



**HAL**  
open science

# A local uniform boundedness result for an elliptic equation

Samy Skander Bahoura

► **To cite this version:**

Samy Skander Bahoura. A local uniform boundedness result for an elliptic equation. 2020. hal-02495444v2

**HAL Id: hal-02495444**

**<https://hal.science/hal-02495444v2>**

Preprint submitted on 18 Mar 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A local uniform boundedness result for an elliptic equation.

Samy Skander Bahoura\*

Equipe d'Analyse Complexe et Géométrie.

Université Pierre et Marie Curie, 75005 Paris, France.

## Abstract

We derive a local uniform boundedness result for an equation with logarithmic weight having interior singularity.

Keywords:  $C^0$  weight, interior singularity, a priori estimate, sup, inf,  $\sup u = f(\inf u)$ , maximum principle.

MSC: **35J60, 35B44, 35B45, 35B50**

## 1 Introduction and Main Results

We set  $\Delta = \partial_{11} + \partial_{22}$  on open bounded domain  $\Omega$  of  $\mathbb{R}^2$ .

We consider the following equation:

$$(P) \begin{cases} -\Delta u = \frac{1}{-\log \frac{|x|}{2d}} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

Here:

$$0 < a \leq V \leq b < +\infty, u \in L_{loc}^\infty(\Omega),$$

and,

$$d = \text{diam}(\Omega), 0 \in \Omega$$

This equation is defined in the sense of the distributions. Equations of the previous type were studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1–20], where one can find some existence and compactness results.

Among other results, we can see in [12] the following important Theorem

**Theorem A** (*Brezis-Merle [12]*). *If  $(u_i)$  is a sequence of solutions of problem (P) with  $(V_i)$  satisfying  $0 < a \leq V_i \leq b < +\infty$  and without the term  $\frac{1}{-\log \frac{|x|}{2d}}$ ,*

*then, for any compact subset  $K$  of  $\Omega$ , it holds:*

---

\*e-mails: samybahoura@yahoo.fr, samybahoura@gmail.com

$$\sup_K u_i \leq c,$$

with  $c$  depending on  $a, b, K, \Omega$

One can find in [12] an interior estimate if we assume  $a = 0$ , but we need an assumption on the integral of  $e^{u_i}$ , namely, we have:

**Theorem B** (Brezis-Merle [12]). For  $(u_i)_i$  and  $(V_i)_i$  two sequences of functions relative to the problem (P) without the term  $\frac{1}{-\log \frac{|x|}{2d}}$  and with,

$$0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C,$$

then for all compact set  $K$  of  $\Omega$  it holds;

$$\sup_K u_i \leq c,$$

with  $c$  depending on  $b, C, K$  and  $\Omega$ .

If we assume  $V$  with more regularity, we can have another type of estimates, a sup + inf type inequalities. It was proved by Shafrir see [19], that, if  $(u_i)_i$  is a sequence of functions solutions of the previous equation without assumption on the boundary and without the term  $\frac{1}{-\log \frac{|x|}{2d}}$  and with  $V_i$  satisfying  $0 < a \leq V_i \leq b < +\infty$ , then we have a sup + C inf inequality.

Here, we have:

**Theorem** For sequences  $(u_i)_i$  and  $(V_i)_i$  of the Problem (P), for all compact subset  $K$  of  $\Omega$  we have:

$$\|u_i\|_{L^\infty(K)} \leq c(a, b, K, \Omega).$$

Consider a positive number  $M > 0$ , if we assume  $u_i \geq -M$ , we can extend the previous result to any function  $u$  bounded from below by  $-M$ . If we consider the function  $v_i = u_i + M$ , then  $v_i$  satisfies all the condition of the previous theorem.

