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Sammy Khalife, Yann Ponty, Laurent Bulteau

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1 Sequence graphs realizations and ambiguity in 2 language models

3 **Sammy Khalife**

4 LIX, CNRS, Ecole Polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau, France
5 khalife@lix.polytechnique.fr

6 **Yann Ponty**

7 LIX, CNRS, Ecole Polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau, France
8 yann.ponty@lix.polytechnique.fr

9 **Laurent Bulteau**

10 LIGM, CNRS, Université Gustave Eiffel, 77454 Marne-la-Vallée, France
11 laurent.bulteau@univ-eiffel.fr

12 — Abstract —

13 Several language models rely on an assumption modeling each local context as a (potentially oriented)
14 bag of words, and have proven to be very efficient baselines. Sequence graphs are the natural
15 structures encoding their information. However, a sequence graph may have several realizations
16 as a sequence, leading to a degree of ambiguity. In this paper, we study such degree of ambiguity
17 from a combinatorial and computational point of view. In particular, we present theoretical results
18 concerning the family of sequence graphs. Several combinatorial problems are presented, depending
19 on three levels of generalisation (window size, graph orientation, and weights), and whether some
20 of these are NP-complete is left opened. We establish different algorithms, including an integer
21 program and a dynamic programming formulation to respectively recognize a sequence graph and to
22 count the number of its distinct realizations. This allows us to show that this model assumption can
23 induce an important number of sentences to have the same representations. We empirically compare
24 the representations obtained with a recurrent neural networks for different realizations of sequence
25 graphs.

26 **2012 ACM Subject Classification** Mathematics of computing → Combinatoric problems; Mathem-
27 atics of computing → Combinatorics on words; Mathematics of computing → Graph algorithms;
28 Theory of computation → Complexity classes; Theory of computation → Problems, reductions and
29 completeness

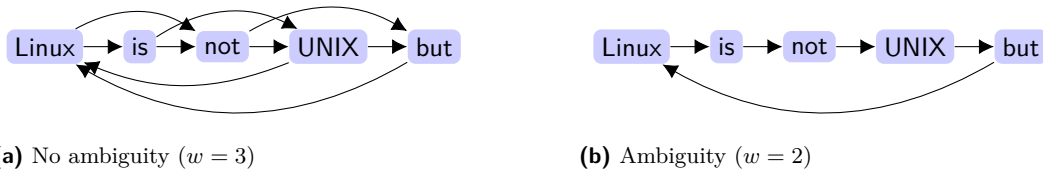
30 **Keywords and phrases** Graphs, Sequences, Combinatorics, Inverse problem, Complexity class

31 **1** Introduction

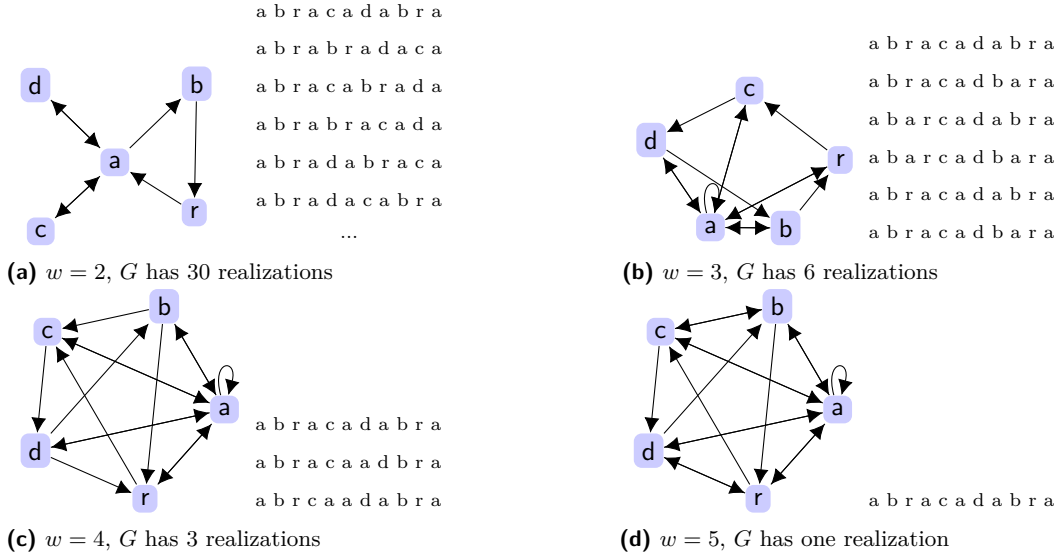
32 The automated treatment of familiar objects, either natural or artifacts, always relies on a
33 translation into entities manageable by computer programs. However, the correspondence
34 between the object to be treated and "its" representation is not necessarily one-to-one. The
35 representations used for learning algorithms are no exception to this rule. In particular,
36 natural language words and textual documents representations are essential for several tasks,
37 including document classification [23], role labelling [19], and named entity recognition
38 [16]. The traditional models based on pointwise mutual information, or graph-of-words
39 (GOW), [9, 17, 20], supplement the content of bag-of-words (TF, TFIDF) with statistics
40 of co-occurrences within a **window** of fixed size w , introduced to mitigate the degree of
41 ambiguity. Several models [2, 14, 18, 21] also use the same type of information and constitute
42 strong baselines for natural language processing.

43 While these representations are more precise than the traditional bag-of-words (e.g Parikh
44 vectors), they still induce some level of ambiguity, *i.e.* a given graph can represent several
45 sequences. Our study is thus motivated by a quantification of the level of ambiguity, seen

Sequence graphs realizations and ambiguity in language models



■ **Figure 1** Sequence graphs (or *graphs-of-words*) built for the sentence “Linux is not UNIX but Linux” using window sizes 3 (a) and 2 respectively (b). In the second case, the sequence graph is ambiguous, since any circular permutation of the words admits the same representation.



■ **Figure 2** Sequence graphs (or *graphs-of-words*) built for the sentence “a b r a c a d a b r a” using window sizes 2 (a), 3 (b), 4 (c) and 5 (d).

46 as an algorithmic problem, coupled with an empirical assessment of the consequences of
47 ambiguity for the representations.

48 After introducing in Section 2 the formal definition of a sequence graph and the descriptions
49 of our main problems, we establish in Section 3.1 complexity aspects of deciding the existence
50 and counting sequences in GOWs associated with a window size $w = 2$. Then we consider
51 in Section 3.2 the general case $w \geq 3$, and propose a integer program and a dynamic
52 programming algorithm to respectively recognize a sequence graph and count admissible
53 sequences. Finally, we assess the prevalence of ambiguity within a synthetic dataset, and
54 observe that sequences invariant with respect to the GOW representation do not lead to
55 invariance with respect to recurrent neural networks such as Long Short Term Memory
56 networks (LSTMs).

57 Related work

58 Sequence graphs encode the information of several co-occurrences based models [2, 15, 18]. To
59 the best of our knowledge, the ambiguity and realizability questions addressed in this work
60 were never systematically addressed by prior work in computational linguistics. Furthermore,
61 we believe the problems studied in this paper are interesting from an algorithmic point of
62 view, and appear to be devoid of reduction to other well-known problems.

63 However, some similarities exist between our problem and others studied in the Distance
 64 Geometry (DG) literature. In distance geometry, the input consists of a set of pairwise
 65 distances between points, having unknown positions in a d -dimensional space. The problem
 66 then consists in determining (the existence of) a set of positions for the points, satisfying the
 67 distance constraints. Since a position is fully characterized from $d + 1$ constraining neighbors,
 68 the problem can be solved by finding a sequential order for processing points, such that the
 69 assignment of a point is always by at least $d + 1$ among its neighbors [13]. This statement
 70 shares some level of similarity with our problem since an admissible sequence for a window
 71 $w = d + 2$ also represents a linear ordering of its nodes, in which $w - 1 = d + 1$ of the
 72 neighbors have lower value with respect to the order.

73 The reasons for the insufficiency of linear ordering in DG to solve our realizability problem
 74 are threefold. First, each element of the sequence x associated to the protein backbone is
 75 associated a unique vertex. This is not the case we investigate here, since a symbol can be
 76 repeated several times, but only one vertex is created in the graph. This implies that the
 77 vertex associated to the i^{th} element ($i \geq w$) of x can have strictly less than $w - 1$ distinct
 78 neighbors in its predecessors in x . Second, the absence of loops in distance geometry, because
 79 an element is at distance 0 from itself. Finally, the graphs are always undirected in distance
 80 geometry.

81 **2** Definitions and problem statement

82 Let $x = x_1, x_2, \dots, x_p$ be a finite sequence of discrete elements among a finite vocabulary
 83 X . Without loss of generality, we can suppose that $X = \{1, \dots, n\}$. In the following, let
 84 $I_p = \{1, \dots, p\}$. This motivates the following definition:

85 **► Definition 1.** $G = (V, E)$ is the graph of the sequence x with window size $w \in \mathbb{N}^*$ if and
 86 only if $V = \{x_i \mid i \in I_p\}$, and

$$87 \quad (i, j) \in E \iff \exists(k, k') \in I_p^2, |k - k'| \leq w - 1, x_k = i \text{ and } x_{k'} = j \quad (1)$$

88 For digraphs, Eq. (1) is replaced with

$$89 \quad (i, j) \in E \iff \exists(k, k') \in I_p^2, k \leq k' \leq k + w - 1, x_k = i \text{ and } x_{k'} = j. \quad (2)$$

90 Finally, a weighted sequence graph G is endowed with a matrix $\Pi(G) = (\pi_{ij})$ such that

$$91 \quad \pi_{ij} = \text{Card} \{(k, k') \in I_p^2 \mid k \leq k' \leq k + w - 1, x_k = i \text{ and } x_{k'} = j\} \quad (3)$$

92 We say that x is a w -admissible sequence for G (or a realization of G), if G is the graph of
 93 sequence x with window size w .

