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The characterization of the minimal paths in the Christoffel tree according to a second-order balancedness

Andrea Frosini[1] and Lama Tarsissi[2,3]*

[1] Dipartimento di Matematica e Informatica, Università di Firenze,
Viale Morgagni 65, 50134 Firenze, Italy
andrea.frosini@unifi.it
[2] LAMA, Univ Gustave Eiffel, UPEC, CNRS, F-77454 Marne-la-Valle, France
[3] LIGM, Univ Gustave Eiffel, CNRS, ESIEE Paris, F-77454 Marne-la-Valle, France
lama.tarsissi@esiee.fr

Abstract. In 1842, Dirichlet observed that any real number α can be obtained as the limit of a sequence $(\frac{p_n}{q_n})$ of irreducible rational numbers. Few years later, M.Stern (1858) and A.Brocot (1861) defined a tree-like arrangement of all the (irreducible) rational numbers whose infinite paths are the Dirichlet sequences of the real numbers and are characterized by their continued fraction representations. The Stern-Brocot tree is equivalent to the Christoffel tree obtained by ordering the Christoffel words according to their standard factorization. We first prove that the Fibonacci word's prefixes define minimal path in the Christoffel tree with respect to the second order balancedness parameter defined on Christoffel words. Switching back to the Stern-Brocot tree, we generalize this result and we obtain a characterization of all the minimal paths in terms of numerical suffixes of the continued fraction representation of the Christoffel words.

Keywords: Continued fractions· Stern-Brocot tree· Christoffel words· Balance property· Minimal path.

1 Introduction

It appears that balancedness [2] is a crucial notion for the recently developed research area of combinatorics on words. A widely studied class of binary balanced words are the Christoffel words, whose interest arises from their strong connections with geometry, algebra, and number theory (see [2] for the definitions, first properties and extensive bibliography). Each Christoffel word is the discretization of a line segment of rational slope, and it can be uniquely factorized into two different Christoffel words. Such standard factorization allows us to arrange all the Christoffel words in a binary tree structure, called Christoffel tree, whose root is the word $w = 01 = (0, 1)$. The notion of balancedness allows us to define,

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for each binary word w, a balance matrix B_w whose maximal element provides its order of balancedness, noted by $\delta(w)$. The present study focuses on Christoffel words that are balanced words, i.e. whose order of balancedness is 1. In this case, each row of the balance matrix is again a binary word, whose order of balancedness can be computed again, obtaining a second order balance matrix U_w whose maximal element is defined as the second order balance value $\delta^2(w)$. This value can also be obtained using the link between the abelian complexity and balanced words represented by Zamboni ([6], Section 4), using the Parikh vectors.

The matrix U_w has several properties that allow a recursive construction by refering to three $U_{w'}$ matrices, of some specific ancestors w' of w in the Christoffel tree [12]. The authors used this notion to study the distribution of $1'$s in Christoffel words and relate it to the longstanding problem of their synchronization [7, 8].

The correspondence between Christoffel word and continued fraction representation of its rational slope allows us to transfer the notion of balancedness to the Stern-Brocot tree. This tree, indicated hereafter as SB-tree, is a binary tree where the irreducible rational numbers are arranged according to a specific operator called Farey sum, defined independently in [4] and in [11], and that perfectly mimes the notion of Standard decomposition of a Christoffel word. In our study, we first establish a connection between the second order balance value of a Christoffel word and the form of the continued fraction representation of the slope of the related discrete segment. Then, we show that the sequence of the ratios of consecutive Fibonacci numbers constitutes, in the SB-tree, a path which minimizes the growth of the second order balancedness parameter of the related Christoffel words. Finally, relying on this result, we characterize the minimal paths (w.r.t the second order balance value) by mixing algebraic and arithmetic techniques on the continued fraction representations.

The paper is structured as follows: in Section 2, we recall the definitions of Christoffel words, Christoffel tree and the Stern-Brocot tree. In Section 2.4, we introduce the notion of balancedness and we define the balance matrix of a binary word. Then we generalize the notion to the second order balance matrix. Finally, in Section 3, we show how the second order balancedness parameter is spread on the SB-tree, providing the example of the Fibonacci sequence. Here we provide our main result: the characterization of the minimal paths in the SB-tree according to the δ^2 parameter.

2 Definitions and previous results

We refer to the book [5] for the standard terminology in combinatorics of words: *alphabet, word, length of a word, occurrence of a letter, factor, prefix, suffix, period, conjugate, primitive, reversal, palindrome etc.* The related notation will be recalled when used.

Christoffel paths and Christoffel words In discrete geometry, the theory of Christoffel words has been considered during this last few decades and has acquired a prominent role in the study of the discretization of segments and shapes. Concerning the former, let a, b be two co-prime numbers; the *Christoffel path* of slope $\frac{a}{b}$ is defined as the connected path in the discrete plane joining the origin $O(0,0)$ to the point (b,a) such that it is the nearest path below the Euclidean line segment joining these two points, as shown in Figure 1. So, there are no points of the discrete plane between the path and the line segment.