Here we have, if we replace the condition  $u_i \geq 0$  by  $u_i \geq -M$ :

**Corollary** For sequences  $(u_i)_i$  and  $(V_i)_i$  of the first equation of (P), for all compact subset  $K$  of  $\Omega$  we have:

$$\|u_i\|_{L^\infty(K)} \leq c(a, b, M, K, \Omega).$$

**Remarks:** 1) In general, the solutions of the previous equation are not  $C^2$ , but we can add assumptions on  $V$  to have  $C^2$  solutions and  $C^2$  compactness result. Indeed, if we add the conditions,  $V_i \in C^{0,\epsilon}$ ,  $\epsilon > 0$  and  $V_i \rightarrow V$  in the space  $C^{0,\epsilon}$ ,  $\epsilon > 0$ , one can reduce the problem to the Newtonian potential of radial distributions  $f(x) = f(|x|) = \frac{V_i(0)e^{u_i(0)}}{-\log \frac{|x|}{2d}} \eta(|x|)$  with  $\eta$  a radial cutoff

function with compact support and  $\eta \equiv 1$  in a neighborhood of 0, see the book of Dautray-Lions, [16].

2) We have an example of the previous problem with  $u_i \geq 0$  if we add the Dirichlet condition and use the maximum principle (in this case we assume  $u \in W_0^{1,1}(\Omega)$  and  $\frac{1}{-\log \frac{|x|}{2d}} e^u \in L^1(\Omega)$  or  $u \in W_0^{1,2}(\Omega) \Rightarrow e^u \in L^k(\Omega), \forall k \geq 1$

as a solution of a variational problem). From the counterexample constructed in the paper of Brezis and Merle, see [12], one can have an example of solutions with Dirichlet condition (but blowing-up solutions on the boundary).

3) We can replace the assumption  $u \in L_{loc}^\infty(\Omega)$  by :

$$u \in L_{loc}^1(\Omega) \text{ and } \frac{1}{-\log \frac{|x|}{2d}} V e^u \in L_{loc}^1(\Omega)$$

this, imply that  $u \in L_{loc}^\infty(\Omega)$ , (here, because  $0 < a \leq V \leq b < +\infty$  it is sufficient that  $\frac{1}{-\log \frac{|x|}{2d}} e^u \in L_{loc}^1(\Omega)$ ).

Indeed, by solving a Dirichlet Problem and use Theorem 1 of the paper of Brezis and Merle, see [12], and Weyl theorem, one can have  $e^{|u|} \in L_{loc}^k(\Omega)$  with  $k \geq 1$ , the result follows by the elliptic estimates of Agmon.

In the previous corollary we have:

$$\sup_K u \leq c(a, b, \inf_\Omega u, K, \Omega).$$

It is an estimate of the maximum on each compact subset of  $\Omega$  of the solutions by mean of the infimum on  $\Omega$  and  $a, b, K$  and  $\Omega$ . (Also we have an a priori estimate).

We ask the following questions about explicit dependance of  $\sup_K u$  in terms of  $\inf_\Omega u$  and inequality of type  $\sup + \inf$ , as in the work of Shafrir [19] and the work of Tarantello, see [20] and Bartolucci-Tarantello, see [9]:

**Problems.** 1) Consider the Problem (P) without the boundary condition (without Dirichlet condition) and assume that:

$$0 < a \leq V \leq b < +\infty,$$

Does exists constants  $C_1 = C_1(a, b, K, \Omega), C_2 = C_2(a, b, K, \Omega)$  such that:

$$\sup_K u + C_1 \inf_\Omega u \leq C_2,$$

for all solution  $u$  of (P) ?

2) If we add the condition  $\|\nabla V\|_\infty \leq A$ , can we have a sharp inequality:

$$\sup_K u + \inf_\Omega u \leq c(a, b, A, K, \Omega)?$$

## 2 Proof of the Theorem

We have:

$$u_i \in L_{loc}^\infty(\Omega).$$

Thus by the local boundedness elliptic estimates of Agmon and the Sobolev embedding, see [1, 2], we have:

$$u_i \in W_{loc}^{2,k}(\Omega) \cap C^{1,\epsilon}(\Omega), \quad k > 2.$$

*Step 1:* For all  $x_0 \in \Omega$ , there is  $r > 0$  such that  $\int_{B(x_0,r)} \frac{1}{-\log \frac{|x|}{2d}} e^{u_i} dx$  is

bounded.

