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Marc Pegon

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# Large mass minimizers for isoperimetric problems with integrable nonlocal potentials

Marc Pegon

This paper is concerned with volume-constrained minimization problems derived from Gamow's liquid drop model for the atomic nucleus, involving the competition of a perimeter term and repulsive nonlocal potentials. We consider potentials given by general radial nonnegative and nonincreasing kernels which are integrable on  $\mathbb{R}^n$ , such as Bessel potentials. We show that, under reasonable regularity assumptions and a bound on the first moment of such kernels, the problems admit minimizers of arbitrarily large mass. In addition, we prove that, up to renormalization, any sequence of minimizers converges to the ball as the mass goes to infinity. Finally, we study the stability of large balls, and show that our bound on the first moment of the kernels is a sharp stability threshold for large balls. A direct consequence of the instability of large balls above this threshold is that there exist nontrivial compactly supported kernels for which the problems admit minimizers which are not balls, that is, symmetry breaking occurs.

## Contents

<b>1. Introduction</b>	<b>2</b>
<b>2. Motivation and context</b>	<b>6</b>
2.1. No repulsion: the Classical Isoperimetric Problem . . . . .	6
2.2. Slow decay at infinity: Riesz potentials . . . . .	7
2.3. Compactly supported kernels . . . . .	8
2.4. Intermediate case: Bessel kernels . . . . .	8
<b>3. Preliminaries</b>	<b>9</b>
3.1. Nonlocal perimeters & rescaled problem . . . . .	9
3.2. Bessel kernels . . . . .	18
<b>4. Existence and convergence of large mass minimizers</b>	<b>21</b>
4.1. Existence of large mass minimizers . . . . .	21
4.2. Indecomposability of minimizers . . . . .	27
4.3. $\Gamma$ -convergence to the classical perimeter . . . . .	28
4.4. Regularity of minimizers . . . . .	30

<b>5.</b>	<b>Stability of the ball</b>	<b>33</b>
5.1.	First and second variations of perimeters . . . . .	33
5.2.	The stability threshold . . . . .	35
<b>A. Asymptotics for nonlocal seminorms on the sphere</b>		<b>46</b>

## 1. Introduction

We study large mass minimizers for a variant of Gamow’s liquid drop model for the atomic nucleus, in which the repulsive term is given by a general nonnegative, integrable, radial and radially nonincreasing kernel. More precisely, given  $G : \mathbb{R}^n \rightarrow [0, +\infty)$  a measurable nonnegative function, where  $n \geq 2$ , we consider the minimization problem

$$\min \left\{ P(E) + \iint_{E \times E} G(x - y) \, dx \, dy : |E| = m \right\}, \quad (\star)$$

where the minimum is taken over all sets of finite perimeter of volume  $|E| = m$  – which we call the mass – and  $P(E)$  denotes the perimeter of  $E$ .

Let us point out that  $(\star)$  has extensively been studied in the literature when  $G$  is a Riesz kernel or a general integrable kernel with compact support. In the Riesz case, it is known that the problem admits the ball as unique minimizer below a critical mass, and it is conjectured that there is no minimizer above a (possibly different) critical mass. This conjecture has already been proven in a few cases. We will discuss the Riesz case and compact support case further in the next section. In this article, contrarily to Riesz kernels, we are interested in kernels which are integrable in  $\mathbb{R}^n$ , and in order to highlight the differences with the Riesz case, we focus exclusively on the behaviour for large masses. Except in [Section 2](#), we shall *always* assume that  $G$  satisfies the following general hypotheses:

(H1)  $G \in L^1(\mathbb{R}^n) \cap W_{\text{loc}}^{1,1}(\mathbb{R}^n \setminus \{0\})$ , and the integrals

$$\int_{\mathbb{R}^n} |x|G(x) \, dx, \quad \int_{\mathbb{R}^n} |x||\nabla G(x)| \, dx, \quad \text{and} \quad \int_{\mathbb{R}^n} |x|^2|\nabla G(x)| \, dx$$

are finite.

(H2)  $G$  is radial and nonincreasing, that is, there exists a nonnegative and nonincreasing function  $g : (0, +\infty) \rightarrow \mathbb{R}$  such that  $G(x) = g(|x|)$  for  $\mathcal{L}^n - a.e. x \in \mathbb{R}^n$ ;

(H3)  $G \in C^1(\mathbb{R}^n \setminus \{0\})$ .

Starting from [Section 5](#), we add the extra assumption

(H4)  $G(x) = o(|x|^{-(n+1)})$  at infinity, and  $G(x) = o(|x|^{\alpha-n})$  near the origin, for some  $\alpha > 0$ .

As we will show, the so-called Bessel kernels are natural examples of kernels satisfying these assumptions. Observe that this problem exhibits a competition between two terms and is thus nontrivial: the local perimeter term constrains the set  $E$  to concentrate as much as possible, while the nonlocal term acts as a repulsive term, forcing  $E$  to spread. Indeed, it is known that the perimeter is *minimized* by balls under volume constraint, while the nonlocal term is *maximized* by balls (by Riesz’ symmetric rearrangement using e.g. [\[23, Chapter 3.7\]](#), and the fact that  $G$  is equal to its symmetric rearrangement).

Let us first remark that,  $G$  being integrable on  $\mathbb{R}^n$ , for any set  $E$  of mass  $m$ , we may rewrite the nonlocal repulsive term as

$$\iint_{E \times E} G(x-y) \, dx \, dy = m \|G\|_{L^1(\mathbb{R}^n)} - \iint_{E \times E^c} G(x-y) \, dx \, dy,$$

so that  $(\star)$  is in fact strictly equivalent to

$$\min \left\{ P(E) - \text{Per}_G(E) : |E| = m \right\}, \quad (\text{P})$$

where the functional  $\text{Per}_G$  is defined as

$$\text{Per}_G(E) := \iint_{E \times E^c} G(x-y) \, dx \, dy$$

and should be considered as a “nonlocal perimeter”, which behaves in several ways as a standard perimeter term rather than as a volume term.

In the rest of the paper, we shall prefer working with this equivalent form (P) of the problem.

We are interested in the asymptotic behavior of this minimization problem for large masses and give answers to several natural questions: does (P) admit a minimizer? If so, what do minimizers look like, are they regular? Can the ball of volume  $m$  be a minimizer?

To state our main results, we define

$$I_G^{l,q} := \int_{\mathbb{R}^n} |x|^q |\partial_r^l G(x)| \, dx = n |B_1| \int_0^\infty |g^{(l)}(r)| r^{q+n-1} \, dr$$

for every  $l \in \{0, 1\}$  and  $q \in \{0, 1, 2\}$ , where  $B_1$  is the open unit ball in  $\mathbb{R}^n$  and  $\partial_r^l G$  is the  $l$ -th radial derivative of  $G$ , and for every positive natural numbers  $p$  and  $n$ , we denote by  $\mathbf{K}_{p,n}$  the constant defined by

$$\mathbf{K}_{p,n} := \int_{\mathbb{S}^{n-1}} |e \cdot x|^p \, d\mathcal{H}_x^{n-1}, \quad (1.1)$$

which does not depend on  $e \in \mathbb{S}^{n-1}$  by symmetry. Here  $\mathbb{S}^{n-1}$  denotes the unit sphere in  $\mathbb{R}^n$  and  $\mathcal{H}^{n-1}$  the  $(n-1)$ -dimensional Hausdorff measure. We prove that if  $I_G^{0,1}$  is small enough, then there exists a critical mass above which (P) admits a minimizer, and that, up to translations and rescaling, any sequence of minimizers converges to the unit ball as the mass goes to infinity.

**Theorem A.** *Assume  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ . Then there exists  $m_e = m_e(n, G)$  such that, for any  $m > m_e$ , (P) admits a minimizer, and any minimizer  $E$  is, up to a translation, included in  $4[B]_m$  up to a set of vanishing Lebesgue measure, where  $[B]_m$  denotes the ball of volume  $m$  centered at the origin.*

**Theorem B (Corollary 4.9).** *Assume  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ . Let  $(m_k)_{k \in \mathbb{N}}$  be a sequence of positive real numbers going to infinity, and for every  $k \in \mathbb{N}$ , let  $E_k$  be a minimizer of (P) of mass  $m_k$  such that*

$$\int_{E_k} x \, dx = 0.$$

*Then letting  $F_k := \left(\frac{|B_1|}{m_k}\right)^{\frac{1}{n}} E_k$ , the sequence  $(F_k)_{k \in \mathbb{N}}$  of sets of finite perimeter of volume  $|B_1|$  converges to  $B_1$ , the unit ball centered at the origin, w.r.t. to the  $L^1$  norm, that is,*

$$|F_k \Delta B_1| \xrightarrow{k \rightarrow \infty} 0,$$

*where  $F_k \Delta B_1$  denotes the symmetric difference of the sets  $F_k$  and  $B_1$ .*

The main obstacle for proving the existence with the direct method in the calculus of variations is the possibility for a minimizing sequence to have some mass escape at infinity. We solve this problem by showing that, for large masses, a minimizing sequence may be constrained inside a ball via a truncation lemma. This relies heavily on the introduction of the formulation (P) involving the nonlocal perimeter  $\text{Per}_G$ , and on the fact that the latter behaves to some extent as a classical perimeter. Note that the general kernels we consider (and in particular Bessel kernels) do not behave as nicely as Riesz kernels under scaling. For the latter, it only results in the addition of a multiplicative constant before the potential, while here the scaling parameter stays inside the kernel, making computations more involved. The convergence of rescaled minimizers follows from the computation of the  $\Gamma$ -limit of the functional associated with the rescaling of (P) so that the mass  $m$  appears in the functional rather than in the constraint.

Applying results from [30], when  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ , we obtain that minimizers have a  $C^{1,\frac{1}{2}}$  reduced boundary, and we show that they are necessarily connected whenever  $G$  is not compactly supported.

Then we recall a well-suited notion of stability for functionals on sets under volume constraint, and show that the threshold  $\frac{2}{\mathbf{K}_{1,n}}$  for  $I_G^{0,1}$  is a stability threshold for large balls.

**Theorem C** (Theorems 5.6 and 5.9). *Assume that  $G$  satisfies (H4). If  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ , then there exists  $m_s = m_s(n, G) > 0$  such that for any  $m > m_s$ , the ball  $[B]_m$  is stable for the functional of (P). Conversely, if  $I_G^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$ , then there exists  $m_u = m_u(n, G)$  such that for any  $m > m_u$  the ball  $[B]_m$  is unstable for the functional of (P): in particular, it cannot be a minimizer, i.e., symmetry-breaking occurs.*

The proofs for the stability and instability of large balls rely essentially on the two following ingredients:

- (i) the decomposition in spherical harmonics of the Jacobi operator associated with the second variation of the perimeter and of the nonlocal term (given by the so-called Funk-Hekke formula for the latter);
- (ii) results analogue to the one by J. Bourgain, H. Brezis, and P. Mironescu in [7] for Sobolev spaces on spheres, i.e. computation of the limit and of a sharp “asymptotic” upper bound for the quantity

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(x - y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1},$$

where  $(\eta_\varepsilon)_{\varepsilon>0}$  is a  $(n - 1)$ -dimensional approximation of identity, and  $f$  belongs to  $H^1(\mathbb{S}^{n-1})$ .

A particularly interesting consequence of Theorem C is that there exist kernels for which (P) admits nontrivial minimizers, that is, minimizers which are not balls. Indeed, S. Rigot proved in [30] that (P) always admits a minimizer whenever  $G$  is compactly supported. Hence, multiplying any sufficiently smooth, nontrivial, compactly supported kernel  $G$  by a large constant, we may assume  $I_G^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$ , so that minimizers exist but cannot be the ball  $[B]_m$  for  $m$  large enough, since it is unstable.

Theorems A to C directly apply when  $G$  is a so-called Bessel kernel: for every  $\alpha, \kappa > 0$ , we denote by  $\mathcal{B}_{\kappa,\alpha}$  the Bessel kernel of order  $\alpha$  defined as the fundamental solution of the operator

$(I - \kappa\Delta)^{\frac{\alpha}{2}}$ , that is,

$$(I - \kappa\Delta)^{\frac{\alpha}{2}} \mathcal{B}_{\kappa,\alpha} = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \quad (1.2)$$

where  $\delta_0$  is the Dirac distribution at the origin. We then have:

**Application D** (Corollary 3.15 and Lemma 3.16). *For every  $\kappa, \alpha \in (0, +\infty)$ , we consider the problem (P) with  $G = \mathcal{B}_{\kappa,\alpha}$ . Let us define*

$$\kappa_\alpha := \pi \left( \frac{(n+1)\Gamma(\frac{\alpha}{2})}{2\Gamma(\frac{1+\alpha}{2})} \right)^2.$$

*If  $\kappa < \kappa_\alpha$ , there exist  $m_e = m_e(\alpha, \kappa) > 0$  and  $m_s = m_s(\alpha, \kappa) > 0$  such that, for every  $m > m_e$ , (P) admits a minimizer, and for every  $m > m_s$ , the ball  $[B]_m$  is a **stable** critical point of the functional of (P). In addition, consider  $(m_k)_{k \in \mathbb{N}}$  a sequence of positive real numbers going to infinity, and  $(E_k)_{k \in \mathbb{N}}$  an associated sequence of centered minimizers of (P), i.e., for all  $k \in \mathbb{N}$ ,  $E_k$  is a minimizer of mass  $m_k$  satisfying*

$$\int_{E_k} x \, dx = 0.$$

*Then setting  $F_k := \left(\frac{|B_1|}{m_k}\right)^{\frac{1}{n}} E_k$ , the sequence  $(F_k)_{k \in \mathbb{N}}$  of sets of volume  $|B_1|$  converges in  $L^1$  to  $B_1$ , in the sense that*

$$|F_k \triangle B_1| \xrightarrow{k \rightarrow \infty} 0.$$

*Conversely, if  $\kappa > \kappa_\alpha$ , there exists  $m_u = m_u(\alpha, \kappa)$  such that for every  $m > m_u$ ,  $[B]_m$  is an **unstable** critical point of the functional of (P). In particular,  $[B]_m$  cannot be a minimizer.*

In view of Theorems A to C, we conjecture that for  $I_G^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$ , there should be a critical mass  $m_B$  such that, for larger masses  $m > m_B$ , the unique minimizer of (P) is the ball  $[B]_m$ , up to translations. This conjecture will be the subject of a future work.

## Outline of the paper

This paper is organized as follows. In Section 2 we discuss a few variants of Gamow's liquid drop model which have already been studied in the literature, and we motivate the choice of our assumptions (H1) to (H4). We also recall some well-known results on isoperimetric inequalities. In Section 3 we establish basic prerequisites on nonlocal perimeters and on Bessel kernels, which justify Application D. In Section 4 we prove existence of large mass minimizers, i.e., Theorem A. We then compute the  $\Gamma$ -limit of the rescaled functional of (P), and show that if  $I_G^{0,1}$  is small enough, the  $\Gamma$ -limit is a positive multiple of the perimeter, which implies Theorem B. We conclude this section by establishing  $C^{1,\frac{1}{2}}$  regularity (applying directly results from [30]) and connectedness of minimizers. Section 5 is eventually dedicated to the study of the stability of large balls, where we show that  $\frac{2}{\mathbf{K}_{1,n}}$  is a threshold for  $I_G^{0,1}$  for which large balls go from stable to unstable, i.e., Theorem C.

## Notation

**Operations on sets.** For any set  $E \in \mathbb{R}^n$ , we define  $E^c := \mathbb{R}^n \setminus E$ , and we write  $|E|$  for its volume (that is, its Lebesgue measure). Given two sets  $E$  and  $F$ , we denote by  $E \triangle F := (E \setminus F) \sqcup (F \setminus E)$  their symmetric difference. We say that two sets  $E$  and  $F$  in  $\mathbb{R}^n$  are equivalent if  $|E \triangle F| = 0$ .

**Balls and spheres.** We denote by  $B_r(x)$  the open ball in  $\mathbb{R}^n$  of radius  $r$  centered at  $x$ . For simplicity we write  $B_r$  when  $x$  is the origin. The volume of  $B_1$  is  $\omega_n := |B_1| = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ , and the area of the unit sphere  $\mathbb{S}^{n-1}$  is  $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n$ . More generally we denote by  $\mathbb{S}^k$  the  $k$ -dimensional unit sphere, and for simplicity we write  $|\mathbb{S}^k| = \mathcal{H}^k(\mathbb{S}^k)$  its surface area. For any  $m > 0$  and  $x \in \mathbb{R}^n$  we let  $[B]_{x,m}$  be the open ball of volume  $m$  centered at  $x$ , or simply  $[B]_m$  if  $x = 0$ .

**Sets of finite perimeter.** For any open set  $\Omega \subseteq \mathbb{R}^n$ , we denote by  $\text{BV}(\Omega)$  the space of functions with bounded variation in  $\Omega$ , and for any  $f \in \text{BV}(\Omega)$  we let  $|Df|$  be its total variation measure, and set  $[f]_{\text{BV}(\Omega)} := \int_{\Omega} |Df|$ . For a set of finite perimeter  $E$  in  $\Omega$ , we let  $\mathbf{1}_E \in \text{BV}(\Omega)$  be its characteristic function (i.e.,  $\mathbf{1}_E(x) = 1$  if  $x \in E$  and 0 otherwise), and define its perimeter in  $\Omega$  by  $P(E; \Omega) := \int_{\Omega} |D\mathbf{1}_E|$ . If  $\Omega = \mathbb{R}^n$  we simply write  $P(E) := P(E; \mathbb{R}^n)$ . We denote by  $\mu_E$  the Gauss–Green measure associated with the set of finite perimeter  $E$  and  $\nu_E(x)$  the outer unit normal of  $\partial^*E$  at  $x$ , where  $\partial^*E$  stands for the reduced boundary of  $E$ .

## 2. Motivation and context

### 2.1. No repulsion: the Classical Isoperimetric Problem

First let us say a few words about the simplest case for (P), that is, when  $G \equiv 0$ . In that case, (P) is the classical isoperimetric problem which consists in minimizing the perimeter under a volume constraint. It is known that the unique minimizer is the ball, up to translations, which gives the classical isoperimetric inequality

$$P(E) \geq P([B]_m),$$

for any set of finite perimeter  $E$  with volume  $m$ , and can which be rewritten

$$P(E) \geq n\omega_n^{\frac{1}{n}} |E|^{1-\frac{1}{n}}. \quad (2.1)$$

Knowing that balls are solutions to the classical isoperimetric problem, it is natural then to consider the related question: if the perimeter of a set  $E$  of volume  $m$  is close to  $P([B]_m)$ , is  $E$  close to the ball  $[B]_m$  in some sense, and if so, is it possible to quantify it? This question has been answered in [14] (see also [13] for a refinement), in the form of a so-called quantitative version of the isoperimetric inequality, which we recall just below. Given  $E$  such that  $|E| = m$ , we define the *isoperimetric deficit* of  $E$  by

$$D(E) := \frac{P(E) - P([B]_m)}{P([B]_m)},$$

and its *Fraenkel asymmetry* by

$$\alpha(E) := \min \left\{ \frac{|E \Delta [B]_{y,m}|}{m} : y \in \mathbb{R}^n \right\}.$$

The sharp quantitative isoperimetric inequality proven in [14] then states that there exists  $C = C(n)$  such that

$$\alpha(E) \leq C \sqrt{D(E)}, \quad (2.2)$$

and that the  $\frac{1}{2}$  exponent over  $D(E)$  is sharp. In addition to their intrinsic interest, isoperimetric inequalities are a very useful tool to study related isoperimetric problems, and we shall often rely on them in the rest of the paper.

## 2.2. Slow decay at infinity: Riesz potentials

Problems such as  $(\star)$  are essentially inspired by a model for the atomic nucleus introduced by George Gamow in the late 1920s, which is now referred to as Gamow's liquid drop model for the atomic nucleus. This denomination is due to the fact that in this simple model (then refined by Heisenberg, von Weizsäcker and Bohr in the 1930s), the protons and neutrons inside the atomic nucleus are treated as an incompressible and uniformly charged fluid. In this model, the atomic nucleus is represented by a set  $\Omega \subseteq \mathbb{R}^3$  of volume  $m$  (which corresponds to its mass), and its energy is given by

$$P(\Omega) + \frac{1}{8\pi} \iint_{\Omega \times \Omega} \frac{1}{|x - y|} dx dy.$$

The perimeter term represents the energy associated with the attractive short-range nuclear force, while the Coulombic repulsive term is due to the positively charged protons pushing themselves away from each other. This model successfully explained the phenomenon of nuclear fission: indeed, there are two critical masses  $0 < m_1 \leq m_2 < \infty$  such that, below  $m_1$ , the problem admits a minimizer (no fission), and above  $m_2$ , there is no minimizer (fission). In fact, there exists another threshold  $0 < m_0 \leq m_1$  such that, below it, the ball is the unique minimizer (up to translations) These results were first rigorously proven in [20] (see also [19] for the planar case). Many variants and generalizations of this model have been proposed since then, one of the most natural being to replace the Newton potential  $\frac{1}{|x|^{n-2}}$  in dimension 3 with Riesz potentials in arbitrary dimension  $n \geq 2$ , that is

$$G(x) = \frac{1}{|x|^{n-\alpha}}, \quad \alpha \in (0, n).$$

The Newton case  $\alpha = 2$  in dimension  $n \geq 3$  was treated e.g. in [18], the Riesz cases with  $\alpha \in (0, n - 1)$  in [6], and the complete Riesz case  $\alpha \in (0, n)$  in any dimension in [12], where the perimeter  $P(E)$  can also be replaced by the  $s$ -fractional perimeter  $P_s(E)$  with  $s \in (0, 1)$ .

Let us sum up some of what is known in the Riesz case in the following theorems.

**Theorem 2.1** ([19, 20, 18, 6, 12]). *Given  $n \geq 2$  and  $\alpha \in (0, n)$ , there exists  $m_0 = m_0(n, \alpha)$  such that for any  $m < m_0$  the ball  $[B]_m$  is the unique minimizer of  $(\star)$  for  $G(x) = |x|^{-(n-\alpha)}$ , up to translations.*

There are also some nonexistence results.

**Theorem 2.2** ([6, 19, 20, 24]). *Given  $n \geq 2$  and  $\alpha \in (n - 2, n)$ , there exists  $m_1 = m_1(n, \alpha)$  such that for any  $m > m_1$ ,  $(\star)$  admits no minimizer for  $G(x) = |x|^{-(n-\alpha)}$ .*

These nonexistence results for large masses are in a sense not surprising. Indeed, on the one hand, note that without the perimeter term the problem

$$\min \left\{ \iint_{E \times E} \frac{1}{|x - y|^{n-\alpha}} dx dy : |E| = m \right\}$$

admits no minimizer, since it is always better to split a set  $E$  into infinitely many pieces and send them farther from each other at infinity, since Riesz kernels are (strictly) radially decreasing. On the other hand, the relatively slow decay at infinity of the Riesz kernels make them nonintegrable, which would explain why the repulsive potential takes over the perimeter term in  $(\star)$  for large masses, resulting in the nonexistence of minimizers.

As for the thresholds  $m_0$ ,  $m_1$ , and  $m_2$ , physical evidence indicate that in dimension  $n = 3$  at least, they should be equal, but this has yet to be proven.



### 2.3. Compactly supported kernels

An interesting case is when the kernel  $G$  has compact support, i.e., when the long-range interaction fully disappears from some distance. Recalling our informal discussion on nonexistence of minimizers for Riesz potentials, we see that in the compact case, sending disjoint pieces of a set  $E$  at infinity does not decrease the energy of the nonlocal term: when the pieces are far enough, they simply don't interact with each other. Thus we may imagine that it is possible to build a minimizing sequence lying in a fixed ball, and prove the existence of minimizers by the direct method. In dimension  $n = 2$  this strategy can be implemented quite easily (the advantage being that sets of finite perimeter are essentially bounded, i.e., included in a ball), but in higher dimension it is not that simple.

Fortunately, using the link between minimizers of  $(\star)$  and “almost-minimizers” of the perimeter (see Section 4.4), that case was successfully treated by S. Rigot in [30], yielding the following result.

**Theorem 2.3** ([30]). *If  $G$  is compactly supported, then  $(\star)$  always admits minimizers. In addition, for any minimizer  $E$ ,  $\partial^*E$  is a  $C^{1,\frac{1}{2}}$ -hypersurface, and, up to a renormalization,  $E$  has a finite number of connected components  $N$ , where  $N$  can be bounded depending only on  $G$ ,  $n$  and  $m$ .*

Note that Theorem 2.3 stands true even if  $G$  is not radial. A consequence of this theorem and of Theorem C is that we can easily build kernels such that large mass minimizers exist and are nontrivial, as mentioned in the introduction.

### 2.4. Intermediate case: Bessel kernels

Between Riesz kernels, which are slowly decreasing kernels, and compactly supported kernels, it is natural to wonder what happens in the intermediate case of rapidly decreasing kernels such as Bessel kernels. Bessel kernels are usually given by the operators  $(I - \Delta)^{-\frac{\alpha}{2}}$  for  $\alpha \in (0, n)$ , i.e., the Bessel kernel of order  $\alpha$  is the fundamental solution of

$$(I - \Delta)^{\frac{\alpha}{2}} f = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n),$$

where  $\delta_0$  is the Dirac distribution at the origin. In fact, we consider the “generalized” Bessel kernels given by  $(I - \kappa\Delta)^{-\frac{\alpha}{2}}$ , where  $\alpha, \kappa \in (0, +\infty)$ . As far as we know, there is little literature on  $(\star)$  when  $G$  is a Bessel kernel, and especially on the asymptotic behavior for large masses. Compared with Riesz kernels (which are associated with the operators  $(-\Delta)^{-\frac{\alpha}{2}}$ ), Bessel kernels are generally not explicit, in the sense that they only have an integral representation, and they do not behave as nicely as Riesz kernels under scaling. Near the origin, Riesz and Bessel kernels of the same order  $\alpha$  behave similarly, however at infinity Bessel kernels decay much faster. Their decay at infinity is *exponential* (in particular, they are integrable), making them an intermediate case between Riesz kernels and compactly supported kernels.

Physically, this model could apply to diblock copolymer melts, where the long-range interactions are partially screened by fluctuations in the background nuclear fluid density (see [21]).

Note that even though Bessel kernels decay exponentially, the situation is very different from the compact support case: here, there is always a little interaction between pieces of  $E$ , no matter how far they are to one another, thus we cannot use the strategy implemented in [30] to get compactness of minimizing sequences, even in dimension  $n = 2$ .

For small masses, the similarity between Riesz and Bessel kernels near the origin suggests that  $(\star)$  presents the same kind of behavior whether  $G$  is a Riesz or a Bessel kernel of the same order  $\alpha$ , that is, there exists a critical mass below which, up to translations, the ball of volume  $m$  is the unique minimizer. In this “small volume” case, we believe the approach for the Riesz case in [12] can be adapted without major difficulties, but this is not the subject of this paper. We are more interested in the case of large volumes. For Riesz kernels of order  $\alpha \in (n-2, n)$ , it is known that above a critical mass,  $(\star)$  admits no minimizers. Here, the better integrability of the Bessel kernels changes the asymptotic behavior when the mass goes to infinity: if  $\kappa$  is small enough,  $(\star)$  admits large mass minimizers, and up to translations, any sequence of normalized (to unit mass) minimizers converges to the unit ball as the mass goes to infinity. We end this introductory discussion on Bessel kernels here, leaving the more technical reminders for Section 3.2.

### 3. Preliminaries

#### 3.1. Nonlocal perimeters & rescaled problem

First, let us make a few remarks on the assumptions (H1) to (H4) made on  $G$ , and some of their immediate consequences.

*Remark 3.1.* By (H1) and (H2),  $g$  is absolutely continuous on  $(0, +\infty)$ . In addition, since  $G$  is radial and radially nonincreasing,  $\nabla G(x) \cdot x = g'(|x|)|x| = -|\nabla G(x)||x|$ , and

$$\begin{aligned} |\mathbb{S}^{n-1}| \int_0^\infty g(r)r^{n-1} dr &= \int_{\mathbb{R}^n} G(x) dx = I_G^{0,1} < \infty, \\ |\mathbb{S}^{n-1}| \int_0^\infty |g'(r)|r^n dr &= \int_{\mathbb{R}^n} |\nabla G(x)||x| dx = I_G^{1,1} < \infty, \\ |\mathbb{S}^{n-1}| \int_0^\infty |g'(r)|r^{n+1} dr &= \int_{\mathbb{R}^n} |\nabla G(x)||x|^2 dx = I_G^{1,2} < \infty. \end{aligned}$$

By (H2) and (H3),  $\nabla G$  is also continuous in  $\mathbb{R}^n \setminus \{0\}$ . Furthermore (H1) implies that  $G(x) \leq C|x|^{-(n+1)}$  at infinity for some  $C \geq 0$ . Indeed, we know that if  $f \in L^1_{\text{loc}}(0, +\infty)$  and  $f' \in L^1(0, +\infty)$ , then  $f$  has a limit at infinity. Applying this to  $f(r) := r^{n+1}g(r)$ , we have  $f' \in L^1(0, +\infty)$  in view of (H1), thus there exists  $l \geq 0$  such that  $r^{n+1}g(r) \rightarrow l$  which shows that  $g(r) \leq 2lr^{-(n+1)}$  in a neighborhood of infinity. The same reasoning applied to  $f(r) = r^n g(r)$  shows that  $r^n g(r)$  has a limit in 0, and for  $r \mapsto r^{n-1}g(r)$  to be integrable near the origin, this limit must vanish, thus  $g(r) = o(r^{-n})$  near 0. In addition, we have the relation

$$I_G^{1,1} = nI_G^{0,0}. \quad (3.1)$$

Indeed, integrating  $(g(r)r^n)' = g'(r)r^n + ng(r)r^{n-1}$  between  $r$  and  $R$ , we find

$$g(R)R^n - g(r)r^n = \int_r^R g'(s)s^n ds + n \int_r^R g(s)s^{n-1} ds,$$

thus using the fact that  $g(r) = o(r^{-n})$  near the origin and at infinity, letting  $r$  go to 0 and  $R$  go to infinity yields

$$- \int_0^\infty g'(s)s^n ds = n \int_0^\infty g(s)s^{n-1} ds,$$

hence (3.1), since  $g' \leq 0$ .