Still in Figure 1, we can see the coding of a Christoffel path by a binary word, say Christoffel word, whose letters 0 and 1 represent a horizontal and a vertical step in the path, respectively.

The Christoffel word related to the path reaching the point (b,a) is indicated by $C(\frac{a}{b})$, and its slope is $\frac{a}{b} = \frac{|w|_1}{|w|_0}$, where the notation $|w|_x$ stands for the number of occurrences of the letter x in w.

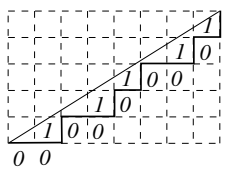


Fig. 1. The Christoffel path of the line segment of slope $\frac{5}{8}$, and the corresponding Christoffel word $C(\frac{5}{8}) = 0010010100101$.

A binary word w is *k-balanced*, with $k > 0$, if, for any two factors w_1 and w_2 of its conjugates such that $|w_1| = |w_2|$, it follows that $||w_1|_1 - |w_2|_1| \leq k$.

One of the most important properties of Christoffel words is that they are 1-balanced, simply balanced, as shown in [3].

2.1 Christoffel tree

Let us recall the definition of the Christoffel tree, i.e. a well-known arrangement of the Christoffel words as a binary tree (see [1]). The root of the Christoffel tree is labeled by the pair $(0,1)$, representing the Christoffel word 01 of slope $\frac{1}{1}$. Each node (u,v) generates two children according to two functions ϕ_0 and ϕ_1 such that: $\phi_0(u,v) = (u,uv)$ produces the left-child, and $\phi_1(u,v) = (uv,v)$ produces the right-child. Each node (u,v) corresponds to the Christoffel word uv and represents its *standard factorization form* [5]. Figure 2 shows the first levels of the Christoffel tree.

Let $w = C(\frac{a}{b})$ be a Christoffel word lying at level k of the tree. The *directive sequence* of w is the word $\Delta(\frac{a}{b}) = i_1 i_2 \ldots i_k$ such that $w = (\phi_{i_k} \circ \ldots \circ \phi_{i_2} \circ \phi_{i_1})(0,1)$. According to the definition of ϕ_0 and ϕ_1, the elements of $\Delta(\frac{a}{b})$ also

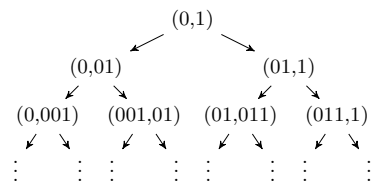


Fig. 2. The first levels of the Christoffel tree.

show, step by step, the directions to reach the word w in the tree starting from its root at level 1. If $i_n = 0$, we must move to the left, otherwise, to the right.

As an example, the directive sequence of the Christoffel word $C(\frac{4}{3}) = 0101011$ at level 4 of the Christoffel tree is $\Delta(\frac{4}{3}) = 100$: according to the definition it holds $(\phi_0 \circ \phi_0 \circ \phi_1)(0,1) = (\phi_0 \circ \phi_0)(01,1) = \phi_0(01,011) = (01,01011)$.

2.2 Stern-Brocot tree

The Christoffel tree is known to be isomorphic to the SB-tree, that was introduced by M.Stern [11] and A.Brocot [4] as a binary-tree arrangement of the irreducible fractions. Such arrangement relies on the Farey sum operator, indicated by $\oplus$, and defined on two generic rational numbers as $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. The root of the SB-tree is labeled with the fraction $\frac{1}{1}$ and each node at level $n > 1$ is labeled with the Farey sum of its nearest left and right ancestors, i.e., the nodes lying on the greatest level of the tree and having k in its left and right subtree, respectively. The left and right ancestors of the root are considered to be the fractions $\frac{0}{1}$ and $\frac{1}{0}$, even if not present in the tree. The first levels of the SB-tree are depicted in Figure 3

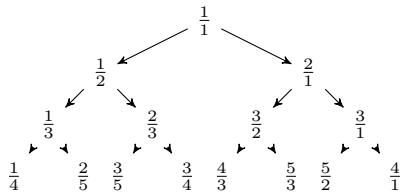


Fig. 3. The first levels of the Stern-Brocot tree.

As an example, we compute the left-child and right-child of the node $\frac{3}{4}$ by Farey summing it with its left-ancestor $\frac{2}{3}$ and with its right-ancestor $\frac{1}{1}$, and obtaining $\frac{5}{7}$ and $\frac{4}{5}$, respectively. The SB-tree contains once each irreducible fraction and over the years has attracted researchers for its many interesting properties.

