}{x^4(x+1)^4} \right]^{(n)} \\ &\rightarrow 0 \end{aligned}$$

as  $x \rightarrow \infty$  for all  $n \geq 0$ . Accordingly, we obtain  $(-1)^n [H_2(x)]^{(n)} \geq 0$  on  $(0, \infty)$  for all  $n \geq 0$ , that is, the function  $H_2(x)$  is completely monotonic on  $(0, \infty)$ . Hence, we have

$$(-1)^n [H_1(x) - H_1(x+1)]^{(n)} = (-1)^n [H_1(x)] - (-1)^n [H_1(x+1)]^{(n)} \geq 0$$

for  $n \geq 0$ . By induction, it follows that

$$\begin{aligned} (-1)^n [H_1(x)]^{(n)} &\geq (-1)^n [H_1(x+1)]^{(n)} \geq (-1)^n [H_1(x+2)]^{(n)} \geq \dots \\ &\geq (-1)^n [H_1(x+m)]^{(n)} \geq \dots \geq \lim_{m \rightarrow \infty} (-1)^n [H_1(x+m)]^{(n)}. \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} [H_1(x)]^{(n)} &= 4 \left[ \left( \frac{1}{x^2} + \frac{2}{x} + 2 \right) \psi'(x) \right]^{(n)} - \left( \frac{4x^3 + 7x^2 + 6x + 2}{x^4} \right)^{(n)} \\ &\quad - [\psi''(x)]^{(n)} - \begin{cases} 2(2x+1)[\psi'(x)]^2 \rightarrow 0, & n=0 \\ 2(2x+1)([\psi'(x)]^2)^{(n)} + 4n([\psi'(x)]^2)^{(n-1)} \rightarrow 0, & n \geq 1 \end{cases} \end{aligned}$$

as  $x \rightarrow \infty$ . This means that  $(-1)^n [H_1(x)]^{(n)} \geq 0$  on  $(0, \infty)$  for all  $n \geq 0$ . In other words,

$$(-1)^n [\mathfrak{H}_2(x) - \mathfrak{H}_2(x+1)]^{(n)} = (-1)^n [\mathfrak{H}_2(x)]^{(n)} - (-1)^n [\mathfrak{H}_2(x+1)]^{(n)} \geq 0$$

which inductively reduces to

$$\begin{aligned} (-1)^n [\mathfrak{H}_2(x)]^{(n)} &\geq (-1)^n [\mathfrak{H}_2(x+1)]^{(n)} \geq (-1)^n [\mathfrak{H}_2(x+2)]^{(n)} \geq \dots \\ &\geq (-1)^n [\mathfrak{H}_2(x+m)]^{(n)} \geq \dots \geq \lim_{m \rightarrow \infty} (-1)^n [\mathfrak{H}_2(x+m)]^{(n)} \end{aligned}$$

for all  $n \geq 0$ . From the integral representation (2.1) and the formulas (5.1) and (5.3), it is immediate that

$$\begin{aligned}
 [\mathfrak{H}_2(x)]^{(n)} &= \psi^{(n+1)}(x) - (-1)^n \int_0^\infty \frac{e^t(2e^t - t - 2)t^{n+1}}{(e^t - 1)^2} e^{-xt} dt \\
 &\quad + 2(-1)^n \sum_{k=0}^n \binom{n}{k} \left[ \int_0^\infty \frac{e^t(e^t - 1 - t)t^k}{(e^t - 1)^2} e^{-xt} dt \right] \\
 &\quad \times \left[ \int_0^\infty \frac{e^t(e^t - 1 - t)t^{n-k}}{(e^t - 1)^2} e^{-xt} dt \right] \\
 &\rightarrow 0
 \end{aligned}$$

as  $x \rightarrow \infty$  for all  $n \geq 0$ . This means that  $(-1)^n[\mathfrak{H}_2(x)]^{(n)} \geq 0$  on  $(0, \infty)$  for all  $n \geq 0$ . In other words, the function  $\mathfrak{H}_2(x)$  is completely monotonic on  $(0, \infty)$ .

When  $\alpha > 2$ , since

$$\mathfrak{H}_\alpha(x) = \mathfrak{H}_2(x) + (\alpha - 2)[x\psi'(x) - 1]^2$$

and, by virtue of (5.1),

$$x\psi'(x) - 1 = \int_0^\infty \frac{e^t(e^t - 1 - t)}{(e^t - 1)^2} e^{-xt} dt$$

is completely monotonic on  $(0, \infty)$ , from the fact that the product of any finitely many completely monotonic functions is still completely monotonic, it follows that, when  $\alpha > 2$ , the function  $\mathfrak{H}_\alpha(x)$  is completely monotonic on  $(0, \infty)$ . As a result, the condition  $\alpha \geq 2$  is sufficient for the function  $\mathfrak{H}_\alpha(x)$  to be completely monotonic on  $(0, \infty)$ .