*Remark 3.2.* The bounds at the origin and at infinity of  $G$  given by (H4) are in fact required only for the study of the stability of the ball conducted in Section 5. The bound at the origin is required in order to be able to use directly the formula for the second variation of the nonlocal term computed in [12] (and it may actually be unnecessary), while the bound at infinity is to ensure that the  $(n - 1)$ -dimensional approximation of identity (see Definition 3.10) given by rescalings of  $r \mapsto rg(r)$  satisfies the extra assumption (A.41) (which can actually be dropped in dimension larger than 2).

As we mentioned in the introduction, the rest of our study relies on the crucial observation that  $(\star)$  is in fact equivalent to (P), i.e., to minimizing  $P - \text{Per}_G$  under volume constraint, where  $\text{Per}_G$  is sometimes called the *nonlocal  $G$ -perimeter of  $E$*  (see e.g. [5, 8]). Since the ball  $[B]_m$  of volume  $m$  maximizes  $\iint_{E \times E} G(x - y) dx dy$ , it minimizes  $\text{Per}_G(E)$  among sets of volume  $m$ . From now on, we set

$$\mathcal{F}_G := P - \text{Per}_G. \quad (3.2)$$

One of the reasons why  $\text{Per}_G$  can be thought of as a perimeter appears if one imagines that the kernel  $G$  goes to infinity at the origin, and decreases quickly away from it. Heuristically in that case the part in  $\text{Per}_G(E)$  that prevails would be when  $x$  and  $y$  are close to each other, so that

$$\text{Per}_G(E) \simeq \iint_{\substack{E \times E^c \\ |x-y| < \varepsilon}} G(x - y) dx dy,$$

for a small  $\varepsilon$ . But notice that the set  $(E \times E^c) \cap \{|x - y| < \varepsilon\}$  is included in  $(\varepsilon\text{-neighb}(\partial E))^2$ , where  $\varepsilon\text{-neighb}(\partial E)$  denotes the  $\varepsilon$ -neighborhood of  $\partial E$ , whenever  $E$  has a smooth boundary. Hence what seems to prevail in  $\text{Per}_G(E)$  is the size of the boundary  $\partial E$ .

In our case  $G$  may not actually be singular at the origin, however we still assume that it is radially decreasing, so that for  $(x, y) \in E \times E^c$ , the closer  $x$  and  $y$  are to each other (and thus to  $\partial E$ ), the more  $G(x - y)$  increases. In addition,  $\text{Per}_G(E)$  can be controlled by the classical perimeter if  $I_G^{0,1} = \int_{\mathbb{R}^n} |x|G(x) dx$  is well defined, using that

$$\|f(h + \cdot) - f\|_{L^1(\mathbb{R}^n)} \leq |h|[f]_{\text{BV}(\mathbb{R}^n)}, \quad \forall f \in \text{BV}(\mathbb{R}^n), \quad \forall h \in \mathbb{R}^n.$$

Indeed, by Fubini's theorem we find

$$\begin{aligned} \text{Per}_G(E) &= \iint_{E \times E^c} G(x - y) dx dy = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\mathbf{1}_E(x + h) - \mathbf{1}_E(x)| G(h) dx dh \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \|\mathbf{1}_E(h + \cdot) - \mathbf{1}_E\|_{L^1(\mathbb{R}^n)} G(h) dh \\ &\leq \frac{1}{2} \int_{\mathbb{R}^n} |h| P(E) G(h) dh = \frac{I_G^{0,1}}{2} P(E). \end{aligned}$$

We can even refine the constant in this inequality using the following proposition, inspired by [7, Theorem 2] (see also [9, Lemma 3]).

**Proposition 3.3.** *Let  $f \in \text{BV}(B_R(x_0))$  for some  $x_0 \in \mathbb{R}^n$  and  $R > 0$ , and let  $\rho : (0, +\infty) \rightarrow [0, +\infty)$  be a measurable function such that  $\int_{\mathbb{R}^n} \rho(|x|) dx = 1$ . Then*

$$\iint_{B_R(x_0) \times B_R(x_0)} \frac{|f(x) - f(y)|}{|x - y|} \rho(|x - y|) dx dy \leq \mathbf{K}_{1,n} \int_{B_R(x_0)} |Df|, \quad (3.3)$$

where  $\mathbf{K}_{1,n}$  is defined by (1.1). As a consequence, if  $f \in \text{BV}(\mathbb{R}^n)$  we have

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|} \rho(|x - y|) dx dy \leq \mathbf{K}_{1,n} \int_{\mathbb{R}^n} |Df|. \quad (3.4)$$

*Proof.* The proof is similar to [9, Proof of Lemma 3], with the difference that we want to obtain an upper bound involving only the  $L^1$  norm of the gradient on  $B_R(x_0)$ . Up to a translation, without loss of generality we may assume that  $x_0 = 0$ . By density of  $C^\infty(B_R) \cap W^{1,1}(B_R)$  in  $\text{BV}(B_R)$  we shall assume that  $f \in C^\infty(B_R) \cap W^{1,1}(B_R)$ . Integrating on lines we have

$$f(x + h) - f(x) = \int_0^1 \nabla f(x + th) \cdot h dt,$$

thus, making the change of variables  $h = x - y$ , and noting that  $h \in B_{2R}$ , using Fubini's theorem we find,

$$\begin{aligned} & \iint_{B_R \times B_R} \frac{|f(x) - f(y)|}{|x - y|} \rho(|x - y|) dx dy \\ &= \int_{B_R} \left( \int_{B_{2R}} \mathbf{1}_{B_R(x)}(h) \frac{|f(x) - f(x - h)|}{|h|} \rho(|h|) dh \right) dx \\ &= \int_{B_{2R}} \left( \int_{B_R(h) \cap B_R} |f(x - h) - f(x)| dx \right) \frac{\rho(|h|)}{|h|} dh \\ &= \int_{B_{2R}} \left( \int_{B_R(h) \cap B_R} \left| \int_0^1 \nabla f(x - th) \cdot h dt \right| dx \right) \frac{\rho(|h|)}{|h|} dh. \end{aligned} \quad (3.5)$$

Applying the coarea formula to (3.5) and then Cauchy-Schwarz inequality and Fubini's theorem gives

$$\begin{aligned} & \iint_{B_R \times B_R} \frac{|f(x) - f(y)|}{|x - y|} \rho(|x - y|) dx dy \\ &= \int_0^{2R} \int_{\mathbb{S}^{n-1}} \left( \int_{B_R(r\sigma) \cap B_R} \left| \int_0^1 \nabla f(x - t\sigma) \cdot \sigma dt \right| dx \right) \rho(r) r^{n-1} d\mathcal{H}_\sigma^{n-1} dr \\ &\leq \int_0^{2R} \int_{\mathbb{S}^{n-1}} \left( \int_0^1 \int_{B_R(r\sigma) \cap B_R} |\nabla f(x - t\sigma) \cdot \sigma| dx dt \right) \rho(r) r^{n-1} d\mathcal{H}_\sigma^{n-1} dr. \end{aligned}$$

Since  $x - t\sigma \in B_R$  whenever  $x \in B_R(r\sigma) \cap B_R$  and  $t \in [0, 1]$  (indeed  $|x - t\sigma| = |t(x - r\sigma) + (1 - t)x| \leq t|x - r\sigma| + (1 - t)|x| < R$ ), a change of variables  $y = x + t\sigma$  yields

$$\begin{aligned} & \iint_{B_R \times B_R} \frac{|f(x) - f(y)|}{|x - y|} \rho(|x - y|) dx dy \\ &\leq \int_0^{2R} \int_{\mathbb{S}^{n-1}} \left( \int_0^1 \int_{B_R} |\nabla f(y) \cdot \sigma| dy dt \right) \rho(r) r^{n-1} d\mathcal{H}_\sigma^{n-1} dr \\ &= \int_0^{2R} \int_{\mathbb{S}^{n-1}} \int_{B_R} |\nabla f(y) \cdot \sigma| \rho(r) r^{n-1} dy d\mathcal{H}_\sigma^{n-1} dr. \end{aligned} \quad (3.6)$$

Using Fubini's theorem once again and the equality

$$\int_{\mathbb{S}^{n-1}} |\nabla f(y) \cdot \sigma| d\mathcal{H}_\sigma^{n-1} = \mathbf{K}_{1,n} |\nabla f(y)|,$$

from (3.6) we obtain

$$\begin{aligned}
& \iint_{B_R \times B_R} \frac{|f(x) - f(y)|}{|x - y|} \rho(|x - y|) \, dx \, dy \\
& \leq \int_0^{2R} \int_{B_R} \int_{\mathbb{S}^{n-1}} |\nabla f(y) \cdot \sigma| \rho(r) r^{n-1} \, d\mathcal{H}_\sigma^{n-1} \, dy \, dr \\
& = |\mathbb{S}^{n-1}| \mathbf{K}_{1,n} \int_0^{2R} \int_{B_R} |\nabla f(y)| \rho(r) r^{n-1} \, dy \, dr = \mathbf{K}_{1,n} \left( \int_{B_R} |\nabla f(y)| \, dy \right) \left( \int_{B_{2R}} \rho(|x|) \, dx \right),
\end{aligned}$$

hence (3.3), since

$$\int_{\mathbb{R}^n} \rho(|x|) \, dx = 1.$$

We deduce (3.4) by letting  $R$  go to infinity in (3.3).  $\square$

*Remark 3.4.* Note that by Proposition 3.11 further below, the constant  $\mathbf{K}_{1,n}$  in Proposition 3.3 is optimal.

Setting  $\rho_G(r) := \frac{rg(r)}{I_G^{0,1}}$  one may rewrite  $\text{Per}_G(E)$  as

$$\begin{aligned}
\text{Per}_G(E) &= \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\mathbf{1}_E(x) - \mathbf{1}_E(y)| G(x - y) \, dx \, dy \\
&= \frac{I_G^{0,1}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\mathbf{1}_E(x) - \mathbf{1}_E(y)|}{|x - y|} \rho_G(|x - y|) \, dx \, dy,
\end{aligned} \tag{3.7}$$

so that a direct application of the previous proposition with  $f = \mathbf{1}_E$  leads to the following control of the nonlocal perimeter by the local perimeter.

**Corollary 3.5.** *For any set of finite perimeter  $E$ , we have*

$$\text{Per}_G(E) \leq \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} P(E).$$

Recall that the classical perimeter is lower semicontinuous with respect to the classical topology of  $L^1(\mathbb{R}^n)$ . Here the nonlocal perimeter is in fact continuous w.r.t. the  $L^1$  convergence, as is shown in the following lemma.

**Lemma 3.6.** *For any sets  $E$  and  $F$  with finite Lebesgue measure, we have*

$$|\text{Per}_G(E) - \text{Per}_G(F)| \leq I_G^{0,0} |E \Delta F|.$$

*Proof.* Let  $E$  and  $F$  be sets with finite (possibly different) Lebesgue measure. Using

$$\text{Per}_G(E) = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\mathbf{1}_E(x) - \mathbf{1}_E(y)| G(x - y) \, dx \, dy,$$

we have

$$\text{Per}_G(E) - \text{Per}_G(F) = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (|\mathbf{1}_E(x) - \mathbf{1}_E(y)| - |\mathbf{1}_F(x) - \mathbf{1}_F(y)|) G(x - y) \, dx \, dy.$$

Thus by the reverse triangle inequality,

$$\begin{aligned} |\text{Per}_G(E) - \text{Per}_G(F)| &\leq \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (|\mathbf{1}_E(x) - \mathbf{1}_F(x)| + |\mathbf{1}_E(y) - \mathbf{1}_F(y)|) G(x - y) \, dx \, dy \\ &= \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} (\mathbf{1}_{E \Delta F}(x) + \mathbf{1}_{E \Delta F}(y)) G(x - y) \, dx \, dy = I_G^{0,0} |E \Delta F|. \end{aligned}$$

□

One of the nice properties of the classical perimeter is also its behavior under scaling. For any set of finite perimeter  $E$  and any  $\lambda > 0$ , we obviously have  $P(\lambda E) = \lambda^{n-1} P(E)$ . This is unfortunately not the case for such nonlocal perimeters. However just as we have  $|P(\lambda E) - P(E)| \leq C|1 - \lambda|P(E)$  when  $\lambda$  is close to 1, for some  $C$  not depending of  $E$ , one can show that the difference between  $\text{Per}_G(\lambda E)$  and  $\text{Per}_G(E)$  is controlled by a perimeter term of the same form  $C|1 - \lambda|P(E)$  thanks to assumptions (H1) to (H3).

**Lemma 3.7.** *For any set of finite perimeter  $E$ , the function  $t \mapsto \text{Per}_G(tE)$  is  $C^1$  in  $(0, +\infty)$ , and*

$$\frac{d}{dt} [\text{Per}_G(tE)] = \frac{2n}{t} \text{Per}_G(tE) + \frac{1}{t} \iint_{(tE) \times (tE)^c} \nabla G(x - y) \cdot (x - y) \, dx \, dy. \quad (3.8)$$

In particular, we have

$$|\text{Per}_G(tE) - \text{Per}_G(E)| \leq C|1 - t|P(E), \quad \forall t \in (\frac{1}{2}, 2), \quad (3.9)$$

for some  $C = C(n, I_G^{1,0}, I_G^{1,2})$ .

*Proof.* This is just a matter of derivating under the integral using the assumptions on  $G$ . Unfortunately, we cannot use directly usual theorems to derivate under the integral, since we cannot bound  $x \mapsto |\nabla G(tx)|$  by an integrable function independently of  $t$ , for  $t$  in a neighborhood of 0 or  $t$  in a neighborhood of infinity.

Up to considering  $F := E^c$ , we can assume without loss of generality that  $|E| < +\infty$ . By scaling we have

$$\text{Per}_G(tE) = t^{2n} \iint_{E \times E^c} G(t(x - y)) \, dx \, dy \quad (3.10)$$

thus  $t \mapsto \text{Per}_G(tE) \in C^1(0, +\infty)$  if and only if  $f : t \mapsto \iint_{E \times E^c} G(t(x - y)) \, dx \, dy \in C^1(0, +\infty)$ ,

and we have

$$\begin{aligned} \frac{d}{dt} [\text{Per}_G(tE)] &= 2nt^{2n-1} \iint_{E \times E^c} G(t(x - y)) \, dx \, dy + t^{2n} f'(t) \\ &= \frac{2n}{t} \text{Per}_G(tE) + t^{2n} f'(t). \end{aligned}$$

Let us show that  $f \in C^0(0, +\infty)$ . Since  $G$  is  $C^0$  away from the origin by Remark 3.1, then for  $\mathcal{L}^{2n}$ -a.e.  $(x, y) \in E \times E^c$ ,  $t \mapsto G(t(x - y)) \in C^0(0, +\infty)$ . In addition, since  $G$  is positive and radially nonincreasing, for any  $t > t_0 > 0$ , we have

$$|G(t(x - y))| \leq G(t_0(x - y))$$

and  $(x, y) \mapsto G(t_0(x - y))$  is integrable on  $E \times E^c$ , thus by the theorem of continuity under the integral,  $f$  is continuous on  $(t_0, +\infty)$ , for all  $t_0 > 0$ , hence  $f \in C^0(0, +\infty)$ .

Given  $0 < t_0 < t_1$  we show that  $f \in C^1(t_0, t_1)$  and compute its derivative. Let  $t \in (t_0, t_1)$ , and  $h_0 > 0$  so that  $t_0 < t - h_0 < t + h_0 < t_1$ . For any  $h \neq 0$  such that  $|h| < h_0$ , integrating on lines, we write

$$\frac{f(t+h) - f(t)}{h} = \iint_{E \times E^c} \int_0^1 \nabla G((t+sh)(x-y)) \cdot (x-y) \, ds \, dx \, dy. \quad (3.11)$$

Given  $\varepsilon > 0$ , we claim that there exists  $R > 0$  such that, for any  $h \in [-h_0, h_0]$ , we have

$$\iint_{\substack{E \times E^c \\ |x-y| < \frac{1}{R} \text{ or } |x-y| > R}} \int_0^1 |\nabla G((t+sh)(x-y)) \cdot (x-y)| \, ds \, dx \, dy \leq \varepsilon. \quad (3.12)$$

Indeed, on the one hand, using Fubini's theorem and changing variables we have

$$\begin{aligned} & \iint_{\substack{E \times E^c \\ |x-y| > R}} \int_0^1 |\nabla G((t+sh)(x-y)) \cdot (x-y)| \, ds \, dx \, dy \\ & \leq \int_0^1 (t+sh)^{-(2n+1)} \iint_{\substack{((t+sh)E) \times ((t+sh)E)^c \\ |x-y| > R/(t+sh)}} |\nabla G(x-y) \cdot (x-y)| \, dx \, dy \, ds \\ & \leq \int_0^1 (t+sh)^{-(2n+1)} \int_{(t+sh)E} \int_{|x-y| > R/(t+sh)} |\nabla G(x-y) \cdot (x-y)| \, dx \, dy \, ds \\ & \leq |E| t_1^n t_0^{-(2n+1)} \int_{B_{\frac{R}{t_1}}^c} |\nabla G(x) \cdot x| \, dx \leq \varepsilon, \end{aligned} \quad (3.13)$$

for any  $R$  large enough independently of  $h$ , since  $\nabla G(x) \cdot x$  belongs to  $L^1(\mathbb{R}^n)$ . On the other hand, proceeding in the same way, we find

$$\iint_{\substack{E \times E^c \\ |x-y| < \frac{1}{R}}} \int_0^1 |\nabla G((t+sh)(x-y)) \cdot (x-y)| \, ds \, dx \, dy \leq |E| t_1^n t_0^{-(2n+1)} \int_{B_{\frac{1}{Rt_0}}} |\nabla G(x) \cdot x| \, dx \leq \varepsilon, \quad (3.14)$$

for  $R$  large enough, hence (3.12). Similar computations also lead to

$$\iint_{\substack{E \times E^c \\ |x-y| < \frac{1}{R} \text{ or } |x-y| > R}} |\nabla G(t(x-y)) \cdot (x-y)| \, dx \, dy \leq \varepsilon, \quad (3.15)$$

for  $R$  large enough. Combining (3.11), (3.12) and (3.15) yields

$$\begin{aligned}
& \left| \frac{f(t+h) - f(t)}{h} - \iint_{E \times E^c} \nabla G(t(x-y)) \cdot (x-y) \, dx \, dy \right| \\
& \leq \left| \iint_{\substack{E \times E^c \\ \frac{1}{R} \leq |x-y| \leq R}} \int_0^1 \nabla G((t+sh)(x-y)) \cdot (x-y) \, ds \, dx \, dy \right. \\
& \quad \left. - \iint_{\substack{E \times E^c \\ 1/R \leq |x-y| \leq R}} \nabla G(t(x-y)) \cdot (x-y) \, dx \, dy \right| + 2\varepsilon,
\end{aligned} \tag{3.16}$$

for some large  $R$  independent of  $h$ . Since  $\nabla G(x) \cdot x$  is continuous away from the origin by (H3), there exists  $C = C(G, R, t_0, t_1) > 0$  such that

$$|\nabla G((t+sh)(x-y)) \cdot (x-y)| \leq C, \quad \text{whenever } 1/R \leq |x-y| \leq R \text{ and } |h| < |h_0|. \tag{3.17}$$

In addition

$$\int_0^1 \nabla G((t+sh)(x-y)) \cdot (x-y) \, ds \xrightarrow{h \rightarrow 0} \nabla G(t(x-y)) \cdot (x-y)$$

for  $\mathcal{L}^{2n}$ -a.e.  $(x, y) \in E \times E^c$ , and  $\mathcal{L}^{2n}((E \times E^c) \cap \{\frac{1}{R} \leq |x-y| \leq R\}) < +\infty$ , thus with (3.17), by dominated convergence we find

$$\begin{aligned}
& \iint_{\substack{E \times E^c \\ \frac{1}{R} \leq |x-y| \leq R}} \int_0^1 |\nabla G((t+sh)(x-y)) \cdot (x-y)| \, ds \, dx \, dy \\
& \xrightarrow{h \rightarrow 0} \iint_{\substack{E \times E^c \\ \frac{1}{R} \leq |x-y| \leq R}} |\nabla G(t(x-y)) \cdot (x-y)| \, dx \, dy.
\end{aligned} \tag{3.18}$$

Combining (3.16) and (3.18) eventually yields

$$f'(t) = \iint_{E \times E^c} \nabla G(t(x-y)) \cdot (x-y) \, dx \, dy$$

for any  $t \in (t_0, t_1)$ , by the arbitrariness  $\varepsilon$ . Then we prove the continuity of  $f'$  in  $(t_0, t_1)$ . For  $t_0 < t < t_1$  and  $h_0$  as before, for any  $h \in [-h_0, h_0]$  we write

$$\begin{aligned}
& |f'(t+h) - f'(t)| \\
& \leq \iint_{\substack{E \times E^c \\ \frac{1}{R} \leq |x-y| \leq R}} |\nabla G((t+h)(x-y)) - \nabla G(t(x-y))| |x-y| \, dx \, dy \\
& \quad + \iint_{\substack{E \times E^c \\ |x-y| < \frac{1}{R} \text{ or } |x-y| > R}} |\nabla G((t+h)(x-y)) - \nabla G(t(x-y))| |x-y| \, dx \, dy.
\end{aligned} \tag{3.19}$$



As before, given  $\varepsilon > 0$ , for  $R$  large enough we have

$$\iint_{\substack{E \times E^c \\ |x-y| < \frac{1}{R} \text{ or } |x-y| > R}} |\nabla G((t+h)(x-y)) - \nabla G(t(x-y))| |x-y| \, dx \, dy \leq \varepsilon, \quad (3.20)$$

for any  $h \in [-h_0, h_0]$ , and using the continuity of  $\nabla G$  (in particular its boundedness) away from the origin, by dominated convergence we find

$$\iint_{\substack{E \times E^c \\ \frac{1}{R} \leq |x-y| \leq R}} |\nabla G((t+h)(x-y)) - \nabla G(t(x-y))| |x-y| \, dx \, dy \xrightarrow{h \rightarrow 0} 0. \quad (3.21)$$

Combining (3.19), (3.20) and (3.21) gives the continuity of  $f'$  in  $(t_0, t_1)$ . Hence by arbitrariness of  $t_0, t_1$ , and in view of (3.10),  $t \mapsto \text{Per}_G(tE) \in C^1(0, +\infty)$  with

$$\begin{aligned} \frac{d}{dt} [\text{Per}_G(tE)] &= \frac{2n}{t} \text{Per}_G(tE) + t^{2n} \iint_{E \times E^c} \nabla G(t(x-y)) \cdot (x-y) \, dx \, dy \\ &= \frac{2n}{t} \text{Per}_G(tE) + \frac{1}{t} \iint_{(tE) \times (tE)^c} \nabla G(x-y) \cdot (x-y) \, dx \, dy, \end{aligned}$$

which gives (3.8). There remains to show (3.9). Let  $t \in (\frac{1}{2}, 2)$ . Integrating in  $t$  and using (3.8), we find

$$\begin{aligned} |\text{Per}_G(tE) - \text{Per}_G(E)| &\leq \int_{[1,t]} \left| \frac{d}{ds} \text{Per}_G(sE) \right| \, ds \\ &\leq 4n \int_{[1,t]} \text{Per}_G(sE) \, ds \\ &\quad + 2 \int_{[1,t]} \iint_{(sE) \times (sE)^c} |\nabla G(x-y)| |x-y| \, dx \, dy \, ds. \end{aligned} \quad (3.22)$$

By Corollary 3.5, we have the estimates

$$\text{Per}_G(sE) \leq \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} P(sE)$$

and

$$\iint_{(sE) \times (sE)^c} |\nabla G(x-y)| |x-y| \, dx \, dy \leq \frac{I_G^{1,2} \mathbf{K}_{1,n}}{2} P(sE),$$

thus with (3.22) it follows

$$|\text{Per}_G(tE) - \text{Per}_G(E)| \leq \left( 2nI_G^{0,1} + I_G^{1,2} \right) P(E) \int_{[1,t]} s^{n-1} \, ds \leq C|1-t^n|P(E) \leq C|1-t|P(E),$$

for some  $C = C(n, I_G^{0,1}, I_G^{1,2})$ .  $\square$

*Remark 3.8.* Note that we could easily show that  $\text{Per}_G$  also compares with a volume term, in the sense that

$$|\text{Per}_G(tE) - \text{Per}_G(E)| \leq C|1-t||E| \quad \forall t \in (\frac{1}{2}, 2),$$

for every set of finite perimeter  $E$ , where  $C$  depends only on  $n$  and  $I_G^{0,0}$ . However it will be necessary for us to have an upper bound for this difference which scales as  $m^{\frac{n-1}{n}}$  if  $E$  is close to a ball rather than as  $m$ , where  $m$  is the volume of  $E$ .

To study the behavior of (P) as  $m$  goes to infinity, we like to move the volume constraint into the functional, and work with sets of fixed volume  $|B_1|$  instead. Thus we consider the equivalent rescaled problem, defined in the following proposition.

**Proposition 3.9** (Rescaled Problem). *Let  $m > 0$ . Setting  $\lambda := \left(\frac{m}{\omega_n}\right)$  and denoting by  $G_\lambda$  the kernel defined by  $G_\lambda(\cdot) := \lambda^{n+1}G(\lambda\cdot)$ , (P) is equivalent to*

$$\min \left\{ P(F) - \text{Per}_{G_\lambda}(F) : |F| = |B_1| \right\}, \quad (\text{RP})$$

in the sense that (P) admits a minimizer if and only if (RP) does, and  $E$  is a minimizer of (P) if and only if  $F := \lambda^{-1}E$  is a minimizer of (RP).

In addition, denoting by  $\rho_G : (0, +\infty) \rightarrow [0, +\infty)$  the function defined by  $\rho_G(r) := \frac{rg(r)}{I_G^{0,1}}$ , and defining, for any  $\varepsilon > 0$ , its rescaling  $\rho_{G,\varepsilon}(r) := \varepsilon^{-n}\rho_G(\varepsilon^{-1}r)$ , we may rewrite  $\text{Per}_{G_\lambda}$  as

$$\text{Per}_{G_\lambda}(F) = \frac{I_G^{0,1}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\mathbf{1}_F(x) - \mathbf{1}_F(y)|}{|x - y|} \rho_{G,1/\lambda}(|x - y|) dx dy. \quad (3.23)$$

*Proof.* Given  $E$  such that  $|E| = m$ ,  $F = \lambda^{-1}E$  with  $\lambda = \left(\frac{m}{\omega_n}\right)^{\frac{1}{n}}$ , we have  $|F| = \omega_n = |B_1|$ . Making the change of variables  $x = \lambda x'$ ,  $y = \lambda y'$  we find

$$\begin{aligned} P(E) - \text{Per}_G(E) &= \lambda^{n-1} \left( P(F) - \iint_{F \times F^c} \lambda^{n+1} G(\lambda(x - y)) dx dy \right) \\ &= \lambda^{n-1} \left( P(F) - \frac{I_G^{0,1}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\mathbf{1}_F(x) - \mathbf{1}_F(y)|}{|x - y|} \rho_{G,1/\lambda}(|x - y|) dx dy \right), \end{aligned}$$

which gives the result.  $\square$

The advantage of rewriting  $\text{Per}_{G_\lambda}$  as in (3.23) is that the family of functions  $(\rho_{G,\varepsilon})_{\varepsilon>0}$  constitutes a  $n$ -dimensional approximation of identity, which we define just below.

**Definition 3.10** (Approximation of identity). For any  $k \in \mathbb{N} \setminus \{0\}$ , we say that a family of measurable functions  $(\rho_\varepsilon)_{\varepsilon>0}$  from  $(0, +\infty)$  into  $[0, +\infty)$  is a  $k$ -dimensional approximation of identity if, for all  $\varepsilon > 0$ , we have

$$(i) \quad |\mathbb{S}^{k-1}| \int_0^\infty \rho_\varepsilon(r) r^{k-1} dr = 1; \quad (ii) \quad \lim_{\varepsilon \rightarrow 0} \int_\delta^\infty \rho_\varepsilon(r) r^{k-1} dr = 0, \quad \forall \delta > 0.$$

The following proposition will be crucial to shed a light on the behavior of (RP) when  $\lambda$  goes to infinity.

**Proposition 3.11** ([9]). *Let  $f \in \text{BV}(\mathbb{R}^n)$ , and  $(\rho_\varepsilon)_{\varepsilon>0}$  be a  $n$ -dimensional approximation of identity. Then we have*

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|} \rho_\varepsilon(|x - y|) dx dy \xrightarrow{\varepsilon \rightarrow 0} \mathbf{K}_{1,n} \int_{\mathbb{R}^n} |Df|,$$

where  $\mathbf{K}_{1,n}$  is defined by (1.1).

Taking  $f = \mathbf{1}_E$  in [Proposition 3.11](#) we get the following immediate corollary.

**Corollary 3.12.** *For any set of finite perimeter  $E$  in  $\mathbb{R}^n$ , we have*

$$\text{Per}_{G_\lambda}(E) \xrightarrow{\lambda \rightarrow \infty} \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} P(E),$$

where  $\text{Per}_{G_\lambda}$  is defined by [\(3.23\)](#).

This shows that the functional minimized in [\(RP\)](#) converges pointwise to a multiple of the classical perimeter  $P$  when  $\lambda$  goes to infinity. Thus we may guess that if this multiple is positive, [\(RP\)](#) will reduce to minimizing the classical perimeter under the volume constraint  $|B_1|$  when  $\lambda$  is large.

### 3.2. Bessel kernels

For any  $\kappa$  and  $\alpha$  in  $(0, +\infty)$ , the Bessel kernel  $\mathcal{B}_{\kappa,\alpha}$  is the fundamental solution  $f$  of

$$(I - \kappa \Delta)^{\frac{\alpha}{2}} f = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

When  $\kappa = 1$ , we write  $\mathcal{B}_\alpha := \mathcal{B}_{1,\alpha}$ . The following proposition sums up some basic properties of  $\mathcal{B}_\alpha$  (see e.g. [\[17, Chapter I.2.2\]](#), [\[33, Chapter V.3\]](#) and [\[3, Chapter II.3\]](#)).

