Nonlinear boundary value problems relative to harmonic functions

Oussama Y. Boukarabila, Laurent Veron

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To Shair with high esteem and sincere friendship

Abstract

We study the problem of finding a function u verifying $-\Delta u = 0$ in Ω under the boundary condition $\frac{\partial u}{\partial \mathbf{n}} + g(u) = \mu$ on $\partial\Omega$ where $\Omega \subset \mathbb{R}^N$ is a smooth domain, $\mathbf{n}$ the normal unit outward vector to Ω, μ is a measure on $\partial\Omega$ and g a continuous nondecreasing function. We give sufficient condition on g for this problem to be solvable for any measure. When $g(r) = |r|^{p-1}r$, $p > 1$, we give conditions in order an isolated singularity on $\partial\Omega$ be removable. We also give capacitary conditions on a measure μ in order the problem with $g(r) = |r|^{p-1}r$ to be solvable for some μ. We also study the isolated singularities of functions satisfying $-\Delta u = 0 \quad$ in Ω and $\frac{\partial u}{\partial \mathbf{n}} + g(u) = 0$ on $\partial\Omega \setminus \{0\}$.

Key Words: Dirichlet to Neumann operator; Laplace-Beltrami operator; Singularities; Limit set; Radon Measure.

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Contents

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Contents

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ such that $0 \in \partial\Omega$ and $g : \mathbb{R} \mapsto \mathbb{R}$ a continuous nondecreasing function such that $rg(r) \geq 0$. The aim of this article is to study the following nonlinear problem

$$\begin{array}{lll} -\Delta u + u = 0 & & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} + g(u) = \mu & & \text{in } \partial\Omega, \end{array} \tag{1.1}$$

where μ is a Radon measure on $\partial\Omega$ and $\mathbf{n}$ the outward normal unit vector on $\partial\Omega$. An associated model problem on which we can develop sharp estimate is the following equation in the upper half-space $\mathbb{R}^N_+ := \{x = (x_1, ..., x_N) \in \mathbb{R}^N : x_N > 0\}$,

$$\begin{array}{lll} -\Delta u = 0 & & \text{in } \mathbb{R}^N_+ \\ -\dfrac{\partial u}{\partial x_N} + |u|^{p-1}u = 0 & & \text{in } \partial\mathbb{R}^N_+ \setminus \{0\} \end{array} \tag{1.2}$$

where $p > 1$. These two problems are by essence non-local and actually, the second problem can be expressed by introducing the square root of the Laplacian in $\mathbb{R}^{N-1}$ under the form

$$(-\Delta_{N-1})^{\frac{1}{2}}\tilde{u} + \tilde{u}^p = 0 \qquad \text{in } \mathbb{R}^{N-1} \setminus \{0\}, \tag{1.3}$$

where $\Delta_{N-1} = \displaystyle\sum_{j=1}^{N-1} \frac{\partial^2}{\partial x_j^2}$ and $\tilde{u}(x_1, ..., x_{N-1}) = u(x_1, ..., x_{N-1}, 0)$. The second equation is equivariant under the scaling transformation T_k $(k > 0)$ defined by

$$T_k[u](x) = k^{\frac{1}{p-1}} u(kx). \tag{1.4}$$

Therefore it is natural to look for self-similar solutions i.e. solutions satisfying $T_k[u] = u$ for any $k > 0$. Introducing the spherical coordinates $(r, \sigma) \in (0, \infty \times S^{N-1})$, then a self-similar solution endows the form

$$u(x) = u(r, \sigma) = r^{-\frac{1}{p-1}}\omega(\sigma), \tag{1.5}$$

and ω satisfies

$$\begin{array}{ll} \Delta'\omega + \ell_{N,p}\omega = 0 & \text{in } S_+^{N-1} \\ \dfrac{\partial\omega}{\partial\nu} + |\omega|^{p-1}\omega = 0 & \text{in } \partial S_+^{N-1}, \end{array} \tag{1.6}$$

where Δ' is the Laplace-Beltrami operator on the unit sphere S^{N-1}, ν is the outward normal unit vector to ∂S_+^{N-1} tangent to S^{N-1} and

$$\ell_{N,p} = \left(\frac{1}{p-1}\right)\left(\frac{1}{p-1} + 2 - N\right). \tag{1.7}$$

This problem points out the existence of critical values of p. We denote by $\mathcal{E}$ the set of solutions of (1.6) and $\mathcal{E}_+ = \{\omega \in \mathcal{E} : \omega \geq 0\}$. This set has the following structure:

Theorem A. *1- If*

$$p \geq \frac{N-1}{N-2}, \tag{1.8}$$

then $\mathcal{E} = \{0\}$.

2- If

$$1 < p \leq \frac{N}{N-1}, \tag{1.9}$$

then $\mathcal{E}_+ = \{0\}$.

3- If

$$\frac{N}{N-1} < p < \frac{N-1}{N-2}, \tag{1.10}$$

then $\mathcal{E} = \{\omega_s, -\omega_s, 0\}$ *where* ω_s *is the unique positive solution of* (1.6).

When $1 < p < \frac{N}{N-1}$, we show that there exist signed solutions to (1.6).

Theorem B. *Let* $\Omega \subset \mathbb{R}^N$, $N \geq 2$, *be a bounded* C^2 *domain such that* $0 \in \partial\Omega$ *and* $g : \mathbb{R} \mapsto \mathbb{R}$ *a continuous function which satisfies* $sg(s) \geq 0$. *Then any function* $u \in C^1(\overline{\Omega} \setminus \{0\})$ *solution of*

$$\begin{array}{ll} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} + g(u) = 0 & \text{in } \partial\Omega \setminus \{0\}, \end{array} \tag{1.11}$$

satisfying near $x = 0$ *either* $u(x) = o(|x|^{2-N})$ *if* $N \geq 3$ *or* $u(x) = o(\ln|x|)$ *if* $N = 2$, *is constant and* $g(u) = 0$.

The set $\mathcal{E}$ plays a fundamental role in the characterization of boundary isolated singularities of solutions of

$$\begin{array}{ll} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} + |u|^{p-1}u = 0 & \text{in } \partial\Omega \setminus \{0\}. \end{array} \tag{1.12}$$

Theorem C. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain such that $0 \in \partial\Omega$. Assume $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a nonnegative function satisfying* (1.12) *and such that $|x|^{\frac{1}{p-1}}u(x)$ is bounded.*

1- If $p \geq \frac{N-1}{N-2}$, then $u = 0$

2- If $\frac{N}{N-1} < p < \frac{N-1}{N-2}$, we have the following alternative:
2-(i) either

$$\lim_{r\to 0} r^{\frac{1}{p-1}}u(r,\sigma) = \omega_s(\sigma) \quad \textit{locally uniformly on } S^{N-1}_+, \tag{1.13}$$

2-(ii) or there exists a nonnegative real number k such that there holds,

$$\begin{array}{lll} a) & \lim\limits_{|x|\to 0} |x|^{2-N}u(x) = k & \textit{if } N \geq 3, \\ b) & \lim\limits_{|x|\to 0} (-\ln|x|)^{-1}u(x) = k & \textit{if } N = 2. \end{array} \tag{1.14}$$

The assumption on the boundedness of $|x|^{\frac{1}{p-1}}u(x)$ seems necessary since no Keller-Osserman universal estimate [22], [27] appears to hold. Actually, if u satisfies (1.2), the function $\tilde{u}$ defined in whole $\mathbb{R}^N$ by

$$\tilde{u}(x_1,...,x_N) = \begin{cases} u(x_1,...,x_N) & \text{if } x_N > 0 \\ u(x_1,...,-x_N) & \text{if } x_N < 0, \end{cases} \tag{1.15}$$

satisfies

$$-\Delta\tilde{u} + 2|u|^{p-1}u\mathcal{H}_{\partial\mathbb{R}^N_+} = 0 \qquad \text{in } \mathbb{R}^N_+ \setminus \{0\}, \tag{1.16}$$

where $\mathcal{H}_{\partial\mathbb{R}^N_+}$ is the (N-1)-dimensional Lebesgue measure supported by $\partial\mathbb{R}^N_+$. Hence the coercivity due to the nonlinear term is localized on $\partial\mathbb{R}^N_+$. Such problems with measure valued nonlinear potential are studied in [29]. Notice also that when $p > \frac{N-1}{N-2}$, then the assumption $u(x) = O(|x|^{-\frac{1}{p-1}})$ implies that $u(x) = o(|x|^{2-N})$, hence Theorem B implies Theorem C.