The complete monotonicity of  $\mathfrak{H}_2(x)$  implies  $\mathfrak{H}_2(x) > 0$ , which is equivalent to the left hand side of the double inequality (1.9), on  $(0, \infty)$ .

Employing the recurrent formula (3.14) and direct computing arrive at

$$\begin{aligned}
 \mathfrak{H}_1(x + 1) - \mathfrak{H}_1(x) &= \frac{1}{x^4} + \frac{4}{x^3} + \frac{4}{x^2} + \frac{2}{x} - 2\left(\frac{1}{x^2} + \frac{2}{x} + 2\right)\psi'(x) \\
 &\quad + (2x + 1)[\psi'(x)]^2 + \psi''(x).
 \end{aligned}$$

Let  $G_1(x) = \mathfrak{H}_1(x + 1) - \mathfrak{H}_1(x)$ . Then

$$\begin{aligned}
 G_1(x) - G_1(x + 1) &= \frac{2(2x^3 + 5x^2 + 4x + 2)}{x^2(x + 1)^2} \psi'(x) - 2[\psi'(x)]^2 \\
 &\quad - \frac{2x^6 + 10x^5 + 19x^4 + 22x^3 + 18x^2 + 8x + 2}{x^4(x + 1)^4}.
 \end{aligned}$$

Let  $G_2(x) = G_1(x) - G_1(x+1)$ . Then

$$G_2(x) - G_2(x+1) = \frac{2}{(x+1)^2(x+2)^2} \left[ \frac{4}{x} + \frac{1}{x^2} - \frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{x+2} + \frac{1}{2(x+2)^2} - \psi'(x) \right].$$

Let  $G_3(x) = \frac{(x+1)^2(x+2)^2}{2} [G_2(x) - G_2(x+1)]$ . Then, by virtue of (2.1) for  $n = 1$  and the formula (5.5), we acquire

$$\begin{aligned} G_3(x) &= \int_0^\infty \left( 4 + t - e^{-t} + \frac{t}{2} e^{-t} + e^{-2t} + \frac{t}{2} e^{-2t} - \frac{t}{1 - e^{-t}} \right) e^{-xt} dt \\ &= \frac{1}{2} \int_0^\infty \frac{8e^{3t} - (t+10)e^{2t} + 4e^t - t - 2}{e^t - 1} e^{-(x+2)t} dt \\ &= \int_0^\infty \left[ 3t + 2 \sum_{k=2}^\infty \frac{2^{k+1} \left[ \left(\frac{3}{2}\right)^k - \frac{k+5}{4} \right] + 1}{k!} t^k \right] \frac{e^{-(x+2)t}}{e^t - 1} dt \\ &= \int_0^\infty \left( 3t + 8t^2 + \frac{32t^3}{3} + \frac{115t^4}{12} + \frac{129t^5}{20} + \frac{139t^6}{40} + \dots \right) \frac{e^{-(x+2)t}}{e^t - 1} dt. \end{aligned}$$

By calculus, we can easily verify that  $\left(\frac{3}{2}\right)^k - \frac{k+5}{4} > 0$  for  $k \geq 2$ . Therefore, the function  $G_3(x)$  is completely monotonic on  $(0, \infty)$ . Hence, the difference  $G_2(x) - G_2(x+1)$  is completely monotonic on  $(0, \infty)$ . As discussed on the functions  $H_2(x)$ ,  $H_1(x)$ , and  $\mathfrak{H}_2(x)$  above, we can turn out that the functions  $G_2(x)$ ,  $G_1(x)$ , and  $-\mathfrak{H}_1(x)$  are all completely monotonic on  $(0, \infty)$ . Furthermore, as discussed on  $\mathfrak{H}_\alpha$  for  $\alpha > 2$  above, we can prove that the function  $\mathfrak{H}_\alpha$  is completely monotonic on  $(0, \infty)$  for all  $\alpha < 1$ . Consequently, the condition  $\alpha \leq 1$  is sufficient for the function  $-\mathfrak{H}_\alpha$  to be completely monotonic on  $(0, \infty)$ .

The complete monotonicity of  $-\mathfrak{H}_1(x)$  implies  $\mathfrak{H}_1(x) < 0$ , which is equivalent to the right hand side of the double inequality (1.9), on  $(0, \infty)$ .

The sharpness of the double inequality (1.9) on  $(0, \infty)$  is concluded from the limits (3.5) and (3.6). The proof of Theorem 4 is complete.

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## Conflict of interest

The author declares that he has no conflict of interest.