**Proposition 3.13.** *The Bessel kernel of order  $\alpha \in (0, +\infty)$  in  $\mathbb{R}^n$  is given by*

$$\mathcal{B}_\alpha(x) = \frac{1}{(4\pi)^{\frac{\alpha}{2}}} \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-\frac{\pi|x|^2}{t}} e^{-\frac{t}{4\pi}} t^{\frac{\alpha-n}{2}} \frac{dt}{t}. \quad (3.24)$$

The kernel  $\mathcal{B}_\alpha$  is radial, radially nonincreasing, and  $C^1$  away from the origin. In addition

$$I_{\mathcal{B}_\alpha}^{0,0} = 1, \quad I_{\mathcal{B}_\alpha}^{0,1} = n \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \frac{\Gamma(\frac{1+n}{2})}{\Gamma(1+\frac{n}{2})}, \quad I_{\mathcal{B}_\alpha}^{1,1} = n, \quad \text{and} \quad I_{\mathcal{B}_\alpha}^{1,2} < +\infty.$$

The asymptotic behavior of  $\mathcal{B}_\alpha$  is

$$\mathcal{B}_\alpha(x) \underset{0}{\sim} \begin{cases} \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \frac{1}{|x|^{n-\alpha}} & \text{if } 0 < \alpha < n \\ \frac{-\log(|x|)}{2^{n-1} \pi^{\frac{n}{2}} \Gamma(\frac{n}{2})} & \text{if } \alpha = n \\ \frac{\Gamma(\frac{\alpha-n}{2})}{2^n \pi^{\frac{n}{2}}} & \text{if } n < \alpha, \end{cases}$$

and

$$\mathcal{B}_\alpha(x) \underset{\infty}{\sim} \frac{1}{2^{\frac{n+\alpha-1}{2}} \pi^{\frac{n-1}{2}} \Gamma(\frac{\alpha}{2})} |x|^{\frac{\alpha-n-1}{2}} e^{-|x|}.$$

By scaling, for  $\alpha, \kappa > 0$ , the (generalized) Bessel kernel  $\mathcal{B}_{\kappa,\alpha}$  is given by

$$\mathcal{B}_{\kappa,\alpha}(x) = \frac{1}{\kappa^{\frac{n}{2}}} \mathcal{B}_\alpha\left(\frac{x}{\sqrt{\kappa}}\right), \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad (3.25)$$

thus

$$I_{\mathcal{B}_{\kappa,\alpha}}^{0,0} = 1, \quad I_{\mathcal{B}_{\kappa,\alpha}}^{0,1} = \kappa^{\frac{1}{2}} I_{\mathcal{B}_\alpha}^{0,1}, \quad \text{and} \quad I_{\mathcal{B}_{\kappa,\alpha}}^{1,1} = n.$$

*Proof.* The integral representation (3.24) and the asymptotics can be found respectively in [17] and [3], and the fact that  $I_{\mathcal{B}_\alpha}^{0,0} = \|\mathcal{B}_\alpha\|_{L^1(\mathbb{R}^n)} = 1$  is well known, so we detail only the computations of  $\int_{\mathbb{R}^n} |x| \mathcal{B}_\alpha(x) dx$ ,  $\int_{\mathbb{R}^n} |x| |\nabla \mathcal{B}_\alpha(x)| dx$ , and the fact that  $\int_{\mathbb{R}^n} |x|^2 |\nabla \mathcal{B}_\alpha(x)| dx$  is finite. By (3.24), using Fubini's theorem, we find

$$I_{\mathcal{B}_\alpha}^{0,1} = \int_{\mathbb{R}^n} |x| \mathcal{B}_\alpha(x) dx = \frac{1}{(4\pi)^{\frac{\alpha}{2}}} \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-\frac{t}{4\pi} t^{\frac{\alpha-n}{2}}} \left( \int_{\mathbb{R}^n} |x| e^{-\frac{\pi|x|^2}{t}} dx \right) \frac{dt}{t}. \quad (3.26)$$

Changing variables, we compute

$$\begin{aligned} \int_{\mathbb{R}^n} |x| e^{-\frac{\pi|x|^2}{t}} dx &= n\omega_n \left( \frac{t}{\pi} \right)^{\frac{n+1}{2}} \int_0^\infty r^n e^{-r^2} dr \\ &= n\omega_n \left( \frac{t}{\pi} \right)^{\frac{n+1}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{2} = \frac{nt^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{4\pi} \Gamma\left(1 + \frac{n}{2}\right)}. \end{aligned} \quad (3.27)$$

Injecting (3.27) into (3.26) yields

$$\begin{aligned} I_{\mathcal{B}_\alpha}^{0,1} &= \frac{1}{(4\pi)^{\frac{1+\alpha}{2}}} \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{n\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1 + \frac{n}{2}\right)} \int_0^\infty e^{-\frac{t}{4\pi} t^{\frac{\alpha+1}{2}}} \frac{dt}{t} \\ &= \frac{1}{(4\pi)^{\frac{1+\alpha}{2}}} \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{n\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1 + \frac{n}{2}\right)} (4\pi)^{\frac{1+\alpha}{2}} \Gamma\left(\frac{1+\alpha}{2}\right) = n \frac{\Gamma\left(\frac{1+\alpha}{2}\right) \Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(1 + \frac{n}{2}\right)}. \end{aligned}$$

Now we show that  $\mathcal{B}_\alpha$  is  $C^1$  away from the origin, and compute its gradient. Note that the integrand of (3.24) is  $C^1$  in  $x$  for almost every  $t \in (0, +\infty)$ . Let  $r > 0$ . For every  $x \in \mathbb{R}^n \setminus \overline{B_r}$  we have

$$\left| e^{-\frac{\pi|x|^2}{t}} e^{-\frac{t}{4\pi} t^{\frac{\alpha-n}{2}}} \frac{1}{t} \right| \leq e^{-\frac{\pi r^2}{t}} e^{-\frac{t}{4\pi} t^{\frac{\alpha-n}{2}-1}} \in L^1(0, +\infty),$$

thus  $\mathcal{B}_\alpha \in C^0(\mathbb{R}^n \setminus \overline{B_r})$  by the theorem of continuity under the integral. By arbitrariness of  $R$ ,  $\mathcal{B}_\alpha$  is then continuous away from the origin. Now let  $0 < r < R$ . For every  $x \in B_R \setminus \overline{B_r}$ , we have

$$\begin{aligned} \left| \frac{d}{dx} \left( e^{-\frac{\pi|x|^2}{t}} e^{-\frac{t}{4\pi} t^{\frac{\alpha-n}{2}}} \frac{1}{t} \right) \right| &= \left| -\frac{2\pi x}{t} e^{-\frac{\pi|x|^2}{t}} e^{-\frac{t}{4\pi} t^{\frac{\alpha-n}{2}}} \frac{1}{t} \right| \\ &\leq 2\pi R e^{-\frac{\pi r^2}{t}} e^{-\frac{t}{4\pi} t^{\frac{\alpha-n}{2}-2}} \in L^1(0, +\infty), \end{aligned}$$

thus by the theorem of derivation under the integral,  $\mathcal{B}_\alpha \in C^1(B_R \setminus \overline{B_r})$ , and for every  $x \in B_R \setminus \overline{B_r}$ ,

$$\nabla \mathcal{B}_\alpha(x) = \frac{1}{(4\pi)^{\frac{\alpha}{2}}} \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty -\frac{2\pi x}{t} e^{-\frac{\pi|x|^2}{t}} e^{-\frac{t}{4\pi} t^{\frac{\alpha-n}{2}}} \frac{dt}{t}. \quad (3.28)$$

By arbitrariness of  $r$  and  $R$ ,  $\mathcal{B}_\alpha$  is  $C^1$  away from the origin, and (3.28) holds for any  $x \in \mathbb{R}^n \setminus \{0\}$ . In view of (3.1), if  $I_{\mathcal{B}_\alpha}^{1,1} = \int_{\mathbb{R}^n} |\partial_r \mathcal{B}_\alpha(x)| |x| dx$  and  $I_{\mathcal{B}_\alpha}^{1,2} = \int_{\mathbb{R}^n} |\partial_r \mathcal{B}_\alpha(x)| |x|^2 dx$  are finite, then  $\mathcal{B}_\alpha$  satisfies (H1) and (H2), and we automatically have

$$I_{\mathcal{B}_\alpha}^{1,1} = n I_{\mathcal{B}_\alpha}^{0,0} = n,$$

thus we need only prove that the moments  $I_{\mathcal{B}_\alpha}^{1,1}$  and  $I_{\mathcal{B}_\alpha}^{1,2}$  are finite, without having to compute them. For the rest of the proof  $C$  denotes a constant, possibly changing from line to line,

depending only on  $n$  and  $\alpha$ . By Fubini's theorem, we have

$$I_{\mathcal{B}_\alpha}^{1,1} \leq C \int_0^\infty e^{-\frac{t}{4\pi}t^{\frac{\alpha-n}{2}}} \frac{2\pi}{t} \left( \int_{\mathbb{R}^n} |x|^2 e^{-\frac{\pi|x|^2}{t}} dx \right) \frac{dt}{t}. \quad (3.29)$$

Changing variables, we compute

$$\int_{\mathbb{R}^n} |x|^2 e^{-\frac{\pi|x|^2}{t}} dx = Ct^{\frac{n+2}{2}} \int_0^\infty r^{n+1} e^{-r^2} dr = Ct^{1+\frac{n}{2}}, \quad (3.30)$$

thus injecting (3.30) into (3.29) yields

$$I_{\mathcal{B}_\alpha}^{1,1} \leq C \int_0^\infty e^{-\frac{t}{4\pi}t^{\frac{\alpha}{2}-1}} dt < +\infty.$$

There remains to show that  $I_{\mathcal{B}_\alpha}^{1,2}$  is finite. Once again, a use of Fubini's theorem gives

$$I_{\mathcal{B}_\alpha}^{1,2} \leq \frac{1}{(4\pi)^{\frac{\alpha}{2}}} \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty e^{-\frac{t}{4\pi}t^{\frac{\alpha-n}{2}}} \frac{2\pi}{t} \left( \int_{\mathbb{R}^n} |x|^3 e^{-\frac{\pi|x|^2}{t}} dx \right) \frac{dt}{t},$$

and a change of variables shows

$$\int_{\mathbb{R}^n} |x|^3 e^{-\frac{\pi|x|^2}{t}} dx = Ct^{\frac{n+3}{2}} \int_0^\infty r^{n+2} e^{-r^2} dr \leq Ct^{\frac{n+3}{2}},$$

hence

$$I_{\mathcal{B}_\alpha}^{1,2} \leq C \int_0^\infty e^{-\frac{t}{4\pi}t^{\frac{\alpha-n}{2}}} t^{\frac{n+1}{2}} \frac{dt}{t} = C \int_0^\infty e^{-\frac{t}{4\pi}t^{\frac{\alpha-1}{2}}} dt < +\infty,$$

which concludes the proof.  $\square$

*Remark 3.14.* The  $\mathcal{B}_\alpha$  kernel can also be expressed in terms of the modified Bessel functions of the third kind  $K_\nu : (0, +\infty) \rightarrow (0, +\infty)$ , defined for any  $\nu \in \mathbb{R}$  by

$$K_\nu(r) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{r^\nu e^{-r}}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-rt} \left(t + \frac{t^2}{2}\right)^{\nu-\frac{1}{2}} dt \quad \text{if } \nu > -\frac{1}{2}, \quad (3.31)$$

and the relation  $K_\nu = K_{-\nu}$  (see [3, Chapter II.3]). Then by [3, Chapter II.4],  $\mathcal{B}_\alpha$  is given by

$$\mathcal{B}_\alpha(x) = \frac{1}{2^{\frac{n+\alpha-2}{2}} \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \frac{K_{\frac{n-\alpha}{2}}(|x|)}{|x|^{\frac{n-\alpha}{2}}}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \quad (3.32)$$

From (3.31) and (3.32) it is easy to see that when  $\alpha = n - 1$ ,  $\mathcal{B}_\alpha$  takes the explicit form

$$\mathcal{B}_{n-1} = \frac{1}{(4\pi)^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} \frac{e^{-|x|}}{|x|}.$$

In particular, when  $n = 3$  and  $\alpha = 2$ ,  $\mathcal{B}_\alpha(x) = \frac{1}{4\pi} \frac{e^{-|x|}}{|x|}$ . When  $\alpha = n + 1$ , changing variables in (3.24), one can compute  $\mathcal{B}_\alpha$  explicitly as well. Indeed, in that case,

$$\mathcal{B}_{n+1}(x) = \frac{1}{(4\pi)^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)} 2\sqrt{4\pi} \int_0^\infty e^{-t^2 - \frac{|x|^2}{4t^2}} dt = \frac{2\pi}{(4\pi)^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)} e^{-|x|}.$$

We have the following straightforward corollary of [Proposition 3.13](#).

**Corollary 3.15.** *For every  $\alpha, \kappa \in (0, +\infty)$ , the kernel  $\mathcal{B}_{\kappa, \alpha}$  satisfies assumptions (H1) to (H4).*

We can express the constants  $\mathbf{K}_{p,n}$  in terms of the Gamma function as follows, in order to make the assumptions on  $I_{\mathcal{B}_{\kappa, \alpha}}^{0,1}$  in [Theorems A to C](#) fully explicit.

**Lemma 3.16.** *For any  $n \in \mathbb{N}$  and  $p > 0$ , we have*

$$\mathbf{K}_{p,n} = \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{1+p}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n+p}{2}\right)},$$

where  $\mathbf{K}_{p,n}$  is given by [\(1.1\)](#). In particular

$$\mathbf{K}_{1,n} = \frac{2\Gamma\left(1 + \frac{n}{2}\right)}{n\sqrt{\pi} \Gamma\left(1 + \frac{n+1}{2}\right)} \quad \text{and} \quad \mathbf{K}_{2,n-1} = \frac{1}{n}.$$

*Proof.* In the definition of  $\mathbf{K}_{p,n}$ , we may assume  $e = (0, \dots, 0, 1)$  without loss of generality, so that  $e \cdot x = x_n$ . Recall that for every nonnegative  $\mathcal{H}^{n-1}$ -measurable function  $f$  on  $\mathbb{S}^{n-1}$ , we have (see e.g. [\[4, Corollary A.6\]](#))

$$\int_{\mathbb{S}^{n-1}} f d\mathcal{H}^{n-1} = \int_{-1}^1 (1-t^2)^{\frac{n-3}{2}} \int_{\mathbb{S}^{n-2}} f(\sqrt{1-t^2}x, t) d\mathcal{H}_x^{n-2} dt.$$

This way we compute

$$\int_{\mathbb{S}^{n-1}} |x_n|^p d\mathcal{H}^{n-1} = |\mathbb{S}^{n-2}| \int_{-1}^1 |t|^p (1-t^2)^{\frac{n-3}{2}} dt = |\mathbb{S}^{n-2}| \frac{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{n+p}{2}\right)},$$

thus

$$\mathbf{K}_{p,n} = \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \frac{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{n+p}{2}\right)} = \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{1+p}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n+p}{2}\right)}.$$

□

[Application D](#) is then a direct consequence of [Corollary 3.15](#) and [Lemma 3.16](#).

## 4. Existence and convergence of large mass minimizers

### 4.1. Existence of large mass minimizers

In order to prove the existence of minimizers for large masses, we want to use the direct method in the calculus of variations, starting from a minimizing sequence. When  $I_G^{0,1}$  is small enough, we will see that any minimizing sequence is bounded in  $\text{BV}(\mathbb{R}^n)$ , but in order to get compactness in  $L^1(\mathbb{R}^n)$  and pass to the limit, we need to show that no mass escapes at infinity. To do so, we will need to establish a few lemmas. First we show that for large masses, if the energy  $\mathcal{F}_G(E)$  of some set  $E$ , defined by [\(3.2\)](#), is smaller than that of a ball of same mass, then  $E$  is actually close to a ball.

**Lemma 4.1.** Assume that  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ . Then for any set  $E$  of finite perimeter with volume  $m = \omega_n \lambda^n$  such that

$$\mathcal{F}_G(E) \leq \mathcal{F}_G([B]_m),$$

we have

$$|E \Delta [B]_{y,m}| \leq m\eta(\lambda),$$

for some ball  $y \in \mathbb{R}^n$ , where

$$\eta(\lambda) := C \left[ \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} P(B_1) - \text{Per}_{G_\lambda}(B_1) \right]^{\frac{1}{2}},$$

and  $C = C(n, I_G^{0,1})$ .

*Remark 4.2.* Notice that  $\eta(\lambda)$  goes to 0 as  $\lambda$  goes to infinity by [Proposition 3.11](#).

*Proof.* The inequality

$$\mathcal{F}_G(E) \leq \mathcal{F}_G([B]_m)$$

rewrites

$$P(E) - P([B]_m) \leq \text{Per}_G(E) - \text{Per}_G([B]_m).$$

Scaling this inequality with  $F := \lambda^{-1}E$  yields

$$P(F) - P(B_1) \leq \text{Per}_{G_\lambda}(F) - \text{Per}_{G_\lambda}(B_1),$$

where  $\text{Per}_{G_\lambda}$  is defined by [\(3.23\)](#). By [Corollary 3.5](#), this implies

$$\begin{aligned} P(F) - P(B_1) &\leq \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} P(F) - \text{Per}_{G_\lambda}(B_1) \\ &= \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} (P(F) - P(B_1)) + \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} P(B_1) - \text{Per}_{G_\lambda}(B_1), \end{aligned}$$

thus

$$P(F) - P(B_1) \leq C_1 f(\lambda) \tag{4.1}$$

where  $C_1 := \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right)^{-1}$  depends only on  $n$  and  $I_G^{0,1}$ , and

$$f(\lambda) := \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} P(B_1) - \text{Per}_{G_\lambda}(B_1). \tag{4.2}$$

Using the quantitative isoperimetric inequality [\(2.2\)](#) and [\(4.1\)](#), we find  $\alpha(F) \leq C_2 \sqrt{f(\lambda)}$ , where  $C_2 = C_2(n, I_G^{0,1})$ . Hence there exists  $B_1(y)$  such that

$$|F \Delta B_1(y)| \leq C_2 \sqrt{f(\lambda)},$$

which gives the result by [\(4.2\)](#), recalling that  $E = \lambda F$ .  $\square$

We also need a truncation lemma akin to [\[25, Lemma 29.12\]](#) or [\[12, Lemma 4.5\]](#) to quantify by how much the energy decreases after properly cutting a set which is already close to a ball.

**Lemma 4.3.** *Assume  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ . There exist  $C_1, C_2 \in (0, +\infty)$  depending only on  $n$  and  $I_G^{0,1}$  such that the following holds. If  $E$  is a set of finite perimeter satisfying  $|E \setminus B_{r_0}| \leq \eta$ , for some positive constants  $\eta$  and  $r_0$ , then there exists  $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$  such that*

$$\mathcal{F}_G(E \cap B_r) \leq \mathcal{F}_G(E) - \frac{|E \setminus B_r|}{C_2\eta^{\frac{1}{n}}}. \quad (4.3)$$

*Proof.* Let  $C_1, C_2 > 0$  to be fixed later, and  $E$  be a set of finite perimeter such that  $|E \setminus B_{r_0}| \leq \eta$ . We define  $u(r) := |E \setminus B_r|$ , and for now we assume that  $u(r_0 + C_1\eta^{\frac{1}{n}}) > 0$ . Since  $u$  is nonincreasing, we have  $u(r) > 0$ , for all  $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$ . Notice that  $u$  is absolutely continuous, and  $u'(r) = -\mathcal{H}^{n-1}(E \cap \partial B_r)$  for  $\mathcal{L}^1$ -almost every  $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$ . By contradiction, let us assume that

$$P(E) - \text{Per}_G(E) < P(E \cap B_r) - \text{Per}_G(E \cap B_r) + \frac{|E \setminus B_r|}{C_2\eta^{\frac{1}{n}}}, \quad (4.4)$$

for all  $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$ . Recall that for almost every  $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$  we have  $\mathcal{H}^{n-1}(\partial^* E \cap \partial B_r) = 0$  (see e.g. [25, Proposition 2.16]). Given such an  $r$ , note that  $P(E) = P(E; B_r) + P(E; \overline{B_r}^c)$ , and  $P(E \cap B_r) = P(E; B_r) + \mathcal{H}^{n-1}(E \cap \partial B_r)$  (see e.g. [25, Lemma 15.12]). Thus

$$\begin{aligned} P(E) - P(E \cap B_r) &= P(E; B_r) + P(E; \overline{B_r}^c) - P(E; B_r) - \mathcal{H}^{n-1}(E \cap \partial B_r) \\ &= P(E; \overline{B_r}^c) - \mathcal{H}^{n-1}(E \cap \partial B_r) \\ &= P(E \setminus B_r) - 2\mathcal{H}^{n-1}(E \cap \partial B_r) = P(E \setminus B_r) + 2u'(r), \end{aligned} \quad (4.5)$$

where we also used  $P(E \setminus B_r) = P(E; \overline{B_r}^c) + \mathcal{H}^{n-1}(E \cap \partial B_r)$ . On the other hand, noticing that

$$\begin{aligned} \text{Per}_G(E) &= \iint_{(E \setminus B_r) \times E^c} G(x-y) \, dx \, dy + \iint_{(E \cap B_r) \times E^c} G(x-y) \, dx \, dy \\ &= \iint_{(E \setminus B_r) \times (E \setminus B_r)^c} G(x-y) \, dx \, dy - \iint_{(E \setminus B_r) \times (E \cap B_r)} G(x-y) \, dx \, dy \\ &\quad + \iint_{(E \cap B_r) \times E^c} G(x-y) \, dx \, dy \\ &= \text{Per}_G(E \setminus B_r) - \iint_{(E \setminus B_r) \times (E \cap B_r)} G(x-y) \, dx \, dy + \iint_{(E \cap B_r) \times E^c} G(x-y) \, dx \, dy \end{aligned}$$

and

$$\text{Per}_G(E \cap B_r) = \iint_{(E \cap B_r) \times E^c} G(x-y) \, dx \, dy + \iint_{(E \cap B_r) \times (E \setminus B_r)} G(x-y) \, dx \, dy,$$

we find

$$\text{Per}_G(E) - \text{Per}_G(E \cap B_r) = \text{Per}_G(E \setminus B_r) - 2 \iint_{(E \cap B_r) \times (E \cap B_r^c)} G(x-y) \, dx \, dy. \quad (4.6)$$

Injecting (4.5) and (4.6) into (4.4), one gets

$$\begin{aligned} P(E \setminus B_r) - \text{Per}_G(E \setminus B_r) &< -2u'(r) - 2 \iint_{(E \cap B_r) \times (E \cap B_r^c)} G(x-y) \, dx \, dy + \frac{u(r)}{C_2\eta^{\frac{1}{n}}} \\ &\leq -2u'(r) + \frac{u(r)}{C_2\eta^{\frac{1}{n}}}, \end{aligned} \quad (4.7)$$



for almost every  $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$ . Note that by [Corollary 3.5](#), we know that  $\text{Per}_G(E \setminus B_r) \leq \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} P(E \setminus B_r)$ , thus

$$\left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) P(E \setminus B_r) < -2u'(r) + \frac{u(r)}{C_2\eta^{\frac{1}{n}}} \quad \text{for a.e. } r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]. \quad (4.8)$$

Now by the isoperimetric inequality [\(2.2\)](#) we have

$$P(E \setminus B_r) \geq n\omega_n^{\frac{1}{n}} u(r)^{\frac{n-1}{n}}. \quad (4.9)$$

Since  $u$  is nonincreasing and  $u(r_0) = |E \setminus B_{r_0}| \leq \eta$ , we also know that

$$u(r) \leq u(r_0)^{\frac{1}{n}} u(r)^{\frac{n-1}{n}} \leq \eta^{\frac{1}{n}} u(r)^{\frac{n-1}{n}}. \quad (4.10)$$

Plugging [\(4.9\)](#) and [\(4.10\)](#) into [\(4.8\)](#), we deduce that for a.e.  $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$ ,

$$n\omega_n^{\frac{1}{n}} \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) u(r)^{\frac{n-1}{n}} < -2u'(r) + \frac{u(r)^{\frac{n-1}{n}}}{C_2},$$

thus

$$C_3 n u(r)^{\frac{n-1}{n}} < -u'(r), \quad (4.11)$$

where

$$C_3 := \frac{1}{2} \left[ \omega_n^{\frac{1}{n}} \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) - \frac{1}{nC_2} \right].$$

We can choose  $C_2$  large enough, depending only on  $I_G^{0,1}$  and  $n$ , so that  $C_3 > 0$ . Then [\(4.11\)](#) can be rewritten

$$\left(u(r)^{\frac{1}{n}}\right)' = \frac{1}{n} u'(r) u(r)^{\frac{1}{n}-1} < -C_3 \quad \text{for a.e. } r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}],$$

thus integrating between  $r_0$  and  $r_0 + C_1\eta^{\frac{1}{n}}$ , one gets

$$u(r_0 + C_1\eta^{\frac{1}{n}})^{\frac{1}{n}} \leq u(r_0)^{\frac{1}{n}} - C_3 C_1 \eta^{\frac{1}{n}} \leq (1 - C_1 C_3) \eta^{\frac{1}{n}}.$$

Choosing  $C_1$  large enough, depending only on  $n$  and  $I_G^{0,1}$ , we get  $u(r_0 + C_1\eta^{\frac{1}{n}})^{\frac{1}{n}} < 0$ , which is a contradiction. Recall that we assumed  $u(r_0 + C_1\eta^{\frac{1}{n}}) > 0$ , so there are two cases: for this  $C_1$  large enough, either  $u(r_0 + C_1\eta^{\frac{1}{n}}) > 0$ , and then as we have seen there exists  $r \in [r_0, r_0 + C_1\eta^{\frac{1}{n}}]$  such that [\(4.3\)](#) holds for some  $C_2 = C_2(n, I_G^{0,1})$ , or  $u(r_0 + C_1\eta^{\frac{1}{n}}) = 0$ , and then [\(4.3\)](#) holds for any  $C_2$  with  $r = r_0 + C_1\eta^{\frac{1}{n}}$ . In any case, [\(4.3\)](#) holds.  $\square$

We are now able to prove the existence of large mass minimizers.

*Proof of [Theorem A](#). Step 1.* Let us show that there exists  $m_e > 0$  depending only on  $G$  and  $n$  such that the following holds. For any set of finite perimeter  $E$  of mass  $m > m_e$  satisfying  $\mathcal{F}_G(E) \leq \mathcal{F}_G([B]_m)$ , up to a translation, there exists a set of finite perimeter  $\tilde{E}$  of mass  $m$  satisfying

$$\mathcal{F}_G(\tilde{E}) \leq \mathcal{F}_G(E) - \frac{C}{m^{\frac{1}{n}}} |E \setminus (4[B]_m)| \quad \text{and} \quad \tilde{E} \subseteq 4[B]_m,$$

for some  $C > 0$  depending only on  $n$ . Let  $m_e > 0$  to be fixed later, and  $\lambda_e := \left(\frac{m_e}{\omega_n}\right)^{\frac{1}{n}}$ . By [Lemma 4.1](#), if  $E$  satisfies  $|E| = m > m_e$  and  $\mathcal{F}_G(E) \leq \mathcal{F}_G([B]_m)$ , then up to translating  $E$ , we have

$$|E \Delta B_\lambda| = |E \Delta [B]_m| \leq m\eta(\lambda),$$

where  $m = \omega_n \lambda^n$  (and in particular,  $\lambda > \lambda_e$ ). Using [Lemma 4.3](#) with  $\lambda^n \eta(\lambda)$  in place of  $\eta$ , we can find  $r \leq \lambda + C_1 \lambda \eta(\lambda)^{\frac{1}{n}}$  such that

$$\mathcal{F}_G(E \cap B_r) \leq \mathcal{F}_G(E) - \frac{\omega_n^{\frac{1}{n}} |E \setminus B_r|}{C_2 (m\eta(\lambda))^{\frac{1}{n}}}. \quad (4.12)$$

For simplicity, let us define  $u := \frac{|E \setminus B_r|}{m}$  and rewrite [\(4.12\)](#) as

$$\mathcal{F}_G(E \cap B_r) \leq \mathcal{F}_G(E) - \frac{\omega_n}{C_2} \frac{\lambda u}{\eta(\lambda)^{\frac{1}{n}}}. \quad (4.13)$$

To compensate for the loss of some mass, we define the rescaled set  $\tilde{E} := \mu(E \cap B_r)$ , where  $\mu > 0$  is chosen such that  $|\tilde{E}| = m$ , that is,  $\mu := (1 - u)^{-\frac{1}{n}}$ . Then we have

$$\mathcal{F}_G(\tilde{E}) = P(\tilde{E}) - \text{Per}_G(\tilde{E}) = \mu^{n-1} P(E \cap B_r) - \text{Per}_G(\mu(E \cap B_r)). \quad (4.14)$$

Note that  $u \leq \frac{|E \setminus B_\lambda|}{m} \leq \eta(\lambda) \leq 1 - 2^{-n}$  for  $\lambda_e$  (equivalently  $m_e$ ) large enough, depending only on  $G$  and  $n$ , since  $\lambda > \lambda_e$  and  $\eta(\lambda)$  goes to zero at  $\lambda$  goes to infinity. This implies  $1 \leq \mu \leq 2$ . Now by [Lemma 3.7](#) we have

$$\text{Per}_G(\mu(E \cap B_r)) \geq \text{Per}_G(E \cap B_r) - C_3(\mu - 1)P(E \cap B_r) \quad (4.15)$$

for some positive constant  $C_3$  that depends only on  $n$ ,  $I_G^{0,1}$  and  $I_G^{1,2}$ . Injecting [\(4.15\)](#) into [\(4.14\)](#), we find

$$\begin{aligned} \mathcal{F}_G(\tilde{E}) &\leq \mu^{n-1} P(E \cap B_r) - \text{Per}_G(E \cap B_r) + C_3(\mu - 1)P(E \cap B_r) \\ &\leq (1 + C_5 u)P(E \cap B_r) - \text{Per}_G(E \cap B_r) + C_4 u P(E \cap B_r) \\ &= (1 + C_5 u)[P(E \cap B_r) - \text{Per}_G(E \cap B_r)] + C_5 u \text{Per}_G(E \cap B_r) + C_4 u P(E \cap B_r) \\ &= (1 + C_5 u)\mathcal{F}_G(E \cap B_r) + C_5 u \text{Per}_G(E \cap B_r) + C_4 u P(E \cap B_r). \end{aligned} \quad (4.16)$$

where we used the fact that  $u = (1 - \mu^{-n})$ , and introduced some positive constants  $C_4$  and  $C_5$  that depend only on  $n$ ,  $I_G^{0,1}$  and  $I_G^{1,2}$ , since  $\mu \in [1, 2]$ . Injecting [\(4.13\)](#) into [\(4.16\)](#), and using the fact that  $\text{Per}_G(E \cap B_r) \leq \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} P(E \cap B_r)$  by [Corollary 3.5](#), we get

$$\begin{aligned} \mathcal{F}_G(\tilde{E}) &\leq (1 + C_5 u) \left( \mathcal{F}_G(E) - \frac{\omega_n \lambda^{n-1} u}{C_2 \eta(\lambda)^{\frac{1}{n}}} \right) + C_6 u P(E \cap B_r) \\ &\leq \mathcal{F}_G(E) + u \left( C_5 P(E) + C_6 P(E \cap B_r) - \frac{\omega_n \lambda^{n-1}}{C_2 \eta(\lambda)^{\frac{1}{n}}} \right) \\ &\leq \mathcal{F}_G(E) + u \left( (C_5 + C_6) P(E) + C_6 P(B_r) - \frac{\omega_n \lambda^{n-1}}{C_2 \eta(\lambda)^{\frac{1}{n}}} \right), \end{aligned} \quad (4.17)$$

where  $C_6$  depends only on  $n$ ,  $I_G^{0,1}$  and  $I_G^{1,2}$ , and where we used  $P(E \cap B_r) \leq P(E) + P(B_r)$ . Recall that  $\mathcal{F}_G(E) \leq \mathcal{F}_G([B]_m)$ , thus by [Corollary 3.5](#),

$$\left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) P(E) \leq P([B]_m) = \lambda^{n-1} P(B_1), \quad (4.18)$$

which implies  $P(E) \leq C_7 \lambda^{n-1}$  where  $C_7 = C_7(n, I_G^{0,1})$ . Since  $0 \leq r \leq \lambda + C_1 \lambda \eta(\lambda)^{\frac{1}{n}}$  and  $\eta(\lambda)$  vanishes at infinity, we can choose  $\lambda_e$  (recall that  $\lambda \geq \lambda_e$ ) such that  $r \leq 2\lambda$ , thus

$$P(B_r) \leq C_8 \lambda^{n-1} \quad (4.19)$$

for some  $C_8 = C_8(n, I_G^{0,1})$ . Plugging [\(4.18\)](#) and [\(4.19\)](#) into [\(4.17\)](#) yields

$$\mathcal{F}_G(\tilde{E}) \leq \mathcal{F}_G(E) + u \lambda^{n-1} \left( (C_5 + C_6) C_7 + C_6 C_8 - \frac{\omega_n}{C_2 \eta(\lambda)^{\frac{1}{n}}} \right).$$

Recalling  $u = \frac{|E \setminus B_r|}{m}$  and the fact that  $\eta(\lambda)$  vanishes at infinity, we can then choose  $\lambda_e$  (i.e.  $m_e$ ) even larger depending only on  $n$ ,  $I_G^{0,1}$  and  $I_G^{1,2}$  such that

$$\mathcal{F}_G(\tilde{E}) < \mathcal{F}_G(E) - \frac{1}{\lambda} |E \setminus B_r| \leq \mathcal{F}_G(E) - \frac{1}{\lambda} |E \setminus B_{2\lambda}|,$$

where we also used the fact  $r \leq 2\lambda$  for the last inequality. Recall that  $\tilde{E} \subseteq B_{\mu r} \subseteq B_{2r} \subseteq B_{4\lambda}$ , which concludes this step.