When u satisfies (1.14), the problem can be interpreted with a boundary data holding in the sense of distributions,

$$\begin{array}{ll} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} + u^p = k\delta_0 & \text{in } \mathcal{D}'(\partial\Omega). \end{array} \tag{1.17}$$

For more general measures and nonlinearities, we define a solution of problem (1.1) as follows,

Definition. *Let $\Omega \subset \mathbb{R}^N$ be as in Theorem B , $\mu \in \mathfrak{M}(\partial\Omega)$ and $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. A function $u \in L^1(\Omega)$ is a weak solution of* (1.1) *if it admits a boundary trace $u\lfloor_{\partial\Omega}$ which is a Borel function on $\partial\Omega$, $g(u) \in L^1(\partial\Omega)$ and*

$$\int_\Omega u(-\Delta\xi + \xi)dx + \int_{\partial\Omega} g(u)\xi dS = \int_{\partial\Omega} \xi d\mu, \quad \textit{for all } \xi \in \mathcal{C}(\Omega), \tag{1.18}$$

where

$$\mathcal{C}(\Omega)=\left\{\xi\in C^1(\overline{\Omega}):\Delta\xi\in L^\infty(\Omega),\ \frac{\partial\xi}{\partial\mathbf{n}}=0\ \ on\ \partial\Omega\right\}. \tag{1.19}$$

In the next result we give a condition for the unconditionnal solvability of problem (1.1).

Theorem D. *Let $\Omega\subset\mathbb{R}^N$, $N\geq 3$ be a bounded C^2 domain and $g:\mathbb{R}\mapsto\mathbb{R}$ a continuous nondecreasing function such that $g(0)=0$. If g satisfies*

$$\int_1^\infty (g(s)+|g(-s)|)s^{-\frac{2N-3}{N-2}}ds<\infty, \tag{1.20}$$

then for any $\mu\in\mathfrak{M}(\partial\Omega)$, the problem (1.1) *admits a unique solution.*

A nonlinearity which satisfies (1.20) is called *subcritical*. When $N=2$ this notion has to be modified. Following Vàzquez we define the exponential orders of growth of a continuous nondecreasing function $g:\mathbb{R}\mapsto\mathbb{R}$ vanishing at 0 by

$$a_+(g)=\inf\left\{a\geq 0:\int_0^\infty e^{-as}g(s)ds<\infty\right\}, \tag{1.21}$$

and

$$a_-(g)=sup\left\{a\leq 0:\int_{-\infty}^0 e^{as}g(s)ds>-\infty\right\}. \tag{1.22}$$

Theorem E. *Let $\Omega\subset\mathbb{R}^2$ be a bounded C^2 domain and $g:\mathbb{R}\mapsto\mathbb{R}$ a continuous nondecreasing function such that $g(0)=0$.*
1- If $a_+(g)=a_-(g)=0$, then for any $\mu\in\mathfrak{M}(\partial\Omega)$ the problem (1.1) *admits a unique solution,*
2- if $0<a_+(g)<\infty$ and $-\infty<a_-(g)<0$ the problem (1.1) *admits a unique solution with $\mu=\sum_{j=1}^k\alpha_j\delta_{a_j}$, with $a_j\in\partial\Omega$ and $\alpha_j\in\mathbb{R}^*$, provided*

$$\frac{\pi}{a_-(g)}\leq\alpha_j\leq\frac{\pi}{a_+(g)}. \tag{1.23}$$

When $N\geq 3$ and g does not satisfy (1.20), there may not exist solutions for any measure. The problem is well understood if $g(r)=|r|^{p-1}r$. For example, if $p\geq\frac{N-1}{N-2}$, there is no weak solution to the problem

$$\begin{aligned}-\Delta u+u&=0 &&\text{in }\ \Omega\\ \frac{\partial u}{\partial\mathbf{n}}+|u|^{p-1}u&=\mu &&\text{in }\ \mathcal{D}'(\partial\Omega).\end{aligned} \tag{1.24}$$

when $\mu=\alpha\delta_a$ with $a\in\partial\Omega$. As in many similar problems, the condition for a Radon measure in order there exists a weak solution to (1.24) is expressed in terms of Bessel capacities, presently the capacity $C^{1,p'}_{\partial\Omega}$ on the boundary with $p'=\frac{p}{p-1}$.

Theorem F. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded C^2. Then problem* (1.24) *admits a solution with $\mu \in \mathfrak{M}_+(\partial\Omega)$, necessarily unique, if and only if μ vanishes on Borel set $E \subset \partial\Omega$ such that $C^{1,p'}_{\partial\Omega}(E) = 0$.*

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2 Separable solutions

We recall that the upper hemisphere S^{N-1}_+ can be parametrized as follows

$$S^{N-1}_+ = \{\sigma = (\sin\phi\,\sigma, \cos\phi) : \sigma' \in S^{N-2}, \phi \in [0, \tfrac{\pi}{2}]\}, \tag{2.1}$$

and we write $\omega(\sigma) = \omega(\sigma', \phi))$. With this parametrization the Laplace-Beltrami operator on S^{N-1} endows the form

$$\Delta'\omega = \frac{1}{\sin^{N-2}\phi}\left(\sin^{N-2}\phi\,\omega_\phi\right)_\phi + \frac{1}{\sin^2\phi}\Delta''\omega \tag{2.2}$$

where Δ'' is the Laplace-Beltrami operator on S^{N-2}. The surface measure on S^{N-1} induced by the Euclidean metric in $\mathbb{R}^N$ is $dS(\sigma) = \sin^{N-2}\phi dS'(\sigma')d\phi$ where $dS'(\sigma')$ is the surface measure on S^{N-2} induced by the Euclidean metric in $\mathbb{R}^{N-1}$.

2.1 Proof of Theorem A

Proof of assertion 1. If $p \geq \frac{N-1}{N-2}$ then $\ell_{N,p} \leq 0$. If ω is a solution of (1.6), then

$$\int_{S^{N-1}_+}\left(|\nabla'\omega|^2 - \ell_{N,p}\omega^2\right)dS + \int_{\partial S^{N-1}_+}|\omega|^{p+1}dS' = 0.$$

Hence $\omega = 0$.

Proof of assertion 2. Assume (1.9) holds and ω is a positive solution of (1.6). The function $\phi \mapsto \cos\phi$ is the first eigenfunction of $-\Delta'$ in $H^1_0(S^{N-1}_+)$ with corresponding eigenvalue N-1. Multiplying the equation by $\cos\phi$ and integrating yields

$$(\ell_{N,p} + 1 - N)\int_{S^{N-1}_+}\omega\cos\phi dS + \int_{\partial S^{N-1}_+}\omega dS' = 0$$

If $1 < p \leq \frac{N}{N-1}$, then $\ell_{N,p} + 1 - N \geq 0$, hence $\omega\lfloor_{\partial S^{N-1}_+} = 0$, hence $\omega = 0$ by Hopf boundary lemma..