Remark 1. By the construction of the SB-tree, the following hold

i) The tree is divided into two symmetric parts. The left part contains all the reduced fractions less than 1, while the right part contains the reduced fractions bigger than 1.
ii) For each level k of the SB tree, the rational numbers appear in increasing order from left to right, and their values are bounded by $\frac{1}{k}$ and $\frac{k}{1}$.

From our perspective, the SB-tree's appeal reveals in its close relationship with the Christoffel tree: in fact, it is easy to check that if we replace the label of each node (u, v) of this last with the rational number $\frac{|uv|_1}{|uv|_0}$ representing the slope of the corresponding Christoffel word, we obtain the SB-tree. This correspondence is interesting but not surprising once we observe that the slope of the concatenation of two Christoffel words is the Farey sum of their slopes.

2.3 Continued fraction representation

Our study requires to recall a final notion to identify the nodes of the SB-tree. Let $\frac{a}{b}$ be a positive fraction, we define its *continued fraction representation* as the sequence of integers $[a_0, \ldots, a_z]$, with $a_0 \geq 0$, represented below, and such that for each $1 \leq i \leq z$, $a_i \geq 1$:

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots \cfrac{1}{a_{z-1} + \frac{1}{a_z}}}}$$

In order to obtain a unique continued fraction representation of each rational number, it is also commonly required that if $z \geq 2$, then $a_z \geq 2$. The continued fraction representation of a rational number is always finite and it becomes infinite in case the number is irrational (among the vast literature on continued fraction representation, we refer the reader to [10] for the main properties).

Equivalently to the Christoffel tree, the directive sequence of the Christoffel word of slope $\frac{a}{b}$ can be read in the SB-tree by using the terms of the continued fraction representation of $\frac{a}{b}$. The following lemma presents this connection (a proof can be found in [9]).

Lemma 1. *[12] Let w be the Christoffel word of slope $\frac{a}{b}$ whose continued fraction representation is $[a_0, a_1, \ldots, a_z]$. The directive sequence of w has the following form: $\Delta(\frac{a}{b}) = 1^{a_0}0^{a_1}1^{a_2}\ldots p^{a_z-1}$, where $p \in \{0, 1\}$ according to the parity of z.*

Referring again to the continued fraction representation $[0, 1, 2, 2]$ of $\frac{5}{7}$, the directive sequence $\Delta(\frac{5}{7})$ is exactly $1^0 0^1 1^2 0^1$.

2.4 The second order balance matrix

In [12], the author introduced the *balance matrix* B_w, of a binary word w, in order to deepen the study of its balance value. We let $\delta(w)$ denote the balance value of a given binary word $w = w_1 w_2 \ldots w_n$. We first compute the matrix S_w

of dimension $n \times n$ whose generic element $S_w[i,j]$ counts the number of elements 1 in the i-length prefix of the conjugate of w starting in position j. We let $M[i]$ represents the ith row of any matrix M.

The balance matrix B_w is computed from S_w by subtracting the minimum of a given row to each of its element, i.e.,

$$B_w[i,j] = S_w[i,j] - \min\{S_w[i]\}.$$

By construction, the maximum element of B_w equals to the balance value of w and is denoted by $\delta(w)$. A simple example will clarify the construction

Example 1. Let us consider the Christoffel word $w = 00100101$ of slope $\frac{3}{5}$. We provide the matrices S_w and B_w:

$$S_w = \begin{pmatrix} 0\,0\,1\,0\,0\,1\,0\,1 \\ 0\,1\,1\,0\,1\,1\,1\,1 \\ 1\,1\,1\,1\,1\,2\,1\,1 \\ 1\,1\,2\,1\,2\,2\,1\,2 \\ 1\,2\,2\,2\,2\,2\,2\,2 \\ 2\,2\,3\,2\,2\,3\,2\,2 \\ 2\,3\,3\,2\,3\,3\,2\,3 \end{pmatrix}, B_w = \begin{pmatrix} 0\,0\,1\,0\,0\,1\,0\,1 \\ 0\,1\,1\,0\,1\,1\,1\,1 \\ 0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,1\,0\,1\,1\,0\,1 \\ 0\,1\,1\,1\,1\,1\,1\,1 \\ 0\,0\,1\,0\,0\,1\,0\,0 \\ 0\,1\,1\,0\,1\,1\,0\,1 \end{pmatrix}.$$

Since each Christoffel word w is 1-balanced, then the rows of its balance matrix B_w are binary words too. So, we can push the notion of balancedness to a second level by studying the balancedness of the rows of B_w. Let us define the *second order balance matrix* U_w of dimension $(n-1) \times (n-1)$, with $n = |w|$, as $U_w[i,j] = \max(B_{B_w[i]}[j])$. In words, the element $U_w[i,j]$ shows the balancedness of the i-th row of B_w with respect to its proper factors of length j. More details and properties about this matrix can be found in [12]. We let $\delta^2(w)$ denote the maximum element of U_w.