*Step 2.* We prove the existence of minimizers. For  $m \geq m_e$ ,  $\lambda = \left(\frac{m}{\omega_n}\right)^{\frac{1}{n}}$ , consider a minimizing sequence  $(E_k)_{k \in \mathbb{N}}$  for [\(P\)](#). There are two cases: either  $[B]_m$  is a minimizer of  $\mathcal{F}_G$ , and we are done, or  $[B]_m$  is not a minimizer of  $\mathcal{F}_G$ , and up to a subsequence (not relabeled), for all  $k \in \mathbb{N}$ , we have  $\mathcal{F}_G(E_k) \leq \mathcal{F}_G([B]_m)$ . In the latter case, by [Step 1](#) we can build another minimizing sequence  $(\tilde{E}_k)_{k \in \mathbb{N}}$  of sets in  $B_{4\lambda}$  such that  $\mathcal{F}_G(\tilde{E}_k) < \mathcal{F}_G(E_k)$ , for all  $k \in \mathbb{N}$ . Now  $(\mathbf{1}_{\tilde{E}_k})_{k \in \mathbb{N}}$  is also bounded in  $\text{BV}(\mathbb{R}^n)$ . Indeed  $\mathcal{F}_G(\tilde{E}_k) \leq \mathcal{F}_G([B]_m)$  implies

$$\left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) P(\tilde{E}_k) \leq P([B]_m),$$

thus  $[\mathbf{1}_{\tilde{E}_k}]_{\text{BV}(\mathbb{R}^n)}$  is bounded, and  $\|\mathbf{1}_{\tilde{E}_k}\|_{L^1(\mathbb{R}^n)} = m$ . By compactness of  $\text{BV}(\mathbb{R}^n)$  in  $L^1_{\text{loc}}(\mathbb{R}^n)$  and the fact that  $\tilde{E}_k \subseteq B_{4\lambda}$ , up to the extraction of a subsequence (still not relabeled),  $\mathbf{1}_{\tilde{E}_k}$  converges to some function  $f \in \text{BV}(\mathbb{R}^n)$  in  $L^1(\mathbb{R}^n)$  and almost everywhere. The almost everywhere convergence implies that  $f$  is the indicator function of some set of finite perimeter  $E$ , and the  $L^1$  convergence ensures  $|E| = m$ . Now by lower semicontinuity of the perimeter w.r.t. the  $L^1$  convergence, we have  $P(E) \leq \liminf_k P(\tilde{E}_k)$ , and by continuity of the nonlocal perimeter in  $L^1(\mathbb{R}^n)$  shown in [Lemma 3.6](#), it follows that  $\text{Per}_G(\tilde{E}_k)$  converges to  $\text{Per}_G(E)$  as  $k$  goes to infinity. Hence  $\mathcal{F}_G(E) \leq \liminf_k \mathcal{F}_G(\tilde{E}_k)$ , which shows that  $E$  is a minimizer of [\(P\)](#), since  $\tilde{E}_k$  is a minimizing sequence.

*Step 3.* We show that for  $\lambda_e$  chosen as before, for any  $\lambda \geq \lambda_e$  and  $m = \omega_n \lambda^n$ , any minimizer of volume  $m$  for [\(P\)](#) is (up to a Lebesgue negligible set) included in  $B_{4\lambda}$ . Consider  $E$  such a minimizer, then by minimality, we have  $\mathcal{F}_G(E) \leq \mathcal{F}_G([B]_m)$ , thus applying [Step 1](#) there exists a set of finite perimeter  $\tilde{E}$  with mass  $m$  such that

$$\mathcal{F}_G(\tilde{E}) \leq \mathcal{F}_G(E) - C |E \setminus B_{4\lambda}|,$$

where  $C > 0$ . By minimality of  $E$ , necessarily  $|E \setminus B_{4\lambda}| = 0$ , hence the result.  $\square$

## 4.2. Indecomposability of minimizers

The aim of this section is to prove the connectedness of minimizers of (P) whenever the kernel  $G$  is not compactly supported. Since any minimizer is always defined up to a set of vanishing Lebesgue measure, and we do not know yet if there is a precise (partially) regular representative, we work here with a measure theoretic notion of connectedness for sets of finite perimeter (see e.g. [1]), which is referred to as *indecomposability*.

**Definition 4.4.** We say that a set of finite perimeter  $E$  is decomposable if there exist two sets of finite perimeter  $E_1$  and  $E_2$  such that  $E = E_1 \sqcup E_2$ ,  $|E_1| > 0$ ,  $|E_2| > 0$  and  $P(E) = P(E_1) + P(E_2)$ . Naturally, we say that a set of finite perimeter is indecomposable if it is not decomposable.

As with the usual topological notion of connectedness, it is possible to partition a set of finite perimeter  $E$  into indecomposable sets (see [1, Theorem 1]) in a unique way (up to sets of vanishing Lebesgue measure). We call the sets composing this partition the  $\mathcal{M}$ -connected components of  $E$ . We have the following result establishing a link between the  $\mathcal{M}$ -connected components of a set of finite perimeter and the topological connected components.

**Theorem 4.5** ([1, Theorem 2]). *If  $E$  is an open set of finite perimeter such that  $\mathcal{H}^{n-1}(\partial E) = \mathcal{H}^{n-1}(\partial^* E)$ , then the  $\mathcal{M}$ -connected components of  $E$  coincide with their topological connected components.*

**Proposition 4.6.** *If  $G$  is not compactly supported, then any minimizer  $E$  of (P) which is included in a ball  $B_R$ , for some  $R > 0$ , is indecomposable.*

*Proof.* We proceed by contradiction. Assume that there exists a minimizer  $E \subseteq B_R$  and two sets of finite  $E_1$  and  $E_2$  such that  $E = E_1 \sqcup E_2$ ,  $|E_1| > 0$ ,  $|E_2| > 0$  and  $P(E) = P(E_1) + P(E_2)$ . Here we prefer to work with the equivalent formulation ( $\star$ ) of the minimization problem, and show that  $E$  cannot be a minimizer of  $P + \mathcal{V}$  under the volume constraint  $m = |E|$ , where  $\mathcal{V}$  is the functional

$$\mathcal{V}_G(F) := \iint_{F \times F} G(x - y) \, dx \, dy.$$

Note that we have the decomposition

$$P(E) + \mathcal{V}_G(E) = P(E_1) + P(E_2) + \mathcal{V}_G(E_1) + \mathcal{V}_G(E_2) + 2 \iint_{E_1 \times E_2} G(x - y) \, dx \, dy.$$

Let  $M > 0$  and  $h \in \mathbb{R}^n$  such that  $|h| > 2R + M$ . Since  $E_1$  and  $E_2$  are included in  $B_R$ , then for any  $(x, y) \in E_1 \times (E_2 + h)$  we have  $|x - y| \geq M$ . Obviously  $P(E_2) = P(E_2 + h)$  and  $\mathcal{V}_G(E_2) = \mathcal{V}_G(E_2 + h)$  by a change of variables. Now let us define the competitor  $E_h := E_1 \sqcup (E_2 + h)$ , which satisfies  $|E_h| = |E_1| + |E_2| = |E|$ . Since  $|x - y| \geq M$  whenever  $(x, y) \in E_1 \times (E_2 + h)$ , we have  $P(E_h) = P(E_1) + P(E_2 + h)$ , thus we compute

$$\begin{aligned} & P(E_h) + \mathcal{V}_G(E_h) \\ &= P(E_1) + P(E_2 + h) + \mathcal{V}_G(E_1) + \mathcal{V}_G(E_2 + h) + 2 \iint_{E_1 \times (E_2 + h)} G(x - y) \, dx \, dy \\ &= P(E_1) + P(E_2) + \mathcal{V}_G(E_1) + \mathcal{V}_G(E_2) + 2 \iint_{E_1 \times (E_2 + h)} G(x - y) \, dx \, dy \\ &= P(E) + \mathcal{V}_G(E) + 2 \iint_{E_1 \times (E_2 + h)} G(x - y) \, dx \, dy - 2 \iint_{E_1 \times E_2} G(x - y) \, dx \, dy, \end{aligned} \tag{4.20}$$

where we used the fact that  $P(E) = P(E_1) + P(E_2)$  for the last equality. Since  $G$  is radial, positive and radially nonincreasing, we have

$$\int_A G(x) dx > 0,$$

for any set  $A$  such that  $|A| > 0$ . In particular

$$\iint_{E_1 \times E_2} G(x - y) dx dy = \int_{E_1} \left( \int_{x - E_2} G(y) dy \right) dx > 0, \quad (4.21)$$

since  $|E_1| > 0$  and  $|E_2| > 0$ . By a change of variables, we also have

$$\iint_{E_1 \times (E_2 + h)} G(x - y) dx dy \leq \int_{E_1} \left( \int_{B_M^c} G(y) dy \right) dx \leq |E_1| \int_{B_M^c} G(y) dy. \quad (4.22)$$

Since  $G \in L^1(\mathbb{R}^n)$ ,  $\int_{B_M^c} G(y) dy$  goes to zero as  $M$  goes to infinity, thus by (4.22) and (4.21) we can find some  $M$  large enough such that

$$\iint_{E_1 \times (E_2 + h)} G(x - y) dx dy - 2 \iint_{E_1 \times E_2} G(x - y) dx dy < 0$$

which yields, with (4.20),

$$P(E_h) + \mathcal{V}(E_h) < P(E) + \mathcal{V}(E)$$

and contradicts the minimality of  $E$ .  $\square$

*Remark 4.7.* In particular, this shows that if  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$  and  $G$  is not compactly supported, then any minimizer of (P) with  $m > m_e$  is indecomposable (where  $m_e$  is given by Theorem A), since it is included in the ball  $4[B]_m$ . In fact, we can even drop the assumption  $E \subseteq B_R$  for some  $R > 0$  in Proposition 4.6, since it turns out to be always the case, as is recalled in Section 4.4.

### 4.3. $\Gamma$ -convergence to the classical perimeter

Using the results from Section 3.1, we establish a  $\Gamma$ -convergence result for the functional (RP), and deduce that (rescaled) large mass minimizers converge to the unit ball as  $\lambda$  goes to infinity, up to translations. In view of (3.23), for any  $\lambda \in (0, +\infty)$ , let us define on  $L^1(\mathbb{R}^n)$  the functional

$$\mathcal{E}_{G,\lambda}(f) := \begin{cases} \int_{\mathbb{R}^n} |Df| - \frac{I_K^{0,1}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|} \rho_{G,1/\lambda}(x - y) & \text{if there exists a set of finite perimeter } E \text{ s.t. } f = \mathbf{1}_E \text{ and } |E| = |B_1|, \\ +\infty & \text{otherwise,} \end{cases} \quad (4.23)$$

which is well defined and finite whenever  $f = \mathbf{1}_E$  for some set of perimeter  $E$  such that  $|E| = |B_1|$  by Proposition 3.3. It is obviously defined so that it “coincides” with  $\mathcal{F}_{G,\lambda} = P - \text{Per}_{G,\lambda}$  on sets of finite perimeter, in the sense that  $\mathcal{E}_{G,\lambda}(\mathbf{1}_E) = \mathcal{F}_{G,\lambda}(E)$  for every set of finite perimeter  $E$  with volume  $|B_1|$ , and so that minimizers of  $\mathcal{E}_{G,\lambda}$  are precisely those functions which are indicator functions of sets of finite perimeter solving (RP).

**Proposition 4.8.** *Let  $(\lambda_k)_{k \in \mathbb{N}}$  be a sequence of positive real numbers going to infinity. If  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ , then the functionals  $\mathcal{E}_{G,\lambda}$  defined by (4.23)  $\Gamma$ -converges w.r.t. the usual  $L^1$  topology to the functional*

$$\mathcal{E}_{G,\infty}(f) := \begin{cases} \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) \int_{\mathbb{R}^n} |Df| & \text{if there exists a set of finite perimeter } E \text{ s.t. } f = \mathbf{1}_E \text{ and } |E| = |B_1|, \\ +\infty & \text{otherwise.} \end{cases}$$

*Proof.* We shall prove, in that order, that

$$\Gamma - \limsup \mathcal{E}_{G,\lambda}(f) \leq \mathcal{E}_{G,\infty}(f), \quad \text{and} \quad \mathcal{E}_{G,\infty}(f) \leq \Gamma - \liminf \mathcal{E}_{G,\lambda}(f),$$

where

$$\Gamma - \limsup \mathcal{E}_{G,\lambda}(f) := \min \left\{ \limsup_k \mathcal{E}_{G,\lambda_k}(f_k) : f_k \xrightarrow{L^1(\mathbb{R}^n)} f \right\}.$$

and

$$\Gamma - \liminf \mathcal{E}_{G,\lambda}(f) := \min \left\{ \liminf_k \mathcal{E}_{G,\lambda_k}(f_k) : f_k \xrightarrow{L^1(\mathbb{R}^n)} f \right\}.$$

*Step 1.* Let  $f \in L^1(\mathbb{R}^n)$ . If  $f$  is not the indicator function of a set of perimeter of volume  $|B_1|$ ,  $\mathcal{E}_{G,\infty}(f) = +\infty$  so the inequality is trivial. Let us assume  $f = \mathbf{1}_F$  for some set of finite perimeter  $F$  such that  $|F| = |B_1|$ , and consider the constant sequence  $f_k \equiv \mathbf{1}_F$ . Then by [Proposition 3.11](#) we have

$$\mathcal{E}_{G,\lambda_k}(\mathbf{1}_F) \xrightarrow{k \rightarrow \infty} \mathcal{E}_{G,\infty}(\mathbf{1}_F),$$

thus  $\Gamma - \limsup \mathcal{E}_{G,\lambda}(f) \leq \mathcal{E}_{G,\infty}(f)$ .

*Step 2.* Given  $f \in L^1(\mathbb{R}^n)$ , consider a sequence  $f_k \in L^1(\mathbb{R}^n)$  such that  $f_k \xrightarrow{L^1(\mathbb{R}^n)} f$ .

If  $f$  is not given by the indicator function of a measurable set  $F$ , we claim that there exists  $k_0 \in \mathbb{N}$  such that for any  $k \geq k_0$ ,  $f_k$  is also not an indicator function. By contradiction, let us assume that there exists a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  such that for every  $n_k$ ,  $f_{n_k}$  is the indicator function of a set  $F_{n_k}$ . By  $L^1$  convergence, up to a further subsequence (not relabeled), we may assume that  $f_{n_k} = \mathbf{1}_{F_{n_k}}$  converges almost everywhere to  $f$ . But then, for almost every  $x \in \mathbb{R}^n$ ,  $f(x) = \lim_{k \rightarrow \infty} \mathbf{1}_{F_{n_k}}(x) \in \{0, 1\}$ , so that  $f$  is an indicator function, which is a contradiction. Thus for any  $k \geq k_0$ ,  $f_k$  is not an indicator function, hence  $\mathcal{E}_{G,\lambda_k}(f_k) = +\infty$ , and we indeed have

$$+\infty = \liminf_{k \rightarrow \infty} \mathcal{E}_{G,\lambda_k}(f_k) \geq \mathcal{E}_{G,\infty}(f) = +\infty. \quad (4.24)$$

Now we assume that  $f = \mathbf{1}_F$  for some measurable set  $F$ . If  $|F| \neq |B_1|$ , then by  $L^1$  convergence there exists  $k_0$  such that for all  $k \geq k_0$ ,  $\int_{\mathbb{R}^n} f_k(x) dx \neq |B_1|$ , thus  $\mathcal{E}_{G,\lambda_k}(f_k) = +\infty$  and (4.24) holds as well. Hence we may now assume that  $|F| = |B_1|$ . By [Proposition 3.3](#), we have

$$\mathcal{E}_{G,\lambda_k}(f_k) \geq \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) \int_{\mathbb{R}^n} |Df_k|,$$

which trivially holds even if  $f_k \notin \text{BV}(\mathbb{R}^n)$  or  $\int_{\mathbb{R}^n} f_k(x) dx \neq |B_1|$ . Since the BV seminorm is lower semicontinuous with respect to the usual  $L^1$  topology, and  $(1 - (I_G^{0,1} \mathbf{K}_{1,n})/2) > 0$ , we find

$$\liminf_k \mathcal{E}_{G,\lambda_k}(f_k) \geq \left(1 - \frac{I_G^{1,0} \mathbf{K}_{1,n}}{2}\right) \int_{\mathbb{R}^n} |Df| = \mathcal{E}_{G,\infty}(f),$$

where we used the fact that  $\int_{\mathbb{R}^n} f(x) dx = |F| = |B_1|$  for the last equality. Putting these cases together, we get that  $\mathcal{E}_{G,\infty}(f) \leq \Gamma - \liminf \mathcal{E}_{G,\lambda}(f)$ .  $\square$

As usual, the  $\Gamma$ -convergence tells us that any converging sequence of minimizers of the functionals  $\mathcal{E}_{G,\lambda_k}$ , where  $\lambda_k \rightarrow \infty$ , necessarily converges to a minimizer of the  $\Gamma$ -limit, which gives the following corollary.

**Corollary 4.9.** *Let  $(\lambda_k)_{k \in \mathbb{N}}$  be a sequence of positive real numbers going to infinity. If  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ , any sequence of minimizers  $(F_k)_{k \in \mathbb{N}}$  of (RP) for  $\lambda = \lambda_k$  satisfying  $\int_{F_k} x dx = 0$  converges to  $B_1$  for the  $L^1$  norm, that is,*

$$|F_k \triangle B_1| \xrightarrow{k \rightarrow \infty} 0.$$

*Proof.* By minimality of  $F_k$  and Proposition 3.3, we have

$$\left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) P(F_k) \leq \mathcal{F}_{\lambda_k}(F_k) \leq \mathcal{F}_{\lambda_k}(B_1) \leq P(B_1),$$

which yields

$$P(F_k) \leq \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right)^{-1} P(B_1), \quad \forall k \in \mathbb{N},$$

since  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ . Let  $E_k := \lambda_k F_k$ , and  $m_k := \omega_n \lambda_k^n$ . Then by Proposition 3.9,  $(E_k)_{k \in \mathbb{N}}$  is a sequence of minimizers of (P) under the constraint  $|E_k| = m_k$ , where  $m_k$  goes to infinity. For any  $k$  large enough,  $m_k > m_e$ , where  $m_e$  is given by Theorem A, thus  $|E_k \setminus (4[B]_{m_k})| = 0$ , i.e., up to subtracting a negligible set, we can assume  $E_k \subseteq 4[B]_{m_k} = B_{4\lambda_k}$ , hence  $F_k \subseteq B_4$  for all  $k \in \mathbb{N}$ . By compact embedding of  $\text{BV}(\mathbb{R}^n)$  into  $L^1_{\text{loc}}(\mathbb{R}^n)$ , there exists  $f \in \text{BV}(\mathbb{R}^n)$  and a subsequence  $(F_{n_k})_{k \in \mathbb{N}}$  such that  $(\mathbf{1}_{F_{n_k}})_{k \in \mathbb{N}}$  converges in  $L^1(\mathbb{R}^n)$  and almost everywhere to  $f \in \text{BV}(\mathbb{R}^n)$ . The almost-everywhere convergence shows that  $f = \mathbf{1}_F$  for some set of finite perimeter  $F \subseteq B_4$ , and the  $L^1$  convergence shows that  $|F| = |B_1|$ . In addition, still by  $L^1$  convergence, we have  $\int_F x dx = 0$ . Note that for every  $k \in \mathbb{N}$ ,  $\mathbf{1}_{F_{n_k}}$  is a minimizer of  $\mathcal{E}_{G,\lambda_{n_k}}$ , thus, in view of the  $\Gamma$ -convergence result given by Proposition 4.8,  $\mathbf{1}_F$  is a minimizer of the functional  $\mathcal{E}_{G,\infty}$ , that is,  $F$  is a minimizer of the perimeter functional under the constraint  $|F| = |B_1|$ . Since the open unit ball centered at the origin is the unique minimizer (up to a translation) of the perimeter under volume constraint, the facts that  $|F| = |B_1|$  and  $\int_F x dx = 0$  imply  $F = B_1$ . The  $L^1$  convergence of  $\mathbf{1}_{F_{n_k}}$  to  $\mathbf{1}_{B_1}$  simply rewrites  $|F_{n_k} \triangle B_1| \xrightarrow{k \rightarrow \infty} 0$ . Since we could have done the same reasoning for any subsequence of  $(F_k)_{k \in \mathbb{N}}$  from the start, the whole sequence  $(F_k)_{k \in \mathbb{N}}$  actually converges to  $B_1$ .  $\square$

Theorem B follows immediately from Corollary 4.9, by the equivalence with problems (P) and (RP) stated in Proposition 3.9.

#### 4.4. Regularity of minimizers

We address here the question of regularity of minimizers of (P). Applying the extensive uniform regularity theory of *volume-constrained almost-minimizers* for the perimeter developed in [30] (here we prefer the term almost-minimizer over quasi-minimizer, similarly to [12, 13]), we readily obtain uniform  $C^{1,\frac{1}{2}}$ -regularity of the boundary of any minimizer of (P) up to a singular set of Hausdorff dimension at most  $(n - 8)$  (the singular set being empty for  $n \leq 7$ ).



Since sets of finite perimeter are defined up to a Lebesgue negligible set, we shall specify which sense we are giving to the boundary. Here we are not referring to the reduced boundary of  $E$  (which in fact does not have singular points if  $E$  is a minimizer of **(P)**), but rather to the support of the Gauss-Green measure of  $E$ , which is given by

$$\text{spt } \nu_E = \{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < |B_r(x)|, \forall r > 0\}.$$

It is known (see e.g. [25, Proposition 12.19]) that for any set of finite perimeter  $E$ , there exists  $E_0$  an equivalent representative of  $E$  (that is,  $|E \Delta E_0| = 0$ ) such that the topological boundary of  $E_0$  agrees with  $\text{spt } \nu_E = \text{spt } \nu_{E_0}$ . Some authors simply denote  $\text{spt } \nu_E$  by  $\partial E$ , but we refrain from doing so here.

Let us first elaborate on a few notions of minimality and quasi/almost-minimality for the perimeter found in the literature, and recall some of the most remarkable related partial regularity results.

**Definition 4.10.** We say that a set of finite perimeter  $E \subseteq \mathbb{R}^n$  is a local minimizer of the perimeter if for every ball  $B_r(x) \subseteq \mathbb{R}^n$  and for every set of finite perimeter  $F \subseteq \mathbb{R}^n$  such that  $E \Delta F \subset\subset B_r(x)$ , we have

$$P(E; B_r(x)) \leq P(F; B_r(x)).$$

It has been shown in [11], through the framework of area minimizing currents, that for any local minimizer  $E$  of the perimeter,  $\text{spt } \nu_E$  is locally a  $C^{1,\alpha}$ -hypersurface, for some  $\alpha \in (0, 1)$ , outside a set of at most  $(n - 8)$ -Hausdorff dimension (it was already proven before in [28], following [10], that the reduced boundary  $\partial^* E$  of any local minimizer of the perimeter is a  $C^{1,\alpha}$ -hypersurface, and that  $\mathcal{H}^{n-1}(\text{spt } \nu_E \setminus \partial^* E) = 0$ ). Similar partial regularity results were then obtained for sets of finite perimeter with prescribed mean curvature, in a general sense (see [26, 27]), possibly with a volume constraint (see [15, 16]). They were also extended to quasi-minimizers of the perimeter, that is, sets of finite perimeter  $E$  satisfying

$$P(E; B_r(x)) \leq (1 + \omega(r))P(F; B_r(x)),$$

for every ball  $B_r(x) \subseteq \mathbb{R}^n$  and every set of finite perimeter  $F$  such that  $E \Delta F \subset\subset B_r(x)$ , where  $\omega : (0, +\infty) \rightarrow (0, +\infty)$  is an increasing function vanishing in 0.

It was proven in [34] that if  $\omega(r) = Cr^{2\alpha}$  for some  $C > 0$  and  $\alpha \in (0, \frac{1}{2}]$ , then the reduced boundary  $\partial^* E$  of any quasi-minimizer of the perimeter  $E$  w.r.t.  $\omega$  is locally a  $(n-1)$ -dimensional graph of class  $C^{1,\alpha}$ , and the Hausdorff dimension of  $\text{spt } \nu_E \setminus \partial^* E$  is at most  $(n - 8)$  (if  $n \leq 7$ , the singular set is empty). With no other assumption on the function  $\omega$  than the fact that it vanishes at 0, it was then proven in [2] that, outside a singular set of dimension at most  $(n - 8)$ ,  $\text{spt } \nu_E$  is a  $C^{0,\alpha}$ -hypersurface for every  $\alpha \in (0, 1)$ , and that the singular set is still empty for  $n \leq 7$ . This result however says nothing about the  $C^{0,\alpha}$  constant of a quasi-minimizer  $E$ , nor does it say at which scales around a point (outside the singular set) the boundary of  $E$  is the graph of a  $C^{0,\alpha}$  map. Finer, uniform versions of the results of [2] were then obtained by S. Rigot in [31]. It is in particular shown that if  $\omega(r) = Cr^{2\alpha}$  for some  $\alpha \in (0, \frac{1}{2}]$ , for any quasi-minimizer  $E$  w.r.t.  $\omega$ ,  $\text{spt } \nu_E$  is locally a  $(n - 1)$ -dimensional graph of class  $C^{1,\alpha}$  outside a singular set of Hausdorff dimension at most  $(n - 8)$ , and its  $C^{1,\alpha}$ -regularity constant can be bounded depending only on  $n$  and  $\omega$ . In addition, when  $n \leq 7$  there exists  $r_0$  depending only on  $n$  and  $\omega$  such that for every  $x \in \text{spt } \nu_E$ ,  $\text{spt } \nu_E \cap B_r(x)$  is a  $(n - 1)$ -dimensional graph of class  $C^{1,\alpha}$  for some  $r > r_0$ . S. Rigot then studied in [30] regularity of volume-constrained almost-minimizers of the perimeter (defined just below), and as an application obtained regularity



results for minimizers of (P) (using the original equivalent formulation (★)). While in [30] the kernel  $G$  is assumed to be compactly supported, this assumption is only used to get existence of a minimizer, and the regularity results rely only on the integrability of the kernel on  $\mathbb{R}^n$  (not even on the fact that it is radial). Let us now recall just a few of those results.

**Definition 4.11** (Volume-constrained almost-minimizers). Let  $f : [0, +\infty) \rightarrow [0, +\infty)$  be such that  $f(r) = o(r^{\frac{n-1}{n}})$  near 0. We say that a set of finite perimeter  $E$  of volume  $m$  is an *almost-minimizer of the perimeter w.r.t.  $f$*  under the volume constraint  $m$  if for every ball  $B_r(x) \subseteq \mathbb{R}^n$  and every set of finite perimeter  $F$  of volume  $m$  such that  $E \Delta F \subset\subset B_r(x)$ , we have

$$P(E; B_r(x)) \leq P(F; B_r(x)) + f(|E \Delta F|).$$

It is shown in [30, Lemma 5.2.1] that any minimizer of (P) is a volume-constrained almost-minimizer of the perimeter w.r.t.  $f(r) = I_G^{0,0} r$ . It is also a direct consequence of Lemma 3.6. Indeed, we see that if a set  $E$  of volume  $m$  solves (P), then for any set of finite perimeter  $F$  such that  $|F| = m$ , then

$$P(E) \leq P(F) + \text{Per}_G(E) - \text{Per}_G(F) \leq P(F) + I_G^{0,0} |E \Delta F|.$$

It is then proven in [30, Proposition 4.3.1] that the volume constraint can be dropped in the almost-minimality condition, and that any volume-constrained almost-minimizer of the perimeter w.r.t.  $f(r) = Cr^p$ , for  $p > \frac{n-1}{n}$ , is a local quasi-minimizer of the perimeter, in the following sense.