94 The natural integers π_{ij} represent the number of co-occurrences of i and j in a window
 95 of size w . Hence, the graph of sequence is unique. An linear time algorithm to construct a
 96 weighted sequence digraph is presented in Sec. A of the appendix. Other cases are obtained
 97 similarly. The procedure in algorithm 1 defines a correspondence between the sequence set
 98 X^* into the graph set $\mathcal{G} : \phi_w : X^* \rightarrow \mathcal{G}, x \mapsto G_w(x)$. Based on these definitions, we consider
 99 the following problems:

100 **► Problem 1** (Weighted-REALIZABLE (W-REALIZABLE)).

101 **Input:** Possibly directed graph G , matrix weights Π , window size w

102 **Output:** True if (G, Π) is the w -sequence graph of some sequence x , False otherwise.

103 ► **Problem 2** (Unweighted-REALIZABLE (U-REALIZABLE)).

104 *Input:* Possibly directed graph G , window size w

105 *Output:* True if G is the w -sequence graph of some sequence x , False otherwise.

106 We denote D -REALIZABLE (resp. G -) the restricted version of REALIZABLE where the
 107 input graph G is directed (resp. undirected), and W -REALIZABLE (resp. U -) the restricted
 108 version of REALIZABLE where the input graph G is weighted (resp. unweighted), possibly
 109 in combination with the D - or G - variants. We write REALIZABLE_w for the case where w
 110 is a fixed (given) constant. We also consider the variants of W -REALIZABLE, denoted WG -
 111 REALIZABLE and WD -REALIZABLE where the input graph is restricted to be respectively
 112 undirected and directed. We define UG -REALIZABLE and UD -REALIZABLE similarly. Finally,
 113 we write $(WG\text{-}, WD\text{-}, \dots)\text{REALIZABLE}_w$ for the case where w is a fixed strictly positive integer.

114 ► **Problem 3** (Unweighted-NUMREALIZATIONS (U-NUMREALIZATIONS)).

115 *Input:* Possibly directed graph G , window size w

116 *Output:* The number of **realizations** of G , i.e. preimages of G through ϕ_w such that
 117 $|\{x \in X^* \mid \phi_w(x) = G\}|$ if finite, or $+\infty$ otherwise.

118 ► **Problem 4** (Weighted-NUMREALIZATIONS (W-NUMREALIZATIONS)).

119 *Input:* Possibly directed graph G , matrix weights Π , window size w

120 *Output:* The number of **realizations** of G in the weighted sense.

121 Similarly, we use the same prefix for the directed or undirected versions of (D -, G -, i.e.
 122 DU - for directed and unweighted). We also denote NUMREALIZATIONS_w for the case where
 123 w is a fixed strictly positive integer. Note that NUMREALIZATIONS strictly generalizes the
 124 previous one, as REALIZABLE can be solved by testing the nullity of the number of suitable
 125 realization computed by NUMREALIZATIONS .

126

DW Directed weighted	DU Directed unweighted
GW Undirected weighted	GU Undirected unweighted

127 **3 Theoretical results**

128 In this section, we present our main theoretical results. Due to length limitations, some of
 129 the proofs are left in the appendix.

130 **3.1 A complete characterization of 2-sequence graphs**

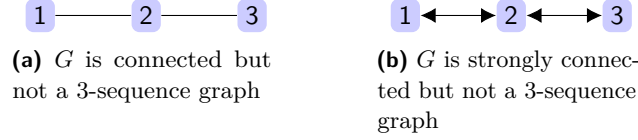
131 A graph has a sequential realization with $w = 2$ when there exists a path visiting every vertex
 132 and covering all of its edges (at least once for the unweighted case and exactly π_e for the
 133 edge e in the weighted case). This characterization enables relatively simple characterization
 134 and algorithmic treatment, leading to the results summarized in Table 2.

■ **Table 1** Complexity for various instances of our problems ($w = 2$)

Data Instance	NUMREALIZATIONS ₂		REALIZABLE ₂	
	Complexity	#Sequences	Complexity	Characterization
Unweighted graph	P	$\{0, +\infty\}$	P	G connected
Weighted graph	#P-hard	$\{0, 1\} \cup 2\mathbb{N}^*$	P	$\psi(G)$ (semi) Eulerian
Unweighted digraph	P	$\{0, 1, +\infty\}$	P	Theorem 14
Weighted digraph	P	\mathbb{N} (BEST Theorem)	P	$\psi(G)$ (semi) Eulerian

135 3.2 General sequence graphs and $\text{REALIZABLE}_{w \geq 3}$

136 The characterization of more general sequence graphs, such as 3-graphs is not the same for
 137 2-graphs, as shows the counter-example in Fig 3a: the depicted graph has no self-edge so
 138 there must at least one clique of size 3. Similarly, Fig. 3b depicts a counter example for
 139 directed graphs: G does not have loops, so if it had a 3-admissible sequence, such sequence
 140 must be of the form $\{1231\dots, 1321\dots, 2312\dots, 3213\dots, 2132\dots\}$ but then $(2, 1)$ would form
 141 an edge.



■ Figure 3 Counter examples for $w = 3$

142 3.3 A polynomial time algorithm for GU-REALIZABLE_w

143 Similarly to the procedure in Sec. B, we will use an auxiliary graph built on G . Let
 144 $H(G) = (E, H_E)$ be the new graph obtained with the following procedure. Two edges
 145 $e = (v_1, v_2)$, $f = (v_3, v_4)$ of E are connected in $H(G)$ if and only if:

$$146 \quad v_2 = v_3 \text{ and } (v_1, v_4) \in E \quad (4)$$

147 This defines an injective function $\tilde{h} : E_H \rightarrow V^3$: an edge of $H(G)$ can be seen as an
 148 unique triplet v_1, v_2, v_3 where $(v_1, v_2), (v_1, v_3)$ and $(v_2, v_3) \in E$. Therefore, by definition, a
 149 walk P in $H(G)$ is always of the form:

$$150 \quad P = (t_1, t_2), \dots, (t_{p-1}, t_p) \text{ s.t } \forall i \in \{1, \dots, p-1\}, (t_i, t_{i+1}) \in E \quad (5)$$

151 It is clear that if $H(G)$ is a 2-graph, then G is a 3-graph since there is a walk going
 152 through all edges of $H(G)$ (so visiting every non isolated node and creating all edges of G).
 153 However, the converse is not true as depicted in Fig. 4. In order to determine if $G = (V, E)$
 154 has an admissible sequence in the general case, a procedure is to recursively merge pairs of
 155 vertices, maintaining constraints depending on E . These constraints are similar to Eq. 4. We
 156 adopt the following notations, $u_{i,j} = (u_i, u_j)$ and $u_{1:k} = (u_1, \dots, u_k)$. The iterative procedure
 157 for $w \geq 3$ is summed up in the following equation. Namely, $\forall k \in \{2, \dots, w-2\}$, one has

$$158 \quad E^{(k)} = \{u_{1:k+1} \in V^{k+1} \mid u_{1:k} \in E^{(k-1)}, u_{2:k+1} \in E^{(k-1)} \wedge (u_1, u_{k+1}) \in E\} \quad (6)$$

159 Let $H^{(k)} = (E^{(k)}, E^{(k+1)})$, it can be defined recursively through:

$$160 \quad H^{(0)} = G \quad \forall k \in \mathbb{N}^*, H^{(k)} = f(H^{(k-1)}) \quad (7)$$

162 where f transforms edges into vertices and creates edges between new vertices that verify
 163 Eq. 6.

164 ► **Definition 2.** Let u be a vertex of $H^{(k)}$ for $k \in \mathbb{N}$, $u = (u_1, \dots, u_k, u_{k+1})$. The sequence
 165 u_1, \dots, u_{k+1} is the **authentic** sequence of u . We also call an authentic sequence of a walk on
 166 $H^{(k)}$: $P = (x_1, \dots, x_{k+1}), (x_2, \dots, x_{k+2}), \dots, (x_v, \dots, x_{v+k})$ the sequence x_1, x_2, \dots, x_{v+k} .

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167 In order to obtain admissible sequences of length p , the computation of $H^{(p)}$ requires p
 168 iterations, and the number of vertices and edges of $H^{(k)}$ can increase during iterations (the
 169 complete graph is an example for which these numbers increase exponentially).

170 ► **Proposition 3.** *Let $x = x_1, \dots, x_p$ be a w -admissible sequence of a graph (or digraph)
 171 $G = (V, E)$. If $w \leq p$, x , then x is an authentic sequence of a walk of length $p - w + 1$ on
 172 $H^{(w-2)}$.*

Proof. Due to length limitation, we provide a proof sketch, full proof is left in the appendix.
 The following property by induction on k :

$\forall k \in \{w, \dots, p\}, \exists \text{ walk } P \text{ on } H^{(w-2)} \text{ such that :}$

$$x_{1:k} = P[1]_1, P[2]_1, \dots, P[k-w]_1, P[k-(w-1)]_{1:(w-1)}$$

173 • **Initialisation:** $k = 1$. By construction of $H^{(w-2)}$, x_1 is the first element of the “static
 174 walk”: $x_{1:w-1} \in H^{(w-2)}$.