Proof of assertion 3. Assume (1.10) holds. We first prove that any solution ω of (1.6) depends only on ϕ following a method introduced in [34] and it has constant sign. We set

$$\bar\omega(\phi) = \frac{1}{|S^{N-2}|}\int_{S^{N-2}}\omega(\sigma', \phi)dS'(\sigma').$$

Then

$$\int_{S^{N-2}} \left(|\omega|^{p-1}\omega - \overline{|\omega|^{p-1}\omega}\right)(\omega-\overline{\omega})\,dS' = \int_{S^{N-2}} \left(|\omega|^{p-1}\omega - |\overline{\omega}|^{p-1}\overline{\omega}\right)(\omega-\overline{\omega})\,dS',$$

since

$$\int_{S^{N-2}} \left(\overline{|\omega|^{p-1}\omega} - |\overline{\omega}|^{p-1}\overline{\omega}\right)(\omega-\overline{\omega})\,dS' = \left(\overline{|\omega|^{p-1}\omega} - |\overline{\omega}|^{p-1}\overline{\omega}\right)\int_{S^{N-2}} (\omega-\overline{\omega})\,dS' = 0.$$

Hence

$$\int_{S^{N-2}} \left(|\omega|^{p-1}\omega - \overline{|\omega|^{p-1}\omega}\right)(\omega-\overline{\omega})\,dS' \geq 2^{1-p}\int_{S^{N-2}} |\omega-\overline{\omega}|^{p+1}\,dS'.$$

From the expression (2.2) we get

$$\begin{aligned} -\int_{S_+^{N-1}} |\nabla'(\omega-\overline{\omega})|^2 dS + \ell_{N,p}\int_{S_+^{N-1}} (\omega-\overline{\omega})^2 dS & \\ = \int_{S^{N-2}} \left(|\omega|^{p-1}\omega - \overline{|\omega|^{p-1}\omega}\right)(\omega-\overline{\omega})\,dS' & \\ \geq 2^{1-p}\int_{S^{N-2}} |\omega-\overline{\omega}|^{p+1}\,dS'. & \end{aligned}$$

Since $\overline{\omega}$ is the projection of ω onto the first eigenspace of $-\Delta'$ in $H^1(S_+^{N-1})$ and N-1 the corresponding eigenvalue,

$$-\int_{S_+^{N-1}} |\nabla'(\omega-\overline{\omega})|^2 dS \leq (1-N)\int_{S_+^{N-1}} (\omega-\overline{\omega})^2 dS.$$

Hence

$$(\ell_{N,p}+1-N)\int_{S_+^{N-1}} (\omega-\overline{\omega})^2 dS \geq 2^{1-p}\int_{S^{N-2}} |\omega-\overline{\omega}|^{p+1}\,dS'.$$

If $p \geq \frac{N}{N-1}$, then $\ell_{N,p}+1-N \leq 0$. This implies $\omega = \overline{\omega}$. It follows that ω depends only on the variable $\phi \in (0, \frac{\pi}{2})$ and thus it satisfies

$$\begin{aligned} &\frac{1}{\sin^{N-2}\phi}\left(\sin^{N-2}\phi\,\omega_\phi\right)_\phi + \ell_{N,p}\omega = 0 \qquad \text{in } (0,\tfrac{\pi}{2}) \\ &\omega_\phi(0) = 0\,,\ \left(\omega_\phi + |\omega|^{p-1}\omega\right)\left(\tfrac{\pi}{2}\right) = 0. \end{aligned} \tag{2.3}$$

Next we prove that any solution has constant sign. Let us assume that $\omega(0) > 0$. If ω vanishes at a first point some $\phi_0 \in (0, \frac{\pi}{2}]$, then it is positive on $(0,\phi_0)$ and $\omega_\phi(\phi_0) < 0$ by Cauchy-Lipschitz theorem. If $\phi_0 = \frac{\pi}{2}$, then $\omega_\phi(\phi_0) = 0$ from (2.3), contradiction. Hence $\phi_0 < \frac{\pi}{2}$. This implies that ω is a positive solution of

$$\begin{aligned} &\frac{1}{\sin^{N-2}\phi}\left(\sin^{N-2}\phi\,\omega_\phi\right)_\phi + \ell_{N,p}\omega = 0 \qquad \text{in } (0,\phi_0) \\ &\omega_\phi(0) = 0\,,\ \omega(\phi_0) = 0. \end{aligned} \tag{2.4}$$

Thus ω is a first eigenfunction of $-\Delta'$ in $H_0^1(S_{\phi_0})$ where

$$S_{\phi_0} = \{\sigma = (\sigma', \phi) \in S^{N-2} \times (0,\phi_0)\} \subsetneq S_+^{N-1}.$$

Hence $\ell_{N,p} > N-1$, contradiction.

Then we prove that there exists at most one positive solution ω. Let $\tilde{\omega}$ be another positive solution. A straightformard computation yields

$$\begin{aligned} 0 &= \int_{S_+^{N-1}} \left(\frac{\Delta'\omega}{\omega} - \frac{\Delta'\tilde{\omega}}{\tilde{\omega}} \right) \left(\omega^2 - \tilde{\omega}^2 \right) dS \\ &= -\int_{S_+^{N-1}} \left(\frac{1}{\omega^2} + \frac{1}{\tilde{\omega}^2} \right) |\omega \nabla' \tilde{\omega} - \tilde{\omega} \nabla' \omega|^2 - \int_{\partial S_+^{N-1}} \left(\omega^{p-1} - \tilde{\omega}^{p-1} \right) \left(\omega^2 - \tilde{\omega}^2 \right) dS'. \end{aligned}$$

This implies that $\omega = \tilde{\omega}$.

Finally we prove existence. Set

$$J(\eta) = \frac{1}{2} \int_{S_+^{N-1}} \left(|\nabla' \eta|^2 - \ell_{N,p} \eta^2 \right) dS + \frac{1}{p+1} \int_{\partial S_+^{N-1}} |\eta|^{p+1} dS. \tag{2.5}$$

The functional J is defined in

$$\mathbb{X}_{rad}(S_+^{N-1}) := \left\{ \eta \in H^1(S_+^{N-1}) \cap L^{p+1}(\partial S_+^{N-1}) : \eta \text{ depends only on } \phi \in [0, \tfrac{\pi}{2}] \right\},$$

and it is lower continuous. If $\eta \in \mathbb{X}_{rad}(S_+^{N-1})$, then $\eta = \eta_1 + \eta_0$ where $\eta_0 = \eta(\frac{\pi}{2})$ and $\eta_1 \in H_0^1(S_+^{N-1})$. In particular

$$\begin{aligned} J(\eta) = \frac{1}{2} \int_{S_+^{N-1}} \left(|\nabla' \eta_1|^2 - \ell_{N,p} \eta_1^2 \right) dS - \ell_{N,p} \eta_0 \int_{S_+^{N-1}} \eta_1 dS \\ - \frac{\ell_{N,p} |S_+^{N-1}|}{2} \eta_0^2 + \frac{|S^{N-2}|}{p+1} |\eta_0|^{p+1} \end{aligned}$$

Since (1.10) holds, $0 < \ell_{N,p} \leq N-1$; if we take $\eta(\phi) = \epsilon_0 \in \mathbb{R}$, then

$$J(\eta) = -\frac{\ell_{N,p} |S_+^{N-1}|}{2} \epsilon_0^2 + \frac{|S^{N-2}|}{p+1} |\epsilon_0|^{p+1}.$$

Hence the infimum of J in $\mathbb{X}_{rad}(S_+^{N-1})$ is negative. Since $p > \frac{N}{N-1}$, then $\ell_{N,p} < N-1$, and for $\epsilon = N-1-\ell_{N,p} > 0$ there holds

$$J(\eta) \geq \frac{\epsilon}{2} \int_{S_+^{N-1}} \eta_1^2 dS - \ell_{N,p} \eta_0 \int_{S_+^{N-1}} \eta_1 dS - \frac{\ell_{N,p} |S_+^{N-1}|}{2} \eta_0^2 + \frac{|S^{N-2}|}{p+1} |\eta_0|^{p+1}.$$

By Young's inequality $J(\eta) \to \infty$ when $\|\eta\|_{H^1(S_+^{N-1})} + \|\eta\|_{L^{p+1}(\partial S_+^{N-1})} \to \infty$. Therefore J achieves its minimum in $\mathbb{X}_{rad}(S_+^{N-1})$ at some ω, which can be assume to be positive since $J(\eta) = J(|\eta|)$. If we denote it by ω_s, there holds $\mathcal{E} = \{\omega_s, -\omega_s, 0\}$, which ends the proof. □

The value $p = \frac{N}{N-1}$ is a bifurcation value as it is shown below.