## References

1. M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.
2. H. Alzer and S. Koumandos, *Sub- and super-additive properties of Fejér's sine polynomial*, Bull. London Math. Soc. **38** (2006), no. 2, 261–268; available online at <https://doi.org/10.1112/S0024609306018273>.
3. H. Alzer and S. Ruscheweyh, *A subadditive property of the gamma function*, J. Math. Anal. Appl. **285** (2003), 564–577; available online at [https://doi.org/10.1016/S0022-247X\(03\)00425-6](https://doi.org/10.1016/S0022-247X(03)00425-6).
4. H. Alzer and J. Wells, *Inequalities for the polygamma functions*, SIAM J. Math. Anal. **29** (1998), no. 6, 1459–1466; available online at <https://doi.org/10.1137/S0036141097325071>.
5. C.-P. Chen, F. Qi, and H. M. Srivastava, *Some properties of functions related to the gamma and psi functions*, Integral Transforms Spec. Funct. **21** (2010), no. 2, 153–164; available online at <https://doi.org/10.1080/10652460903064216>.
6. B.-N. Guo and F. Qi, *A completely monotonic function involving the tri-gamma function and with degree one*, Appl. Math. Comput. **218** (2012), no. 19, 9890–9897; available online at <https://doi.org/10.1016/j.amc.2012.03.075>.
7. B.-N. Guo, F. Qi, and Q.-M. Luo, *The additivity of polygamma functions*, Filomat **29** (2015), no. 5, 1063–1066; available online at <https://doi.org/10.2298/FIL1505063G>.
8. A. Le Brigant and S. Puechmorel, *The Fisher-Rao geometry of beta distributions applied to the study of canonical moments*, arXiv preprint (2019), available online at <https://arxiv.org/abs/1904.08247v1>.
9. A. Le Brigant, S. Preston, and S. Puechmorel, *Fisher-Rao geometry of Dirichlet distributions*, arXiv preprint (2020), available online at <https://arxiv.org/abs/2005.05608>.
10. W.-H. Li and F. Qi, *Some Hermite-Hadamard type inequalities for functions whose  $n$ -th derivatives are  $(\alpha, m)$ -convex*, Filomat **27** (2013), no. 8, 1575–1582; available online at <https://doi.org/10.2298/FIL1308575L>.
11. A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and its Applications*, 2nd Ed., Springer Verlag, New York-Dordrecht-Heidelberg-London, 2011; available online at <https://doi.org/10.1007/978-0-387-68276-1>.
12. D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; available online at <https://doi.org/10.1007/978-94-017-1043-5>.
13. C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications: A Contemporary Approach*, Second edition, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2018; available online at <https://doi.org/10.1007/978-3-319-78337-6>.
14. F. Qi, *A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers*, J. Comput. Appl. Math. **351** (2019), 1–5; available online at <https://doi.org/10.1016/j.cam.2018.10.049>.

15. F. Qi, *Complete monotonicity of a function involving the tri- and tetra-gamma functions*, Proc. Jangjeon Math. Soc. **18** (2015), no. 2, 253–264; available online at <https://doi.org/10.17777/pjms.2015.18.2.253>.
16. F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat **27** (2013), no. 4, 601–604; available online at <https://doi.org/10.2298/FIL1304601Q>.
17. F. Qi, *Some properties of several functions involving polygamma functions and originating from the sectional curvature of the beta manifold*, HAL preprint (2020), available online at <https://hal.archives-ouvertes.fr/hal-02496404>.
18. F. Qi and R. P. Agarwal, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. **2019**, Paper No. 36, 42 pages; available online at <https://doi.org/10.1186/s13660-019-1976-z>.
19. F. Qi and C. Berg, *Complete monotonicity of a difference between the exponential and trigamma functions and properties related to a modified Bessel function*, Mediterr. J. Math. **10** (2013), no. 4, 1685–1696; available online at <https://doi.org/10.1007/s00009-013-0272-2>.
20. F. Qi and B.-N. Guo, *Complete monotonicity of divided differences of the di- and tri-gamma functions with applications*, Georgian Math. J. **23** (2016), no. 2, 279–291; available online at <https://doi.org/10.1515/gmj-2016-0004>.
21. F. Qi and B.-N. Guo, *Necessary and sufficient conditions for functions involving the tri- and tetra-gamma functions to be completely monotonic*, Adv. Appl. Math. **44** (2010), no. 1, 71–83; available online at <https://doi.org/10.1016/j.aam.2009.03.003>.
22. F. Qi and B.-N. Guo, *Some properties of extended remainder of Binet's first formula for logarithm of gamma function*, Math. Slovaca **60** (2010), no. 4, 461–470; available online at <https://doi.org/10.2478/s12175-010-0025-7>.
23. F. Qi, S. Guo, and B.-N. Guo, *Complete monotonicity of some functions involving polygamma functions*, J. Comput. Appl. Math. **233** (2010), no. 9, 2149–2160; available online at <https://doi.org/10.1016/j.cam.2009.09.044>.
24. F. Qi and A.-Q. Liu, *Completely monotonic degrees for a difference between the logarithmic and psi functions*, J. Comput. Appl. Math. **361** (2019), 366–371; available online at <https://doi.org/10.1016/j.cam.2019.05.001>.
25. R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; available online at <https://doi.org/10.1515/9783110269338>.
26. D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.
27. Z.-H. Yang, *Some properties of the divided difference of psi and polygamma functions*, J. Math. Anal. Appl. **455** (2017), no. 1, 761–777; available online at <https://doi.org/10.1016/j.jmaa.2017.05.081>.