175 • **Induction:** Verification that if $x_{1:k}$ is a walk of length $k - w + 1$, one can find a walk of
 176 length $(k + 1) - w + 1$ to generate $x_{1:(k+1)}$. ◀

177 ► **Theorem 4.** *Let $w \in \mathbb{N}^*$. GU-REALIZABLE_w is in P .*

178 **Proof.** The case for $w = 1$ is trivial, and $w = 2$ has been treated. For $w \geq 3$, an algorithm
 179 is obtained by going through all the connected components of $H^{(w-2)}$. Let C_1, \dots, C_m the
 180 connected components of $H^{(w-2)}$. On the one hand, it is possible to compute them in
 181 polynomial time. On the other hand, it is possible to construct walks covering all of their
 182 respective edges in polynomial time (for instance iteratively using shortest paths). Let
 183 W_1, \dots, W_m such walks and X_1, \dots, X_m their respective admissible sequences.

184 Using Prop. 3, G is a w -sequence graph if and only if there exists a walk \tilde{W}_{i_0} on some
 185 C_{i_0} creating exactly the edges of G . However, W_{i_0} creates more edges than any walk on C_{i_0}
 186 by construction.

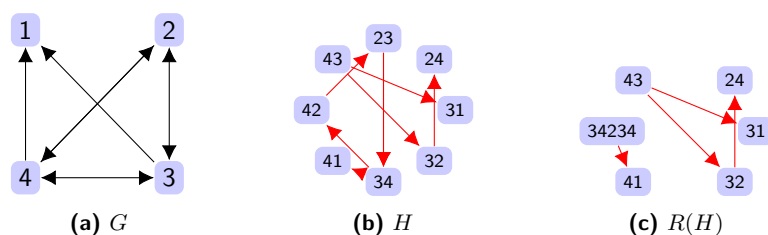
In conclusion, the assertion:

$$\exists i \in \{1, \dots, m\}, \phi_w(X_i) = G$$

187 is a characterization that G is a w -sequence. This assertion is decidable in polynomial time
 188 since for all i , $\phi_w(X_i)$ is computable in polynomial time (cf. Algorithm 1). ◀

189 For digraphs, the analogue of the aforementioned procedure would consist in enumerating
 190 all paths in the DAG $R(H^{(w-2)})$. However, the number of paths can be exponential, even for
 191 a sequence graph. In the next subsection, we will prove that DU-REALIZABLE_w is actually
 192 NP-hard. Finally, if x_1, \dots, x_c are vertices of a strongly component of $H^{(w-2)}$, which order
 193 should be considered to form a new vertex attribute x_C ? The following lemma shows that
 194 this order is not important, as long as it represents a walk in the component. Moreover, it
 195 is possible to reconstruct all admissible sequences from walks on $R(H^{(w-2)})$. With the same
 196 notations:

197 ► **Lemma 5.** *Let x a walk on $H^{(w-2)}$ whose authentic sequence is w -admissible for G . If x
 198 goes through a strongly component C of $H^{(w-2)}$, adding any supplementary path included in
 199 C is stable for w -admissibility. Any graph generated by a walk on $H^{(w-2)}$ can be generated
 200 by a walk on $R(H^{(w-2)})$.*



■ **Figure 4** Procedure to find a 3-admissible sequence. 34234, 41: is 3-admissible, with authentic sequence 342341

201 **Proof.** We present a proof sketch. The first statement concerning stability requires a
 202 straightforward verification using the definition of $H^{(w-2)}$. Second, a procedure to generate
 203 G from a walk on $R(H^{(w-2)})$ using a walk $x_{1:p}$ on $H^{(w-2)}$ is to consider an iterative scheme,
 204 and discuss three cases:

- 205 ■ (i) x_i and x_{i+1} are not in a strongly connected component (SCC)
- 206 ■ (ii) x_i is not in a SCC and x_{i+1} is in a SCC
- 207 ■ (iii) x_i and x_{i+1} are both in SCCs

208 For case (i), we just keep x_i and x_{i+1} . For cases (ii) and (iii), we use the first part result of
 209 the Lemma and add covering walks over the strongly connected components. ◀

210 3.4 Main complexity results

211 In this subsection we present the remaining complexity results, which are summarized in
 212 Table 2. In the previous subsection, we proved that $\text{GU-REALIZABLE}_w \in P, \forall w \geq 3$. Besides,
 213 for GU, the number of realizations of a graph G is either 0 (not realizable), $+\infty$ (realizable
 214 and there exists a cycle in a component of H generating G), or 1 (realizable but no cycle
 215 in any component of H generating G). These three cases can be tested in polynomial time
 216 using our algorithm, showing that $\text{GU-NUMREALIZATIONS}_w \in P, \forall w \geq 3$. In the remaining
 217 of this section, we present the reductions we used for the other instances.

■ **Table 2** Complexity for various instances of our problems ($w \geq 3$). We remind that a para-NP-hard problem does not admit any XP algorithm unless $P=NP$.

Variation	Constant $w, w \geq 3$		Parameter w	
	NUMREALIZATIONS_w Complexity	REALIZABLE_w Complexity	NUMREALIZATIONS Complexity	REALIZABLE Complexity
GU	P	P	W[1]-hard; XP	W[1]-hard; XP
GW	NP-hard	NP-hard	para-NP-hard	para-NP-hard
DU	NP-hard	NP-hard	para-NP-hard	para-NP-hard
DW	NP-hard	NP-hard	para-NP-hard	para-NP-hard

218 ▶ **Proposition 6.** CLIQUE admits a polynomial time parameterized reduction into GU-
 219 REALIZABLE.

220 **Proof.** Let $G = (V, E)$ be a simple graph. Let G' be a graph constructed from G adding
 221 two nodes a and b with loops, such that a and b are connected to each vertex of G . Let k be
 222 a strictly positive integer and $w = k + 1$. We will show that G has a k -clique if and only if
 223 G' is w -realizable.

First, let us suppose that G has a k -clique. Let C be an arbitrary sequence of the vertices of one of its k -clique. Let $v_1, \dots, v_{|V|}$ be the vertices of G and $(u_1, u'_1), \dots, (u_{|E|}, u'_{|E|})$ be its edges. In the following x^w represents the w -repetition of x . Then, the following sequence is a w -realization of G' :

$$a^w u_1 u'_1 a^w u_2 u'_2 a^w \dots a^w u_{|E|} u'_{|E|} a^w C b^w v_1 b^w v_2 b^k \dots b^w v_{|V|}$$

224 Now let us suppose that G' is w -realizable and let $x = x_1, \dots, x_p$ be a w -realization of G' .
 225 Without loss of generality, let us suppose a appears before b in x . Let i_b be the index of
 226 the first appearance of b and let i_a be the largest index of the appearance of a before i_b .
 227 Then $i_b - i_a \geq w$, otherwise there would be an edge between a and b . Furthermore, since
 228 G is simple, there cannot be two repetitions of a vertex in the sequence $x_{i_a+1}, \dots, x_{i_a+w-1}$.
 229 Due to the definition of a sequence graph, all vertices $\{x_{i_a+1}, \dots, x_{i_a+w-1}\}$ are connected,
 230 forming a clique in G of size $w - 1 = k$, which ends the proof. ◀

231 ▶ **Corollary 7.** GU-REALIZABLE is $W[1]$ -hard for parameter w .

232 **DU-Realizable is NP-hard for $w \geq 3$**

233 Consider the following intermediate problem:

234 **OptionalRealizable $_w$** Given a directed unweighted graph $D = (V, A)$, a subset $A' \subseteq A$ of
 235 *compulsory arcs*, two distinguished vertices $s, s' \in V$. Is there a sequence S such that the
 236 graph of S contains only arcs in A and (at least) all arcs in A' .

237 We first prove that this problem is NP-hard, then show how it reduces to DU-Realizable.

238 **OptionalRealizable $_w$, $w \geq 3$ is NP-hard**

239 Given $G = (V, A)$ and a start vertex s , build a directed weighted graph $G' = (V', A')$ as
 240 follows:

- 241 ■ Vertex set: $V = \bigcup_{v \in V} \{v_0 \mid v_1\} \cup \{x_p^i, 1 \leq p \leq 2n + 1, 1 \leq i \leq w - 2\}$
- 242 ■ Arc set,
 - 243 ■ optional arcs $(x_{2p-1}^i, v_0), (v_0, x_{2p}^i), (x_{2p}^i, v_1), (v_1, x_{2p+1}^i)$ for each $v \in V, 1 \leq p \leq n,$
 244 $1 \leq i \leq w - 2$.
 - 245 ■ optional arcs (u_1, v_0) for each (u, v) in A
 - 246 ■ compulsory arcs (v_0, v_1) for each $v \in V$
 - 247 ■ optional arcs (x_p^i, x_p^j) for $i < j$ and (x_p^i, x_{p+1}^j) for $j \leq i$

248 Start vertices are $(x_0^1, \dots, x_0^{k-2}, s)$.

249 G' is a yes-instance $\Leftrightarrow G$ admits a hamiltonian path

250 \Leftarrow Let v^p be the p th vertex of V in the hamiltonian path. Let X^p be the sequence
 251 $x_{2p-1}^0 \dots x_{2p-1}^{w-2} v_0^p x_{2p}^0 \dots x_{2p}^{w-2} v_1^p$. Let $X^{n+1} = x_{2n}^0 \dots x_{2n}^{w-2}$, and S be the concatenation
 252 $X^1 \dots X^{n+1}$. It can be checked that S contains only arcs of A and all compulsory arcs.