Proposition 2.1 *There exists a C^1 curve $\epsilon \mapsto (p_\epsilon, \omega_\epsilon)$ defined in $[0, \epsilon_0]$ with $\epsilon > 0$ such that $(p_0, \omega_0) = (\frac{N}{N-1}, 0)$ where $1 < p_\epsilon < \frac{N}{N-1}$ and ω_ϵ is a nonzero signed solution of*

$$\begin{aligned} &\Delta' \omega_\epsilon + \ell_{N,p_\epsilon} \omega = 0 && \text{in } S_+^{N-1} \\ &\frac{\partial \omega}{\partial \nu} + |\omega|^{p_\epsilon - 1} \omega = 0 && \text{in } \partial S_+^{N-1}. \end{aligned} \tag{2.6}$$

Proof. The linearization of (1.6) at $p = \frac{N}{N-1}$ and $\omega = 0$ yields

$$\begin{aligned} \Delta'\psi + (N-1)\psi &= 0 \qquad \text{in } S_+^{N-1} \\ \frac{\partial \psi}{\partial \nu} &= 0 \qquad \text{in } \partial S_+^{N-1}. \end{aligned} \tag{2.7}$$

If $\mathbb{R}_+^N := \{x = (x_1, ..., x_{_N}) : x_{_N} > 0\}$, then for $j < N$ the restriction to S_+^{N-1} of the function $\psi_j : x \mapsto x_j$ satisfies (2.7). In order to satisfy the simplicity requirement, we consider the functions defined on S_+^{N-1} depending only of the variable $x_j\lfloor_{S_+^{N-1}}$. Then ψ_j is a simple eigenfunction of Δ' associated to the eigenvalue $N-1$. By the classical Crandall-Rabinowitz theorem[17] there exists a C^1 curve $\epsilon \mapsto (p_\epsilon, \omega_\epsilon)$ starting from $(\frac{N}{N-1}, 0)$ such that ω_ϵ is a nonzero solution depending only of the variable $x_j\lfloor_{S_+^{N-1}}$ of the problem

$$\begin{aligned} \Delta'\omega_\epsilon + \ell_{N,p_\epsilon}\omega_\epsilon &= 0 \qquad \text{in } S_+^{N-1} \\ \frac{\partial \omega_\epsilon}{\partial \nu} + |\omega_\epsilon|^{p_\epsilon - 1}\omega_\epsilon &= 0 \qquad \text{in } \partial S_+^{N-1}. \end{aligned} \tag{2.8}$$

Since ω_ϵ depends only on $x_j\lfloor_{S_+^{N-1}}$ and inherits the properties of ψ_j, it changes sign. By Theorem A-1-2, $p_\epsilon < \frac{N}{N-1}$, which ends the proof. □

2.2 Separable solutions in dimension 2

When $N = 2$, (2.4) endows the form

$$\begin{aligned} \omega_{\phi\phi} + \frac{1}{(p-1)^2}\omega &= 0 \qquad \text{in } (0, \pi) \\ \left(-\omega_\phi + |\omega|^{p-1}\omega\right)(0) &= 0 \\ \left(\omega_\phi + |\omega|^{p-1}\omega\right)(\pi) &= 0, \end{aligned} \tag{2.9}$$

and therefore

$$\omega(\phi) = a\cos\left(\frac{\phi}{p-1}\right) + b\sin\left(\frac{\phi}{p-1}\right)$$

for some real numbers a, b. The boundary conditions are the following

$$(i) \qquad -\frac{b}{p-1} + |a|^{p-1}a = 0 \Longleftrightarrow b = (p-1)|a|^{p-1}a,$$

$$\begin{aligned} (ii) \qquad & -\frac{a}{p-1}\sin\left(\frac{\pi}{p-1}\right) + \frac{b}{p-1}\cos\left(\frac{\pi}{p-1}\right) \\ & + \left(a\cos\left(\frac{\pi}{p-1}\right) + b\sin\left(\frac{\pi}{p-1}\right)\right)\left|a\cos\left(\frac{\pi}{p-1}\right) + b\sin\left(\frac{\pi}{p-1}\right)\right|^{p-1} = 0. \end{aligned} \tag{2.10}$$

Theorem 2.2 *If $N = 2$ the set $\mathcal{E}$ is always discrete and more precisely,*
1- If $\frac{1}{p-1} \in \mathbb{N}^*$, then 0 is the unique solution to (2.9).
2- If $\frac{1}{p-1} \notin \mathbb{N}^*$, then (2.10) admits three solutions ω_s, $-\omega_s$ and zero. Furthermore ω_s keeps a constant sign if $p \geq 2$.

Proof. Because of (2.10)-(i) we can assume $a, b > 0$. Set $X = (p-1)a^{p-1}$ and

$$\Phi(X) = -\sin\left(\frac{\pi}{p-1}\right) + X\cos\left(\frac{\pi}{p-1}\right) + X\left(\cos\left(\frac{\pi}{p-1}\right) + X\sin\left(\frac{\pi}{p-1}\right)\right)\left|\cos\left(\frac{\pi}{p-1}\right) + X\sin\left(\frac{\pi}{p-1}\right)\right|^{p-1}.$$

All the separable solutions with $a > 0$ (and similarly with $a < 0$) are obtained with $a^{p-1} = X_0$ and $b = \left(\frac{a}{p-1}\right)^{\frac{1}{p}}$ where X_0 is a positive zero of the function Φ.

(1) If $\frac{\pi}{p-1} = \frac{\pi}{2} + k\pi$ for some $k \in \mathbb{N}$, then $\Phi(X) = (-1)^{k+1}(1 - X^{p+1})$. Hence there exist only three solutions corresponding to

$$(a, b) = (0, 0)$$

$$(a, b) = \left(\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}, \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}\right)$$

$$(a, b) = \left(-\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}, -\left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}\right).$$

(2) If $\frac{\pi}{p-1} = k\pi$ for some $k \in \mathbb{N}$, then $\Phi(X) = 2(-1)^k X$. Hence the only solution is $(a, b) = (0, 0)$.
(3) If $\frac{\pi}{p-1} \neq \frac{k\pi}{2}$ for any $k \in \mathbb{N}$, then

$$\Phi(X) = X - \tan\left(\frac{\pi}{p-1}\right) + X\tan\left(\frac{\pi}{p-1}\right)\left|\sin\left(\frac{\pi}{p-1}\right)\right|^{p-1}\left|\cot\left(\frac{\pi}{p-1}\right) + X\right|^{p-1}\left(\cot\left(\frac{\pi}{p-1}\right) + X\right).$$

Hence

$$\Phi'(X) = 1 + \tan\left(\frac{\pi}{p-1}\right)\left|\sin\left(\frac{\pi}{p-1}\right)\right|^{p-1}\left|\cot\left(\frac{\pi}{p-1}\right) + X\right|^{p-1}\left(\cot\left(\frac{\pi}{p-1}\right) + (p+1)X\right).$$