**Definition 4.12** ( $r_0$ -quasi-minimizers of the perimeter). Given  $r_0 > 0$  and  $\omega : (0, r_0) \rightarrow (0, +\infty]$  an increasing function such that  $\omega(r) \xrightarrow{r \rightarrow 0} 0$ , we say that a set of finite perimeter is a  *$r_0$ -quasi minimizer of the perimeter w.r.t.  $\omega$*  if for every ball  $B_r(x) \subseteq \mathbb{R}^n$  such that  $r \in (0, r_0)$  and every set of finite perimeter  $F$  satisfying  $E \Delta F \subset\subset B_r(x)$ , we have

$$P(E; B_r(x)) \leq (1 + \omega(r))P(F; B_r(x)).$$

**Proposition 4.13** ([30, Proposition 4.3.1]). *If  $f(r) = C_1 r^p$  for some  $p > \frac{n-1}{n}$ , and  $E$  is a volume-constrained almost-minimizer of the perimeter of volume  $m$  w.r.t.  $f$ , then there exist  $r_0 = r_0(n, C_1, p, m)$  and  $C_2 = C_2(n, C_1, p, m)$  such that  $E$  is a  $r_0$ -quasi-minimizer of the perimeter w.r.t. to  $\omega(r) := C_2 r^{2\alpha}$ , where  $\alpha := \min(\frac{np - (n-1)}{2}, \frac{1}{2})$ .*

*Remark 4.14.* We could also apply directly results from [35] to get rid of the volume constraint. However, the approach differs from the one in [30] and does not give any control on the radius  $r_0$  of quasi-minimality, which may depend on the minimizer  $E$  considered.

From this are deduced partial  $C^{1,\alpha}$ -regularity results when  $n \geq 8$  and  $C^{1,\alpha}$ -regularity everywhere when  $n \leq 7$  (see [30, Theorems 1.4.8 & 1.4.9], which are consequences of [31, Theorems 2.6 & 6.4]; see also [2, Theorems 4.7 & 4.10]), with regularity constants depending only on  $n, m$  and  $f$ . In our case,  $p = 1$ , and we obtain  $C^{1, \frac{1}{2}}$  regularity. Let us sum up some of the regularity results we obtain in the end for minimizers of (P) (and equivalently of (RP)) in the following theorem.

**Theorem 4.15.** *Let  $E$  be a minimizer<sup>1</sup> of (P) or (RP). Then  $\partial^* E$  is locally a  $(n-1)$ -dimensional graph of class  $C^{1, \frac{1}{2}}$ . In addition, defining*

$$E_0 := \{x \in \mathbb{R}^n : \text{there exists } r > 0 \text{ s.t. } |B_r(x) \cap E| = |B_r(x)|\},$$

<sup>1</sup>If one exists, no matter whether  $I_G^{0,1}$  is lower than  $\frac{2}{\mathbf{k}_{1,n}}$  or not.

$E_0$  is an open set equivalent to  $E$  whose topological connected components coincide with the  $\mathcal{M}$ -connected components of  $E$ , and it is included in some ball  $B_R$ , where  $R$  depends only on  $n$ ,  $m$  (or  $\lambda$ ) and  $I_G^{0,0}$ . If  $n \leq 7$ , then  $\partial E_0 = \partial^* E$ , making the topological boundary of  $E_0$  a  $C^{1,\frac{1}{2}}$ -hypersurface, and if  $n \geq 8$ , then  $\dim_{\mathcal{H}}(\partial E_0 \setminus \partial^* E) \leq n - 8$ . Furthermore, if  $G$  does not have a compact support, then  $E_0$  is connected.

*Proof.* The fact that  $E_0$  is an open set equivalent to  $E$  such that  $\partial E_0 = \text{spt } \nu_E$  is due to [30, Proposition 2.2.1] (see also [31, Lemma 3.6]). By Proposition 4.13 with  $f(r) = I_G^{0,0} r$  (i.e.  $p = 1$ ),  $E$  is a  $r_0$ -quasi-minimizer of the perimeter w.r.t. to  $\omega(r) = Cr$  for some  $C$  and  $r_0$  depending only on  $n$ ,  $m$  and  $I_G^{0,0}$ . Using [34, Theorem 1] (see also [25, Theorem 28.1]), we know that  $\partial^* E$  is a  $C^{1,\frac{1}{2}}$ -hypersurface, with  $\dim_{\mathcal{H}}(\partial E_0 \setminus \partial^* E) \leq n - 8$  whenever  $n > 8$ , and  $\partial E_0 = \partial^* E$  whenever  $n \leq 7$ . By the results of [30, 31] we know that there exists a singular set  $\Sigma(E)$  (which is defined by a condition of mean-flatness, see [31, Section 6] or [2, Definition 4.6]) such that  $\partial E_0 \setminus \Sigma(E)$  is locally made of  $(n - 1)$ -dimensional graphs of class  $C^{1,\frac{1}{2}}$ , where  $\dim_{\mathcal{H}}(\Sigma(E)) \leq n - 8$  for  $n > 8$  and  $\Sigma(E) = \emptyset$  for  $n < 7$  (see [2, Theorem 4.10]). In addition, a look at the proofs in [2] or at [31] shows that the  $C^{1,\frac{1}{2}}$ -regularity constants are bounded depending only on  $n$ ,  $m$  and  $I_G^{0,0}$ . By definition of this singular set and  $C^{1,\frac{1}{2}}$  regularity of  $\partial^* E$ , we know that  $\Sigma(E) \subseteq \partial E_0 \setminus \partial^* E$ , so that the constant of  $C^{1,\frac{1}{2}}$  regularity of  $\partial^* E$  is obviously bounded depending only on  $n$ ,  $m$  and  $I_G^{0,0}$  as well. Since  $E_0$  is open and  $\mathcal{H}^{n-1}(\partial E_0) = \mathcal{H}^{n-1}(\partial^* E)$ , [1, Theorem 2] implies that the  $\mathcal{M}$ -connected components of  $E$  coincide with the topological connected components of  $E_0$ . The fact that  $E_0$  is included in a ball  $B_R$  such that  $R = R(n, m, I_G^{0,0})$  comes from the density estimate

$$|E \cap B_r(x)| \geq c|B_r(x)|, \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in E \text{ and every } 0 < r < r_1, \quad (4.25)$$

where  $c = c(n, m, I_G^{0,0})$ ,  $r_1 = r_1(n, m, I_G^{0,0})$ , which is a consequence of the results in [30]. More precisely, a uniform version of [30, Lemma 2.1.3] is obtained in [30, Section 4.1] (see in particular paragraph 4.1.3 therein), which readily implies (4.25). Eventually, if  $G$  does not have a compact support, by Proposition 4.6,  $E$  has only one  $\mathcal{M}$ -connected component, thus  $E_0$  is connected.  $\square$

## 5. Stability of the ball

### 5.1. First and second variations of perimeters

In this subsection we recall formulas for the first and second variations of the classical and nonlocal perimeters, which can be found e.g. in [12, Section 6]. In all this subsection  $E$  denotes an open set of finite parameter such that  $\partial E$  is a  $C^2$  hypersurface. First we define some terminology.

Given a vector field  $X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ , we define the flow induced by  $X$  as the solution in  $t \in \mathbb{R}$  of the ODEs

$$\begin{cases} \partial_t \Phi_t(x) = X(\Phi_t(x)) \\ \Phi_0(x) = x. \end{cases}$$

It is well known that  $\Phi_t(x)$  is well defined for every  $t \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ , and that  $(\Phi_t)_{t \in \mathbb{R}}$  is a one-parameter group of smooth diffeomorphisms on  $\mathbb{R}^n$ , i.e.  $\Phi_t \circ \Phi_s = \Phi_{s+t}$  for all  $s, t \in \mathbb{R}$ , and  $\Phi_0 = \text{id}_{\mathbb{R}^n}$ . Given  $\Phi_t$  a flow induced by  $X$ , we let  $E_t := \Phi_t(E)$ .

**Definition 5.1.** We say that a vector field  $X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$  induces a volume-preserving flow on  $E$  if there exists  $\delta > 0$  such that  $|E_t| = |E|$  for all  $|t| < \delta$ .

Given a functional  $\mathcal{F}$  on sets of finite perimeter such that  $t \mapsto \mathcal{F}(E_t) \in C^2(-\delta, \delta)$  for some  $\delta > 0$ , we define the first and second variations of  $\mathcal{F}$  at  $E$  in the direction  $X \in C_c^\infty(\mathbb{R}^n)$  by

$$\delta\mathcal{F}(E)[X] := \left[ \frac{d}{dt} \mathcal{F}(E_t) \right]_{|t=0}, \quad \delta^2\mathcal{F}(E)[X] := \left[ \frac{d^2}{dt^2} \mathcal{F}(E_t) \right]_{|t=0}.$$

Then we define the notion of volume-constrained stationary sets for a functional.

**Definition 5.2.** We say that  $E$  is a volume-constrained stationary set for the functional  $\mathcal{F}$  if  $\delta\mathcal{F}(E)[X] = 0$  for every  $X \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$  inducing a volume-preserving flow on  $E$ .

We are interested in the variations of the classical perimeter  $P$  and of the nonlocal perimeter  $\text{Per}_G$ , which we will deduce from the variations of the nonlocal term

$$\mathcal{V}_G(E) := \iint_{E \times E} G(x-y) dx dy = |E|I_G^{0,1} - \text{Per}_G(E).$$

For the classical perimeter, it is known that  $t \mapsto P(E_t)$  is smooth in  $(-\delta, \delta)$  whenever  $E$  is a set of finite perimeter, and if  $\partial E$  is a  $C^2$ -hypersurface, the first variation is

$$\delta P(E)[X] = \int_{\partial E} H_{\partial E} \zeta d\mathcal{H}^{n-1},$$

where  $\nu_E$  is the outer unit normal to  $E$ ,  $\zeta := X \cdot \nu_E$ , and  $H_{\partial E}$  is the scalar mean curvature of  $\partial E$ . The second variation is given by

$$\delta^2 P(E)[X] = \int_{\partial E} |\nabla_\tau \zeta|^2 - c_{\partial E}^2 \zeta^2 d\mathcal{H}^{n-1} + \int_{\partial E} H_{\partial E} ((\text{div } X)\zeta - \text{div}_\tau(\zeta X_\tau)) d\mathcal{H}^{n-1},$$

where  $c_{\partial E}^2(x)$  is the sum of the squares of the principal curvatures of  $\partial E$  at  $x$ ,  $X_\tau := X - \zeta \nu_E$ , and  $\nabla_\tau$  and  $\text{div}_\tau$  denote respectively the tangential gradient and divergence on  $\partial E$ . In addition, if  $E$  is a volume-constrained stationary set for the perimeter, and  $X$  induces a volume-preserving flow on  $E$ , then the second variation of the perimeter takes the simpler form

$$\delta^2 P(E)[X] = \int_{\partial E} |\nabla_\tau \zeta|^2 - c_{\partial E}^2 \zeta^2 d\mathcal{H}^{n-1}. \quad (5.1)$$

Indeed, the fact that  $t \mapsto |E_t|$  is constant in a neighborhood of 0 implies

$$0 = \left[ \frac{d}{dt} |E_t| \right]_{|t=0} = \int_{\partial E} \zeta d\mathcal{H}^{n-1} \quad \text{and} \quad 0 = \left[ \frac{d^2}{dt^2} |E_t| \right]_{|t=0} = \int_{\partial E} (\text{div } X)\zeta d\mathcal{H}^{n-1}.$$

As for the second variation of  $\mathcal{V}_G$ , note that  $G \in C^1(\mathbb{R}^n \setminus \{0\})$ ,  $G(x) = o(|x|^{\alpha-n})$  near the origin, for some  $\alpha > 0$ , and  $G(x) = o(|x|^{-(n+1)})$  at infinity by (H4), thus  $G$  satisfies the assumption of [12, (6.7)], and, since  $E$  is an open set with finite volume such that  $\partial E$  is a  $C^2$ -hypersurface, we can apply [12, Theorem 6.1] directly to get

$$\begin{aligned} \delta\mathcal{V}_G(E)[X] &= \int_{\partial E} H_{G,\partial E}^* \zeta d\mathcal{H}^{n-1}, \\ \delta^2\mathcal{V}_G(E)[X] &= - \iint_{\partial E \times \partial E} G(x-y) |\zeta(x) - \zeta(y)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} + \int_{\partial E} c_{G,\partial E}^2 \zeta^2 d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial E} H_{G,\partial E}^* ((\text{div } X)\zeta - \text{div}_\tau(\zeta X_\tau)) d\mathcal{H}^{n-1}, \end{aligned} \quad (5.2)$$

where

$$c_{G,\partial E}^2(x) := \int_{\partial E} G(x-y) |\nu_E(x) - \nu_E(y)|^2 d\mathcal{H}_y^{n-1}, \quad \forall x \in \partial E,$$

and  $H_{G,\partial E}^*$ , which plays the role of the mean curvature for the nonlocal perimeter  $\text{Per}_G$ , is defined by

$$H_{G,\partial E}^*(x) := 2 \int_E G(x-y) dy, \quad \forall x \in \partial E. \quad (5.3)$$

Note that all the integrals in (5.2) are finite whenever  $\partial E$  is a  $C^2$ -hypersurface by the assumptions on  $G$ , since  $X_\tau$ ,  $\zeta$  and  $\nu_E$  are bounded and  $C^1$  functions. Similarly to the perimeter functional, if  $E$  is a volume-constrained stationary set for  $\mathcal{V}_G$ , and  $X$  induces a volume-preserving flow on  $E$ , the fact that  $t \mapsto |E_t|$  is constant in a neighborhood of 0 implies that the second variation of  $\mathcal{V}_G$  is simply given by

$$\delta^2 \mathcal{V}_G(E)[X] = - \iint_{\partial E \times \partial E} G(x-y) |\zeta(x) - \zeta(y)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} + \int_{\partial E} c_{G,\partial E}^2 \zeta^2 d\mathcal{H}^{n-1}.$$

Recalling that,

$$\text{Per}_G(E_t) = I_G^{0,1} |E_t| - \mathcal{V}_G(E_t), \quad \forall t \in (-\delta, \delta),$$

we see that  $E$  is a volume-constrained stationary set for  $\mathcal{V}_G$  if and only if it is such a set for  $\text{Per}_G$ , and in that case, if  $X$  induces a volume-preserving flow,  $|E_t| = |E|$ , thus the second variation of  $\text{Per}_G$  is given by

$$\begin{aligned} \delta^2 \text{Per}_G(E)[X] &= -\delta^2 \mathcal{V}_G(E)[X] \\ &= \iint_{\partial E \times \partial E} G(x-y) |\zeta(x) - \zeta(y)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} - \int_{\partial E} c_{G,\partial E}^2 \zeta^2 d\mathcal{H}^{n-1}. \end{aligned} \quad (5.4)$$

We end this section by recalling the definition of stability in that setting.

**Definition 5.3.** We say that  $E$  is a volume-constrained stable set for a function  $\mathcal{F}$  if  $E$  is a volume-constrained stationary set for  $\mathcal{F}$ , and  $\delta^2 \mathcal{F}(E)[X] \geq 0$  for every vector field  $X \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$  inducing a volume-preserving flow on  $E$ .

Obviously, if  $E$  is a volume-constrained minimizer for a functional  $\mathcal{F}$ , it is a volume-constrained stable set for  $\mathcal{F}$  in this sense.

## 5.2. The stability threshold

We are interested in the stability of the ball  $[B]_m$  for (P). As before, given  $m > 0$ , we let  $\lambda = \left(\frac{m}{\omega_n}\right)^{\frac{1}{n}}$ .

**Proposition 5.4.** *Let  $m > 0$ . The ball  $[B]_m$  is a volume-constrained stationary set for  $\mathcal{F}_G$ , and the unit ball  $B_1$  is a volume-constrained stationary set for  $\mathcal{F}_{G_\lambda}$ , where  $G_\lambda$  is defined in Proposition 3.9. In addition,  $[B]_m$  is a volume-constrained stable set for  $\mathcal{F}_G$  if and only if*

$B_1$  is stable for  $\mathcal{F}_{G_\lambda}$ . Furthermore, for any vector field  $X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$  inducing a volume-preserving flow on  $B_1$ , the second variation of  $\mathcal{F}_{G_\lambda}$  at  $B_1$  in the direction  $X$  is given by

$$\begin{aligned} & \delta^2 \mathcal{F}_{G_\lambda}(B_1)[X] \\ &= \int_{\mathbb{S}^{n-1}} |\nabla_\tau \zeta|^2 d\mathcal{H}^{n-1} - \frac{I_G^{0,1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|\zeta(x) - \zeta(y)|^2}{|x - y|^2} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \quad + \left( \frac{I_G^{0,1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} c_{G,\lambda,\mathbb{S}^{n-1}}^2 - c_{\partial B_1}^2 \right) \int_{\mathbb{S}^{n-1}} \zeta^2 d\mathcal{H}^{n-1}. \end{aligned}$$

where  $c_{\mathbb{S}^{n-1}}^2 = n - 1$  is the sum of the squares of the principal curvatures of  $\mathbb{S}^{n-1}$ ,

$$c_{G,\lambda,\mathbb{S}^{n-1}}^2 := \int_{\mathbb{S}^{n-1}} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_y^{n-1},$$

and where for all  $r \in (0, +\infty)$  and  $\varepsilon > 0$ , we defined

$$\eta_G(r) := \frac{|\mathbb{S}^{n-1}|}{I_G^{0,1} |\mathbb{S}^{n-2}|} r^2 g(r), \quad \eta_{G,\varepsilon}(r) := \varepsilon^{-(n-1)} \eta_G(\varepsilon^{-1} r).$$

*Proof.* Since balls are minimizers of the perimeter under volume constraint, they are volume-constrained stationary sets for  $P$ . Recall that  $\mathcal{V}_G$  is maximized by balls under volume constraint, so that  $\text{Per}_G$  is minimized by balls under volume constraint, thus balls are stationary sets for  $\text{Per}_G$ <sup>2</sup>. By scaling of  $\text{Per}_G$ , the unit ball minimizes  $\text{Per}_{G_\lambda}$  under volume constraint as well. In the end,  $[B]_m$  is a volume-constrained stationary set for  $\mathcal{F}_G$ , and  $B_1$  is a volume-constrained stationary set for  $\mathcal{F}_{G_\lambda}$ . Applying (5.4) to  $\text{Per}_{G_\lambda}$ , we find

$$\begin{aligned} \delta^2 \text{Per}_{G_\lambda}(B_1)[X] &= \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \lambda^{n+1} G(\lambda(x - y)) |\zeta(x) - \zeta(y)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \quad - \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \lambda^{n+1} G(\lambda(x - y)) |x - y|^2 |\zeta(x)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \end{aligned} \quad (5.5)$$

for every  $X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$  inducing a volume-preserving flow on  $B_1$ . Scaling back, we get

$$\begin{aligned} & \lambda^{n-1} \delta^2 \text{Per}_{G_\lambda}(B_1)[X] \\ &= \iint_{\partial B_\lambda \times \partial B_\lambda} G(x - y) |\lambda X(\lambda^{-1} x) \cdot \nu_{\partial B_\lambda}(x) - \lambda X(\lambda^{-1} y) \cdot \nu_{\partial B_\lambda}(y)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \quad - \iint_{\partial B_\lambda \times \partial B_\lambda} G(x - y) |\nu_{\partial B_\lambda}(x) - \nu_{\partial B_\lambda}(y)|^2 (\lambda X(\lambda^{-1} x) \cdot \nu_{\partial B_\lambda}(x))^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= \iint_{\partial B_\lambda \times \partial B_\lambda} G(x - y) |(X_\lambda \cdot \nu_{\partial B_\lambda})(x) - (X_\lambda \cdot \nu_{\partial B_\lambda})(y)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \quad - \iint_{\partial B_\lambda \times \partial B_\lambda} G(x - y) |\nu_{\partial B_\lambda}(x) - \nu_{\partial B_\lambda}(y)|^2 (X_\lambda \cdot \nu_{\partial B_\lambda})^2(x) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= \delta^2 G_G([B]_m)[X_\lambda], \end{aligned}$$

where  $X_\lambda$  is the vector field defined by  $X_\lambda(x) = \lambda X(\lambda^{-1} x)$ . Obviously we have as well

$$\lambda^{n-1} \delta^2 P(B_1)[X] = \delta^2 P([B]_m)[X_\lambda],$$

<sup>2</sup>One can also notice from (5.3) that  $H_{G,\mathbb{S}^{n-1}}^*$  is constant by symmetry, which directly gives the stationarity of  $B_1$  in view of the expression of the first variation of  $\mathcal{V}_G$  given by (5.2).

thus

$$\delta^2 \mathcal{F}([B]_m)[X_\lambda] = \lambda^{n-1} \delta^2 \mathcal{F}_{G_\lambda}(B_1)[X]. \quad (5.6)$$

Observe that  $X$  induces a volume-preserving flow on  $B_1$  if and only if  $X_\lambda$  induces a volume-preserving flow on  $[B]_m$ . Indeed, the flow induced by  $X_\lambda$ , denoted by  $\Phi_\lambda$ , is given by  $\Phi_{\lambda,t}(x) = \lambda \Phi_t(\lambda^{-1}x)$ , where  $\Phi$  is the flow induced by  $X$ , and it is then easy to see that  $\Phi_{\lambda,t}([B]_m) = \lambda \Phi_t(B_1)$ . Hence with (5.6) we see that  $B_1$  is a volume-constrained stable set for  $\mathcal{F}_{G_\lambda}$  if and only if  $[B]_m$  is stable for  $\mathcal{F}$ . In addition, we can rewrite (5.5) in terms of  $\eta_{G,1/\lambda}$  by

$$\begin{aligned} \delta^2 \text{Per}_{G_\lambda}(B_1)[X] &= \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|\zeta(x) - \zeta(y)|^2}{|x - y|^2} \lambda^{n-1} (\lambda|x - y|)^2 g(\lambda|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\quad - \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \lambda^{n-1} (\lambda|x - y|)^2 g(\lambda|x - y|) \zeta^2(x) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= \frac{I_G^{0,1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \left( \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|\zeta(x) - \zeta(y)|^2}{|x - y|^2} \eta_{G,1/\lambda}(x - y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \right. \\ &\quad \left. - \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \eta_{G,1/\lambda}(x - y) \zeta^2(x) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \right). \end{aligned}$$

Note that

$$c_{G,\lambda,\mathbb{S}^{n-1}}^2 = \int_{\mathbb{S}^{n-1}} \eta_{G,1/\lambda}(x - y) d\mathcal{H}_y^{n-1}$$

does not depend on  $x \in \mathbb{S}^{n-1}$ , since  $\eta_{G,1/\lambda}$  is invariant under rotations, thus by Fubini's theorem we find

$$\begin{aligned} \delta^2 \text{Per}_{G_\lambda}(B_1)[X] &= \frac{I_G^{0,1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \left( \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|\zeta(x) - \zeta(y)|^2}{|x - y|^2} \eta_{G,1/\lambda}(x - y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \right. \\ &\quad \left. - c_{G,\lambda,\mathbb{S}^{n-1}}^2 \int_{\mathbb{S}^{n-1}} \zeta^2 d\mathcal{H}^{n-1} \right), \end{aligned}$$

which concludes the proof.  $\square$

*Remark 5.5.* In view of the expression of  $\delta^2 \mathcal{F}_{G_\lambda}(B_1)[X]$ , it would be natural to define the quadratic functional  $\mathcal{Q}\mathcal{F}_{G_\lambda}(u) := \mathcal{Q}P(u) - \mathcal{Q}\text{Per}_{G_\lambda}(u)$ , where

$$\mathcal{Q}P(u) := \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1} - (n-1) \int_{\mathbb{S}^{n-1}} |u|^2 d\mathcal{H}^{n-1},$$

and

$$\begin{aligned} \mathcal{Q}\text{Per}_{G_\lambda}(u) &:= \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \lambda^{n+1} G(\lambda(x - y)) |u(x) - u(y)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\quad - \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \lambda^{n+1} G(\lambda(x - y)) |x - y|^2 |u(x)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}, \end{aligned}$$

on the vector space of functions  $u \in H^1(\mathbb{S}^{n-1})$  such that  $\int_{\mathbb{S}^{n-1}} u d\mathcal{H}^{n-1} = 0$ , since  $\zeta = X \cdot \nu_{B_1}$  is null-averaged on  $\mathbb{S}^{n-1}$  whenever  $X$  induces a volume-preserving flow on  $B_1$ . Instead of defining the stability of the unit ball by the nonnegativity of the quantity  $\delta^2 \mathcal{F}_{G_\lambda}(B_1)[X]$  for every flow  $X$  inducing a volume-preserving flow on  $B_1$ , we could have defined it by the nonnegativity of the quadratic functional  $\mathcal{Q}\mathcal{F}_{G_\lambda}$  on null-averaged functions in  $H^1(\mathbb{S}^{n-1})$ . In fact, it is interesting to remark that, by the proof of [12, Theorem 7.1], those two notions of stability coincide.

Let us point out that  $\eta_{G,1/\lambda}$  is chosen in such a way that

$$|\mathbb{S}^{n-2}| \int_0^\infty \eta_{G,1/\lambda}(r) r^{n-2} dr = \frac{|\mathbb{S}^{n-1}|}{I_G^{0,1}} \int_0^\infty r g(r) r^{n-1} dr = 1.$$

Similarly to the family  $(\rho_{G,\varepsilon})_{\varepsilon>0}$  introduced in [Proposition 3.9](#) for rewriting  $\text{Per}_{G,\lambda}$ , we will see that the family  $(\eta_{G,\varepsilon})_{\varepsilon>0}$  is an approximation of identity as well, as defined in [Definition 3.10](#) – however, here it is a  $(\mathbf{n} - \mathbf{1})$ -dimensional approximation of identity.

We wish to pass to the limit, but since we integrate  $\mathbb{S}^{n-1}$  instead of  $\mathbb{R}^n$ , we cannot use [Proposition 3.11](#). In [[22](#), Theorem 1.1], an equivalent to this proposition is given for smooth Riemannian manifolds, unfortunately the requirements on the family  $(\eta_{G,\varepsilon})_{\varepsilon>0}$  are too strong to be applicable in our case (in particular, the monotonicity of  $\eta_{G,\varepsilon}$ , for all  $\varepsilon$ , which we do not assume). In [Appendix A](#), we prove the required counterpart of [Proposition 3.11](#) for the sphere, which allows us to prove that large balls may be unstable.

**Theorem 5.6.** *If  $I_G^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$ , there exists  $m_u = m_u(n, G)$  such that for any  $m > m_u$  the ball  $[B]_m$  is an **unstable** critical point of the functional  $\mathcal{F}_G$ .*

*Proof.* As we have seen in [Proposition 5.4](#), given  $m$  and  $\lambda$  positive such that  $\omega_n \lambda^n = m$ , the ball  $[B]_m$  is stable for  $\mathcal{F}_G$  if and only if the unit ball  $B_1$  is stable for  $\mathcal{F}_{G,\lambda}$ . Consider a vector field  $X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$  inducing a volume-preserving flow on  $B_1$  such that  $\delta^2 P(B_1)[X] > 0$ . Let  $\zeta = X \cdot \nu_{B_1}$ . First let us check that the family  $(\eta_{G,\varepsilon})_{\varepsilon>0}$  is indeed a  $(n - 1)$ -dimensional approximation of identity. Since  $g$  is nonnegative, we have  $\eta_{G,\varepsilon} \geq 0$ , and we have already checked

$$|\mathbb{S}^{n-2}| \int_0^\infty \eta_{G,\varepsilon}(r) r^{n-2} dr = 1.$$

In addition, for any  $R > 0$  and some  $C$  depending only on  $n$  and  $g$ , we have

$$\int_R^\infty \eta_{G,\varepsilon}(r) r^{n-2} dr = C \int_{\frac{R}{\varepsilon}}^\infty g(r) r^n dr \xrightarrow{\varepsilon \rightarrow 0} 0,$$

since  $r \mapsto r^n g(r) \in L^1(0, \infty)$  by [\(H1\)](#) and [\(H2\)](#). Thus  $(\eta_{G,\varepsilon})_{\varepsilon>0}$  is a  $(n - 1)$ -dimensional approximation of identity. Let us show that it also satisfies assumption [\(A.41\)](#). Let  $K$  be a compact subset of  $(0, +\infty)$ , and  $a, b > 0$  such that  $K \subseteq (a, b)$ . Then for every  $s \in K$ , using the monotonicity of  $g$ , we have

$$\eta_{G,\varepsilon}(s) = \varepsilon^{-(n+1)} s^2 g(\varepsilon^{-1} s) \leq \varepsilon^{-(n+1)} b^2 g(\varepsilon^{-1} a) = a^{-(n+1)} b^2 \left(\frac{a}{\varepsilon}\right)^{n+1} g\left(\frac{a}{\varepsilon}\right).$$

Recalling that  $g(s) = o(s^{-(n+1)})$  at infinity by [\(H4\)](#), we then see that  $\sup_{s \in K} \eta_{G,\varepsilon}(s)$  goes to 0 as  $\varepsilon$  goes to 0, i.e., the family  $(\eta_{G,\varepsilon})_{\varepsilon>0}$  satisfies [\(A.41\)](#). Thus we can apply [Proposition A.4](#), which gives

$$\lim_{\lambda \rightarrow \infty} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|\zeta(x) - \zeta(y)|^2}{|x - y|^2} \eta_{G,1/\lambda}(x - y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} = \mathbf{K}_{2,n-1} \int_{\mathbb{S}^{n-1}} |\nabla_\tau \zeta|^2 d\mathcal{H}^{n-1}. \quad (5.7)$$

Similarly, we compute

$$\begin{aligned}
c_{G,\lambda,\mathbb{S}^{n-1}}^2 &= \frac{1}{|\mathbb{S}^{n-1}|} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \eta_{G,1/\lambda}(x-y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&= \frac{1}{|\mathbb{S}^{n-1}|} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|x-y|^2}{|x-y|^2} \eta_{G,1/\lambda}(x-y) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&\xrightarrow{\lambda \rightarrow \infty} \frac{\mathbf{K}_{2,n-1}}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} |\nabla \tau x|^2 d\mathcal{H}_x^{n-1} = \mathbf{K}_{2,n-1} c_{\mathbb{S}^{n-1}}^2.
\end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8) with Proposition 5.4, we find

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} \delta^2 \mathcal{F}_{G_\lambda}(B_1)[X] &= \left(1 - I_G^{0,1} \frac{\mathbf{K}_{2,n-1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\right) \left(\int_{\partial E} |\nabla_\tau \zeta|^2 - c_{\partial E}^2 \zeta^2 d\mathcal{H}^{n-1}\right) \\
&= \left(1 - I_G^{0,1} \frac{\mathbf{K}_{2,n-1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\right) \delta^2 P(B_1)[X],
\end{aligned}$$

where we used (5.1) for the last equality. Now by Lemma 3.16 we see that in fact

$$\frac{\mathbf{K}_{2,n-1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} = \frac{\mathbf{K}_{1,n}}{2},$$

thus

$$\lim_{\lambda \rightarrow \infty} \delta^2 \mathcal{F}_{G_\lambda}(B_1)[X] = \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) \delta^2 P(B_1)[X] < 0,$$

since  $I_G^{0,1} > \frac{2}{\mathbf{K}_{1,n}}$  and  $\delta^2 P(B_1)[X] > 0$ . This shows that there exists  $\lambda_u > 0$  such that for any  $\lambda > \lambda_u$ , the unit ball is unstable for  $\mathcal{F}_{G_\lambda}$ , which concludes the proof, setting  $m_u := \omega_n \lambda_u^n$ .  $\square$

We have just seen that if  $I_G^{0,1}$  is above the threshold  $\frac{2}{\mathbf{K}_{1,n}}$ , then balls are unstable for large masses. Conversely, we prove below that if  $I_G^{0,1}$  is below it, then balls are actually stable for large masses, which shows that this stability hreshold is sharp. In order to prove this, we use the decomposition of the Jacobi operator associated with the second variation of  $\mathcal{F}_{G_\lambda}$  in spherical harmonics, and express the second variation in terms of their eigenvalues, showing that for large masses they are all nonnegative.