253 \Rightarrow Consider a sequence S , an occurrence of x_p^i in S for some $1 \leq i \leq w - 2, 1 \leq p \leq n$
 254 (note that $p \neq n + 1$), and let S' be the subsequence of S containing the $w - 1$ characters
 255 following x_{2p+1}^i . Let $T = x_p^{i+1} \dots x_p^{w-2}$ and $U = x_{p+1}^1 \dots x_{p+1}^i$ (note that T is possibly
 256 empty). T and U are seen both as strings and as sets of vertices. The out-neighborhood
 257 of x_p^i contains all vertices of $T \cup U$, as well as all vertices v_q for $v \in V$, where $q = 0$ if p is
 258 odd and $q = 1$ if p is even. Since there are $k - 2$ vertices in $T \cup U$, and no vertex has a
 259 self-loop, then by the pigeon-hole principle string S' must contain at least one vertex v^q ,
 260 $v \in V$. Since there are no arc (v^q, v'^q) for $v, v' \in V$, S' contains exactly one vertex v^q , thus

261 it also contains all vertices of $T \cup U$. Based on the direction of the arcs in $T \cup U \cup \{v^q\}$, it
 262 follows that $S' = T \cdot v^q \cdot U$.

Let X_p be the string $x_p^1 \dots x_p^{w-2}$. From the arguments above, and the fact that S starts
 with X_1 , there exist indices $i_1, j_1, \dots, i_n, j_n$ such that

$$S = X_1 v_{i_1}^0 X_2 v_{j_1}^1 X_3 v_{i_2}^0 X_4 v_{j_3}^1 X_5 \dots X_{2n+1}$$

263 From the window size w , there must exist an arc $(v_{i_p}^0, v_{j_p}^1)$ for each p , so by construction
 264 $i_p = j_p$. Furthermore, these arcs are compulsory for each vertex v^0 , so (i_1, \dots, i_n) is a
 265 permutation of $\{1, \dots, n\}$. Finally, there also exist an arc $(v_{j_p}^1, v_{i_{p+1}}^0)$ in G' , so there exists
 266 an arc $(v_{i_p}, v_{i_{p+1}})$ in G . Thus, $(v_{i_1}, \dots, v_{i_n})$ is a hamiltonian path in G .

267 **DU-Realizable_w is NP-hard**

268 By reduction from **OptionalRealizable_w**. Given a directed unweighted graph $G = (V, A)$, a
 269 subset $A' \subseteq A$ of compulsory arcs (let $A'' = A \setminus A'$ be the set of optional arcs), an integer w ,
 270 and $w - 1$ distinguished vertices $s_1 \dots s_{w-1} \in V$.

Let $m = |A''|$, write $A'' = \{(u_1, v_1), \dots, (u_m, v_m)\}$. Create G' by adding $w(m + 1)$
 separator vertices y_p^i , $1 \leq p \leq m + 1$ and $1 \leq i \leq w$ and m vertices z_p . Build the strings

$$Z = \left(\prod_{p=1}^m (y_p^1 \dots y_p^w u_p z_p v_p) \right) y_{m+1}^1 \dots y_{m+1}^w$$

$$Z' = Z s_1 \dots s_{w-1}$$

271 . Add all arcs realized by Z' involving y_p^i and/or z_p to G' .

272 G has a realization with optional arcs $\Leftrightarrow G'$ has a realization

273 \Rightarrow Build a realization for G' by concatenating Z with the realization for G starting with
 274 $s_1 \dots s_{w-1}$. All optional arcs of G' are realized in Z , all compulsory arcs of G' are realized
 275 in the suffix (the realization of G'), and all arcs involving a separator are realized in Z' . No
 276 forbidden arc is realized.

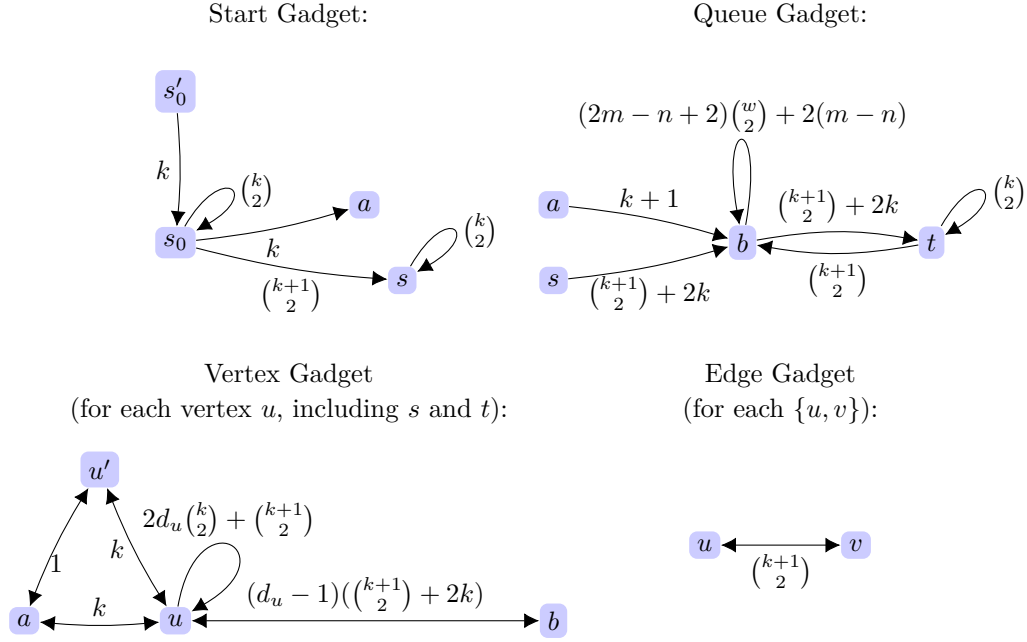
277 \Leftarrow Let S be a realization of G' . The set of in-neighbors of any separator has size at most
 278 $w - 1$ and induce a tournament in G' (this is clear for all arcs involving separators, it is also
 279 true for a potential pair of vertices (u_i, v_i) of G since G has no length-2 cycle. So the $w - 1$
 280 characters before any separator are ordered as in Z . Furthermore each separator (except
 281 y_1^1) contains at least one other separator in each in-neighborhood, so any occurrence of a
 282 separator is actually the last character of a substring of S equal to a prefix of Z . Since y_1^1
 283 has in-degree 0, it may only appear as the first character of S , and any prefix of Z in S
 284 is also a prefix of S . Moreover since y_{m+1}^w must appear in S , we have $S = ZS'$ with no
 285 separator appearing in S' . Thus S' realizes only arcs from G . From the out-neighborhood of
 286 $y_{m+1}^1, \dots, y_{m+1}^w$, we have that S starts with s_1, \dots, s_{w-1} . Moreover no compulsory arc of G is
 287 realized in Z , nor with one vertex in Z and one in S' (since such arcs start with a separator),
 288 so all compulsory arcs are realized in S' . Overall, G is a yes-instance of **OptionalRealizable_w**
 289 with sequence S' .

290 **GW-Realizable_w, DW-Realizable_w are NP-hard for all $w \geq 3$**

291 By reduction from a variant of hamiltonian path:

292 **Input:** Undirected graph G with two degree-1 vertices.

293 **Question:** Does G have a hamiltonian path?



■ **Figure 5** Subgraphs used in the reduction from Hamiltonian Path to DW-Realizable_3 . Weights on double arcs apply to both directions. Note that some arcs appear in different gadgets, in which case the weights should be summed (in particular, so loops on s and t have total weight $2d_u \binom{k}{2} + \binom{k+1}{2}$)

294 Note that this variant of HP is easily shown to be NP-hard from Hamiltonian cycle via
 295 the following reduction: given a graph G on which we need to find a hamiltonian cycle,
 296 pick any vertex v , duplicate it into v_1, v_2 (each edge $\{u, v\}$ becomes two edges $\{u, v_1\}$ and
 297 $\{u, v_2\}$), and add pending vertices s and t connected to v_1 and v_2 respectively.

298 **Reduction for DW-Realizable**

299 Given $G = (V, E)$ with degree-1 vertices s and t , build a directed weighted graph
 300 $G' = (V', A)$ as follows:

301 **Vertex set.** For each $u \in V$, create a vertex denoted u' . Create two additional dummy
 302 vertices a and b . Let $V' := \{a, b, s_0, s'_0\} \cup \bigcup_{u \in V} \{u, u'\}$. The arcs are given in Figure 5, as
 303 the union of the start gadget, the queue gadget, and the vertex and edge gadgets respectively
 304 for each vertex and edge of G .

305 **Reduction for GW-Realizable**

306 Build the directed graph G' as above, and let G'_u be the undirected version of G' : remove
 307 arc orientations, for $u \neq v$ the weight of $\{u, v\}$ is the sum of the weight of (u, v) and (v, u) in
 308 G' (the weight of loops is unchanged).