If $\tan\left(\frac{\pi}{p-1}\right) > 0$, then $\Phi'(X) > 0$ and since $\Phi(0) < 0$, Φ admits a unique root $X_0 > 0$. Hence there exist only three solutions, $\omega_s, -\omega_s, 0$.
If $\tan\left(\frac{\pi}{p-1}\right) < 0$, then $\Phi(0) > 0$. Moreover

$$\Phi''(X) = p\tan\left(\frac{\pi}{p-1}\right)\left|\sin\left(\frac{\pi}{p-1}\right)\right|^{p-1}\left|\cot\left(\frac{\pi}{p-1}\right) + X\right|^{p-3} \times \left(\cot\left(\frac{\pi}{p-1}\right) + X\right)\left((p+1)X + 2\cot\left(\frac{\pi}{p-1}\right)\right).$$

Hence Φ'' is negative in the interval $(0, -\frac{2}{p+1}\cot\left(\frac{\pi}{p-1}\right))$, positive in $(-\frac{2}{p+1}\cot\left(\frac{\pi}{p-1}\right), -\cot\left(\frac{\pi}{p-1}\right))$ and negative in $(-\cot\left(\frac{\pi}{p-1}\right), \infty)$. A standard study shows that Φ' is positive on $(0, X_*)$ for some $X_* > -\cot\left(\frac{\pi}{p-1}\right)$, vanishes at X_* and is negative on (X_*, ∞). Finally, Φ is increasing on $(-\infty, X_*)$ with a positive maximum and negative on (X_*, ∞). As a consequence Φ admits a unique zero at $X_0 > -\cot\left(\frac{\pi}{p-1}\right)$ and there exist again only three solutions $\omega_s, -\omega_s$ and 0. This ends the proof. □

3 Isolated singularities

3.1 Regularity results

We assume that $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded smooth domain such that $0 \in \partial\Omega$. We have the following basic estimate the proof of which is based upon Moser's iterative scheme.

Proposition 3.1 *Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function such that $rg(r) \geq 0$ on $\mathbb{R}$. Then any function $u \in C^1(\overline{\Omega} \setminus \{0\})$ which verifies*

$$\begin{array}{ll} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} + g(u) = 0 & \text{in } \partial\Omega \setminus \{0\}, \end{array} \tag{3.1}$$

satisfies for any $a > 1$ and some $c_a > 0$,

$$\|u\|_{L^\infty(\Omega \cap B^c_{2r})} \leq \frac{c_a}{r^{\frac{N}{a}}} \|u\|_{L^a(\Omega \cap B^c_r)} \qquad \text{for all } r \in (0, r_0], \tag{3.2}$$

where $r_0 > 0$ depends on Ω. In particular, if u is nonnegative, then for any $\epsilon > 0$ there exists $c_\epsilon > 0$ such that

$$\|u\|_{L^\infty(\Omega \cap B^c_{2r})} \leq \frac{c_\epsilon}{r^{N-1+\epsilon}} \int_{\partial\Omega} d\lambda, \tag{3.3}$$

where $\lambda \in \mathfrak{M}_+(\partial\Omega)$ is the boundary trace of u.

Proof. Let $\zeta \in C^{1,1}(\mathbb{R}^N)$ such that $0 \leq \zeta \leq 1$, $\zeta(x) = 1$ if $|x| \geq s$, $\zeta(x) = 0$ if $|x| \leq r$ for some $0 < r < s$, and $|\nabla\zeta(x)| \leq \frac{2}{s-r}\chi_{\Gamma^s_r}(x)$ where $\Gamma^s_r = \{x \in \mathbb{R}^N : r \leq |x| \leq s\}$. For $\alpha > 0$ we have from (3.1),

$$\int_\Omega \langle \nabla u, \nabla(\zeta^2 |u|^{\alpha-1} u)\rangle dx + \int_{\partial\Omega} \zeta^2 g(u)|u|^{\alpha-1} u \, dS = 0.$$

Then

$$\begin{aligned}\int_\Omega \langle \nabla u.\nabla(\zeta^2|u|^{\alpha-1}u)\rangle dx &= \frac{4\alpha}{(\alpha+1)^2}\int_\Omega \left|\nabla|u|^{\frac{\alpha+1}{2}}\right|^2 \zeta^2 dx \\ &+ \frac{4\alpha}{\alpha+1}\int_\Omega \zeta|u|^{\frac{\alpha-1}{2}}u\nabla|u|^{\frac{\alpha+1}{2}}.\nabla\zeta dx \\ &\geq \frac{4\alpha}{(\alpha+1)^2}\int_\Omega \left|\nabla|u|^{\frac{\alpha+1}{2}}\right|^2 \zeta^2 dx \\ &- \frac{4\alpha}{\alpha+1}\left(\int_\Omega |u|^{\alpha+1}|\nabla\zeta|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega \left|\nabla|u|^{\frac{\alpha+1}{2}}\right|^2 \zeta^2 dx\right)^{\frac{1}{2}}\end{aligned}$$

Put

$$X = \left(\int_\Omega \left|\nabla|u|^{\frac{\alpha+1}{2}}\right|^2 \zeta^2 dx\right)^{\frac{1}{2}},\ Y = \left(\int_\Omega |u|^{\alpha+1}|\nabla\zeta|^2 dx\right)^{\frac{1}{2}}$$

and $A = \int_{\partial\Omega} \zeta^2 g(u)|u|^{\alpha-1}u\, dS$, then

$$4\alpha X^2 - 4(\alpha+1)XY + (\alpha+1)^2A^2 \leq 0 \Longrightarrow X \leq \frac{\alpha+1}{\alpha}Y. \tag{3.4}$$

The discriminant of this equation in X is necessarily nonnegative, therefore

$$Y \leq \alpha A^2 \Longleftrightarrow \int_{\partial\Omega} \zeta^2 g(u)|u|^{\alpha-1}u\, dS \leq \frac{1}{\alpha}\int_\Omega |u|^{\alpha+1}|\nabla\zeta|^2 dx. \tag{3.5}$$

Since $\zeta\nabla|u|^{\frac{\alpha+1}{2}} = \nabla\left(\zeta|u|^{\frac{\alpha+1}{2}}\right) - |u|^{\frac{\alpha+1}{2}}\nabla\zeta$, we deduce from (3.4) with the help of Young's inequality,

$$\frac{\alpha}{(\alpha+1)^2}\int_\Omega \left|\nabla\left(\zeta|u|^{\frac{\alpha+1}{2}}\right)\right|^2 dx + \int_{\partial\Omega} \zeta^2 g(u)|u|^{\alpha-1}u\, dS \leq \frac{13}{\alpha}\int_\Omega |u|^{\alpha+1}|\nabla\zeta|^2 dx,$$

which leads to

$$\begin{aligned}\frac{\alpha}{(\alpha+1)^2}\left\|\zeta|u|^{\frac{\alpha+1}{2}}\right\|^2_{W^{1,2}(\Omega)} &+ \int_{\partial\Omega} \zeta^2 g(u)|u|^{\alpha-1}u\, dS \\ &\leq \frac{13}{\alpha}\int_\Omega |u|^{\alpha+1}|\nabla\zeta|^2 dx + \frac{\alpha}{(\alpha+1)^2}\left\|\zeta|u|^{\frac{\alpha+1}{2}}\right\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.6}$$

We first assume $N \geq 3$ and set $\theta = \frac{N}{N-2}$. If $s - r \leq 1$, we obtain, using Gagliardo-Nirenberg's inequality,

$$\frac{\alpha}{(\alpha+1)^2}\|u\|^{\alpha+1}_{L^{\theta(\alpha+1)}(\Omega\cap B_s^c)} + \int_{\partial\Omega\cap B_s^c} g(u)|u|^{\alpha-1}u\, dS \leq \frac{c_N^2}{\alpha(s-r)^2}\|u\|^{\alpha+1}_{L^{\alpha+1}(\Omega\cap B_r^c)}. \tag{3.7}$$