Example 2. Let us consider the 1-balanced word w of Example 2. The fourth row of the matrix U_w is obtained by computing the balance value of the fourth row of B_w with respect to all the possible factors' lengths from 1 to 8. We explicitly determine the element $U_w[4,3]$: the eight factors of the conjugates of $B_w[4]$ of length 3 are $(001, 010, 101, 011, 110, 101, 101, 100)$, and the maximum difference of the number of elements 1 between any two of them is 1, so $U_w[4,3] = 1$. Doing an analogous computation for all the lengths from 1 to 7, we obtain $U_w[4] = (1,2,1,2,1,2,1)$.

So, the second order of balancedness of a Christoffel word w provides an idea of how uniformly the factors of each abelian class, as defined in [6], are distributed in w.

3 Path Minimality in SB-tree

In this section, we prove some new properties of the second order balance parameter for Christoffel words and we show how it is distributed on the Christoffel

tree. Finally, we characterize the paths of the tree where the parameter's growth is minimal. So, let us consider the following two ancestors of $\frac{a}{b} = [a_0, \ldots, a_z]$ in the SB-tree:

- $\frac{u}{v}$ whose continued fraction expansion is $= [a_0, \ldots, a_{z-1} + 1]$
- $\frac{\rho}{\theta}$ whose continued fraction expansion is either $[a_0, \ldots, a_z - 2]$ if $a_z > 2$, or $[a_0, \ldots, a_{z-2}]$ if $a_z = 2$.

In [12], Section 6.1, the authors proved that the matrix U_w, with $w = C(\frac{a}{b})$, can be decomposed into blocks belonging either to U_{w_1} or to $U_{w_2} + 1$, with $w_1 = C(\frac{u}{v})$ and $w_2 = C(\frac{\rho}{\theta})$. From this result it immediately follows:

Theorem 1. *The second order balance value of $C(\frac{a}{b})$ is:*

$$\delta^2\left(C\left(\frac{a}{b}\right)\right) = \max\left(\delta^2\left(C\left(\frac{u}{v}\right)\right), \delta^2\left(C\left(\frac{\rho}{\theta}\right)\right) + 1\right).$$

Using the previous result, we set a lower bound to the growth of the second order balanced value inside the Christoffel tree. The next lemma states that $\delta^2(C(\frac{a}{b}))$ increases according to the distance between $\frac{a}{b}$ and $\frac{u}{v}$ in the SB-tree.

Lemma 2. *Let $\frac{a}{b} = [a_0, \ldots, a_z]$ and belongs to the level k on the SB-tree, where $\delta^2(C(\frac{u}{v})) = t$ and $\delta^2(C(\frac{\rho}{\theta})) = n$.*

i) If $a_z = 2$, then $\delta^2(C(\frac{a}{b})) \in \{t; t+1\}$.
ii) If $a_z > 2$, then $\delta^2(C(\frac{a}{b})) = n + 1$, i.e it is increased by one value each 2 levels after $\frac{u}{v}$.

Proof. Let us consider these two rational numbers $\frac{a}{b} = [a_0, \ldots, a_z]$ and $u/v = [a_0, \ldots, a_{z-1} + 1]$, where $\delta^2(C(\frac{u}{v})) = t$.
If $a_z = 2$, then $\frac{\rho}{\theta} = [a_0, \ldots, a_{z-2}]$, in this case, its second order value is equal to $n \leq t$. In all the cases, from Theorem 1, $\delta^2(C(\frac{a}{b})) = \max(\delta^2(C(\frac{u}{v})), \delta^2(C(\frac{\rho}{\theta})) + 1)$ that can be either t or $t + 1$.
If $a_z \geq 3$, then we must consider two cases for the rational number $\frac{\rho}{\theta}$.

- If $a_z = 3$, in this case $\frac{\rho}{\theta} = \frac{u}{v}$ and $n = t$, then:
  $\delta^2(C(\frac{a}{b})) = \max(\delta^2(C(\frac{u}{v})), \delta^2(C(\frac{\rho}{\theta}) + 1) = n + 1$.
- If $a_z \geq 4$, in this case $\frac{\rho}{\theta} = [a_0, a_1, \cdots, a_z - 2]$, and $n \geq t$ then:
  $\delta^2(C(\frac{a}{b})) = \delta^2(C(\frac{\rho}{\theta})) + 1 = n + 1$.

Finally, we provide a lower bound, denoted by δ_k^2, for the second order balance value of any element at level k in the Christoffel tree.