309 **Main claims**

310 We prove the following three claims:

- 311 (i) G hamiltonian $\Rightarrow G'$ has a realization
- 312 (ii) G' has a realization $\Rightarrow G'_u$ has a realization
- 313 (iii) G'_u has a realization $\Rightarrow G$ hamiltonian

314 All together, they show the correctness of the reductions for both GW-Realizable and
 315 DW-Realizable since they yield :

316 G hamiltonian $\Leftrightarrow G'$ has a realization

317 G hamiltonian $\Leftrightarrow G'_u$ has a realization

318

319 **Proof of Claim (i).** G has a hamiltonian path, let $(u_1 = s, u_2, \dots, u_n = t)$ be its hamiltonian
 320 path and $(v_1, w_1), \dots, (v_{m'}, w_{m'})$ be the pairs of connected verices except pairs (u_i, u_{i+1}) (i.e.
 321 the set $\bigcup_{\{u,v\} \in E} \{(u,v), (v,u)\} \setminus \{(u_i, u_{i+1}) \mid 1 \leq i < n\}$. Note that $m' = 2m - (n - 1)$.
 322 Define sequence S as follows.

$$S := \left| \begin{array}{l} s'_0 s_0^k a s^k s' s^k a u_2^k u'_2 u_2^k a \dots a u_{n-1}^k u'_{n-1} u_{n-1}^k a t^k t' t^k a \\ b^w v_1^k b w_1^k b^w v_2^k b w_2^k \dots b^w v_{m-n}^k b w_{m-n}^k b^w \end{array} \right.$$

323 Note that a sequence of the form $x^k a y^k$ yields $\binom{k}{2}$ loops for x , $\binom{k}{2}$ loops for y , as well
 324 as $\binom{k+1}{2}$ arcs (x, y) (indeed, there are $1 + 2 + \dots + w - 2 = \binom{k+1}{2}$ such arcs). A sequence
 325 of the form $b x^k b^w$ yields in particular an arc (b, x) of weight k and arc (x, b) of weight
 326 $\binom{k+1}{2} + k$. ◀

327 **Proof of Claim (ii).** Clear, any realization for G' is a realization for G'_u . ◀

328 **Proof of Claim (iii).** Pick a realization S of G'_u . Define the weight of a vertex in G_u as the
 329 sum of the weights of its incident edges (counting loops twice). From the construction, we
 330 obtain the following weights for a selection of vertices:

- 331 ■ s'_0 has weight $w - 1$
- 332 ■ u' has weight $2(w - 1)$ for $u \in V$
- 333 ■ a has weight $2(n + 1)(w - 1)$

334 From the weight of s'_0 , it follows that this vertex must be an endpoint of S (wlog, S
 335 starts with s'_0). It follows that for any other vertex v with weight $2i(w - 1)$, v must have
 336 exactly i occurrences in S (in general it can be either i or $i + 1$, but if v has $i + 1$ occurrences
 337 it must be both the first and last character of S , i.e. $v = s'_0$: a contradiction). Thus each u'
 338 occurs once and a occurs $n + 1$ times in S .

339 Each u' occurs once, so order vertices of V according to their occurrence in S (i.e.
 340 $V = \{u_1, \dots, u_n\}$ with u'_1 appearing before u'_2 , etc.). For each i , the neighborhood of u'_i in
 341 S contains a twice, one a on each side (since there is no (a, a) loop). Other neighbors of
 342 u'_i may only be occurrences of u_i , so each u'_i belongs to a factor, denoted X_i , of the form
 343 $au_i^* u'_i u_i^* a$. Two consecutive factors X_i, X_{i+1} may overlap by at most one character (a), and
 344 if they do, then there exists an edge $\{u_i, u_{i+1}\}$ (since $w \geq 3$) in G . There are n such factors
 345 X_{u_i} , and only $n + 1$ occurrences of a , so all as except extreme ones belong to the overlap of
 346 two consecutive X_j s, and there exists an edge $\{u_i, u_{i+1}\}$ for each i . Thus (u_1, \dots, u_n) is a
 347 hamiltonian path of G . ◀

348 4 Effective general algorithms

349 4.1 REALIZABLE_w Linear integer programming formulation

350 Let $G = (V, E)$ be a graph with integer weights $\pi_{e \in E}$. In this model, we represent a sequence
 351 x over the alphabet $\{1, \dots, n\}$, as a $(0 - 1)$ matrix $X \in \mathbb{M}_{n,p}(\{0, 1\})$ encoding the sequence x :

$$352 X_{i,j} = \begin{cases} 1 & \text{if } x_j = i \\ 0 & \text{otherwise} \end{cases}$$

It should be noted that the set sequence of sequences over the alphabet $\{1, \dots, n\}$ is exactly represented by the $(0 - 1)$ matrices such that

$$\forall j \in \{1, \dots, p\} \quad \sum_{i=1}^n X_{i,j} = 1$$

353 Given a window size w , a unit of $\pi_{e=(v_1, v_2)}$ corresponds to the appearance of two elements
 354 v_1, v_2 at a distance $i \in \{1, \dots, w - 1\}$ in the sequence. Now, let us consider a fixed distance i ,
 355 and a starting index $j \in \{1, \dots, p - i\}$, we use a intermediary slack variable $y_j^e(i) \in \{0, 1\}$ to
 356 model the presence of such appearance using the constraint:

$$357 \quad X_{v_1, j} X_{v_2, j+i} = y_j^e(i) \quad (8)$$

358 Then, the Boolean variable $y_j^e(i)$ is equal to 1 when v_1 is located at position j and v_2 at
 359 position $j + i$. We linearise Eq. 8 as:

$$360 \quad \begin{aligned} -X_{v_1, j} \quad \quad \quad + y_j^e(i) &\leq 0 \\ -X_{v_2, j+i} + y_j^e(i) &\leq 0 \\ X_{v_1, j} + X_{v_2, j+i} - y_j^e(i) &\leq 1 \end{aligned} \quad (9)$$

Each slack variable $y_k^e(i)$ is attributed to an edge e , a relative distance $i \in \{1, \dots, w - 1\}$ and a starting position $k \in \{1, \dots, p - i\}$. Given our constraint formulation, every slack variable is attributed 3 constraints. For a digraph, the number of possible pair positions for a unit of $\pi_{e=(v_1, v_2)}$ is given by:

$$C = \sum_{i=1}^{w-1} (p - i) = p(w - 1) - \frac{w(w - 1)}{2} = (w - 1)(p - \frac{w}{2})$$

361 Therefore, in our model, C corresponds to the number of slack variables attributed to
 362 constraints for an edge of the graph.

363 On the contrary, the absence of an edge $e = (v_1, v_2)$, corresponding to $\pi_e = 0$, can be
 364 modeled for a distance $i \in \{1, \dots, w - 1\}$ and a starting position $j \in \{1, \dots, p - i\}$ as:

$$365 \quad X_{v_1, j} + X_{v_2, j+i} \leq 1$$

366 Then, REALIZABLE_w can be formulated as the following linear integer program:

$$367 \quad \min_{X \in \{0, 1\}^{p \times n}, y \in \{0, 1\}^{|E| \times C}} \sum_{e \in E} \sum_{i \in \{1, \dots, w-1\}} y_1^e(i) + \dots + y_{p-i}^e(i)$$

368 under the constraints

$$369 \quad \forall j \in \{1, \dots, p\} \quad \sum_{i=1}^n X_{i,j} = 1$$

$$370 \quad \forall e = (v_1, v_2) \in E \quad \begin{cases} -X_{v_1, 1} \quad \quad \quad + y_1^e(i) \leq 0 \\ \quad \quad \quad - X_{v_2, 1+i} + y_1^e(i) \leq 0 \\ X_{v_1, 1} + X_{v_2, 1+i} - y_1^e(i) \leq 1 & X_{v'_1, 1} + X_{v'_2, 1+i} \leq 1 \\ \quad \quad \quad \vdots & \quad \quad \quad \vdots \\ -X_{v_1, p-i} \quad \quad \quad + y_{p-i}^e(i) \leq 0 & X_{v'_1, p-i} + X_{v'_2, p} \leq 1 \\ \quad \quad \quad - X_{v_2, p} + y_{p-i}^e(i) \leq 0 \\ X_{v_1, p-i} + X_{v_2, p} - y_{p-i}^e(i) \leq 1 \end{cases}$$

$$\text{and } \forall e \in E \quad \sum_{i \in \{1, \dots, w-1\}} y_1^e(i) + \dots + y_{p-i}^e(i) \geq \pi_e$$

If the objective function reaches $\sum_{e \in E} \pi_e$ at its minimum then the output of $\text{REALIZABLE}_w(G, \Pi)$ is True, and False otherwise.

4.2 NUMREALIZATIONS_w Dynamic programming formulation

We did not present a way to count admissible sequences in the general case. Although the tractability of our problems (NP-hardness of REALIZABLE_w , #P-hardness of NUMREALIZATIONS_w) currently remains open for some cases, we present in this subsection a method based on dynamic programming valid for all cases.