We fix $r > 0$ and define the sequences for $n \in \mathbb{N}^*$

$$\begin{array}{ll} p_n = \theta p_{n-1} & \text{with } p_0 = \alpha+1 = a > 1 \\ r_n = r(2-2^{-n}) & \text{with } r_0 = r \\ s_n = r(2-2^{-n-1}), & \end{array}$$

thus $s_n - r_n = 2^{-n-1}r$. We obtain from (3.7)

$$\frac{p_n-1}{p_n^2}\left\|u\right\|^{p_n}_{L^{p_{n+1}}(\Omega\cap B^c_{s_n})}+\int_{\partial\Omega\cap B^c_{s_n}} g(u)|u|^{p_n-2}u\,dS\le\frac{c_{_N}}{(p_n-1)(s_n-r_n)^2}\left\|u\right\|^{p_n}_{L^{p_n}(\Omega\cap B^c_{r_n})}. \tag{3.8}$$

Therefore

$$\left\|u\right\|_{L^{p_{n+1}}(\Omega\cap B^c_{s_n})}\le\left(\frac{c_{_N}2^{n+1}(\alpha+1)}{\alpha r}\right)^{\frac{2}{p_n}}\left\|u\right\|_{L^{p_n}(\Omega\cap B^c_{r_n})} \tag{3.9}$$

Because $s_n\to 2r$ when $n\to\infty$, we obtain by an easy induction

$$\left\|u\right\|_{L^\infty(\Omega\cap B^c_{2r})}\le\frac{c_{N,\alpha}}{r^{\frac{N}{a}}}\left\|u\right\|_{L^a(\Omega\cap B^c_r)}. \tag{3.10}$$

We notice that we have neglected the boundary integral in (3.8). Indeed, the same induction yields

$$\left\|u\right\|_{L^\infty(\partial\Omega\cap B^c_{2r})}\le\frac{c'_{N,\alpha}}{r^{\frac{N}{a}}}\left\|u\right\|_{L^a(\Omega\cap B^c_r)}. \tag{3.11}$$

If $N=2$ we use the interpolation inequality

$$\left\|\zeta|u|^{\frac{\alpha+1}{2}}\right\|^2_{W^{\frac12,2}(\Omega)}\le c_1\left\|\zeta|u|^{\frac{\alpha+1}{2}}\right\|_{W^{1,2}(\Omega)}\left\|\zeta|u|^{\frac{\alpha+1}{2}}\right\|_{L^2(\Omega)}. \tag{3.12}$$

Combining it with the imbedding inequality

$$\left\|\zeta|u|^{\frac{\alpha+1}{2}}\right\|^2_{L^{\frac{2N}{N-1}}(\Omega)}\le c_2\left\|\zeta|u|^{\frac{\alpha+1}{2}}\right\|^2_{W^{\frac12,2}(\Omega)}, \tag{3.13}$$

we obtained that (3.7) is replaced by

$$\frac{\alpha}{(\alpha+1)^2}\left\|u\right\|^{\alpha+1}_{L^{\tilde\theta(\alpha+1)}(\Omega\cap B^c_s)}+\frac12\int_{\partial\Omega\cap B^c_s}g(u)|u|^{\alpha-1}u\,dS\le\frac{\tilde c^2_{_N}}{\alpha(s-r)^2}\left\|u\right\|^{\alpha+1}_{L^{\alpha+1}(\Omega\cap B^c_r)}. \tag{3.14}$$

with $\tilde\theta=\frac{N}{N-1}$. *Mutatis mutandis*, the end of the proof follows easily.

Next we assume that $u\ge 0$. Then it admits a boundary trace (see e.g. [24]) which is a nonnegative Radon λ measure on $\partial\Omega$ and the Riesz-Herglotz representation formula in terms of Poisson potential of the measure λ holds,

$$u(x)=\mathbf{P}^\Omega[\lambda]:=\int_{\partial\Omega}P^\Omega(x,y)d\lambda(y)\qquad\text{for all } x\in\Omega, \tag{3.15}$$

where P^Ω is the Poisson kernel defined in $\Omega\times\partial\Omega$. Furthermore u belongs to the Lorentz space $L^{\frac{N}{N-1},\infty}(\Omega)$ and $L^{\frac{N+1}{N-1},\infty}_\rho(\Omega)$, where $\rho(x)=\operatorname{dist}(x,\partial\Omega)$ (see e.g. [20]). Furthermore

$$\left\|u\right\|_{L^{\frac{N}{N-1},\infty}(\Omega)}+\left\|u\right\|_{L^{\frac{N+1}{N-1},\infty}_\rho(\Omega)}\le c_\Omega\left\|\lambda\right\|_{\mathfrak{M}(\partial\Omega)}. \tag{3.16}$$

For any $\epsilon>0$ there exists $c_\epsilon>0$ such that

$$\left\|u\right\|_{L^{\frac{N}{N-1+\epsilon},\infty}(\Omega)}\le c_\epsilon\left\|u\right\|_{L^{\frac{N}{N-1},\infty}(\Omega)}$$

If we apply (3.11) with $a = \frac{N}{N-1+\epsilon}$ we infer

$$\|u\|_{L^\infty(\Omega\cap B^c_{2r})} \le \frac{c'_\epsilon}{r^{N-1+\epsilon}} \|\lambda\|_{\mathfrak{M}(\partial\Omega)}. \tag{3.17}$$

This ends the proof. □

Remark. A natural question is whether (3.3) is valid with $\epsilon = 0$. Notice that using the standard estimates on the Poisson kernel we have,

$$\|u\|_{L^\infty(\Omega_r)} \le \frac{c}{r^{N-1}} \|\lambda\|_{\mathfrak{M}(\partial\Omega)} \quad \text{for all } r > 0, \tag{3.18}$$

where $\Omega_r = \{x \in \Omega : \rho(x) \ge r\}$ and $c = c(\Omega) > 0$.

3.2 Linear estimates

We assume that Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \ge 2$.

Proposition 3.2 *Let $a \ge 0$ be a constant and λ and μ be two bounded Radon measures on Ω and $\partial\Omega$ respectively. Then there exists a unique weak solution $u \in L^1(\Omega)$ of*

$$\begin{aligned} -\Delta u + au &= \lambda && \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} &= \mu && \text{in } \Omega. \end{aligned} \tag{3.19}$$

Furthermore there exists $c = c(\Omega) > 0$ such that

$$\|u\|_{L^{\frac{N}{N-2},\infty}(\Omega)} + \|\nabla u\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \le c\left(\|\lambda\|_{\mathfrak{M}(\Omega)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}\right) + b_a \|u\|_{L^1(\Omega)}, \tag{3.20}$$

if $N > 2$ with $b_a \ge 0$, $b_a > 0$ if $a = 0$, and

$$\|u\|_{L^r(\Omega)} + \|\nabla u\|_{L^{2,\infty}(\Omega)} \le c(r)\left(\|\lambda\|_{\mathfrak{M}(\Omega)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}\right) + b_a \|u\|_{L^1(\Omega)}, \tag{3.21}$$

for any $r < \infty$, if $N = 2$.