Theorem 2. *For each level k in the SB tree, we have: $\delta_k^2 \geq \lceil \frac{k}{3} \rceil$.*

Proof. By induction on the levels of the Stern-Brocot tree, we have $\delta_1^2 = \delta_2^2 = 1$; and $\delta_3^2 \in \{1, 2\}$ while $\delta_4^2 = 2$. Hence the minimal value of δ_3^2 (denoted $\min \delta_3^2$) is equal to 1 and $\min \delta_4^2 = 2$. Which means that $\delta_i^2 \geq \lceil \frac{i}{3} \rceil$ with $1 \leq i \leq 4$.
Suppose that it is true for all the levels till $k = 3k'$; i.e $\min \delta_k^2 = \min \delta_{k-1}^2 = \min \delta_{k-2}^2 = k'$. We prove that $\delta_{k+1}^2 \geq \lceil \frac{k+1}{3} \rceil = k' + 1$.

All the children of the fractions on level k, with $\delta^2 > k'$, have $\delta^2 \geq k'+1$. It remains to study the case where $\delta^2_k = k'$. For that, we let $\frac{u}{v}$ be a fraction at level k with $\delta^2(C(\frac{u}{v})) = k'$. By contradiction, we let $\frac{c}{d}$ be the fraction at level $k+1$ with $\delta^2(C(\frac{c}{d})) = k'$. This fraction is either in the same, or opposite direction of $\frac{u}{v}$. If $\frac{c}{d}$ is in the same direction of $\frac{u}{v}$, then by Lemma 2, we get $\delta^2(C(\frac{a}{b})) = k'-1$, where $\frac{a}{b}$ is the fraction at level $k-1$. If $\frac{c}{d}$ is in the opposite direction, then the fraction $\frac{u}{v}$ can be either in the same direction or opposite to $\frac{a}{b}$. In this case, we let $\frac{m}{n}$ be the fraction at level $(k-2)$ and we have:

- If $u/v = (a/b)$ then by Lemma 2, $\delta^2(C(\frac{m}{n})) = k'-1$.
- If $u/v = (a/b)$, then if $\delta^2(C(\frac{a}{b})) = k'$, we get: $\delta^2(C(\frac{c}{d})) = max(\delta^2(C(\frac{u}{v})), \delta^2(C(\frac{m}{n}))+1) = k'$. If $\delta^2(C(\frac{m}{n})) = k'$ then $\delta^2(C(\frac{c}{d})) = k'+1$; and if $\delta^2(C(\frac{m}{n})) = k'-1$ we get a contradiction since $\delta^2_{(k-2)} \geq \lceil \frac{k-2}{3} \rceil = k'$.

Therefore $\delta^2 \geq k'+1$ for all the fractions on the level $k+1$ thus

$$\delta^2_{k+1} \geq \lceil \frac{k+1}{3} \rceil = \lceil \frac{3k'+1}{3} \rceil = k'+1.$$

□

The remaining part of the section, is devoted to the study of the paths of the SB-tree whose related Christoffel words realize the lower bound. The symmetry of the tree allows us to consider, without loss of generality, its left part only, i.e., those fractions less than or equal to 1 and whose continued fractions have $a_0 = 0$. We denote the subtree SB_L.

3.1 Minimal paths in the SB-tree

Let us assign to each fraction of SB_L the δ^2 value of its related Christoffel word starting from the root $\frac{1}{1}$: Figure 4 shows the computation till the fourth level of the tree.

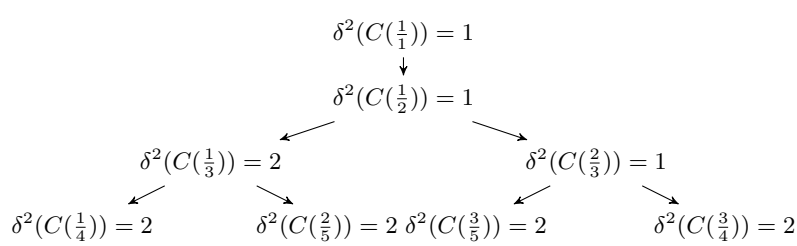


Fig. 4. The second order balancedness of the rational numbers belonging to the first four levels of SB_L.

Let us indicate by $P(\frac{a}{b})$ the sequence of all the fractions in the path from the root of SB_L to the element $\frac{a}{b}$, and by $PS(\frac{a}{b})$ the sequence of the related δ^2 values. From Lemma 2, it follows that $PS(\frac{a}{b})$ is a weakly increasing integer sequence.

As an example, let us consider the fraction $\frac{7}{12}$, lying in the sixth level of SB_L. We have $P(\frac{7}{12}) = (\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{7}{12})$, and the related δ^2 sequence is $PS(\frac{7}{12}) = (1, 1, 1, 2, 2, 2)$.

We give now the definition of a *minimal path* at a certain level k of SB_L.

Definition 1. *For each level k of the SB_L, the minimal path MP_k is given by:* $(1, 1, 1, 2, 2, 2, 3, \ldots, \lceil \frac{k}{3} \rceil)$.