For  $k \in \mathbb{N}$ , we denote by  $\mathcal{S}_k$  the finite dimensional subspace of  $L^2(\mathbb{S}^{n-1})$  made of the spherical harmonics of degree  $k$ , and  $(Y_k^i)_{1 \leq i \leq d(k)}$  an orthonormal basis of  $\mathcal{S}_k$  in  $L^2(\mathbb{S}^{n-1})$ . When there can be no confusion, we denote by  $Y_k$  a generic element of  $\mathcal{S}_k$ . Let us recall that  $Y_k \in \mathcal{S}_k$  and  $Y_l \in \mathcal{S}_l$  are orthogonal in  $L^2(\mathbb{S}^{n-1})$  whenever  $k \neq l$ , and that the family

$$(Y_k^i)_{\substack{0 \leq k < \infty, \\ 1 \leq i \leq d(k)}}$$

is total, i.e., it is an orthonormal basis of  $L^2(\mathbb{S}^{n-1})$ . For any  $u \in L^2(\mathbb{S}^{n-1})$ , we denote by  $a_k^i(u)$  its  $(i, k)$ -coordinate in the basis, that is,

$$u = \sum_{k=0}^{\infty} \sum_{i=1}^{d(k)} a_k^i(u) Y_k^i. \tag{5.9}$$



For any  $\lambda > 0$ , let us define the operator

$$\mathcal{R}_{G,\lambda}(u)(x) := 2 \int_{\mathbb{S}^{n-1}} \frac{u(x) - u(y)}{|x - y|^2} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_y^{n-1}, \quad \forall u \in C^1(\partial B_1). \quad (5.10)$$

Recall that  $\frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|^2} = Cg(\lambda|x-y|)$  for some  $C = C(n, I_G^{0,1})$ , and that by (H4),  $g(r) = o(r^{\alpha-n})$  at 0 for some  $\alpha > 0$ . We distinguish the two cases  $\alpha \in (1, +\infty)$  and  $\alpha \in (0, 1]$ :

*Case 1.* If  $\alpha \in (1, +\infty)$ , we can see that (5.10) actually defines an operator from  $L^2(\mathbb{S}^{n-1})$  into  $L^2(\mathbb{S}^{n-1})$ . Indeed, in view of Lemma A.3, we have

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|^2} d\mathcal{H}_y^{n-1} &\leq C\lambda^{n+1} \int_{\mathbb{S}^{n-1}} g(\lambda|x-y|) d\mathcal{H}_y^{n-1} \\ &= C\lambda^{n+1} \int_0^2 \left(1 - \frac{r^2}{4}\right)^{\frac{n-3}{2}} g(\lambda r) r^{n-2} dr \\ &\leq C\lambda^{n+1} \int_0^1 r^{n-2} g(\lambda r) dr + C\lambda^{n+1} \|g\|_{L^\infty(\lambda, 2\lambda)}, \end{aligned} \quad (5.11)$$

whose right-hand side is finite since  $r^{n-2}g(\lambda r) = o(r^{\alpha-2})$  is integrable in a neighborhood of 0. Thus, using Cauchy-Schwarz inequality, for any  $u \in L^2(\mathbb{S}^{n-1})$  we find

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} \left( \int_{\mathbb{S}^{n-1}} |u(y)| \frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|^2} d\mathcal{H}_y^{n-1} \right)^2 d\mathcal{H}_x^{n-1} \\ &\leq \int_{\mathbb{S}^{n-1}} \left( \int_{\mathbb{S}^{n-1}} |u(y)|^2 \frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|^2} d\mathcal{H}_y^{n-1} \right) \left( \int_{\mathbb{S}^{n-1}} \frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|^2} d\mathcal{H}_y^{n-1} \right) d\mathcal{H}_x^{n-1} \\ &\leq C \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |u(y)|^2 \frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|^2} d\mathcal{H}_y^{n-1} \\ &\leq C \int_{\mathbb{S}^{n-1}} |u(y)|^2 \left( \int_{\mathbb{S}^{n-1}} \frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|^2} d\mathcal{H}_x^{n-1} \right) d\mathcal{H}_y^{n-1} \leq C \int_{\mathbb{S}^{n-1}} |u(y)|^2 d\mathcal{H}_y^{n-1}. \end{aligned} \quad (5.12)$$

As a consequence, by Fubini's theorem, the integral in (5.10) converges for  $\mathcal{H}^{n-1}$ -almost every  $x \in \mathbb{S}^{n-1}$ , and  $\mathcal{R}_{G,\lambda}(u) \in L^2(\mathbb{S}^{n-1})$  whenever  $u \in L^2(\mathbb{S}^{n-1})$ .

*Case 2.* If  $\alpha \in (0, 1]$ , then  $r^{n-2}g(\lambda r)$  is not necessarily integrable at the origin, so that the corresponding kernel is *hypersingular*, and  $\mathcal{R}_{G,\lambda}(u)$  may not be well defined for every  $u \in L^2(\mathbb{S}^{n-1})$ . Nonetheless, when  $u \in C^1(\mathbb{S}^{n-1})$ , we have

$$\frac{|u(x) - u(y)|}{|x - y|^2} \eta_{G,1/\lambda}(|x - y|) \leq C \|u\|_{C^1(\mathbb{S}^{n-1})} \frac{\eta_{G,1/\lambda}(|x - y|)}{|x - y|},$$

for every  $x, y \in \mathbb{S}^{n-1}$ , so that  $\mathcal{R}_{G,\lambda}(u)(x)$  is still well defined for every  $x \in \mathbb{S}^{n-1}$ , and  $\mathcal{R}_{G,\lambda}(u) \in L^2(\mathbb{S}^{n-1})$ . Indeed, the integral

$$\int_{\mathbb{S}^{n-1}} \frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|} d\mathcal{H}_y^{n-1} \leq C\lambda^{n+1} \int_0^1 r^{n-1} g(\lambda r) dr + C\lambda^{n+1} \|g\|_{L^\infty(\lambda, 2\lambda)}$$

is finite, since  $r^{n-1}g(\lambda r) = o(r^{\alpha-1})$  is integrable in a neighborhood of 0.

In addition, it is easy to see that we have

$$\int_{\mathbb{S}^{n-1}} u \mathcal{R}_{G,\lambda}(u) d\mathcal{H}^{n-1} = \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|u(x) - u(y)|^2}{|x - y|^2} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \quad (5.13)$$

whenever  $u \in C^1(\mathbb{S}^{n-1})$ . Let us recall the so-called Funk-Hecke formula, from which we deduce that spherical harmonics are actually eigenfunctions for the operator  $\mathcal{R}_{G,\lambda}$  in both cases  $\alpha \in (0, 1]$  and  $\alpha \in (1, +\infty)$ .

**Theorem 5.7** (Funk-Hecke formula [32, Theorem 1.7]). *Let  $f : (-1, 1) \rightarrow \mathbb{R}$  be such that*

$$\int_{-1}^1 (1 - t^2)^{\frac{n-3}{2}} |f(t)| dt < \infty.$$

*Then for every  $Y_k \in \mathcal{S}_k$  and every  $x \in \mathbb{S}^{n-1}$ , we have*

$$\int_{\mathbb{S}^{n-1}} f(x \cdot y) Y_k(y) d\mathcal{H}_y^{n-1} = \mu_k Y_k(x), \quad (5.14)$$

where  $\mu_k$  is given by

$$\mu_k = |\mathbb{S}^{n-1}| \int_{-1}^1 P_{n,k}(t) f(t) (1 - t^2)^{\frac{n-3}{2}} dt,$$

and  $P_{n,k}$  denotes the Legendre polynomial of degree  $k$  in dimension  $n$ , that is,

$$P_{n,k}(t) = (-1)^k \frac{\Gamma\left(\frac{n-1}{2}\right)}{2^k \Gamma\left(k + \frac{n-1}{2}\right)} (1 - t^2)^{-\frac{n-3}{2}} \left(\frac{d}{dt}\right)^k (1 - t^2)^{k + \frac{n-3}{2}}.$$

**Corollary 5.8.** *Spherical harmonics are eigenfunctions of the operators  $\mathcal{R}_{G,\lambda}$ . More precisely, for every  $k \in \mathbb{N}$ , there exists  $\mu_{G,\lambda,k} \in \mathbb{R}$  such that*

$$\mathcal{R}_{G,\lambda}(Y_k) = \mu_{G,\lambda,k} Y_k, \quad \forall Y_k \in \mathcal{S}_k.$$

As a consequence, for every  $u \in C^1(\mathbb{S}^{n-1})$ , we have

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|u(x) - u(y)|^2}{|x - y|^2} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} = \sum_{k=0}^{\infty} \sum_{i=1}^{d(k)} \mu_{G,\lambda,k} a_k^i(u)^2, \quad (5.15)$$

and  $\mu_{G,\lambda,k}$  is given by

$$\mu_{G,\lambda,k} = \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|Y_k(x) - Y_k(y)|^2}{|x - y|^2} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}, \quad (5.16)$$

for any  $Y_k \in \mathcal{S}_k$ .

*Proof.* Recall that by Remark 3.1, a consequence of (H1) to (H3) is that  $g(r) = o(r^{-n})$  in 0. The situation being easier when we assume that (H4) holds for some  $\alpha > 1$ , i.e., that  $g(r) = o(r^{\alpha-n})$  at the origin, for pedagogical purposes we first prove that spherical harmonics are eigenfunctions for the operator  $\mathcal{R}_{G,\lambda}$  under this assumption, and we drop it completely only in a second step. In a third step we easily deduce (5.15) and (5.16).

*Step 1.* Here we assume  $g(r) = o(r^{\alpha-n})$  near 0, for some  $\alpha \in (1, +\infty)$ . Then we can consider the operator

$$\mathcal{R}_{G,\lambda}(u)(x) := \int_{\mathbb{S}^{n-1}} \frac{u(y)}{|x-y|^2} \eta_{G,1/\lambda}(|x-y|) d\mathcal{H}_y^{n-1},$$

and proceeding as above in (5.11) and (5.12), we see that it is a bounded linear operator from  $L^2(\mathbb{S}^{n-1})$  into  $L^2(\mathbb{S}^{n-1})$ . Now let us see that we can just use the Funk-Hekke formula. Recalling that  $|x-y| = \sqrt{2(1-x \cdot y)}$ , let us define

$$f_{G,\lambda}(t) := \frac{\eta_{G,1/\lambda}(\sqrt{2(1-t)})}{2(1-t)}, \quad (5.17)$$

so that for any  $Y_k \in \mathcal{S}_k$ , we have

$$\mathcal{R}_{G,\lambda}(Y_k)(x) = \int_{\mathbb{S}^{n-1}} f_{G,\lambda}(x \cdot y) Y_k(y) d\mathcal{H}_y^{n-1}.$$

Changing variables, we have

$$\begin{aligned} \int_{-1}^1 (1-t^2)^{\frac{n-3}{2}} |f_{G,\lambda}(t)| dt &= \int_0^2 \lambda^{n+1} \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} g(\lambda s) s^{n-2} ds \\ &\leq C \lambda^{n+1} \int_0^1 r^{n-2} g(\lambda r) dr + C \lambda^{n+1} \|g\|_{L^\infty(\lambda, 2\lambda)} < +\infty, \end{aligned} \quad (5.18)$$

thus [Theorem 5.7](#) applies and shows that spherical harmonics are eigenfunctions for the operator  $\mathcal{R}_{G,\lambda}$ . In addition, for every  $k \in \mathbb{N}$ , all the  $Y_k \in \mathcal{S}_k$  share the same eigenvalue, which we denote by  $\mu_{G,\lambda,k}^*$ . Taking  $Y_k = 1$  in (5.14), one gets in particular

$$\mu_{G,\lambda,0}^* = \int_{\mathbb{S}^{n-1}} \frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|^2} d\mathcal{H}_y^{n-1}, \quad \forall x \in \mathbb{S}^{n-1},$$

so that  $\mathcal{R}_{G,\lambda}$  is simply given by

$$\mathcal{R}_{G,\lambda} = 2 \left( \mu_{G,\lambda,0}^* \text{Id} - \mathcal{R}_{G,\lambda} \right).$$

Whence we deduce that spherical harmonics are also eigenfunctions for the operator  $\mathcal{R}_{G,\lambda}$ , and the eigenvalue associated with any  $Y_k \in \mathcal{S}_k$  is

$$\mu_{G,\lambda,k} := 2(\mu_{G,\lambda,0}^* - \mu_{G,\lambda,k}^*).$$

*Step 2.* When we only know that  $g(r) = o(r^{-n})$  (by [Remark 3.1](#)), the integral on the right-hand side of (5.18) is not necessarily finite, so that we cannot apply the Funk-Hekke formula directly on  $f_{G,\lambda}$  to prove that spherical harmonic are eigenfunctions of the operator  $\mathcal{R}_{G,\lambda}$ . In fact,  $\mathcal{R}_{G,\lambda}(u)$  is not even well defined for every  $u \in C^\infty(\mathbb{S}^{n-1})$ , since

$$\int_{\mathbb{S}^{n-1}} \frac{\eta_{G,1/\lambda}(|x-y|)}{|x-y|^2} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}$$

may not converge. Instead, we proceed similarly to [32, Lemmas 6.25 & 6.26], defining the operators

$$\begin{aligned} \mathcal{R}_{G,\lambda}^\varepsilon(u)(x) &:= 2 \int_{\mathbb{S}^{n-1} \cap \{|x-y| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^2} \eta_{G,1/\lambda}(|x-y|) d\mathcal{H}_y^{n-1} \\ &= 2 \int_{\mathbb{S}^{n-1}} (u(x) - u(y)) f_{G,\lambda}^\varepsilon(x \cdot y) d\mathcal{H}_y^{n-1}, \end{aligned} \quad (5.19)$$

for every  $\varepsilon \in (0, 1)$ , where

$$f_{G,\lambda}^\varepsilon(t) := f_{G,\lambda}(t)\mathbf{1}_{(0,1-\frac{\varepsilon}{2})}(t),$$

and  $f_{G,\lambda}$  is given by (5.17). Now by introducing this cutoff we have removed the singularity making  $f_{G,\lambda}$  nonintegrable on  $(-1, 1)$  and the operator hypersingular, so that we may now use the Funk-Hekke formula with  $f_{G,\lambda}^\varepsilon$ , which gives

$$\mathcal{R}_{G,\lambda}^\varepsilon(Y_k)(x) = 2\mu_{G,\lambda,k}^\varepsilon Y_k(x), \quad \forall Y_k \in \mathcal{S}_k, \quad \forall x \in \mathbb{S}^{n-1}, \quad (5.20)$$

where

$$\mu_{G,\lambda,k}^\varepsilon = |\mathbb{S}^{n-1}| \int_{-1}^{1-\frac{\varepsilon}{2}} (1 - P_{n,k}(t)) f_{G,\lambda}(t) (1-t^2)^{\frac{n-3}{2}} dt, \quad (5.21)$$

and  $P_{n,k}$  is given by Theorem 5.7. It is known (see references in the proof of [32, Lemma 6.25]) that

$$|P_{n,k}(t) - 1| \leq \frac{k(k+n-2)}{n-1} (1-t), \quad \forall t \in (-1, 1),$$

thus

$$\left| (1 - P_{n,k}(t)) f_{G,\lambda}(t) (1-t^2)^{\frac{n-3}{2}} \right| \leq \frac{k(k+n-2)}{n-1} f_{G,\lambda}(t) (1+t)^{\frac{n-3}{2}} (1-t)^{\frac{n-1}{2}}, \quad (5.22)$$

for all  $t \in (-1, 1)$ . On the one hand, recall that  $f_{G,\lambda}$  is continuous in  $[-1, 1)$ , and  $(1+t)^{\frac{n-3}{2}} \leq C(1+t)^{-\frac{1}{2}}$  in a neighborhood of  $-1$ , for some  $C = C(n)$ , since  $n \geq 2$ , thus the right-hand side of (5.22) is integrable in a neighborhood of  $-1$ . On the other hand,  $f_{G,\lambda}(t) = Cg(\lambda\sqrt{2(1-t)}) = o((1-t)^{-\frac{n}{2}})$  near  $t = 1$ , so that

$$f_{G,\lambda}(t) (1+t)^{\frac{n-3}{2}} (1-t)^{\frac{n-1}{2}} = o((1-t)^{-\frac{1}{2}}) \quad \text{in a neighborhood of } 1,$$

thus the right-hand side of (5.22) is integrable near  $t = 1$  as well. Since this right-hand side is also continuous in  $(-1, 1)$ , in the end it belongs to  $L^1(-1, 1)$ , and by dominated convergence, letting  $\varepsilon$  go to 0 in (5.21) gives

$$\lim_{\varepsilon \rightarrow 0} \mu_{G,\lambda,k}^\varepsilon = |\mathbb{S}^{n-1}| \int_{-1}^1 (1 - P_{n,k}(t)) f_{G,\lambda}(t) (1-t^2)^{\frac{n-3}{2}} dt =: \mu_{G,\lambda,k}.$$

Hence, combining this with (5.20), and passing to the limit in (5.19) (with  $u = Y_k$ ) by dominated convergence yields

$$\mathcal{R}_{G,\lambda}(Y_k) = \mu_{G,\lambda,k} Y_k,$$

which shows that spherical harmonics are eigenfunctions of the operator  $\mathcal{R}_{G,\lambda}$  in that case too. *Step 3.* Using the decomposition of  $u \in L^2(\mathbb{S}^{n-1})$  in the orthonormal basis of spherical harmonics given by (5.9), and the spectral representation of  $\mathcal{R}_{G,\lambda}$  given the previous step, from (5.13) we deduce

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|u(x) - u(y)|^2}{|x - y|^2} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} = \sum_{k=0}^{\infty} \sum_{i=1}^{d(k)} \mu_{G,\lambda,k} a_k^i(u)^2$$

for every  $u \in C^1(\mathbb{S}^{n-1})$ , which is precisely (5.15). Taking  $u = Y_k$  in (5.15) gives (5.16).  $\square$

**Theorem 5.9.** Assume  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ . Then there exists  $m_s = m_s(n, G) > 0$  such that for any  $m > m_s$ , the ball  $[B]_m$  is a volume-constrained stable set for the functional  $\mathcal{F}_G$ .

*Proof.* By Proposition 5.4, given  $m$  and  $\lambda$  positive such that  $\omega_n \lambda^n = m$ , the ball  $[B]_m$  is stable for  $\mathcal{F}_G$  if and only if the unit ball  $B_1$  is stable for  $\mathcal{F}_{G_\lambda}$ . We shall prove that the latter holds provided  $\lambda$  is large enough. Consider a vector field  $X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$  inducing a volume-preserving flow on  $B_1$ . Recall that the second variation of  $\mathcal{F}_{G_\lambda}$  is given by

$$\begin{aligned} & \delta^2 \mathcal{F}_{G_\lambda}(B_1)[X] \\ &= \int_{\mathbb{S}^{n-1}} |\nabla_\tau \zeta|^2 d\mathcal{H}^{n-1} - \frac{I_G^{0,1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|\zeta(x) - \zeta(y)|^2}{|x - y|^2} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \quad + \left( \frac{I_G^{0,1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} c_{G,\lambda,\mathbb{S}^{n-1}}^2 - c_{\partial B_1}^2 \right) \int_{\mathbb{S}^{n-1}} \zeta^2 d\mathcal{H}^{n-1}. \end{aligned} \tag{5.23}$$

It is well known (see e.g. [29]) that for any  $u \in H^1(\mathbb{S}^{n-1})$ , we have

$$\int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1} = \sum_{k=1}^{\infty} \sum_{i=1}^{d(k)} l_k a_k^i(u)^2, \tag{5.24}$$

where

$$l_k = k(k + n - 2), \quad \forall k \in \mathbb{N}.$$

Recall that the coordinate functions  $x \mapsto x_i$  are spherical harmonic of degree 1, thus inserting these functions into (5.16) and summing over  $i$  gives

$$\begin{aligned} \mu_{G,\lambda,1} &= \frac{1}{|\mathbb{S}^{n-1}|} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= \int_{\mathbb{S}^{n-1}} \eta_{G,1/\lambda}(|x - y|) d\mathcal{H}_y^{n-1} = c_{G,\lambda,\mathbb{S}^{n-1}}^2. \end{aligned} \tag{5.25}$$

Furthermore, since  $X$  induces a volume-preserving flow on  $B_1$ , we have

$$\int_{\mathbb{S}^{n-1}} \zeta d\mathcal{H}^{n-1} = 0.$$

Writing

$$\zeta = \sum_{k=0}^{\infty} \sum_{i=1}^{d(k)} a_k^i(\zeta) Y_k^i$$

and using the well-known fact that

$$\int_{\mathbb{S}^{n-1}} Y_k d\mathcal{H}^{n-1} = 0$$

for all spherical harmonic  $Y_k$  of degree  $k > 0$ , it follows that  $a_0^1(\zeta) = 0$ . Combining this with (5.23) to (5.25), and recalling  $c_{\mathbb{S}^{n-1}} = n - 1 = l_1$ , using Corollary 5.8 we find

$$\begin{aligned} \delta^2 \mathcal{F}_{G_\lambda}(B_1)[X] &= \sum_{k=1}^{\infty} \sum_{i=1}^{d(k)} \left( l_k - l_1 - \frac{I_G^{0,1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} (\mu_{G,\lambda,k} - \mu_{G,\lambda,1}) \right) a_k^i(\zeta)^2 \\ &= \sum_{k=2}^{\infty} \sum_{i=1}^{d(k)} \left( l_k - \frac{I_G^{0,1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mu_{G,\lambda,k} + \frac{I_G^{0,1} |\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mu_{G,\lambda,1} - l_1 \right) a_k^i(\zeta)^2. \end{aligned} \tag{5.26}$$

Notice that  $s^{-2}\eta_{G,1/\lambda}(s) = \frac{|\mathbb{S}^{n-1}|}{I_G^{0,1}|\mathbb{S}^{n-2}|}g(s)$ , thus  $s \mapsto s^{-2}\eta_{G,1/\lambda}(s)$  is nonincreasing. Hence by (5.16) and Corollary A.2, for  $\varepsilon > 0$  to be fixed later, there exists  $\lambda_s$  such that, for any  $\lambda > \lambda_s$ , we have

$$\mu_{G,\lambda,k} \leq (\mathbf{K}_{2,n-1} + \varepsilon) \int_{\mathbb{S}^{n-1}} |\nabla_\tau Y_k|^2 d\mathcal{H}^{n-1} = (\mathbf{K}_{2,n-1} + \varepsilon) l_k. \quad (5.27)$$

Using (5.27) and Lemma 3.16, we then have

$$l_k - \frac{I_G^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mu_{G,\lambda,k} \geq \left(1 - \frac{I_G^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mathbf{K}_{2,n-1}(1 + \varepsilon)\right) l_k = \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}(1 + \varepsilon)\right) l_k. \quad (5.28)$$

As we have seen in the proof of Theorem 5.6, (5.8) shows that

$$\mu_{G,\lambda,1} = c_{G,\lambda,\mathbb{S}^{n-1}}^2 = \int_{\mathbb{S}^{n-1}} \eta_{G,1/\lambda}(x-y) d\mathcal{H}_y^{n-1} \xrightarrow{\lambda \rightarrow \infty} \mathbf{K}_{2,n-1} c_{\mathbb{S}^{n-1}}^2 = \mathbf{K}_{2,n-1} l_1,$$

thus up to choosing  $\lambda_s$  even larger, we have

$$\mu_{G,\lambda,1} \geq \mathbf{K}_{2,n-1} l_1(1 - \varepsilon), \quad \forall \lambda > \lambda_s. \quad (5.29)$$

Combining (5.28) and (5.29), it follows

$$\begin{aligned} l_k - \frac{I_G^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mu_{G,\lambda,k} + \frac{I_G^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mu_{G,\lambda,1} - l_1 \\ \geq \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}(1 + \varepsilon)\right) l_k + \left(\frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}(1 - \varepsilon) - 1\right) l_1 \end{aligned} \quad (5.30)$$

for all  $\lambda > \lambda_s$ , where  $\lambda_s$  depends only on  $n, G$  and  $\varepsilon$ . Since  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ , we may choose  $\varepsilon$  small enough depending only on  $n$  and  $I_G^{0,1}$  so that

$$1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}(1 + \varepsilon) > 0,$$

thus, noting that  $l_k$  is bounded from below by  $l_2 > 0$  for all  $k \geq 2$ , because  $(l_k)_{k \in \mathbb{N}}$  is an increasing sequence, (5.30) yields

$$\begin{aligned} l_k - \frac{I_G^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mu_{G,\lambda,k} + \frac{I_G^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mu_{G,\lambda,1} - l_1 \\ \geq \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}(1 + \varepsilon)\right) l_2 + \left(\frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}(1 - \varepsilon) - 1\right) l_1 \\ = \left(1 - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2}\right) (l_2 - l_1) - \frac{I_G^{0,1} \mathbf{K}_{1,n}}{2} (l_1 + l_2) \varepsilon, \end{aligned}$$

whenever  $\lambda > \lambda_s$ , where  $\lambda_s = \lambda_s(n, G, \varepsilon)$ . Whence, choosing  $\varepsilon$  even smaller if needed, depending only on  $n$  and  $I_G^{0,1}$ , we have

$$l_k - \frac{I_G^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mu_{G,\lambda,k} + \frac{I_G^{0,1}|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \mu_{G,\lambda,1} - l_1 > 0,$$

for all  $k \geq 2$  and every  $\lambda > \lambda_s$ , where  $\lambda_s = \lambda_s(n, G)$ , hence  $\delta^2 \mathcal{F}_{G_\lambda}(B_1)[X] \geq 0$  in view of (5.26), and the unit ball  $B_1$  is a volume-constrained stable set for  $\mathcal{F}_{G_\lambda}$ . Setting  $m_s := \omega_n \lambda_s^n$  then gives the result.  $\square$

**Theorem C** is eventually a straightforward consequence of **Theorems 5.6** and **5.9**.

Let us emphasize that **Theorem 5.6** relies heavily on the computation, in **Appendix A**, of the limit of the quantity

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(x - y) dx dy d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}$$

as  $\varepsilon$  goes to 0, for every  $f \in H^1(\mathbb{S}^{n-1})$  and every  $(n - 1)$ -dimensional approximation of identity  $(\eta_\varepsilon)_{\varepsilon > 0}$ , but that this is not enough for proving **Theorem 5.9**. For the latter, we need to obtain an *upper bound* for this quantity, and we need this bound to be *uniform and asymptotically sharp*, in the sense that the multiplicative constant in front of  $\|\nabla_\tau f\|_{H^1(\mathbb{S}^{n-1})}^2$  converges to  $\mathbf{K}_{2,n-1}$  independently of  $f$  and  $\varepsilon$  vanishes.

## A. Asymptotics for nonlocal seminorms on the sphere

In this appendix, we prove Bourgain-Brezis-Mironescu-type results similar to the ones in **Propositions 3.3** and **3.11**, in the case where  $\mathbb{R}^n$  is replaced by the  $(n - 1)$ -dimensional sphere  $\mathbb{S}^{n-1} = \mathbb{S}^{n-1}$ . In [22] the case of a general Riemannian manifold is considered, yet the monotonicity of the radial kernels is required, which is too strong to be applicable in our case. In addition, it does not provide a satisfying bound on the quantity

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(x - y) dx dy d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}, \quad (\text{A.1})$$

which is necessary to conclude that large balls are stable whenever  $I_G^{0,1} < \frac{2}{\mathbf{K}_{1,n}}$ .

Let us remark that we could have treated the general case  $W^{1,p}(\mathbb{S}^{n-1})$  for any  $p \in (1, +\infty)$  the same way. We focused on the case  $p = 2$  since this is the only case we need, and computations are slightly simpler.