Let us consider  $x_0 \in \Omega$  and set  $\phi_1$  the first eigenfunction of the Laplacian with Dirichlet condition and with corresponding eigenvalue  $\lambda_1 > 0$  for the ball  $B(x_0, 2r) \subset \Omega$ . We use integration by parts between  $u_i$  and  $\phi_1$ . The Stokes formula gives:

$$\int_{B(x_0,2r)} \frac{1}{-\log \frac{|x|}{2d}} V_i e^{u_i} \phi_1 dx = \lambda_1 \int_{B(x_0,2r)} u_i \phi_1 dx - \int_{\partial B(x_0,2r)} \partial_\nu \phi_1 u_i dx,$$

In the first part of the right hand side, we write ( $\phi_1 \geq 0$ ):

$$u_i \phi_1 = \sqrt{\left(-\log \frac{|x|}{2d}\right)} \phi_1 \times \sqrt{\frac{\phi_1}{-\log \frac{|x|}{2d}}} u_i,$$

and, we apply the Holder inequality. Again, we use the fact that:  $\partial_\nu \phi_1 \geq 0$ ,  $u_i \geq 0$  and  $V_i \geq a > 0$  to have:

$$\int_{B(x_0,2r)} \frac{1}{-\log \frac{|x|}{2d}} e^{u_i} \phi_1 dx \leq C,$$

since  $\phi_1 > 0$  on  $B(x_0, 2r)$ , the result follows.

*Step 2:*  $u_i^+ = u_i$  is locally bounded in  $L^1$ .

We write:

$$u_i = \sqrt{-\log \frac{|x|}{2d}} \times \sqrt{\frac{1}{-\log \frac{|x|}{2d}}} u_i,$$

We use the Holder inequality to have:

$$\|u_i\|_{L^1(B(x_0,r_0))} \leq \left\| \sqrt{-\log \frac{|x|}{2d}} \right\|_{L^2(B(x_0,r_0))} \times \left\| \sqrt{\frac{1}{-\log \frac{|x|}{2d}}} u_i \right\|_{L^2(B(x_0,r_0))},$$

Thus,

$$\begin{aligned} \|u_i\|_{L^1(B(x_0, r_0))} &\leq \left( \left\| -\log \frac{|x|}{2d} \right\|_{L^1(B(x_0, r_0))} \right)^{1/2} \times \left( \left\| \frac{1}{-\log \frac{|x|}{2d}} u_i^2 \right\|_{L^1(B(x_0, r_0))} \right)^{1/2} \leq \\ &\leq C \int_{B(x_0, r)} \frac{1}{-\log \frac{|x|}{2d}} e^{u_i} dx \leq C. \end{aligned}$$

Thus,

$$\|u_i\|_{L^1(B(x_0, r_0))} \leq C.$$

Since,

$$\int_{B(y, r)} \frac{1}{-\log \frac{|x|}{2d}} V_i e^{u_i} dx \leq C,$$

We have a convergence to a nonnegative measure  $\mu$ :

$$\int_{B(y, r)} \frac{1}{-\log \frac{|x|}{2d}} V_i e^{u_i} \phi dx \rightarrow \int_{B(y, r)} \phi d\mu, \quad \forall \phi \in C_c(B(y, r)).$$

We set  $S$  the following set:

$$S = \{x \in B(y, r), \exists (x_i) \in \Omega, x_i \rightarrow x, u_i(x_i) \rightarrow +\infty\}.$$

We say that  $x_0$  is a regular point of  $\mu$  if there function  $\psi \in C_c(B(y, r))$ ,  $0 \leq \psi \leq 1$ , with  $\psi = 1$  in a neighborhood of  $x_0$  such that:

$$\int \psi d\mu < 4\pi. \quad (1)$$

We can deduce that a point  $x_0$  is non-regular if and only if  $\mu(x_0) \geq 4\pi$ .