The recursion proceeds by extending a partial sequence, initially set to be empty, keeping track of for represented edges along the way. Namely, consider $N_w[\Pi, p, \mathbf{u}]$ to be the number of w -admissible sequences of length p for the graph $G = (V, E)$, respecting a weight matrix $\Pi = (\pi_{ij})_{i,j \in V^2}$, preceded by a sequence of nodes $\mathbf{u} := (u_1, \dots, u_{|\mathbf{u}|}) \in V^*$. It can be shown that, for all $\forall p \geq 1$, $\Pi \in \mathbb{N}^{|V|^2}$ and $\mathbf{u} \in V^{\leq w}$, $N_w[\Pi, p, \mathbf{u}]$ obeys the following formula, using the notations of Section 3.2:

$$N_w[\Pi, p, \mathbf{u}] = \sum_{v \in V} \begin{cases} N_w[\Pi'_{(\mathbf{u},v)}, p-1, (u_1, \dots, u_{|\mathbf{u}|}, v)] & \text{if } |\mathbf{u}| < w-1 \\ N_w[\Pi'_{(\mathbf{u},v)}, p-1, (u_2, \dots, u_{w-1}, v)] & \text{if } |\mathbf{u}| = w-1 \end{cases} \quad (10)$$

with $\Pi'_{(\mathbf{u},v)} := (\pi_{ij} - |\{k \in [1, |\mathbf{u}|] \mid (u_k, v) = (i, j)\}|)_{(i,j) \in V^2}$. The base case of this recurrence corresponds to $p = 0$, and is defined as

$$\forall \Pi, N_w[\Pi, 0, \mathbf{u}] = \begin{cases} 1 & \text{if } \Pi = (0)_{(i,j) \in V^2} \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

The total number of admissible sequences is then found in $N_w[\Pi, p, \varepsilon]$, *i.e.* setting \mathbf{u} to the empty prefix ε , allowing the sequence to start from any node.

The recurrence can be computed in $\mathcal{O}(|V|^w \times \prod_{i,j \in V^2} (\pi_{i,j} + 1))$ time using memoization, for p the sequence length. The complexity can be refined by noting that:

$$\sum_{i,j \in V^2} \pi_{i,j} \leq w \times p$$

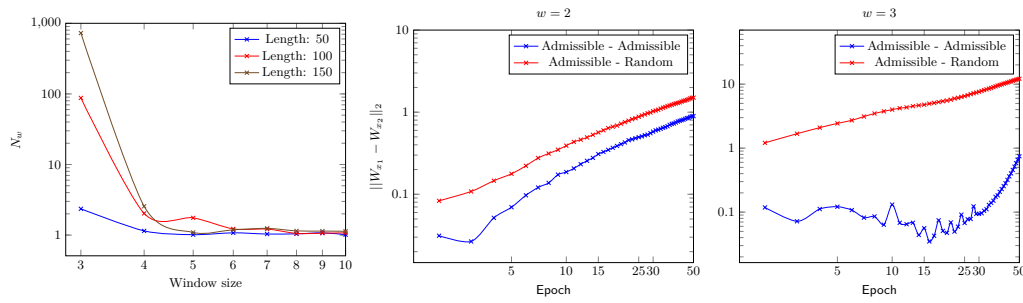
To investigate the worst case scenario, we can consider the optimisation problem:

$$\max_{\Pi} \prod_{i,j \in V^2} (\pi_{i,j} + 1) \text{ such that } \sum_{i,j} \pi_{i,j} = w p. \quad (12)$$

This problem is equivalent to maximise a product under a budget constraint. When $n^2 \geq w \times p$, which is the case in practice, the maximum is reached for a Boolean matrix $\Pi = (\pi_{i,j}) \in \{0, 1\}^{|V|^2}$, verifying the constraint. This property can be deduced from the inequality:

$$\begin{aligned} 1 \leq a < b-1 &\implies \log a + \log b < \log(a+1) + \log(b-1) \\ &\implies ab < (a+1)(b-1) \end{aligned}$$

It follows that, in the worst-case scenario, $\prod_{i,j \in V^2} (\pi_{i,j} + 1) \in \mathcal{O}(2^{w p})$. Thus, despite the, apparently extreme complexity of our algorithm, it is still possible to compute $N_w[\Pi, p, u_{1:w}]$ for “reasonable” values of p and w .



■ **Figure 6** Left plot: Average lower bound (N_w) on the average number of sequences. Two right plots: $\|W_{x_1} - W_{x_2}\|_2 = f(\text{Epoch})$ - LSTM, log - log scale

402 5 Application to sequential models

403 5.1 Number of equivalent sequences for weighted sequence digraphs

404 Since the dynamic programming method in Sec. 4.2 is exponential in the worst case, we
 405 provide results for relatively short sequences generated from text data (a dump of English
 406 Wikipedia, 2016) of 500 documents, each of them having a length $p \in \{50, 100, 150\}$. Each
 407 document contained a minimum of $\frac{3}{4}p$ distinct words. For each $w \in 3 \rightarrow 10$, we estimate the
 408 number of admissible sequences yielding the same representations for a set of documents
 409 and different window size using the procedure described in Sec. 4.2 to compute N_w . Due
 410 to memory limitations, it should be noted that N_w is a lower bound of the number of total
 411 admissible sequences, since a starting pattern (first w tokens) is fixed.

412 Results are reported in the left plot of Fig. 6. For $w = 2$, the number of sequences
 413 (obtained using Prop. 19) was significantly larger ($> 10^5$), so not reported in the figure
 414 for clarity. As expected, the number of distinct admissible sequences tends to 1 when the
 415 window size increases. This suggests that window sizes used in skip gram models should
 416 be usually larger or equal to 5. In natural language processing, a frequent configuration is
 417 $w = 10$ [18, 21]. However, some examples with different realizations exist, even for $w = 10$
 418 and $p = 50$.

419 5.2 Comparison with a recurrent neural network

420 The second experiment we consider is to evaluate the difference of the parameters between
 421 a sequential model trained on two admissible sequences of a given graph. The sequential
 422 model we are considering are a class of recurrent dynamical recurrent models, referred to as
 423 long short term memory (LSTM) networks [12]. These models have attracted new interest
 424 due to experimental progress for time series prediction [11] and natural language processing
 425 [3, 6, 25]. Given a window size w , the task we consider is to predict the next element of the
 426 sequence given the $w - 1$ previous ones. If the sequences were equivalent for the sequential
 427 model, the weights should numerically converge after training.

428 To generate pairs of admissible sequences encoding for non trivial graphs (i.e not the
 429 complete graph), we used algorithm based on Lemma 5. We generate w -admissible sequence
 430 (thousand tokens long), for $w \in \{2, 3\}$, but could not provide other pairs for $w > 3$ due
 431 to computational time. We compare the pairs of admissible sequences with a pair of one
 432 of the sequence, and a sequence generated randomly uniformly on the same vocabulary.
 433 We implemented the LSTM network using the Python library Keras [7], a high-level API

434 running over TensorFlow [1]. In order to remove randomness from the training algorithm,
435 we froze the seed generating initial weights (the optimization directions being fixed by the
436 data). We chose tanh as main activation function, sigmoid for the recurrent activations, with
437 2 units. The number of units is chosen relatively low in order to obtain a reasonable number
438 of weights (in this case 16).

439 Two right plots of Fig. 6 reports the results for $w \in \{2, 3\}$, W_x represents all the weights
440 of the network for a sequence x . For $w = 2$, there the difference of the weights for 2 admissible
441 sequences is lower than with one of the sequence and a random one, but this proximity does
442 not appear to be significant compared with a random sequence. For $w = 3$, the recurrent
443 network has relatively close weights for two admissible sequences when compared with a
444 random one.

445 **6 Conclusion**

446 In this study, we revisited a series of problems related to the ambiguity of sequence graphs
447 representations, which are popular in the context of text mining and natural language
448 processing. We derived theoretical properties and practical algorithms for the family of
449 sequence graphs.

450 This study can be of use used for several sequential models, such as continuous bag of
451 words (CBOW), skip-grams ([10, 14, 24]), pointwise mutual information models [2, 18, 21].

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526 **A Additional figures**

■ **Algorithm 1** Construction of a sequence digraph

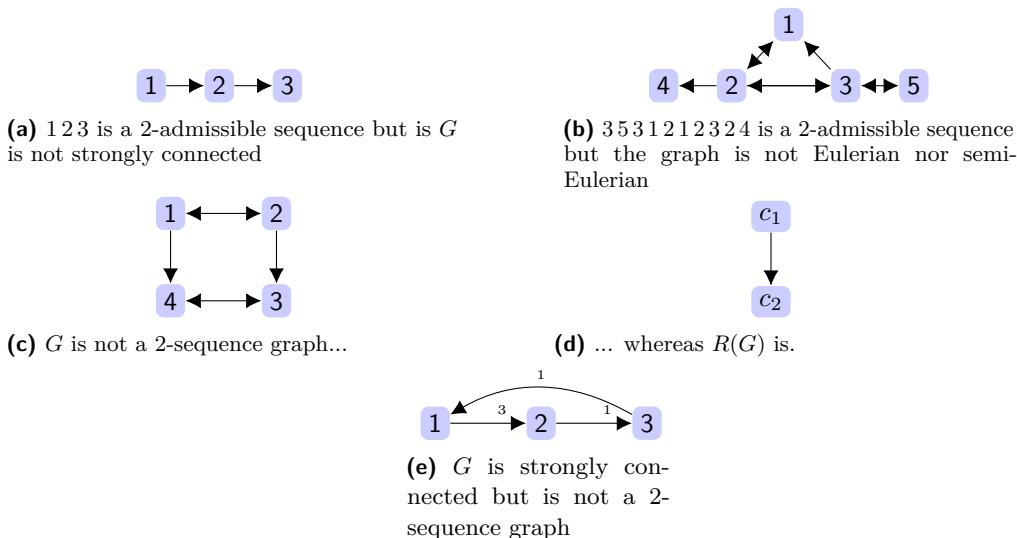
Input: Sequence x of length p , window size w , $p \geq w \geq 2$