Proof. We first consider the case $\Omega = B_R^+ := \{x = (x', x_N) \in B_R : x_N > 0\}$. We set

$$\tilde{u}(x', x_N) = \begin{cases} u(x', x_N) & \text{if } x_N > 0 \\ u(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Then $\tilde{u}$ satisfies

$$\begin{aligned} -\Delta\tilde{u} + a\tilde{u} &= \tilde{\lambda} + 2\mathcal{H}_{\partial\mathbb{R}^N_+}\mu && \text{in } B_R \\ \frac{\partial\tilde{u}}{\partial\mathbf{n}} &= \tilde{\mu}\lfloor_{\partial B_R} && \text{in } \partial B_R, \end{aligned} \tag{3.22}$$

where $\mathcal{H}_{\partial\mathbb{R}^N_+}$ is the (N-1)-dimensional Hausdorff measure and $\tilde{\lambda}$ and $\tilde{\mu}$ are defined accordingly to $\tilde{u}$ by an even reflexion through $\partial\mathbb{R}^N_+$. Then $\tilde{u}$ satisfies locally (3.20) in the sense that for any $0 < R' < R$ there holds

$$\|\tilde{u}\|_{L^{\frac{N}{N-2},\infty}(B_{R'})} + \|\nabla\tilde{u}\|_{L^{\frac{N}{N-1},\infty}((B_{R'})} \le c\left(\left\|\tilde{\lambda}\right\|_{\mathfrak{M}(B_R)} + \|\mu\|_{\mathfrak{M}(\partial B^+_{R'})}\right), \tag{3.23}$$

when $N > 2$, with straightforward modification if $N = 2$. This implies

$$\|u\|_{L^{\frac{N}{N-2},\infty}(B^+_{R'})} + \|\nabla u\|_{L^{\frac{N}{N-1},\infty}((B^+_{R'})} \leq c\left(\left\|\tilde{\lambda}\right\|_{\mathfrak{M}(B_R)} + \|\mu\|_{\mathfrak{M}(\partial B^+_R)}\right); \tag{3.24}$$

For a general domain Ω, consider a point $a \in \partial\Omega$. There exists $r_a > 0$ such that we can perform an even reflexion though $\partial\Omega \cap B_{r_a}(a)$ following the normal vector to $\partial\Omega$ as in [6, Lemma 2.4], with the modification that we use an even reflection and not the odd one which is therein adapted to zero boundary data. If we denote by $\tilde{u}$ the reflected function defined in $B_{r_a}(a)$, it satisfies

$$-\sum_j \frac{\partial}{\partial x_j}\tilde{A}_j(x, \nabla\tilde{u}) + a\tilde{u} = \tilde{\lambda} + 2\mathcal{H}_{\partial\mathbb{R}^N_+}\mu \qquad \text{in } B_{r_a}(a), \tag{3.25}$$

where the A_j are C^1 functions satisfying the standard ellipticity and boundedness conditions. The local regularity theory yields

$$\|\tilde{u}\|_{L^{\frac{N}{N-2},\infty}(B_{r'_a}(a))} + \|\nabla\tilde{u}\|_{L^{\frac{N}{N-1},\infty}((B_{r'_a}(a))} \leq c\left(\left\|\tilde{\lambda}\right\|_{\mathfrak{M}(B_{r_a}(a))} + \|\mu\|_{\mathfrak{M}(\partial B^+_{r_a}(a))}\right). \tag{3.26}$$

for any $0 < r'_a < r_a$, where c depends on Ω and $r_a - r'_a$. We obtain (3.20) by a compactness argument. The proof of (3.21) is similar. Uniqueness is straightforward. □

Remark. These results are not new. However they show that the estimates are local which will be useful later on in the sense that for any compact set $K \subset \overline{\Omega}$ and any $\epsilon > 0$ there holds

$$\|u\|_{L^{\frac{N}{N-2},\infty}(K)} + \|\nabla\tilde{u}\|_{L^{\frac{N}{N-1},\infty}(K)} \leq c\left(\left\|\tilde{\lambda}\right\|_{\mathfrak{M}(\Omega\cap K_\epsilon)} + \|\mu\|_{\mathfrak{M}(\partial\Omega\cap K_\epsilon)}\right), \tag{3.27}$$

where $K_\epsilon = \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) \leq \epsilon\}$ and where c is a positive constant depending on Ω, K, ϵ.

Remark. A more general global statement of existence and regularity with a more involved proof can be found in [26, Theorems 1, 2]. The same estimates holds up to replacing $b_a \|u\|_{L^1(\Omega)}$ by $b_a \|u\|_{L^1(\partial\Omega)}$ in (3.20)-(3.21) if (3.19) is replaced by

$$\begin{aligned} -\Delta u &= \lambda && \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} + au &= \mu && \text{in } \Omega. \end{aligned} \tag{3.28}$$

Lemma 3.3 *Let $\lambda \in L^1(\Omega)$, $\mu \in L^1(\partial\Omega)$ and $u \in L^1(\Omega)$ be the weak solution of (3.19). Then we have for all $\zeta \in \mathcal{C}(\Omega)$, $\zeta \geq 0$,*

$$\int_\Omega |u|(-\Delta\zeta + a\zeta)dx \leq \int_\Omega \lambda\zeta \operatorname{sign}_0(u)dx + \int_{\partial\Omega} \mu\zeta \operatorname{sign}_0(u)dS \tag{3.29}$$

where $\operatorname{sign}_0 = \chi_{(0,\infty)} - \chi_{(-\infty,0)}$, *and*

$$\int_\Omega u_+(-\Delta\zeta + a\zeta)dx \leq \int_\Omega \lambda\zeta \operatorname{sign}_0^+(u)dx + \int_{\partial\Omega} \mu\zeta \operatorname{sign}_0^+(u)dS \tag{3.30}$$

where $\operatorname{sign}_0^+ = \chi_{(0,\infty)}$.

Proof. We first assume that u is a smooth function. Let $\{\gamma_k\} \subset C_0^\infty(\mathbb{R})$ be a sequence of nonnegative functions such that $0 \le \gamma_k \le 1$, $\gamma_k = 0$ on $(-\infty, 0]$, $\gamma_k' \ge 0$, $\gamma_k = 1$ on $[k^{-1}, \infty)$, and let $\zeta \in \mathcal{C}(\Omega)$, $\zeta \ge 0$. Then

$$\int_\Omega \left(\gamma_k'(u)|\nabla u|^2\zeta + \gamma_k(u)\langle \nabla u, \nabla\zeta\rangle\right) dx + a\int_\Omega u\gamma_k(u)dx \\ = \int_\Omega \gamma_k(u)\zeta\lambda dx + \int_{\partial\Omega} \gamma_k(u)\zeta\mu dS.$$

Set $j_k(r) = \int_0^r \gamma_k(s)ds$, then

$$\int_\Omega \langle \nabla j_k(u), \nabla\zeta\rangle dx + a\int_\Omega u\gamma_k(u)dx \le \int_\Omega \gamma_k(u)\zeta\lambda dx + \int_{\partial\Omega} \gamma_k(u)\zeta\mu dS.$$

Since $\zeta \in \mathcal{C}(\Omega)$,

$$\int_\Omega \left(-j_k(u)\Delta\zeta + au\gamma_k(u)\right) dx \le \int_\Omega \gamma_k(u)\zeta\lambda dx + \int_{\partial\Omega} \gamma_k(u)\zeta\mu dS. \tag{3.31}$$

Letting $k \to \infty$, we infer

$$\int_\Omega \left(-\Delta\zeta + a\zeta\right) u_+ dx \le \int_\Omega sign_+(u)\zeta\lambda dx + \int_{\partial\Omega} sign_+(u)\zeta\mu dS.$$

In the same way, we prove

$$\int_\Omega \left(-\Delta\zeta + a\zeta\right) |u| dx \le \int_\Omega sign(u)\zeta\lambda dx + \int_{\partial\Omega} sign(u)\zeta\mu dS.$$

In the general case, let $\{\lambda_\ell\}$, $\{\mu_\ell\}$ be two sequences converging in $L^1(\Omega)$ and $L^1(\partial\Omega)$ to λ and μ respectively. Then the sequence of solutions $\{u_\ell\}$ of

$$\begin{aligned} -\Delta u_\ell + au_\ell &= \lambda_\ell \qquad \text{in } \Omega \\ \frac{\partial u_\ell}{\partial \mathbf{n}} &= \mu_\ell \qquad \text{in } \Omega, \end{aligned} \tag{3.32}$$

converges to the solution u of (3.32) in $L^{\frac{N}{N-2},\infty}(\Omega)$ (any $L^r(\Omega)$ with $1 < r < \infty$ if $N = 2$) and $\{\nabla u_\ell\}$ converges to ∇u in $\left(L^{\frac{N}{N-1},\infty}(\Omega)\right)^N$. This implies that (3.31) holds, hence (3.29) and (3.30) follow. □

The following general regularity result proved in [28, Theorem 6] will be used later on.