A consequence of this fact is that if  $x_0$  is a regular point then:

$$\exists R_0 > 0 \text{ such that one can bound } (u_i) = (u_i^+) \text{ in } L^\infty(B_{R_0}(x_0)). \quad (2)$$

We deduce (2) from corollary 4 of Brezis-Merle paper, because we have by step 2 locally the inequality:

$$\|u_i^+\|_1 = \|u_i\|_1 \leq c.$$

We denote by  $\Sigma$  the set of non-regular points.

*Step 3:*  $S = \Sigma$ .

We have  $S \subset \Sigma$ . Let's consider  $x_0 \in \Sigma$ . Then we have:

$$\forall R > 0, \lim \|u_i^+\|_{L^\infty(B_R(x_0))} = +\infty. \quad (3)$$

Suppose contrary that:

$$\|u_i^+\|_{L^\infty(B_{R_0}(x_0))} \leq C.$$

Then:

$$\|e^{u_{i_k}}\|_{L^\infty(B_{R_0}(x_0))} \leq C, \text{ and}$$

$$\int_{B_R(x_0)} \frac{1}{-\log \frac{|x|}{2d}} V_{i_k} e^{u_{i_k}} = o(1).$$

For  $R$  small enough, which imply (1) for a function  $\psi$  and  $x_0$  will be regular, contradiction. Then we have (3). We choose  $R_0 > 0$  small such that  $B_{R_0}(x_0)$  contain only  $x_0$  as non-regular point.  $\Sigma$ . Let's  $x_i \in B_R(x_0)$  such that:

$$u_i^+(x_i) = \max_{B_R(x_0)} u_i^+ \rightarrow +\infty.$$

We have  $x_i \rightarrow x_0$ . Else, there exists  $x_{i_k} \rightarrow \bar{x} \neq x_0$  and  $\bar{x} \notin \Sigma$ , i.e.  $\bar{x}$  is a regular point. It is impossible because we would have (2).

Since the measure is finite, if there are blow-up points, or non-regular points,  $S = \Sigma$  is finite.

*Step 4:*  $\Sigma = \{\emptyset\}$ .

Now: suppose contrary that there exists a non-regular point  $x_0$ . We choose a radius  $R > 0$  such that  $B_R(x_0)$  contain only  $x_0$  as non-regular point. Thus outside  $\Sigma$  we have local uniform boundedness of  $u_i$ , also in  $C^1$  norm. Also, we have weak \*-convergence of  $V_i$  to  $V \geq 0$  with  $V \leq b$ .

Let's consider (by a variational method):

$$z_i \in W_0^{1,2}(B_R(x_0)),$$

$$-\Delta z_i = f_i = \frac{1}{-\log \frac{|x|}{2d}} V_i e^{u_i} \text{ in } B_R(x_0), \text{ et } z_i = 0 \text{ on } \partial B_R(x_0).$$

By a duality theorem:

$$z_i \in W_0^{1,q}(B_R), \|\nabla z_i\|_q \leq C_q.$$

By the maximum principle,  $u_i \geq z_i$  in  $B_R(x_0)$ .

$$\int \frac{1}{-\log \frac{|x|}{2d}} e^{z_i} \leq \int \frac{1}{-\log \frac{|x|}{2d}} e^{u_i} \leq C. \quad (4)$$

On the other hand,  $z_i \rightarrow z$  a.e. (uniformly on compact sets of  $B_R(x_0) - \{x_0\}$ ) with  $z$  solution of :

$$-\Delta z = \mu \text{ in } B_R(x_0), \text{ et } z = 0 \text{ on } \partial B_R(x_0).$$

Also, we have up to a subsequence,  $z_i \rightarrow z$  in  $W_0^{1,q}(B_R(x_0))$ ,  $1 \leq q < 2$  weakly, and thus  $z \in W_0^{1,q}(B_R(x_0))$ .