Parameter: Optional list of parameters

Output: $(G_w(x), \Pi)$

```

1:  $V \leftarrow \emptyset$ 
2: Initiate  $\Pi = (\pi_{i,j})$  to  $d \times d$  matrix of zeros
3: for  $i = 1 \rightarrow p - 1$  do
4:    $V \leftarrow V \cup \{x_i, x_{i+1}\}$ 
5:   for  $j = i + 1 \rightarrow \min(i + w - 1, p)$  do
6:      $\pi_{x_i, x_j} \leftarrow \pi_{x_i, x_j} + 1$ 
7:   end for
8: end for
9: return solution
    
```



■ **Figure 7** Counter examples for $w = 2$

527 **B Results and proofs Sec. 3.1 ($w = 2$)**

528 In this section, we present the results for digraphs and $w = 2$. Obviously, the simplest case
 529 concerns undirected graphs as stated in:

530 ► **Proposition 8.** *If $G = (V, E)$ is unweighted and undirected, with $|V| > 1$, the following
 531 are equivalent:*

- 532 (i) G is connected
- 533 (ii) G has a 2-admissible sequence
- 534 (iii) G admits an infinite number of 2-admissible sequences

535 In these conditions, a 2-admissible sequence can start and end at any vertex.

536 **Proof.** Let us suppose G has an admissible sequence u . Let a, b two distinct vertices of G .
 537 Then using the definition, a and b must appear at least once in the sequence u , i.e $u_{i_a} = a$
 538 and $u_{i_b} = b$. If $i_a < i_b$, then the sequence $s = (u_i \mid i_a \leq i \leq i_b)$ defines a path from a to b
 539 since $\forall i, e_{s_i s_{i+1}} \in E$. The case $i_b > i_a$ is dealt similarly. ◀

540 The previous characterization is wrong for digraphs, even with strongly connectivity. A
 541 counter example is depicted in Fig. 7a. However, strong connectivity remains a sufficient
 542 condition:

543 ▶ **Proposition 9.** *Let $G = (V, E)$ a unweighted digraph. If G is strongly connected then*
 544 *$G \in \text{Im } \phi_2$. A 2-admissible sequence can start or end at any given vertex of G .*

545 **Proof.** This can be proved similarly to (i) \implies (ii) for proposition 9 by replacing connectivity
 546 with strong connectivity. ◀

547 ▶ **Proposition 10.** *Let $G = (V, E)$ an unweighted digraph. If G is Eulerian or semi-Eulerian,*
 548 *then $G \in \text{Im } \phi_2$.*

549 **Proof.** If G is Eulerian or semi-Eulerian, there exists a walk going through all edges, this
 550 walk defines a 2-admissible sequence. ◀

551 Again the converse of Prop. 10 does not hold as depicted in Fig. 7b. The characterization
 552 of sequence digraph is more subtle. As a start, it is natural to consider directed acyclic
 553 graphs (DAGs):

554 ▶ **Proposition 11.** *Let $G = (V, E)$ a DAG. G is a 2-sequence graph if and only if it is a*
 555 *directed path, i.e G is a directed tree where each node has at most one child and at most one*
 556 *parent. In this case, G has a unique 2-admissible sequence.*

557 **Proof.** If G is directed path, since G is finite, it admits a source node. Therefore a 2-
 558 admissible sequence is obtained by simply going through all vertices from the source node.
 559 This is obviously the only one.

560 Conversely, let us suppose G is a DAG and a 2-sequence graph. If G is not a directed
 561 path, there are two cases: either there exists a vertex having two children, or two parents.
 562 Let s be a vertex having 2 distinct children c_1 and c_2 . This is not possible since there cannot
 563 be a walk going through (s, c_1) and (s, c_2) : G would have a cycle otherwise. Finally a vertex
 564 v cannot have two parents p_1 and p_2 : if a 2-admissible sequence existed, it would have to go
 565 through (p_1, v) and (p_2, v) , creating a cycle, hence the contradiction. ◀

566 Every directed graph G is a DAG of its strongly connected components. In the following,
 567 let $R(G)$ be the DAG obtained by contracting the strongly connected components of G .

568 ▶ **Proposition 12.** *Let $G = (V, E)$ a digraph. If G is a 2-sequence graph then $R(G)$ is a*
 569 *2-sequence graph.*

570 **Proof.** Let G be a 2-sequence graph, and let us suppose that $R(G)$ is not a 2-sequence graph.
 571 Since $R(G)$ is a (weakly) connected DAG, then using Prop. 11, it cannot be a directed path,
 572 so $R(G)$ has either a node having two children or two parents. Let S be a node of $R(G)$
 573 having at least 2 distinct children C_1 and C_2 . This means that there exist three distinct
 574 corresponding nodes in V , s , v_1 and v_2 such that $(s, v_1) \in E$ and $(s, v_2) \in E$. Since G is a
 575 2-sequence graph, there exists a walk covering (s, v_1) and (s, v_2) , such walk would make S ,
 576 C_1 and C_2 the same node in $H(G)$, hence the contradiction. The case for which a vertex has
 577 two parents is dealt with similarly. ◀

578 The converse of Prop. 12 does not hold as depicted in Fig. 7c, 2. However, let us add a
579 weight compatibility in $R(G)$ as follows:

580 ► **Definition 13.** *Let G be a digraph, and $R^+(G)$ be the weighted DAG obtained from $R(G)$,
581 such that the weight of an edge is attributed the number of distinct arcs from two strongly
582 connected components in G .*

583 ► **Theorem 14.** *Let $G = (V, E)$ be an unweighted digraph.*

584 *G is a 2-sequence graph if and only if $R^+(G)$ is a directed path and its weights are all
585 equal to 1.*

586 **Proof.** If G is a 2-sequence graph, $R(G)$ is a 2-sequence graph using Prop. 12. Also Prop. 11
587 implies that $R(G)$ and $R^+(G)$ are directed paths. Moreover, if $R^+(G)$ had a weight strictly
588 greater than 1, then there would be strictly more than one edge between two connected
589 components C_1 and C_2 . All these edges go in the same direction otherwise $C_1 \cup C_2$ would
590 form a strongly connected component. This is a contradiction since any 2-admissible sequence
591 would have to go from C_1 to C_2 and then come back to C_1 (or conversely) which would
592 would make $C_1 \cup C_2$ a strongly connected component.

593 Conversely, let us suppose $R^+(G)$ is a directed path and its weights are equal to one.
594 First, there exists a walk x_1, \dots, x_p covering all edges of $R^+(G)$ verifying: (i) $\forall i, x_i \in V$ or x_i
595 represents a strongly connected component of G , (ii) there is only one edge in G between
596 from x_i to x_{i+1} and (iii) x has no repetition, i.e there is no common vertex in G between x_i
597 and x_{i+1} . We construct a 2-admissible sequence y for G by means of the following procedure.

598 **Initialisation:** If $x_1 \in V$, we simply set $y \leftarrow x_1$. Otherwise, x_1 corresponds to a strongly
599 connected component C_1 of G and we add to y any 2-admissible sequence of C_1 .

600 For $i \in \{1, \dots, p-1\}$:

- 601 • If $(x_i, x_{i+1}) \in E$: we add x_{i+1} to the sequence y .
- 602 • If $x_i \in V$ and x_{i+1} is a strongly connected component C_i of G : By assumption, there
603 exists only one edge of G from x_i to a vertex of C_i , say c_0^i . Since C_i is strongly connected,
604 using Prop. 9, C_i has a walk going through all of its edges and starting in c_0^i , say c_0^i, \dots, c_p^i .
605 We add c_0^i, \dots, c_p^i to y .
- 606 • If x_i corresponds to a strongly connected component C_i and $x_{i+1} \in V$: we perform
607 similar operations by stopping on the single node of C_i that has an edge to x_{i+1} (this is
608 possible thanks to Prop. 9).
- 609 • x_i and x_{i+1} both correspond to strongly connected components C_i and C_{i+1} , there
610 exists only one edge between in E between C_i and C_{i+1} , say $e_i = (v_i, v_{i+1})$. We can complete
611 y by a walk from the last vertex visited which belong to C_i and v_i , and then by a 2-admissible
612 sequence through C_{i+1} starting in v_i and ending in v_{i+1} .

613 End For

614 The process stops when $i = p-1$, and all edges are covered by the sequence y . ◀

615 Therefore, an algorithm to decide if a digraph is a 2-sequence is obtained by extract its
616 connected components (there exist linear time algorithms e.g [22]), and to count the number
617 of distinct edges between these.

618 ► **Corollary 15.** *Let G an unweighted digraph. The possible numbers of 2-admissible sequences
619 for G is exactly $\{0, 1, +\infty\}$. Moreover, G admits a unique 2-admissible sequence if and only
620 if G is a directed path.*

621 **Proof.** Let G a 2-sequence graph and let us show that G has either a unique or an infinite
622 number of 2-admissible sequence. G verifies characterization of Theorem 14. If $R(G)$ has

623 a vertex representing a strongly connected component of G (or a vertex with a self loop),
 624 then by adding an arbitrary number of cycles to y , the obtained walk is still admissible.
 625 Otherwise, if every vertex of $R(G)$ is in V without self-loops in E , then G is a DAG. Using
 626 Prop. 11, y is the unique 2-admissible sequence. ◀

627 Weighted 2-sequence graphs

628 The weighted case cannot be treated similarly due to the weight constraints implying that a
 629 weighted graph has a finite number of admissible sequences. A counter example is depicted
 630 in Fig. 7e.

631 ▶ **Definition 16.** Let $\psi(G)$ be the multigraph with the same vertices as $G = (V, E)$ and with
 632 π_{ij} edges between $(i, j) \in V^2$.