In order to give an example, we consider the Fibonacci sequence $\{f_k\}_{k=0}^{\infty} = 1, 1, 2, 3, 5, 8, 13, \ldots$, and define the *zig-zag* path to be the sequence of rational numbers obtained by the ratio of two consecutive elements. It is well known that the golden number is the limit of the sequence of the rational numbers $\frac{f_k}{f_{k+1}}$, with $k \geq 0$. The continued fraction representations of these rational numbers are of the form $[0, 1, 1 \ldots, 1, 2]$, and they constitute a (infinite) path in SB_L. This path realizes the lower bound stated in Theorem 2 assuring that, for each k, $PS(\frac{f_k}{f_{k+1}})$ is minimal, and it equals MP_k.

Now we focus our study on the questions: For each level k, is the *zig-zag* path the only minimal one? If no, can we characterize all the minimal paths in SB_L?

3.2 A connection between δ^2 value and the continued fraction representation

From Lemma 2, we know that $\delta^2\left(C\left(\frac{a}{b}\right)\right)$ is related to $\delta^2\left(C\left(\frac{u}{v}\right)\right)$ and $\delta^2\left(C\left(\frac{\rho}{\theta}\right)\right)$. In the final part of the section, we will show how this relation reflects on their continued fraction representations.

Theorem 3. *Let $\frac{a}{b} = [a_0, a_1, \cdots, a_z]$, we have:*

$$\delta^2\left(C\left(\frac{a}{b}\right)\right) = \begin{cases} \delta^2\left(C\left(\frac{u}{v}\right)\right) & if\ a_{z-1} \geq 2\ and\ a_z = 2 \\ \delta^2\left(C\left(\frac{\rho}{\theta}\right)\right) + 1 & elsewhere. \end{cases}$$

Proof. Let us consider the following continued fraction representations: $\frac{a}{b} = [a_0, a_1, \cdots, a_z]$; $\frac{u}{v} = [a_0, a_1, \cdots, a_{z-1} + 1]$; $\frac{p}{q} = [a_0, \cdots, a_{z-2} + 1]$; $\frac{s}{t} = [a_0, \cdots, a_{z-2}]$; and $\frac{\rho}{\theta} = \begin{cases} [a_0, a_1, \cdots, a_z - 2] & if\ a_z \geq 4 \\ [a_0, a_1, \cdots, a_{z-1} + 1] & if\ a_z = 3 \\ [a_0, a_1, \cdots, a_{z-2}] & if\ a_z = 2. \end{cases}$

Without loss of generality and in order to enlighten the notation, we denote $\delta^2\left(C\left(\frac{a}{b}\right)\right)$ by D_a, $\delta^2\left(C\left(\frac{u}{v}\right)\right) = D_u$, $\delta^2\left(C\left(\frac{\rho}{\theta}\right)\right) = D_r$, $\delta^2\left(C\left(\frac{p}{q}\right)\right) = D_p$ and $\delta^2\left(C\left(\frac{s}{t}\right)\right) = D_s$. Equivalently, by symmetry, we consider z to be an even number which means that the last element of $\Delta(\frac{a}{b})$ is equal to 1. We start by considering separately the cases $a_z \geq 4$ and $a_z = 3$, then we discuss the case $a_z = 2$.

1. If $a_z \geq 4$, we let $D_u = d$, then $D_r \geq d$ since $\frac{\rho}{\theta} > \frac{u}{v}$. Hence $D_a = \max(D_u, D_r + 1) = D_r + 1$ (see Figure 5, on the left).
2. If $a_z = 3$, in this case the fractions $\frac{u}{v}$ and $\frac{\rho}{\theta}$ are the same and we have: $D_a = \max(D_u, D_r + 1) = \max(D_r, D_r + 1) = D_r + 1$ (see Figure 5, on the right).

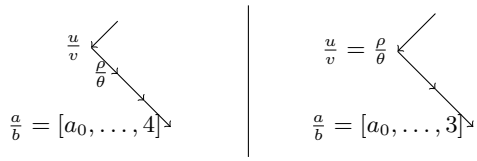
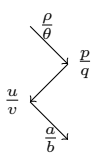


Fig. 5. Positions of the fractions $\frac{a}{b}, \frac{u}{v}, \frac{\rho}{\theta}$ on SB_L in case where the last element of the continued fraction of $\frac{a}{b} = a_z = 4$ or $a_z = 3$ respectively.

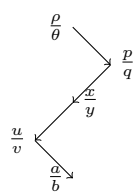
3. If $a_z = 2$, in this case we get several sub-cases depending on the values of a_{z-1} and a_{z-2}.

   (a) If $a_{z-1} = 1$, we get 4 sub-cases that are summarized in the table of Figure 6, where fraction $\frac{u}{v} = [a_0, \cdots, 2]$ and $D_r = d$.