In the following, we will often use the following to integrate  $\mathbb{S}^{n-1}$  on slices (see e.g. [4, Corollary A.6]): a  $\mathcal{H}^{n-1}$ -measurable function  $f$  is integrable on  $\mathbb{S}^{n-1}$  if and only  $(x, t) \mapsto (1 - t^2)^{\frac{n-3}{2}} f(\sqrt{1 - t^2}x, t)$  is integrable on  $\mathbb{S}^{n-2} \times (-1, 1)$ , and in that case we have

$$\int_{\mathbb{S}^{n-1}} f d\mathcal{H}^{n-1} = \int_{-1}^1 (1 - t^2)^{\frac{n-3}{2}} \int_{\mathbb{S}^{n-2}} f(\sqrt{1 - t^2}x, t) d\mathcal{H}_x^{n-2} dt. \quad (\text{A.2})$$

**Proposition A.1.** *Let  $\eta : (0, +\infty) \rightarrow [0, +\infty)$  be a nonnegative measurable function such that  $s \mapsto s^{-k}\eta(s)$  is nonincreasing, for some  $k \in \mathbb{N}$ . Then for any  $\varepsilon > 0$ , there exists  $r > 0$  depending only on  $\varepsilon$ ,  $k$  and  $n$  such that*

$$\begin{aligned} & \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ & \leq \left( (1 + \varepsilon) |\mathbb{S}^{n-2}| \int_0^{2r} \eta(s) s^{n-2} ds + \frac{C}{r^2} \|\eta\|_{L^\infty(r,2)} \right) \mathbf{K}_{2,n-1} \|\nabla_\tau f\|_{L^2(\mathbb{S}^{n-1})}^2, \end{aligned} \quad (\text{A.3})$$

for all  $f \in H^1(\mathbb{S}^{n-1})$ , where  $C$  depends only on  $n$ .

*Proof.* Without loss of generality we may assume  $\varepsilon \in (0, \frac{1}{2})$ . By using the density of  $C^1(\mathbb{S}^{n-1})$  in  $H^1(\mathbb{S}^{n-1})$  and Fatou's lemma, we need only find some  $r > 0$  such that (A.3) holds for any  $f \in C^1(\mathbb{S}^{n-1})$ , thus from now on we assume that  $f \in C^1(\mathbb{S}^{n-1})$ . We proceed in 3 steps.

*Step 1.* Let  $r > 0$  to be determined later. We cut the integral into the following two parts

$$\begin{aligned}
& \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&= \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&\quad + \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| \geq r}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} =: I + II.
\end{aligned} \tag{A.4}$$

Defining

$$(f)_{\mathbb{S}^{n-1}} := \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(x) d\mathcal{H}^{n-1},$$

note that

$$\begin{aligned}
II &\leq r^{-2} \|\eta\|_{L^\infty(r,2)} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} |f(x) - f(y)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&= r^{-2} \|\eta\|_{L^\infty(r,2)} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} |f(x) - (f)_{\mathbb{S}^{n-1}} + (f)_{\mathbb{S}^{n-1}} - f(y)|^2 d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
&\leq 2\|\eta\|_{L^\infty(r,2)} |\mathbb{S}^{n-1}| \|f - (f)_{\mathbb{S}^{n-1}}\|_{L^2(\mathbb{S}^{n-1})}^2 r^{-2} \\
&\leq \frac{C}{r^2} \|\eta\|_{L^\infty(r,2)} \|\nabla_\tau f\|_{L^2(\mathbb{S}^{n-1})}^2
\end{aligned} \tag{A.5}$$

for some  $C = C(n)$ , where we used Poincaré's inequality on the sphere for the last inequality. We now focus on  $I$ . We will make use of the stereographic projection  $\Pi : \mathbb{S}^{n-1} \setminus \{e\} \rightarrow \mathbb{R}^{n-1}$  from the north pole  $N := (0, \dots, 0, 1) \in \mathbb{S}^{n-1}$  on the hyperplane  $x_n = -1$ , given by

$$\Pi(x_1, \dots, x_n) = \left( \frac{2x_1}{1 - x_n}, \dots, \frac{2x_{n-1}}{1 - x_n} \right),$$

and its inverse  $\Pi^{-1} : \mathbb{R}^{n-1} \rightarrow \mathbb{S}^{n-1} \setminus \{e\}$ , given by

$$\Pi^{-1}(\xi_1, \dots, \xi_{n-1}) = \left( \frac{\xi_1}{1 + |\xi|^2}, \dots, \frac{\xi_{n-1}}{1 + |\xi|^2}, \frac{|\xi|^2 - 1}{2(1 + |\xi|^2)} \right).$$

It is well known that  $\Pi$  is a smooth diffeomorphism from  $\mathbb{S}^{n-1} \setminus \{e\}$  into  $\mathbb{R}^{n-1}$ , and that the Jacobian determinant of  $\Pi^{-1}$  is given by

$$\mathbf{J}_{\Pi^{-1}}(x) = \frac{1}{(1 + |x|^2)^{n-1}}.$$

Here we use the notation  $B_R(x)$  for balls in  $\mathbb{R}^n$  and  $D_R(x)$  for balls in  $\mathbb{R}^{n-1}$ . Denoting by  $S := (0, \dots, 0, -1) \in \mathbb{S}^{n-1}$  the south pole of  $\mathbb{S}^{n-1}$ , we have  $\Pi(S) = S$ , and for any  $R > 0$ ,  $\Pi$  maps



$B_R(S) \cap \mathbb{S}^{n-1}$  to the open ball  $D_{\alpha(R)} \subseteq \mathbb{R}^{n-1}$ , where  $\alpha(R)$  is a smooth function which goes to 0 as  $R$  goes to 0. Since we may choose  $r$  arbitrarily small, we will see that to estimate  $I$  we can focus on the behavior near the south pole  $S$ , where the stereographic projection is “almost” the identity. Indeed,  $\Pi(S) = S$ , and in addition the differential of  $\Pi^{-1} \in \mathcal{L}(\mathbb{R}^{n-1}, \mathbb{R}^n)$  at 0 is

$$L_0 := D(\Pi^{-1})(0) = \left( \mathbb{R}^{n-1} \ni (h_1, \dots, h_{n-1}) \mapsto (h_1, \dots, h_{n-1}, 0) \right). \quad (\text{A.6})$$

Since  $\Pi^{-1}$  is smooth, there exists  $r_0$  small enough so that

$$\sup_{x \in D_{4r_0}} \|D(\Pi^{-1})(x) - L_0\| \leq \varepsilon, \quad \sup_{x \in D_{4r_0}} \mathbf{J}_{\Pi^{-1}}(x) \geq 1 - \varepsilon, \quad (\text{A.7})$$

and

$$\Pi(B_\rho(S) \cap \mathbb{S}^{n-1}) = D_{\alpha(\rho)} \subseteq D_{2\rho} \subseteq D_{4r_0}, \quad \forall \rho \in (0, 2r_0), \quad (\text{A.8})$$

where for any  $L \in \mathcal{L}(\mathbb{R}^{n-1}, \mathbb{R}^n)$ , we choose the norm

$$\|L\| := \sup_{|h| \leq 1} \frac{|L(x)|}{|x|}.$$

We now split the sphere  $\mathbb{S}^{n-1}$  into small disjoint spherical caps of size smaller than  $r_0$ . Since we will need to extend a little bit those spherical caps, and control by how much they overlap, we want to work only with a finite number of them. Of course, we cannot cover exactly  $\mathbb{S}^{n-1}$  by a finite number of spherical caps, but we can arbitrarily choose the amount of area not covered. Indeed, by Vitali’s covering theorem for the Hausdorff measures, given the family of balls

$$\mathcal{F} := \left\{ B_\rho(x) : \rho \in (0, r_0) \text{ and } x \in \mathbb{S}^{n-1} \right\},$$

there exists a countable family of disjoint balls in  $\mathcal{F}$ , denoted by  $(B_{\rho_j})_{j \in \mathbb{N}}$ , covering  $\mathbb{S}^{n-1}$  in the measure theoretical sense, i.e.,

$$\mathcal{H}^{n-1} \left( \mathbb{S}^{n-1} \setminus \left( \bigsqcup_{j \in \mathbb{N}} B_{\rho_j} \right) \right) = 0.$$

Note that for every  $j \in \mathbb{N}$ ,  $\rho_j < r_0$ . Now we may choose  $M \in \mathbb{N}$  large enough so that

$$\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \setminus U) \leq \varepsilon, \quad (\text{A.9})$$

where

$$U := \bigsqcup_{j=1}^M (B_{\rho_j} \cap \mathbb{S}^{n-1}). \quad (\text{A.10})$$

For this finite “almost” covering of  $\mathbb{S}^{n-1}$ , for any  $\delta > 0$ , we consider

$$V_\delta := ((B_\delta + U) \cap \mathbb{S}^{n-1}) \setminus U \quad (\text{A.11})$$

its “ $\delta$ -overlap” in  $\mathbb{S}^{n-1}$ . By continuity from below of measures, we have  $\mathcal{H}^{n-1}(V_\delta) \rightarrow 0$  as  $\delta$  goes to 0, thus for  $\delta_0$  small enough we have

$$\mathcal{H}^{n-1}(V_{\delta_0}) \leq \frac{|\mathbb{S}^{n-1}|}{M} \varepsilon. \quad (\text{A.12})$$

Let us now choose  $r := \min(r_0, \frac{\delta_0}{2})$  (and note that it does not depend on  $f$ ).

Now that we have built a finite collection of small disjoint spherical caps almost covering  $\mathbb{S}^{n-1}$  in the sense of the Hausdorff measure  $\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1}$ , and have chosen  $\delta_0$  such that the Lebesgue measure of their  $\delta_0$ -overlap is small, we also want to control both the size of  $\mathbb{S}^{n-1} \setminus U$  w.r.t. the measure  $|\nabla_\tau f|^2 \# (\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1})$ , and the size of the  $\delta_0$ -overlap of the covering w.r.t. to the measure

$$\frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}.$$

Unfortunately, the variations of  $f$  may concentrate on those sets, however by an averaging argument, for each  $f$ , we can find a good rotation  $\mathcal{R}_0$  such that the rotated almost-covering captures most of the variations of  $f$ . Indeed, let  $\mu$  be the Haar measure on  $SO(n)$ , the group of rotations on the sphere. Since the application

$$\nu_1 : \mathbb{S}^{n-1} \supseteq A \mapsto \int_{SO(n)} \int_{\mathcal{R}(A)} |\nabla_\tau f(x)|^2 d\mathcal{H}_x^{n-1} d\mu(\mathcal{R})$$

is a Radon measure on  $\mathbb{S}^{n-1}$  invariant under rotations, by uniqueness of the Haar measure on  $\mathbb{S}^{n-1}$  (which is a positive multiple of  $\mathcal{H}^{n-1} \llcorner \mathbb{S}^{n-1}$ ), there exists a constant  $C$  such that

$$\nu_1(A) = C \mathcal{H}^{n-1}(A), \quad \forall A \subseteq \mathbb{S}^{n-1}.$$

Taking  $A = \mathbb{S}^{n-1}$ , we find  $C = \frac{\|\nabla_\tau f\|_{L^2(\mathbb{S}^{n-1})}^2}{|\mathbb{S}^{n-1}|}$ , so that

$$\nu_1(A) = \frac{\mathcal{H}^{n-1}(A)}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} |\nabla_\tau f|^2 d\mathcal{H}^{n-1}, \quad \forall A \subseteq \mathbb{S}^{n-1}, \quad (\text{A.13})$$

hence by (A.12),

$$\begin{aligned} \nu_1(V_{\delta_0}) &= \int_{SO(n)} \int_{\mathcal{R}(V_{\delta_0})} |\nabla_\tau f(x)|^2 d\mathcal{H}_x^{n-1} d\mu(\mathcal{R}) = \frac{\mathcal{H}^{n-1}(V_{\delta_0})}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} |\nabla_\tau f|^2 d\mathcal{H}^{n-1} \\ &\leq \frac{\varepsilon}{M} \int_{\mathbb{S}^{n-1}} |\nabla_\tau f|^2 d\mathcal{H}^{n-1}. \end{aligned} \quad (\text{A.14})$$

Similarly, the application

$$\nu_2 : \mathbb{S}^{n-1} \supseteq A \mapsto \int_{SO(n)} \iint_{\mathcal{R}(A) \times \mathcal{R}(A)} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} d\mu(\mathcal{R}) \quad (\text{A.15})$$

is a Radon measure on  $\mathbb{S}^{n-1}$  invariant by rotations: the fact that  $f \in C^1(\mathbb{S}^{n-1})$  implies that  $|f(x) - f(y)| < \|\nabla_\tau f\|_{L^\infty(\mathbb{S}^{n-1})} |x - y|$ , which justifies that the integral on the right-hand side of (A.15) is always finite (in fact  $\nu_2$  is a finite measure). Thus

$$\nu_2(A) = \frac{\mathcal{H}^{n-1}(A)}{|\mathbb{S}^{n-1}|} \int_{SO(n)} \iint_{\mathcal{R}(A) \times \mathcal{R}(A)} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} d\mu(\mathcal{R}),$$

for all  $A \subseteq \mathbb{S}^{n-1}$ . In particular, by (A.9), we have

$$\begin{aligned} \nu_2(\mathbb{S}^{n-1} \setminus U) &= \int_{SO(n)} \iint_{\mathcal{R}(\mathbb{S}^{n-1} \setminus U) \times \mathcal{R}(\mathbb{S}^{n-1} \setminus U)} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} d\mu(\mathcal{R}) \\ &\leq \varepsilon \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}. \end{aligned} \quad (\text{A.16})$$

In view of (A.14), by Markov's inequality we have

$$\mu \left( \mathcal{R} \in SO(n) : \int_{\mathcal{R}(V_{\delta_0})} |\nabla_\tau f|^2 d\mathcal{H}^{n-1} \geq t \int_{\mathbb{S}^{n-1}} |\nabla_\tau f|^2 d\mathcal{H}^{n-1} \right) \leq \frac{\nu_1(V_{\delta_0})}{t} \leq \frac{\varepsilon}{Mt}$$

for every  $t > 0$ . Taking  $t = \frac{4\varepsilon}{M}$ , this implies

$$\mu \left( \mathcal{R} \in SO(n) : \int_{\mathcal{R}(V_{\delta_0})} |\nabla_\tau f|^2 d\mathcal{H}^{n-1} < \frac{4\varepsilon}{M} \int_{\mathbb{S}^{n-1}} |\nabla_\tau f|^2 d\mathcal{H}^{n-1} \right) \geq \frac{3}{4}. \quad (\text{A.17})$$

Similarly, (A.16) implies that

$$\begin{aligned} \mu \left( \mathcal{R} \in SO(n) : \iint_{\mathcal{R}(\mathbb{S}^{n-1} \setminus U) \times \mathcal{R}(\mathbb{S}^{n-1} \setminus U)} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \right. \\ \left. < 4\varepsilon \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \right) \geq \frac{3}{4}. \end{aligned} \quad (\text{A.18})$$

Combining (A.17) and (A.18) we may thus find some  $\mathcal{R}_0 \in SO(n)$  such that

$$\int_{\mathcal{R}_0(V_{\delta_0})} |\nabla_\tau f|^2 d\mathcal{H}^{n-1} < \frac{4\varepsilon}{M} \int_{\mathbb{S}^{n-1}} |\nabla_\tau f|^2 d\mathcal{H}^{n-1} \quad (\text{A.19})$$

and

$$\begin{aligned} \iint_{\mathcal{R}_0(\mathbb{S}^{n-1} \setminus U) \times \mathcal{R}_0(\mathbb{S}^{n-1} \setminus U)} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ < 4\varepsilon \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}. \end{aligned} \quad (\text{A.20})$$

As a consequence,

$$\begin{aligned} I &\leq (1 - 4\varepsilon)^{-1} \iint_{\substack{\mathcal{R}_0(U) \times \mathcal{R}_0(U) \\ |x-y| < r}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= (1 - 4\varepsilon)^{-1} \sum_{j=1}^M \int_{\mathcal{R}_0(B_{\rho_j}(x_j) \cap \mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1} \cap \{|x-y| < r\}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_y^{n-1} d\mathcal{H}_x^{n-1}. \end{aligned} \quad (\text{A.21})$$

Now we want to estimate the double integral on each of the spherical caps in the sum of (A.21).

*Step 2.* We show that there exists some  $C$  depending only on  $n$  and  $k$  such that, for every  $j \in \{1, \dots, M\}$ , we have

$$\begin{aligned} \int_{\mathcal{R}_0(B_{\rho_j}(x_j) \cap \mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1} \cap \{|x-y| < r\}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_y^{n-1} d\mathcal{H}_x^{n-1} \\ \leq (\mathbf{K}_{2,n-2} + C\varepsilon) \left( |\mathbb{S}^{n-2}| \int_0^{2r} \eta(s) s^{n-2} ds \right) \left( \int_{\mathcal{R}_0(B_{\rho_j+\delta_0}(x_j) \cap \mathbb{S}^{n-1})} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} \right). \end{aligned} \quad (\text{A.22})$$

Given a fixed  $j \in \{1, \dots, M\}$ , there exists  $\mathcal{R}_j \in SO(n)$  such that  $\mathcal{R}_j(\mathcal{R}_0(B_{\rho_j}(x_j) \cap \mathbb{S}^{n-1}) = B_{\rho_j}(S) \cap \mathbb{S}^{n-1}$  (i.e.  $\mathcal{R}_j$  maps the spherical cap of radius  $\rho_j$  around  $\mathcal{R}_0(x_j)$  to the spherical cap of radius  $\rho_j$  around the south pole). Making the change of variables  $x' = \mathcal{R}_j(x)$  and  $y' = \mathcal{R}_j(y)$ , we find

$$\begin{aligned} & \int_{\mathcal{R}_0(B_{\rho_j}(x_j) \cap \mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1} \cap \{|x-y| < r\}} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta(|x-y|) d\mathcal{H}_y^{n-1} d\mathcal{H}_x^{n-1} \\ &= \int_{B_{\rho_j}(S) \cap \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1} \cap \{|x-y| < r\}} \frac{|g_j(x) - g_j(y)|^2}{|x-y|^2} \eta(|x-y|) d\mathcal{H}_y^{n-1} d\mathcal{H}_x^{n-1}, \end{aligned} \quad (\text{A.23})$$

where  $g_j := f \circ \mathcal{R}_j^{-1}$ . Now making the change of variables  $x = \Pi^{-1}(\xi)$  and  $y = \Pi^{-1}(\zeta)$ , it follows

$$\begin{aligned} & \int_{B_{\rho_j}(S) \cap \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1} \cap \{|x-y| < r\}} \frac{|g_j(x) - g_j(y)|^2}{|x-y|^2} \eta(|x-y|) d\mathcal{H}_y^{n-1} d\mathcal{H}_x^{n-1} \\ &= \int_{D_{\alpha(\rho_j)}} \int_{|\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)| < r} \frac{|g_j \circ \Pi^{-1}(\xi) - g_j \circ \Pi^{-1}(\zeta)|^2}{|\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)|^2} \frac{\eta(|\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)|)}{(1 + |\xi|^2)^{n-1} (1 + |\zeta|^2)^{n-1}} d\xi d\zeta \\ &\leq \int_{D_{\alpha(\rho_j)}} \int_{|\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)| < r} \frac{|g_j \circ \Pi^{-1}(\xi) - g_j \circ \Pi^{-1}(\zeta)|^2}{|\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)|^2} \eta(|\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)|) d\xi d\zeta, \end{aligned} \quad (\text{A.24})$$

where  $\alpha(\rho_j)$  is such that  $D_{\alpha(\rho_j)} = \Pi(B_{\rho_j}(S) \cap \mathbb{S}^{n-1}) \subseteq D_{2\rho_j} \subseteq D_{4r_0}$  by (A.8). Since  $|\Pi^{-1}(\zeta) - S| < r + \rho_j < 2r_0$  whenever  $\Pi^{-1}(\xi) \in B_{\rho_j}(S) \cap \mathbb{S}^{n-1}$  and  $|\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)| < r < r_0$ , we also have  $\zeta \in D_{4r_0}$  by (A.8), thus  $\xi$  and  $\zeta$  both belong to  $D_{4r_0}$ . Given  $\xi, \zeta \in D_{4r_0}$ , by Taylor's theorem we have

$$\Pi^{-1}(\xi) = \Pi^{-1}(\zeta) + \int_0^1 D(\Pi^{-1})(\zeta + t(\xi - \zeta))(\xi - \zeta) dt$$

thus, using the reverse triangle inequality

$$\begin{aligned} \left| |\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)| - |L_0(\xi - \zeta)| \right| &\leq \int_0^1 |D(\Pi^{-1})(\zeta + t(\xi - \zeta))(\xi - \zeta) - L_0(\xi - \zeta)| dt \\ &\leq \left( \sup_{x \in D_{4r_0}} \|(D(\Pi^{-1})(x) - L_0)\| \right) |\xi - \zeta| \leq \varepsilon |\xi - \zeta|, \end{aligned}$$

in view of (A.7), where  $L_0$  is the linear map defined in (A.6). As a consequence, we have

$$(1 - \varepsilon)|\xi - \zeta| \leq |\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)| \leq (1 + \varepsilon)|\xi - \zeta|, \quad \forall \xi, \zeta \in D_{4r_0}. \quad (\text{A.25})$$

By assumption, recall that  $s \mapsto s^{-k} \eta(s)$  is nonincreasing, thus for every  $\xi, \zeta \in D_{4r_0}$  such that  $\xi \neq \zeta$ , using (A.25), it follows

$$\begin{aligned} \eta(|\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)|) &\leq |\Pi^{-1}(\xi) - \Pi^{-1}(\zeta)|^k ((1 - \varepsilon)|\xi - \zeta|)^{-k} \eta((1 - \varepsilon)|\xi - \zeta|) \\ &\leq \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right)^k \eta((1 - \varepsilon)|\xi - \zeta|). \end{aligned} \quad (\text{A.26})$$

Plugging (A.25) and (A.26) into (A.24) yields

$$\begin{aligned}
& \int_{B_{\rho_j}(S) \cap \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1} \cap \{|x-y| < r\}} \frac{|g_j(x) - g_j(y)|^2}{|x-y|^2} \eta(|x-y|) d\mathcal{H}_y^{n-1} d\mathcal{H}_x^{n-1} \\
& \leq \frac{(1+\varepsilon)^k}{(1-\varepsilon)^{k+2}} \int_{D_{\alpha(\rho_j)}} \int_{(1-\varepsilon)|\xi-\zeta| < r} \frac{|g_j \circ \Pi^{-1}(\xi) - g_j \circ \Pi^{-1}(\zeta)|^2}{|\xi-\zeta|^2} \eta((1-\varepsilon)|\xi-\zeta|) d\xi d\zeta \quad (\text{A.27}) \\
& \leq \frac{(1+\varepsilon)^k}{(1-\varepsilon)^{k+2}} \int_{D_{\alpha(\rho_j)}} \int_{|\xi-\zeta| < 2r} \frac{|g_j \circ \Pi^{-1}(\xi) - g_j \circ \Pi^{-1}(\zeta)|^2}{|\xi-\zeta|^2} \eta((1-\varepsilon)|\xi-\zeta|) d\xi d\zeta.
\end{aligned}$$

Since  $g_j \in C^1(\mathbb{S}^{n-1})$  and  $\Pi^{-1}$  is smooth, by Taylor's theorem we have

$$\begin{aligned}
|g_j \circ \Pi^{-1}(\xi) - g_j \circ \Pi^{-1}(\zeta)|^2 & \leq \int_0^1 |\nabla_\tau g_j(\Pi^{-1}(\xi + t(\zeta - \xi))) \\
& \quad \cdot (D(\Pi^{-1})(\xi + t(\zeta - \xi)) \cdot (\xi - \zeta))|^2 dt. \quad (\text{A.28})
\end{aligned}$$

Writing temporarily  $z = \xi + t(\zeta - \xi) \in D_{4r_0}$ , notice that

$$\begin{aligned}
& |\nabla_\tau g_j(\Pi^{-1}(z)) \cdot (D(\Pi^{-1})(z) \cdot (\xi - \zeta))|^2 \\
& = |\nabla_\tau g_j(\Pi^{-1}(z)) \cdot ((D(\Pi^{-1})(z) - L_0 + L_0)(\xi - \zeta))|^2 \\
& = \left| \nabla_\tau g_j(\Pi^{-1}(z)) \cdot (\xi - \zeta) \right|^2 + \left| \nabla_\tau g_j(\Pi^{-1}(z)) \cdot ((D(\Pi^{-1})(z) - L_0)(\xi - \zeta)) \right|^2 \\
& \quad - 2 \left[ \nabla_\tau g_j(\Pi^{-1}(z)) \cdot ((D(\Pi^{-1})(z) - L_0)(\xi - \zeta)) \right] \left[ \nabla_\tau g_j(\Pi^{-1}(z)) \cdot (\xi - \zeta) \right] \\
& \leq \left| \nabla_\tau g_j(\Pi^{-1}(z)) \cdot (\xi - \zeta) \right|^2 + 3\varepsilon |\nabla_\tau g_j(\Pi^{-1}(z))|^2 |\xi - \zeta|^2
\end{aligned} \quad (\text{A.29})$$

where we used (A.7) for the last inequality. Plugging (A.28) and (A.29) into (A.27) gives

$$\begin{aligned}
& \int_{B_{\rho_j}(S) \cap \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1} \cap \{|x-y| < r\}} \frac{|g_j(x) - g_j(y)|^2}{|x-y|^2} \eta(|x-y|) d\mathcal{H}_y^{n-1} d\mathcal{H}_x^{n-1} \\
& \leq \frac{(1+\varepsilon)^k}{(1-\varepsilon)^{k+2}} (J + JJ), \quad (\text{A.30})
\end{aligned}$$

where

$$J := \int_0^1 \int_{D_{\alpha(\rho_j)}} \int_{|\xi-\zeta| < 2r} \left| \nabla_\tau g_j(\Pi^{-1}(\xi + t(\zeta - \xi))) \cdot \frac{\zeta - \xi}{|\zeta - \xi|} \right|^2 \eta((1-\varepsilon)|\xi - \zeta|) d\zeta d\xi dt \quad (\text{A.31})$$

and

$$JJ := 3\varepsilon \int_0^1 \int_{D_{\alpha(\rho_j)}} \int_{|\xi-\zeta| < 2r} |\nabla_\tau g_j(\Pi^{-1}(\xi + t(\zeta - \xi)))|^2 \eta((1-\varepsilon)|\xi - \zeta|) d\zeta d\xi dt. \quad (\text{A.32})$$

Making the change of variables  $h = \zeta - \xi$  (for fixed  $\xi$ ), and then  $z = \xi + th$  in  $\xi$  (for fixed  $t$

and  $h$ ), (A.32) becomes

$$\begin{aligned}
JJ &\leq 3\varepsilon \int_0^1 \int_{D_{\alpha(\rho_j)}} \int_{|h|<2r} |\nabla_\tau g_j(\Pi^{-1}(\xi + th))|^2 \eta((1-\varepsilon)|h|) dh d\xi dt \\
&\leq 3\varepsilon \int_0^1 \int_{|h|<2r} \int_{D_{\alpha(\rho_j)+2r}} |\nabla_\tau g_j(\Pi^{-1}(z))|^2 \eta((1-\varepsilon)|h|) dz dh dt \\
&\leq \frac{3\varepsilon}{1-\varepsilon} \left( \int_{|h|<2r} \eta((1-\varepsilon)|h|) dh \right) \left( \int_{D_{\alpha(\rho_j)+2r}} |\nabla_\tau g_j(\Pi^{-1}(z))|^2 \mathbf{J}_{\Pi^{-1}}(z) dz \right) \\
&= \frac{3\varepsilon}{(1-\varepsilon)^n} \left( \int_{|h|<2(1-\varepsilon)r} \eta(|h|) dh \right) \left( \int_{\Pi^{-1}(D_{\alpha(\rho_j)+2r})} |\nabla_\tau g_j(x)|^2 d\mathcal{H}^{n-1} \right),
\end{aligned} \tag{A.33}$$

where we used (A.7) for the second to last inequality and the change of variables  $h \mapsto (1-\varepsilon)h$  and  $z \mapsto \Pi^{-1}(z)$  for the last equality. For any  $y_1 \in \Pi^{-1}(D_{\alpha(\rho_j)+2r})$ , there exist  $x_1 \in D_{\alpha(\rho_j)}$  and  $x_2 \in D_{2r}$  such that  $y_1 = \Pi^{-1}(x_1 + x_2)$ . Then by definition of  $\alpha(\rho_j)$ ,  $\Pi^{-1}(D_{\alpha(\rho_j)}) = B_{\rho_j}(S) \cap \mathbb{S}^{n-1}$ , so that  $\Pi^{-1}(x_1) \in B_{\rho_j}(S) \cap \mathbb{S}^{n-1}$ , and we have

$$\begin{aligned}
|y_1 - S| &= |\Pi^{-1}(x_1 + x_2) - S| \\
&\leq |\Pi^{-1}(x_1) - S| + |\Pi^{-1}(x_1 + x_2) - \Pi^{-1}(x_1)| \leq \rho_j + |x_2| \leq \rho_j + 2r,
\end{aligned}$$

since  $\Pi^{-1}$  is 1-Lipschitz. Thus  $\Pi^{-1}(D_{\alpha(\rho_j)+2r}) \subseteq B_{\rho_j+2r}(S) \cap \mathbb{S}^{n-1}$ , and from (A.33) it follows

$$JJ \leq \frac{3\varepsilon}{(1-\varepsilon)^n} \left( |\mathbb{S}^{n-2}| \int_0^{2r} \eta(s) s^{n-2} ds \right) \left( \int_{B_{\rho_j+\delta_0}(S) \cap \mathbb{S}^{n-1}} |\nabla_\tau g_j(x)|^2 d\mathcal{H}^{n-1} \right), \tag{A.34}$$

since  $2r < \delta_0$ . To estimate  $J$ , we make the same change of variables as for  $JJ$ , i.e.,  $h = \zeta - \xi$  (for fixed  $\xi$ ) and then  $z = \xi + th$  in  $\xi$  (for fixed  $t$  and  $h$ ), which gives

$$\begin{aligned}
J &= \int_0^1 \int_{D_{\alpha(\rho_j)}} \int_{D_{2r}} \left| \nabla_\tau g_j(\Pi^{-1}(\xi + th)) \cdot \frac{h}{|h|} \right|^2 \eta((1-\varepsilon)|h|) dh d\xi dt \\
&\leq \int_{D_{2r}} \int_{D_{\alpha(\rho_j)+2r}} \left| \nabla_\tau g_j(\Pi^{-1}(z)) \cdot \frac{h}{|h|} \right|^2 \eta((1-\varepsilon)|h|) dz dh.
\end{aligned} \tag{A.35}$$

Then using the coarea formula w.r.t. to the  $h$  variable, from (A.35) and by definition of  $\mathbf{K}_{2,n-1}$  in (1.1), we infer

$$\begin{aligned}
J &\leq |\mathbb{S}^{n-2}| \int_{D_{\alpha(\rho_j)+2r}} \int_0^{2r} \left( \int_{\mathbb{S}^{n-2}} \left| \nabla_\tau g_j(\Pi^{-1}(z)) \cdot \sigma \right|^2 d\mathcal{H}_\sigma^{n-2} \right) s^{n-2} \eta((1-\varepsilon)s) ds dz \\
&= \mathbf{K}_{2,n-2} |\mathbb{S}^{n-2}| \int_{D_{\alpha(\rho_j)+2r}} \int_0^{2r} \left| \nabla_\tau g_j(\Pi^{-1}(z)) \right|^2 s^{n-2} \eta((1-\varepsilon)s) ds dz \\
&\leq \frac{\mathbf{K}_{2,n-2}}{(1-\varepsilon)^{n-1}} \left( |\mathbb{S}^{n-2}| \int_0^{2(1-\varepsilon)r} \eta(s) s^{n-2} ds \right) \left( \int_{D_{\alpha(\rho_j)+2r}} \left| \nabla_\tau g_j(\Pi^{-1}(z)) \right|^2 dz \right).
\end{aligned} \tag{A.36}$$

Proceeding as we did for  $JJ$ , making the change of variables  $x = \Pi^{-1}(z)$ , using (A.7) and recalling that  $2r < \delta_0$ , (A.36) becomes

$$J \leq \frac{\mathbf{K}_{2,n-2}}{(1-\varepsilon)^n} \left( |\mathbb{S}^{n-2}| \int_0^{2s} \eta(s) r^{n-2} ds \right) \left( \int_{B_{\rho_j+\delta_0}(S) \cap \mathbb{S}^{n-1}} |\nabla_\tau g_j(x)|^2 d\mathcal{H}^{n-1} \right). \quad (\text{A.37})$$

Now combining the estimates on  $J$  and  $JJ$  given respectively by (A.37) and (A.34), from (A.23) and (A.30) it follows

$$\begin{aligned} & \int_{\mathcal{R}_0(B_{\rho_j}(x_j) \cap \mathbb{S}^{n-1})} \int_{\partial B_1 \cap \{|x-y| < r\}} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta(|x-y|) d\mathcal{H}_y^{n-1} d\mathcal{H}_x^{n-1} \\ & \leq \frac{(1+\varepsilon)^k}{(1-\varepsilon)^{n+k+2}} (\mathbf{K}_{2,n-2} + 3\varepsilon) \left( |\mathbb{S}^{n-2}| \int_0^{2r} \eta(s) s^{n-2} ds \right) \\ & \quad \left( \int_{B_{\rho_j+\delta_0}(S) \cap \mathbb{S}^{n-1}} |\nabla_\tau g_j(x)|^2 d\mathcal{H}^{n-1} \right) \\ & \leq (\mathbf{K}_{2,n-2} + C\varepsilon) \left( |\mathbb{S}^{n-2}| \int_0^{2r} \eta(s) s^{n-2} ds \right) \left( \int_{\mathcal{R}_0(B_{\rho_j+\delta_0}(x_j) \cap \mathbb{S}^{n-1})} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} \right), \end{aligned} \quad (\text{A.38})$$

for some constant  $C$  depending only on  $n$  and  $k$  (since we assumed  $\varepsilon \in (0, \frac{1}{2})$ ), and where we made the change of variables  $x = \mathcal{R}_j(y)$  for the last inequality. This is precisely (A.22), which concludes this step.