633 Due to the previous study, the characterization of weighted 2-sequence graphs using $\psi(G)$ is
 634 immediate.

635 ▶ **Theorem 17.** If G is a weighted graph (directed or not), with $\Pi(G) \in \mathcal{M}_d(\mathbb{N})$, then:
 636 $G \in \text{Im } \phi_2 \iff \psi(G)$ is connected and semi-Eulerian.

637 **Proof.** $G \in \text{Im } \phi_2$ means that there is a trail going through each edge $(i, j) \in E$ exactly $\pi_{i,j}$
 638 times. This trail corresponds to a semi-Eulerian path in $\psi(G)$. ◀

639 ▶ **Lemma 18.** Let $G = (V, E)$ a weighted 2-sequence graph (possibly oriented). Let \mathcal{E} be the
 640 set of Eulerian paths of $\psi(G)$ and \mathcal{S} be the set of 2-realizations of G . Then

$$641 \quad \mathcal{E} = (\#\mathcal{S}) \prod_{e \in E} \pi_e! \quad (13)$$

642 **Proof.** We will first prove it for digraphs. If $e = (v_1, v_2)$ is an edge of a digraph, we will
 643 represent the source and target vertex of e as $e(s)$ and $e(t)$. Let (e_1, e_2, \dots, e_h) be a Eulerian
 644 path of $\psi(G)$ defined as a sequence of its edges. Then $\forall (i, j) \in \{1, \dots, h\}^2$, $e_i \neq e_j$ and
 645 $\forall i \in \{1, \dots, h-1\}$, $e_i(t) = e_{i+1}(s)$. Let us consider the transformation:

$$646 \quad \begin{aligned} \mathcal{E} &\longrightarrow \mathcal{S} \\ (e_1, e_2, \dots, e_h) &\mapsto (e_1(s), e_2(s), \dots, e_{p-1}(s), e_h(t)) \end{aligned} \quad (14)$$

647 We have already shown this transformation is surjective: any 2-sequence of G can be
 648 obtained with a Eulerian path of $\psi(G)$. We will now consider the action of \mathfrak{S}_h on \mathcal{E} .
 649 For a Eulerian path, let us suppose that two edges of $\psi(G)$ have been permuted, say
 650 e_1 and e_{i_0} without loss of generality. If the two corresponding sequences are the same:
 651 $(e_{i_0}(s), e_2(s), \dots, e_h(t)) = (e_1(s), e_2(s), \dots, e_{i_0}(s), \dots, e_h(t))$. Obviously, $e_{i_0}(s) = e_1(s)$. Also
 652 $e_1(t) = e_2(s)$ implies $e_{i_0}(t) = e_1(t)$. This shows that e_{i_0} and e_1 are associated to the
 653 same edge in E . Therefore, given a 2-sequence, the choice of a corresponding Eulerian
 654 path correspond to the choice of $\sigma = (\tau_1, \dots, \tau_{|E|})$ where τ_e is a permutation of $\{1, \dots, \pi_e\}$.
 655 Therefore $\#\mathcal{E} = (\#\mathcal{S}) \prod_{e \in E} \pi_e!$.

656 If G is undirected, the proof is still valid, but the operators $e \mapsto e(s)$ and $e \mapsto e(t)$ are
 657 now induced the natural direction of the considered Eulerian path. ◀

658 Counting the number of Eulerian paths in a undirected graph has been proven to be a
 659 $\#P$ -complete problem [4]. Since $G \mapsto \psi(G)$ is bijective, counting the number of 2-admissible
 660 sequences is also $\#P$ -complete. Finally, counting Eulerian trails of weighted digraphs has
 661 been well studied, hence the following proposition:

662 ► **Proposition 19.** If $G = (V, E)$ is a weighted digraph, with $\Pi(G) \in \mathcal{M}_d(\mathbb{N})$. Then, if $\deg(v)$
 663 is the indegree of a vertex v , the number p_2 of 2-admissible sequences is

$$664 \quad p_2 = \frac{t(\psi(G))}{\prod_{e \in E} \pi_e!} \prod_{v \in V} (\deg_{\psi(G)}(\psi(v)) - 1)! \quad (15)$$

665 where $t(G)$ is the number of spanning trees of a graph G . If L is the Laplacian matrix of G ,
 666 then $t(G)$ is given by

$$667 \quad t(G) = \prod_{\substack{\lambda_i \in Sp(L) \\ \lambda_i \neq 0}} \lambda_i \quad (16)$$

668 **Proof.** Direct consequence of BEST Theorem [8]) and Matrix tree theorem [5] ◀

669 To use formula 15, $\deg_{\psi(G)}(\psi(v))$ can be obtained using the following formula: $\deg_{\psi(G)}(\psi(v)) =$
 670 $\sum_{n \in V} \pi_{nv} + \sum_{n \in V} \pi_{vn}$.

671 **C Proofs section 3.2 ($w \geq 3$)**

672 **Proof of Proposition 3.** Let $x = x_1, \dots, x_p$ be a w -admissible sequence of G . Let P be a walk
 673 on $H^{(w-2)}$, and $P[i]$ be the i -th element of P , $P[i] \in H^{(w-2)}$: $P[i] = (P[i]_1, \dots, P[i]_{w-1})$.

Let us suppose that $w \leq p$ (which we can always do), and let us show the following property by induction on k :

$\forall k \in \{w-1, \dots, p\}$, \exists walk P on $H^{(w-2)}$ such that :

$$x_{1:k} = P[1]_1, P[2]_1, \dots, P[k - (w-1)]_1, P[k+1 - (w-1)]_{1:(w-1)}$$

- Initialisation: $k = w-1$. By construction of $H^{(w-2)}$, $x_{1:w-1}$ is the authentic sequence of “static walk”: $P = P[1] = x_{1:w-1} \in H^{(w-2)}$.
- Induction: let us suppose the property is verified for $k \in \{w-1, \dots, p-1\}$, i.e there exists a walk P on $H^{(w-2)}$ such that:

$$x_{1:k} = P[1]_1, P[2]_2, \dots, P[k - (w-1)]_1, P[k+1 - (w-1)]_{1:(w-1)}$$

Since x is w -admissible, then by definition:

$$\forall i \in \{k+1 - (w-1), \dots, k\}, \forall j \in \{i+1, \dots, \min\{k+1, i+w-1\}\} : (x_i, x_j) \in E$$

674 Therefore, by definition of $H^{(w-2)}$, $\xi^{k+1} = x_{k+1-(w-1)}, \dots, x_{k+1} \in H^{(w-2)}$.

Let $P[k+2 - (w-1)] \hat{=} \xi^{k+1}$, then

$$P[k+2 - (w-1)]_{1:(w-1)} = x_{k+1-(w-1)}, \dots, x_{k+1}$$

Besides, from the induction assumption: $\forall i \in \{1, \dots, k - (w-1)\}$, $P[i]_1 = x_i$. This ensures that:

$$x_{1:(k+1)} = P[1]_1, P[2]_1, \dots, P[k+1 - (w-1)]_1, P[k+2 - (w-1)]_{1:(w-1)}$$

675 which ends the induction and the proof. ◀

676 **Proof of Lemma 5.** Let $P = P[1], \dots, P[r]$ a walk on $H^{(w-2)}$ going through a strongly
 677 connected component C , with an arbitrary ordering of its vertices, i.e $C = \{c_1, \dots, c_m\}$. This
 678 means $\exists (m_0, i_0) \in \{1, \dots, m\} \times \{1, \dots, r-1\}$ s.t $P[i_0] = c_{m_0}$ and $(c_{m_0}, P[i_0+1]) \in E^{(w-2)}$.
 679 Let $\mathcal{P}_C = c_{m_0}, c_{j_1}, \dots, c_{j_v}$ be a path in C with $(c_{j_v}, P[i_0+1]) \in E^{(w-2)}$. Let Q be the new

680 path: $Q = P[1], \dots, P[i_0], c_{j_1}, \dots, c_{j_v}, P[i_0 + 1], \dots, P[r]$. By construction of $H^{(w-2)}$, the edges
 681 (between elements of V) created by any walk on $H^{(w-2)}$ are in E , so Q is still admissible.

682 Let us label every node of $R(H^{(w-2)})$ representing a strongly connected component of
 683 $H^{(w-2)}$ by any 2-admissible sequence (one exists thanks to Prop. 9). A walk on $H^{(w-2)}$:
 684 x_1, \dots, x_p can be met by a walk on $R(H^{(w-2)})$ using the following procedure:

685 For $i \in \{1, \dots, p-1\}$:

- 686 ■ if $x_i, x_{i+1} \in E^{(w-2)}$, we keep x_i and x_{i+1}
- if x_i is a vertex of $H^{(w-2)}$ and x_{i+1} is in a strongly connected component of $H^{(w-2)}$ (but
 a node of $R(H^{(w-2)})$), represented by c_1, \dots, c_{C_i} , then a path from x_{i+1} to c_1 exists since
 the component is strongly connected: $x_{i+1}, p_1, \dots, p_m, c_1$. We keep

$$x_i, x_{i+1}, p_1, \dots, p_m, c_1, \dots, c_{C_i}$$

687 Using the aforementioned result, this does not perturb admissibility.

- 688 ■ if x_{i+1} is a vertex of $H^{(w-2)}$ and x_i is in a strongly connected component of $H^{(w-2)}$, we
 689 proceed similarly (x_i and x_{i+1} are swapped).
- 690 ■ if both x_{i+1} and x_i are strongly connected components of $H^{(w-2)}$, we add intermediary
 691 nodes to both components similarly.

692