Then by Fatou lemma:

$$\int \frac{1}{-\log \frac{|x|}{2d}} e^z \leq C. \quad (5)$$

As  $x_0 \in S$  is not regular point we have  $\mu(\{x_0\}) \geq 4\pi$ , which imply that,  $\mu \geq 4\pi\delta_{x_0}$  and by the maximum principle in  $W_0^{1,1}(B_R(x_0))$  (obtained by Kato's inequality)

$$z(x) \geq 2 \log \frac{1}{|x - x_0|} + O(1) \text{ if } x \rightarrow x_0.$$

Because,

$$z_1 \equiv 2 \log \frac{1}{|x - x_0|} + 2 \log R \in W_0^{1,s}(B_R(x_0)), \quad 1 \leq s < 2.$$

Thus,

$$\frac{1}{-\log \frac{|x|}{2d}} e^z \geq \frac{C}{-|x - x_0|^2 \log \frac{|x|}{2d}}, \quad C > 0.$$

Both in the cases  $x_0 = 0$  and  $x_0 \neq 0$  we have:

$$\int_{B_R(x_0)} \frac{1}{-\log \frac{|x|}{2d}} e^z = \infty.$$

However, by (5):

$$\int \frac{1}{-\log \frac{|x|}{2d}} e^z \leq C.$$

which a contradiction.

## References

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, 1998.
- [2] S. Agmon. Lectures on Elliptic Boundary Values Problems. Van Nostrand 1965.
- [3] C. Bandle. Isoperimetric Inequalities and Applications. Pitman, 1980.
- [4] Bahoura, S.S. About Brezis Merle problem with Lipschitz condition. ArXiv:0705.4004.
- [5] Bartolucci, D. A "sup+Cinf" inequality for Liouville-type equations with singular potentials. Math. Nachr. 284 (2011), no. 13, 1639-1651.
- [6] Bartolucci, D. A "sup+Cinf" inequality for the equation  $-\Delta u = V e^u / |x|^{2\alpha}$ . Proc. Roy. Soc. Edinburgh Sect. A 140 (2010), no. 6, 1119-1139.

- [7] Bartolucci, D. A sup+inf inequality for Liouville type equations with weights. *J. Anal. Math.* 117 (2012), 29-46.
- [8] Bartolucci, D. A sup  $\times$  inf-type inequality for conformal metrics on Riemann surfaces with conical singularities. *J. Math. Anal. Appl.* 403 (2013), no. 2, 571-579.
- [9] Bartolucci, D. Tarantello. G. The Liouville equation with singular data: a concentration-compactness principle via a local representation formula, *Journal of Differential Equations* 185 (2002), 161-180.
- [10] L. Boccardo, T. Gallouet. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.* 87 no 1, (1989), 149-169.
- [11] H. Brezis, YY. Li and I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. *J.Funct.Anal.*115 (1993) 344-358.
- [12] H. Brezis, F. Merle. Uniform estimates and Blow-up behavior for solutions of  $-\Delta u = V(x)e^u$  in two dimension. *Commun. in Partial Differential Equations*, 16 (8 and 9), 1223-1253(1991).
- [13] H. Brezis, W. A. Strauss. Semi-linear second-order elliptic equations in L1. *J. Math. Soc. Japan* 25 (1973), 565-590.
- [14] W. Chen, C. Li. A priori estimates for solutions to nonlinear elliptic equations. *Arch. Rational. Mech. Anal.* 122 (1993) 145-157.
- [15] C-C. Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in  $\mathbb{R}^2$ . *Commun. Anal. Geom.* 6, No.1, 1-19 (1998).
- [16] R. Dautray. J-L. Lions. Part 2, Laplace operator.
- [17] YY. Li, I. Shafrir. Blow-up analysis for solutions of  $-\Delta u = Ve^u$  in dimension two. *Indiana. Math. J.* Vol 3, no 4. (1994). 1255-1270.
- [18] YY. Li. Harnack Type Inequality: the method of moving planes. *Commun. Math. Phys.* 200,421-444 (1999).
- [19] I. Shafrir. A sup+inf inequality for the equation  $-\Delta u = Ve^u$ . *C. R. Acad.Sci. Paris Sér. I Math.* 315 (1992), no. 2, 159-164.
- [20] G. Tarantello. A Harnack inequality for Liouville-type equation with Singular sources. *Indiana University Mathematics Journal.* Vol 54, No 2 (2005). pp 599-615.