*Step 3.* Now that we have estimated the nonlocal energy on each small spherical cap composing  $U$ , we want to sum over them to retrieve an estimation of the energy on the whole sphere. Recalling that  $(B_{\rho_j+\delta_0}(x_j) \setminus B_{\rho_j}(x_j)) \cap \mathbb{S}^{n-1} \subseteq V_{\delta_0}$  (by definition of  $V_\delta$  in (A.11)), and using (A.19), we have

$$\begin{aligned} & \int_{\mathcal{R}_0(B_{\rho_j+\delta_0}(x_j) \cap \mathbb{S}^{n-1})} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} \\ & = \int_{\mathcal{R}_0(B_{\rho_j}(x_j) \cap \mathbb{S}^{n-1})} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} + \int_{\mathcal{R}_0(B_{\rho_j+\delta_0}(x_j) \setminus B_{\rho_j}(x_j)) \cap \mathbb{S}^{n-1}} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} \\ & \leq \int_{\mathcal{R}_0(B_{\rho_j}(x_j) \cap \mathbb{S}^{n-1})} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} + \frac{4\varepsilon}{M} \int_{\mathbb{S}^{n-1}} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1}. \end{aligned} \quad (\text{A.39})$$

Then recalling (A.10), summing (A.39) over  $j \in \{1, \dots, M\}$  gives

$$\begin{aligned} \sum_{j=1}^M \int_{\mathcal{R}_0(B_{\rho_j+\delta_0}(x_j) \cap \mathbb{S}^{n-1})} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} & \leq \int_U |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} + 4\varepsilon \int_{\mathbb{S}^{n-1}} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} \\ & \leq (1 + 4\varepsilon) \int_{\mathbb{S}^{n-1}} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1}. \end{aligned} \quad (\text{A.40})$$

Inserting (A.38) and (A.40) into (A.21) leads to

$$\begin{aligned} I &\leq \left( \frac{1+4\varepsilon}{1-4\varepsilon} \right) (\mathbf{K}_{2,n-1} + C\varepsilon) \left( |\mathbb{S}^{n-2}| \int_0^{2r} \eta(s) s^{n-2} ds \right) \left( \int_{\mathbb{S}^{n-1}} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} \right) \\ &\leq (\mathbf{K}_{2,n-1} + C\varepsilon) \left( |\mathbb{S}^{n-2}| \int_0^{2r} \eta(s) s^{n-2} ds \right) \left( \int_{\mathbb{S}^{n-1}} |\nabla_\tau f(x)|^2 d\mathcal{H}^{n-1} \right), \end{aligned}$$

for some  $C = C(n, k)$ . Thus with (A.4) and (A.5), we obtain

$$\begin{aligned} &\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\leq \left( (1 + C_1\varepsilon) |\mathbb{S}^{n-2}| \int_0^{2r} \eta(s) s^{n-2} ds + \frac{C_2}{r^2} \|\eta\|_{L^\infty(r,2)} \right) \mathbf{K}_{2,n-1} \|\nabla_\tau f\|_{L^2(\partial B_1)}^2, \end{aligned}$$

for some constants  $C_1 = C_1(n, k)$  and  $C_2 = C_2(n)$ , hence the result since  $C_1$  depends only on  $n$  and  $k$ .  $\square$

We also have the following useful corollary for  $(n-1)$ -dimensional approximations of identity.

**Corollary A.2.** *Let  $(\eta_\varepsilon)_{\varepsilon>0}$  be a  $(n-1)$ -dimensional approximation of identity satisfying*

$$\sup_K \eta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \text{for every compact set } K \subseteq (0, +\infty), \quad (\text{A.41})$$

and such that  $s \mapsto s^{-k} \eta_\varepsilon(s)$  is nonincreasing for some  $k \in \mathbb{N}$  and for every  $\varepsilon > 0$ . Then for every  $\delta > 0$ , there exists  $\varepsilon_0 > 0$  such that, for every  $0 < \varepsilon < \varepsilon_0$  and every  $f \in H^1(\mathbb{S}^{n-1})$ , we have

$$\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(|x - y|) \leq (\mathbf{K}_{2,n-1} + \delta) \int_{\mathbb{S}^{n-1}} |\nabla_\tau f|^2 d\mathcal{H}^{n-1}, \quad (\text{A.42})$$

In particular, the maps  $N_\varepsilon : H^1(\mathbb{S}^{n-1}) \rightarrow [0, +\infty)$  defined by

$$N_\varepsilon(f) := \left( \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(|x - y|) dx dy \right)^{\frac{1}{2}}$$

are uniformly equicontinuous for every  $\varepsilon < \varepsilon_0$ .

*Proof.* Let  $r = r(n, k, \delta) > 0$  given by Proposition A.1 for  $\varepsilon = \frac{\delta}{2\mathbf{K}_{2,n-1}}$ , so that we have

$$\begin{aligned} &\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\leq \left[ \left( \mathbf{K}_{2,n-1} + \frac{\delta}{2} \right) + \frac{C}{r^2} \|\eta_\varepsilon\|_{L^\infty(r,2)} \right] \|\nabla_\tau f\|_{L^2(\mathbb{S}^{n-1})}^2, \end{aligned}$$

where  $C = C(n)$ . By (A.41), there exists  $\varepsilon_0 > 0$  such that

$$\frac{C}{r^2} \|\eta_\varepsilon\|_{L^\infty(r,2)} \leq \frac{\delta}{2}, \quad \forall \varepsilon \in (0, \varepsilon_0),$$



hence (A.42) holds for every  $\varepsilon < \varepsilon_0$  and every  $f \in H^1(\mathbb{S}^{n-1})$ . As for the continuity of  $N_\varepsilon$ , using the reverse triangle inequality and applying (A.42), we find

$$|N_\varepsilon(g) - N_\varepsilon(h)| \leq N_\varepsilon(g - h) \leq (\mathbf{K}_{2,n-1} + \delta)^{\frac{1}{2}} \|g - h\|_{H^1(\mathbb{S}^{n-1})},$$

for all  $g, h \in H^1(\mathbb{S}^{n-1})$  and all  $\varepsilon < \varepsilon_0$ , hence the uniform equicontinuity of the maps  $N_\varepsilon$ .  $\square$

We can now compute the limit of (A.1), which is a counterpart of Proposition 3.11 for the sphere. But first, let us state and prove the following basic lemma.

**Lemma A.3.** *Let  $f$  be a continuous nonnegative function from  $(0, +\infty)$  such that*

$$\int_0^\infty f(r)r^{n-2} dr < +\infty.$$

*Then for every  $R > 0$  and every  $x \in \partial B_R$ , the map  $F : y \mapsto f(|x - y|)$  belongs to  $L^1(\partial B_R)$  for any  $R > 0$ , and we have*

$$\int_{\partial B_R} f(|x - y|) d\mathcal{H}_y^{n-1} = |\mathbb{S}^{n-2}|R \int_0^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{\frac{n-3}{2}} s^{n-2} f(s) ds.$$

*In addition, if  $n \geq 3$ , we have*

$$\int_{\partial B_R} f(|x - y|) d\mathcal{H}_y^{n-1} \leq |\mathbb{S}^{n-2}|R \int_0^{\sqrt{4R}} s^{n-2} f(s) ds,$$

*and if  $n = 2$ ,*

$$\int_{\partial B_R} f(|x - y|) d\mathcal{H}_y^{n-1} \leq |\mathbb{S}^{n-2}|R \left( \frac{1}{\sqrt{2}} \int_0^{\sqrt{2R}} f(s)s^{n-2} ds + \frac{\pi\sqrt{R}}{2} \|f\|_{L^\infty(\sqrt{2R}, \sqrt{4R})} \right).$$

*Proof.* Up to a change of variables, we can assume that  $x = N = (0, \dots, 0, 1)$  is the ‘‘north pole’’. Our computations will show that  $y \mapsto f(|RN - y|) \in L^1(\partial B_R)$ . Applying (A.2) to  $y \mapsto f(R|N - y|)$  we find

$$\begin{aligned} \int_{\partial B_R} f(|RN - y|) d\mathcal{H}_y^{n-1} &= R^{n-1} \int_{\mathbb{S}^{n-1}} f(R|N - y|) d\mathcal{H}_y^{n-1} \\ &= R^{n-1} \int_{-1}^1 (1 - t^2)^{\frac{n-3}{2}} \int_{\mathbb{S}^{n-2}} f(R|N - (\sqrt{1 - t^2}y, t)|) d\mathcal{H}_y^{n-2} dt \\ &= R^{n-1} \int_{-1}^1 (1 - t^2)^{\frac{n-3}{2}} \int_{\mathbb{S}^{n-2}} f(\sqrt{2R(1 - t)}) d\mathcal{H}_y^{n-2} dt \\ &= |\mathbb{S}^{n-2}|R^{n-1} \int_{-1}^1 (1 - t^2)^{\frac{n-3}{2}} f(\sqrt{2R(1 - t)}) dt. \end{aligned}$$

Changing variables with  $s = \sqrt{2R(1 - t)}$ , it follows

$$\begin{aligned} \int_{\partial B_R} f(|RN - y|) d\mathcal{H}_y^{n-1} &= |\mathbb{S}^{n-2}|R^{n-1} \int_0^{\sqrt{4R}} \left( \left(\frac{s}{R}\right)^2 \left(R - \frac{s^2}{4}\right) \right)^{\frac{n-3}{2}} f(s) \frac{s ds}{R} \\ &= |\mathbb{S}^{n-2}|R \int_0^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{\frac{n-3}{2}} s^{n-2} f(s) ds. \end{aligned}$$

There remains to show that the integral on the right-hand side is finite, with the desired upper bounds. When  $n \geq 3$ , we have

$$\left(1 - \frac{s^2}{4R}\right)^{\frac{n-3}{2}} \leq 1, \quad \forall s \in (0, \sqrt{4R}),$$

which gives the required estimate and shows that the integral is finite, since

$$\int_0^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{\frac{n-3}{2}} s^{n-2} f(s) ds \leq \int_0^\infty s^{n-2} g(s) ds < +\infty.$$

When  $n = 2$ , let us split the integral into two parts

$$\int_0^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s) ds = \int_0^{\sqrt{2R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s) ds + \int_{\sqrt{2R}}^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s) ds. \quad (\text{A.43})$$

On the one hand, we have

$$\int_0^{\sqrt{2R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s) ds \leq \frac{1}{\sqrt{2}} \int_0^{\sqrt{2R}} f(s) ds, \quad (\text{A.44})$$

and on the other hand

$$\begin{aligned} \int_{\sqrt{2R}}^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} f(s) ds &\leq \|f\|_{L^\infty(\sqrt{2R}, \sqrt{4R})} \int_{\sqrt{2R}}^{\sqrt{4R}} \left(1 - \frac{s^2}{4R}\right)^{-\frac{1}{2}} ds \\ &= \|f\|_{L^\infty(\sqrt{2R}, \sqrt{4R})} \sqrt{4R} \int_{\frac{1}{\sqrt{2}}}^1 (1 - s^2)^{-\frac{1}{2}} ds \\ &= \|f\|_{L^\infty(\sqrt{2R}, \sqrt{4R})} \frac{\pi R}{2}. \end{aligned} \quad (\text{A.45})$$

hence the required estimate by combining (A.43) to (A.45).  $\square$

**Proposition A.4.** *Let  $f \in H^1(\mathbb{S}^{n-1})$  and let  $(\eta_\varepsilon)_{\varepsilon>0}$  be a  $(n-1)$ -dimensional approximation of identity satisfying (A.41). Then we have*

$$\lim_{\varepsilon \rightarrow 0} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} = \mathbf{K}_{2,n-1} \int_{\mathbb{S}^{n-1}} |\nabla_\tau f|^2 d\mathcal{H}^{n-1}.$$

*Proof.* We need only prove the result for functions in  $C^2(\mathbb{S}^{n-1})$  by approximation, using the density of  $C^2(\mathbb{S}^{n-1})$  in  $H^1(\mathbb{S}^{n-1})$  and the continuity of  $N_\varepsilon$  in  $H^1(\mathbb{S}^{n-1})$  given by Corollary A.2. Let us assume that  $f \in C^2(\mathbb{S}^{n-1})$ , and consider some  $0 < r < \frac{1}{2}$  to be fixed later. Let us split into two parts the integral

$$\begin{aligned} &\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &= \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\ &\quad + \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| \geq r}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}. \end{aligned} \quad (\text{A.46})$$

Changing variables and using the fact that  $f \in C^2(\mathbb{S}^{n-1})$ , we have

$$\iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| \geq r}} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \leq C \int_{\mathbb{S}^{n-1} \cap \{|x-N| \geq r\}} \eta_\varepsilon(|N-y|) d\mathcal{H}_y^{n-1} \quad (\text{A.47})$$

for some  $C > 0$  depending only on  $n$  and  $\|f\|_{C^2(\mathbb{S}^{n-1})}$ , where  $N = (0, \dots, 0, 1) \in \mathbb{S}^{n-1}$ . Using [Lemma A.3](#) with  $f = \mathbf{1}_{[r, +\infty)} \eta_\varepsilon$ , we find

$$\int_{\mathbb{S}^{n-1} \cap \{|x-N| \geq r\}} \eta_\varepsilon(|N-y|) d\mathcal{H}_y^{n-1} = |\mathbb{S}^{n-1}| \int_r^2 \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} s^{n-2} \eta_\varepsilon(s) ds. \quad (\text{A.48})$$

If  $n \geq 3$ , we have  $\left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \leq 1$ , thus [\(A.47\)](#) and [\(A.48\)](#) give

$$\iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| \geq r}} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \leq C \int_r^\infty s^{n-2} \eta_\varepsilon(s) ds,$$

whose right-hand side goes to 0 as  $\varepsilon$  goes to 0 since  $\eta_\varepsilon$  is a family  $(n-1)$ -dimensional mollifiers. If  $n = 2$ , by [Lemma A.3](#) we have the estimate

$$\int_r^2 \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \eta_\varepsilon(s) ds \leq \left( \frac{1}{\sqrt{2}} \int_r^{\sqrt{2}} \eta_\varepsilon(s) s^{n-2} ds + \frac{\pi}{2} \|\eta_\varepsilon\|_{L^\infty(\sqrt{2}, 2)} \right) \quad (\text{A.49})$$

which also goes to 0 as  $\varepsilon$  goes to 0 by assumption [\(A.41\)](#). Thus for any  $r \in (0, 1)$  we have

$$\lim_{\varepsilon \rightarrow 0} \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| \geq r}} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} = 0. \quad (\text{A.50})$$

Since  $f \in C^2(\mathbb{S}^{n-1})$ , we have

$$|f(x) - f(y) - \nabla_\tau f(x) \cdot (x-y)| \leq C|x-y|^2, \quad \forall x, y \in \mathbb{S}^{n-1}, \quad (\text{A.51})$$

for some  $C$  depending only  $\|f\|_{C^2(\mathbb{S}^{n-1})}$ . Let us write

$$\begin{aligned} |f(x) - f(y)|^2 &= |f(x) - f(y) - \nabla_\tau f(x) \cdot (x-y) + \nabla_\tau f(x) \cdot (x-y)|^2 \\ &= |\nabla_\tau f(x) \cdot (x-y)|^2 + |f(x) - f(y) - \nabla_\tau f(x) \cdot (x-y)|^2 \\ &\quad + 2(\nabla_\tau f(x) \cdot (x-y))(f(x) - f(y) - \nabla_\tau f(x) \cdot (x-y)). \end{aligned} \quad (\text{A.52})$$

Since  $|f(x) - f(y) - \nabla_\tau f(x) \cdot (x-y)|^2 \leq C|x-y|^4$  and  $|(\nabla_\tau f(x) \cdot (x-y))(f(x) - f(y) - \nabla_\tau f(x) \cdot (x-y))| \leq C|x-y|^3$  for some  $C = C(\|f\|_{C^2(\mathbb{S}^{n-1})})$  and every  $x, y \in \mathbb{S}^{n-1}$  by [\(A.51\)](#), [\(A.52\)](#) gives

$$|\nabla_\tau f(x) \cdot (x-y)|^2 - C|x-y|^3 \leq |f(x) - f(y)|^2 \leq |\nabla_\tau f(x) \cdot (x-y)|^2 + C|x-y|^3,$$

for every  $x, y \in \mathbb{S}^{n-1}$ . Thus

$$\begin{aligned}
& \left| \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \right. \\
& \quad \left. - \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} \left| \nabla_\tau f(x) \cdot \frac{x-y}{|x-y|} \right|^2 \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \right| \\
& \leq \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} |x-y| \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}.
\end{aligned} \tag{A.53}$$

Observe that

$$\iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} |x-y| \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \leq Cr \int_{\mathbb{S}^{n-1} \cap \{|N-y| < r\}} \eta_\varepsilon(|N-y|) d\mathcal{H}_y^{n-1},$$

where  $C$  depends only on  $n$  and  $\|f\|_{C^2(\mathbb{S}^{n-1})}$ , thus integrating on slices with [Lemma A.3](#) yields

$$\begin{aligned}
\iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} |x-y| \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} & \leq Cr \int_0^r \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \eta_\varepsilon(s) s^{n-2} ds \\
& \leq Cr \int_0^\infty \eta_\varepsilon(s) s^{n-2} ds \leq Cr,
\end{aligned} \tag{A.54}$$

where we used the fact that  $\left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \leq 2$  for all  $n \geq 2$  and  $s \leq r < 1$ , and where  $C$  denotes a constant depending only on  $n$  and  $\|f\|_{C^2(\mathbb{S}^{n-1})}$ . Once again, let us make a change of variables and integrate on slices using [\(A.2\)](#)

$$\begin{aligned}
& \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} \left| \nabla_\tau f(x) \cdot \frac{x-y}{|x-y|} \right|^2 \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \\
& = \int_{\mathbb{S}^{n-1} \cap \{|N-y| < r\}} \eta_\varepsilon(|N-y|) \left( \int_{\mathbb{S}^{n-1}} \left| \nabla_\tau f(x) \cdot \frac{N-y}{|N-y|} \right|^2 d\mathcal{H}_x^{n-1} \right) d\mathcal{H}_y^{n-1} \\
& = \int_{\{s : \sqrt{2(1-s)} < r\}} \int_{\mathbb{S}^{n-2}} \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \eta_\varepsilon(\sqrt{2(1-s)}) \\
& \quad \left( \int_{\mathbb{S}^{n-1}} \left| \nabla_\tau f(x) \cdot \frac{N - (\sqrt{1-s^2}y, s)}{\sqrt{2(1-s)}} \right|^2 d\mathcal{H}_x^{n-1} \right) d\mathcal{H}_y^{n-2} ds.
\end{aligned} \tag{A.55}$$

Writing

$$\frac{N - (\sqrt{1-s^2}y, s)}{\sqrt{2(1-s)}} = \frac{1}{\sqrt{2}}(\sqrt{1+s}y, \sqrt{1-s}) = \sqrt{\frac{1+s}{2}}(y, 0) + \sqrt{\frac{1-s}{2}}(0_{\mathbb{R}^{n-1}}, 1),$$

we find

$$\begin{aligned} \left| \nabla_\tau f(x) \cdot \frac{N - (\sqrt{1-s^2}y, s)}{\sqrt{2(1-s)}} \right|^2 &= \frac{1+s}{2} |\nabla_\tau f(x) \cdot (y, 0)|^2 + \frac{1-s}{2} |\nabla_\tau f(x) \cdot N|^2 \\ &\quad + \sqrt{1-s^2} (\nabla_\tau f(x) \cdot (y, 0)) (\nabla_\tau f(x) \cdot N) \\ &=: \frac{1+s}{2} |\nabla_\tau f(x) \cdot (y, 0)|^2 + T(x, s) \end{aligned} \quad (\text{A.56})$$

where  $T : \mathbb{S}^{n-1} \times (1 - \frac{r^2}{2}, 1) \rightarrow \mathbb{R}^n$  is a continuous function such that

$$|T(x, s)| \leq C\sqrt{1-s} \leq Cr, \quad \forall s \in (1 - \frac{r^2}{2}, 1), \quad (\text{A.57})$$

for some  $C$  depending only on  $n$  and  $\|f\|_{C^2(\mathbb{S}^{n-1})}$ . In view of (A.55) to (A.57), we have

$$\iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} \left| \nabla_\tau f(x) \cdot \frac{x-y}{|x-y|} \right|^2 \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} = I_\varepsilon + II_\varepsilon \quad (\text{A.58})$$

where

$$\begin{aligned} I_\varepsilon &:= \int_{\{s : \sqrt{2(1-s)} < r\}} \int_{\mathbb{S}^{n-2}} \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \left(\frac{1+s}{2}\right) \eta_\varepsilon(\sqrt{2(1-s)}) \\ &\quad \left(\int_{\mathbb{S}^{n-1}} |\nabla_\tau f(x) \cdot (y, 0)|^2 d\mathcal{H}_x^{n-1}\right) d\mathcal{H}_y^{n-2} ds. \end{aligned} \quad (\text{A.59})$$

and

$$|II_\varepsilon| \leq Cr \int_{\{\sqrt{2(1-s)} < r\}} \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \eta_\varepsilon(\sqrt{2(1-s)}) ds = Cr \int_0^r \left(1 - \frac{t^2}{4}\right)^{\frac{n-3}{2}} \eta_\varepsilon(t) t^{n-2} dt,$$

for some  $C$  depending only on  $n$  and  $\|f\|_{C^2(\mathbb{S}^{n-1})}$ , where we changed variables for the last equality. Since  $r < 1$ , we have  $\left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} < 2$  for any  $n \geq 2$ , thus

$$|II_\varepsilon| \leq Cr \int_0^r \eta_\varepsilon(t) t^{n-2} dt \leq Cr,$$

where we used the fact that  $|\mathbb{S}^{n-2}| \int_0^\infty \eta_\varepsilon(s) s^{n-2} ds = 1$ . Hence, given  $\delta > 0$ , with (A.54) we can choose  $r$  small enough such that

$$|II_\varepsilon| \leq \delta \quad \text{and} \quad \iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| < r}} |x-y| \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \leq \delta \quad (\text{A.60})$$

for every  $\varepsilon > 0$ . Recalling (A.50), we can then choose  $\varepsilon_0 > 0$  such that, for all  $\varepsilon < \varepsilon_0$ , we have

$$\iint_{\substack{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \\ |x-y| \geq r}} \frac{|f(x) - f(y)|^2}{|x-y|^2} \eta_\varepsilon(|x-y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \leq \delta. \quad (\text{A.61})$$

Combining (A.46), (A.53) and (A.58) to (A.61), we find

$$\left| \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} - I_\varepsilon \right| \leq 3\delta, \quad \forall \varepsilon < \varepsilon_0, \quad (\text{A.62})$$

for our choice of  $r$ . Then we compute

$$\begin{aligned} I_\varepsilon &= \int_{\{s : \sqrt{2(1-s)} < r\}} \int_{\mathbb{S}^{n-2}} \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \left(\frac{1+s}{2}\right) \eta_\varepsilon(\sqrt{2(1-s)}) \\ &\quad \int_{\mathbb{S}^{n-1}} |\nabla_\tau f(x) \cdot (y, 0)| d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-2} ds \\ &= \int_{\{s : \sqrt{2(1-s)} < r\}} \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \left(\frac{1+s}{2}\right) \eta_\varepsilon(\sqrt{2(1-s)}) \\ &\quad \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-2}} |\nabla_\tau f(x) \cdot (y, 0)| d\mathcal{H}_y^{n-2} d\mathcal{H}_x^{n-1} ds \\ &= \mathbf{K}_{2,n-1} |\mathbb{S}^{n-2}| \int_{\{s : \sqrt{2(1-s)} < r\}} \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}} \left(\frac{1+s}{2}\right) \eta_\varepsilon(\sqrt{2(1-s)}) \\ &\quad \int_{\mathbb{S}^{n-1}} |\nabla_\tau f(x)|^2 d\mathcal{H}_x^{n-1} ds, \end{aligned}$$

where we used Fubini's theorem and the definition of  $\mathbf{K}_{2,n-1}$ . Changing variables, it follows

$$I_\varepsilon = \mathbf{K}_{2,n-1} \|\nabla_\tau f\|_{L^2(\mathbb{S}^{n-1})}^2 |\mathbb{S}^{n-2}| \int_0^r \left(1 - \frac{s^2}{4}\right)^{\frac{n-3}{2}+1} \eta_\varepsilon(s) s^{n-2} ds.$$

Now we could have chosen  $r$  small enough such that we have as well

$$0 \leq \mathbf{K}_{2,n-1} \|\nabla_\tau f\|_{L^2(\mathbb{S}^{n-1})}^2 |\mathbb{S}^{n-2}| \left(1 - \left(1 - \frac{r^2}{4}\right)^{\frac{n-3}{2}+1}\right) \leq \delta,$$

since  $\frac{n-3}{2} + 1 \geq 0$  whenever  $n \geq 2$ , thus

$$\left| I_\varepsilon - \mathbf{K}_{2,n-1} \|\nabla_\tau f\|_{L^2(\mathbb{S}^{n-1})}^2 |\mathbb{S}^{n-2}| \int_0^r \eta_\varepsilon(s) s^{n-2} ds \right| \leq \delta, \quad (\text{A.63})$$

where we used the fact  $|\mathbb{S}^{n-2}| \int_0^1 \eta_\varepsilon(s) s^{n-2} ds = 1$ . Hence from (A.62) and (A.63), it follows

$$\begin{aligned} &\left| \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \right. \\ &\quad \left. - \mathbf{K}_{2,n-1} \|\nabla_\tau f\|_{L^2(\mathbb{S}^{n-1})}^2 |\mathbb{S}^{n-2}| \int_0^r \eta_\varepsilon(s) s^{n-2} ds \right| \leq 4\delta, \quad \forall \varepsilon < \varepsilon_0. \end{aligned} \quad (\text{A.64})$$

Notice that for every  $r \in (0, 1)$ ,

$$|\mathbb{S}^{n-2}| \int_0^r \eta_\varepsilon(s) s^{n-2} ds = 1 - |\mathbb{S}^{n-1}| \int_r^\infty \eta_\varepsilon(s) s^{n-2} ds \xrightarrow{\varepsilon \rightarrow 0} 1,$$

so that letting  $\varepsilon$  go to 0 in (A.64), the arbitrariness of  $\delta$  implies

$$\lim_{\varepsilon \rightarrow 0} \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|f(x) - f(y)|^2}{|x - y|^2} \eta_\varepsilon(|x - y|) d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} = \mathbf{K}_{2,n-1} \int_{\mathbb{S}^{n-1}} |\nabla_\tau f|^2 d\mathcal{H}^{n-1}.$$

□

In some sense, Corollary A.2 and Proposition A.4 justify that the upper bound given in Proposition A.1 is “asymptotically sharp”.

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