D_r	d			
D_p	d		$d+1$	
D_u	d	$d+1$	$d+1$	$d+1$
D_a	$d+1$	$d+1$	$d+1$	$d+1$

Fig. 6. Position of the fractions $\frac{\rho}{\theta}, \frac{p}{q}, \frac{u}{v}$ and $\frac{a}{b}$, when $a_z = 2$, and $a_{z-1} = 1$, and the table showing the 4 possible subcases



D_r	d	
D_p	d	$d+1$
D_u	$d+1$	$d+2$
D_a	$d+1$	$d+2$

Fig. 7. Position of the fractions $\frac{\rho}{\theta}, \frac{p}{q}, \frac{u}{v}$ and $\frac{a}{b}$, when $a_z = 2$, and $a_{z-1} = 2$ $a_z = 2$ and the table representing the 2 sub-cases.

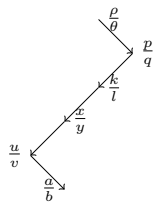
   If $D_p = d$, then D_u can be equal to d or $d + 1$. In the former case, it holds $D_a = \max(D_u, D_r + 1) = d + 1 = D_r + 1$, while, in the latter, it holds then $D_a = \max(d + 1, d + 1) = d + 1 = D_r + 1$.
   Acting similarly in the case $D_p = d + 1$, it holds $D_a = d + 1 = D_r + 1$.

   (b) If $a_{z-1} = 2$, the cases are represented in Figure 7.
   If $D_p = d$, then by Lemma 2, $D_u = d + 1$ and $D_a = d + 1 = D_u$. On the other hand, if $D_p = d + 1$, then $D_u = d + 2$ and $D_a = d + 2 = D_u$.

   (c) If $a_{z-1} = 3$, the rational numbers are represented in Figure 8 where the fraction $\frac{k}{l} = [a_0, \cdots, a_{z-2}, 2]$ with $\delta^2\left(C\left(\frac{k}{l}\right)\right) = D_k$ must be considered.

If $D_k = d$, by Lemma 2, it holds $D_u = d+1$ and $D_a = d+1 = D_u$, while if $D_k = d+1$,it holds $D_u = d+2$ and $D_a = d+2 = D_u$.

(d) Finally, if $a_{z-1} > 3$, it holds $D_u > D_r + 1$ therefore, $D_a = D_u$. □



D_r	d	
D_k	d	$d+1$
D_u	$d+1$	$d+2$
D_a	$d+1$	$d+2$

Fig. 8. Position of the fractions $\frac{\rho}{\theta}$, $\frac{p}{q}$, $\frac{k}{l}$, $\frac{u}{v}$ and $\frac{a}{b}$, when $a_z = 2$, *and* $a_{z-1} = 3$ and the table representing the 2 sub-cases.

3.3 General form of a minimal path

In this section we give the final result of the paper, by providing a characterization on the elements of the continued fraction of $\frac{a}{b}$ in order to consider $PS(\frac{a}{b})$ as a minimal path in SB_L.

At level 4 of the SB_L, only two rational numbers out of 4 belong to a minimal path as we can see in Table 1. We have $PS(\frac{3}{5}) = PS(\frac{3}{4}) = (1,1,1,2)$, where $\frac{3}{5} = [0,1,1,2]$ and $\frac{3}{4} = [0,1,3]$. This insures that the continued fraction of any rational number in SB_L, who belongs to a minimal path must starts with $[0,1]$. At level 5, we note that these two rational numbers generate 4 rational numbers that belong to a minimal path. While at level 6, we get again 4 rational numbers that are also represented in Table 1. By Definition 1, we can notice that three consecutive elements p_i, p_{i+1}, and p_{i+2}, of a minimal path are such that $p_{i+1} - p_i \leq 1$. Hence, all the children of the rational numbers of SB_L belonging to MP_6 maintain the minimality at levels 7 and 8. While at level 9, there exist only 16 $(= 4^2)$ fractions that belong to MP_9 and are represented in Table 2. Considering these first levels, we note that the continued fraction representations of the elements of MP_9 must start with $0, 1$ and end with 2. On the other hand, all the remaining δ^2 values of each element of MP_9 can be grouped into the blocks $(1,1,1)$; $(1,2)$; $(2,1)$ or 3 as the red and blue colored sequences in Table 2 witness. Aware of this, we can characterize the elements of SB_L that belongs to a minimal path.

Theorem 4. *Let $\frac{a}{b}$ be an element of SB_L at a certain level $k \geq 6$ and whose continued fraction representation is $[a_0, a_1, \ldots, a_z]$. The a_i's must respect the following conditions: $a_0 = 0$, $a_1 = 1$, $a_z = 2$ and the elements $a_3, \ldots, a_{z-1}$ are obtained by the concatenation of the blocks $(1,1,1)$, $(2,1)$ $(1,2)$ and (3).*

Proof. By the computation of the first minimal paths of SB_L we have that $a_0 = 0$ and $a_1 = 1$. From Lemma 2, we know that a_z must be equal to 2 in order to realize the minimal growth. Now, from Definition 1, we know that the rational number $\frac{a}{b}$ belongs to a minimal path if the values of $PS(\frac{a}{b})$ are increased by 1, each three steps. Relying on that, we consider all the possible ways to pass from a level t to a level $t+3$ in SB_L keeping the minimal growth of one, and we easily realize that these elements can only have one the following forms:
$(1,1,1)$, $(2,1)$ $(1,2)$ or (3), i.e. all the possible integer decompositions of the number three. Note that the first case, is equivalent to the *zig-zag* path, while the other cases are obtained by Theorem 3 and Lemma 2.

□

Level	Fraction $\frac{a}{b}$	Continued fraction	$P(\frac{a}{b})$	$PS(\frac{a}{b})$
4	$\frac{1}{4}$	$[0,4]$	$(\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4})$	$(1,1,2,2)$
4	$\frac{2}{5}$	$[0,2,2]$	$(\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{2}{5})$	$(1,1,2,2)$
4	$\frac{3}{5}$	$[0,1,1,2]$	$(\frac{1}{1},\frac{1}{2},\frac{2}{3},\frac{3}{5})$	$(1,1,1,2)$
4	$\frac{3}{4}$	$[0,1,3]$	$(\frac{1}{1},\frac{1}{2},\frac{2}{3},\frac{3}{4})$	$(1,1,1,2)$
6	$\frac{8}{13}$	$[0,1,1,1,1,2]$	$(\frac{1}{1},\frac{1}{2},\frac{2}{3},\frac{3}{5},\frac{5}{8},\frac{8}{13})$	$(1,1,1,2,2,2)$
6	$\frac{8}{11}$	$[0,1,2,1,2]$	$(\frac{1}{1},\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{5}{7},\frac{8}{11})$	$(1,1,1,2,2,2)$
6	$\frac{7}{12}$	$[0,1,1,2,2]$	$(\frac{1}{1},\frac{1}{2},\frac{2}{3},\frac{3}{5},\frac{4}{7},\frac{7}{12})$	$(1,1,1,2,2,2)$
6	$\frac{7}{9}$	$[0,1,3,2]$	$(\frac{1}{1},\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{7}{9})$	$(1,1,1,2,2,2)$

Table 1. The rational numbers $\frac{a}{b}$ at level 4 of SB_L with their continued fraction representation, $P(\frac{a}{b})$ and $PS(\frac{a}{b})$. The four rational numbers at level 6 that belong to a minimal path with their continued fraction representation, and sequences P and PS.

$\frac{29}{50}$	$\frac{27}{46}$	$\frac{26}{45}$	$\frac{23}{39}$
$[0,1,1,2,1,1,1,2]$	$[0,1,1,2,2,1,2]$	$[0,1,1,2,1,2,2]$	$[0,1,1,2,3,2]$
$\frac{34}{55}$	$\frac{30}{49}$	$\frac{31}{50}$	$\frac{25}{41}$
$[0,1,1,1,1,1,1,1,2]$	$[0,1,1,1,1,2,1,2]$	$[0,1,1,1,1,1,2,2]$	$[0,1,1,1,1,3,2]$
$\frac{34}{47}$	$\frac{30}{41}$	$\frac{31}{43}$	$\frac{25}{34}$
$[0,1,2,1,1,1,1,2]$	$[0,1,2,1,2,1,2]$	$[0,1,2,1,1,2,2]$	$[0,1,2,1,3,2]$
$\frac{29}{37}$	$\frac{27}{35}$	$\frac{26}{33}$	$\frac{23}{30}$
$[0,1,3,1,1,1,2]$	$[0,1,3,2,1,2]$	$[0,1,3,1,2,2]$	$[0,1,3,3,2]$

Table 2. The 16 rational numbers that belong to MP_9 on SB_L.

Finally, some simple computations lead to the following:

Corollary 1. *Let k be a certain level in SB_L, the number of MP_k is equal to:*

$$\begin{cases} 4^{\frac{k}{3}-1} & if\ k \equiv_3 0 \\ 2 \cdot 4^{\lfloor \frac{k}{3} \rfloor - 1} & if\ k \equiv_3 1 \\ 4 \cdot 4^{\lfloor \frac{k}{3} \rfloor - 1} & if\ k \equiv_3 2 \end{cases}$$

This last theorem determines the form of the minimal paths inside the SB-tree according to the δ^2-paths, and from a geometrical perspective, it defines a new family of Christoffel words that deserves to be investigated. A further generalization of the order of balancedness may also identify special paths of the SB-tree showing a fixed point property.

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