Proposition 3.4 *Let $a \ge 0$ be a constant and u be the weak solution of* (3.19) *with $\lambda = 0$ and $\mu \in L^m(\partial\Omega)$, $m \ge 1$. Let $q \in [m, \infty]$. The following regularity results hold:*

1- If $\frac{1}{m} - \frac{1}{q} < \frac{1}{N-1}$, then $u \in L^q(\partial\Omega)$.

2- If $\frac{1}{m} - \frac{N}{(N-1)q} < \frac{1}{N-1}$, then $u \in L^q(\Omega)$.

3- If $\frac{1}{m} - \frac{N}{(N-1)q} < 0$, then $u \in W^{1,q}(\Omega)$.

Remark. In each case of the above proposition there holds

$$\|u\|_{\mathbb{X}} \leq c \|\mu\|_{L^m(\partial\Omega)} + b_a \|u\|_{L^1(\Omega)}, \tag{3.33}$$

where $\mathbb{X}$ is either $L^q(\partial\Omega)$ either $L^q(\Omega)$ or $W^{1,q}(\Omega)$ and $b_a \geq 0$ is as in Proposition 3.2. From this result we obtain higher regularity according to the regularity of the boundary data.

Proposition 3.5 *Let $a \geq 0$ be a constant and u be the weak solution of* (3.19) *with $\lambda = 0$. If $\mu \in W^{1,m}(\partial\Omega)$, $m \geq 1$. Let $q \in [m, \infty]$ be such that $\frac{1}{m} - \frac{N}{(N-1)q} < 0$. Then $u \in W^{2,q}(\Omega)$. Moreover*

$$\|u\|_{W^{2,q}(\Omega)} \leq c \|\mu\|_{W^{1,m}(\partial\Omega)} + b_a \|u\|_{L^1(\Omega)}. \tag{3.34}$$

Proof. For the sake of simplicity we assume that $\Omega = B_1$, the unit ball in $\mathbb{R}^N$. In spherical coordinates u satisfies

$$\begin{aligned} -u_{rr} - \frac{N-1}{r} u_r - \frac{1}{r^2}\Delta' u = 0 \qquad & \text{in } (0,1) \times S^{N-1} \\ u_r(1,.) = \mu(.) \qquad & \text{in } S^{N-1}, \end{aligned} \tag{3.35}$$

where Δ' is the Laplace-Beltrami operator on S^{N-1}. Let A be a skew-symmetric matrix in $\mathbb{R}^N$, $X_t := \exp(tA)$ the group of isometries that it generates and L_A the Lie derivative defined by

$$L_A w(\sigma) = \frac{d}{dt} w(X_t \sigma) \lfloor_{t=0}$$

Since L_A commutes with Δ', the function $(r, \sigma) \mapsto v(r, \sigma) = L_A u(r, \sigma)$ satisfies

$$\begin{aligned} -v_{rr} - \frac{N-1}{r} v_r - \frac{1}{r^2}\Delta' v = 0 \qquad & \text{in } (0,1) \times S^{N-1} \\ v_r(1,.) = L_A \mu(.) \qquad & \text{in } S^{N-1}. \end{aligned}$$

We deduce

$$\|v\|_{W^{1,q}(\Omega)} \leq c \|L_A \mu\|_{L^m(\partial\Omega)} + b_a \|u\|_{L^1(\Omega)} \leq c \|\mu\|_{W^{1,m}(\partial\Omega)} + b_a \|u\|_{L^1(\Omega)}.$$

This implies firstly that

$$\|\nabla' v\|_{W^{1,q}(\Omega)} \leq c' \|\mu\|_{W^{1,m}(\partial\Omega)} + N b_a \|u\|_{L^1(\Omega)},$$

which is an estimate for all the tangential derivatives of v and we obtained the final estimate with the normal derivative using the equation. □

Interating this method and using interpolation techniques, we obtain

Proposition 3.6 *Let $a \geq 0$ be a constant and u be the weak solution of* (3.19). *If $\mu \in W^{k+s,m}(\partial\Omega)$, $m \geq 1$, $k \in \mathbb{N}^*$, $s \in (0,1)$. Let $q \in [m, \infty]$ be such that $\frac{1}{m} - \frac{N}{(N-1)q} < 0$. If $\lambda \in W^{k-1+s,q}(\Omega)$ then $u \in W^{k+1+s,q}(\Omega)$ and*

$$\|u\|_{W^{k+1+s,q}(\Omega)} \leq c \left(\|\mu\|_{W^{k+s,m}(\partial\Omega)} + \|\lambda\|_{W^{k-1+s,q}(\Omega)} \right) + b_a \|u\|_{L^1(\Omega)}. \tag{3.36}$$

The next local version of the previous results will be used later on.

Proposition 3.7 *Let $a \geq 0$ be a constant, $N \subsetneq \partial\Omega$ be compact and u be a nonnegative weak solution of (3.19) with $\lambda = 0$ and $\mu \in W^{k+s,m}_{loc}(\partial\Omega\backslash N)$, $m \geq 1$, $k \in \mathbb{N}^*$, $s \in (0,1)$. Let $q \in [m, \infty]$ be such that $\frac{1}{m} - \frac{N}{(N-1)q} < 0$. If for $\epsilon > 0$ small enough we set $N_\epsilon = \{x \in \overline{\Omega} : \operatorname{dist}(x, N) \leq \epsilon\}$, then $u \in W^{k+1+s,q}(\Omega \setminus N_\epsilon)$ and*

$$\|u\|_{W^{k+1+s,q}(\Omega\backslash N_{2\epsilon})} \leq c \|\mu\|_{W^{k+s,m}(\partial\Omega\backslash N_\epsilon)} + b_a \|u\|_{L^1(\Omega\backslash N_\epsilon)}, \tag{3.37}$$

with $c = c(\epsilon) > 0$.

Proof. Let $\zeta \in C_0^\infty(\mathbb{R}^N)$, $\zeta \geq 0$, vanishing in a neighborhood of N and $v = \zeta u$, then

$$\begin{aligned} -\Delta v &= \zeta\lambda + u\Delta\zeta + 2\langle\nabla u, \nabla\zeta\rangle && \text{in } \Omega \\ \frac{\partial v}{\partial \mathbf{n}} &= \zeta\mu - u\frac{\partial\zeta}{\partial\mathbf{n}} && \text{in } \partial\Omega \end{aligned} \tag{3.38}$$

Since u is positive harmonic, it belongs to $L^{\frac{m}{2-m}}(\Omega)$ and $|\nabla u| \in L^m(\Omega)$ for any $m \in (1, \frac{N}{N-1})$. Then $\zeta\lambda + u\Delta\zeta + 2\nabla u.\nabla\zeta \in L^m(\Omega)$. Combining the trace theorem and Sobolev imbedding theorem, $u\frac{\partial\zeta}{\partial\mathbf{n}} \in W^{\frac{m-1}{m},m}(\partial\Omega) \subset L^{\frac{m(N-1)}{N-m}}(\partial\Omega)$. Since the solution w of

$$\begin{aligned} -\Delta w &= \zeta\lambda + u\Delta\zeta + 2\langle\nabla u, \nabla\zeta\rangle && \text{in } \Omega \\ \frac{\partial w}{\partial \mathbf{n}} &= 0 && \text{in } \partial\Omega \end{aligned}$$

with zero average belongs to $W^{2,m}(\Omega) \subset W^{1,\frac{mN}{N-m}}$, it follows from Proposition 3.4 that $v \in W^{1,q}(\Omega)$ for any $q < \frac{mN}{N-m}$. We iterate this process by setting

$$m_0 = m < \frac{N}{N-1} \quad \text{and} \quad \frac{1}{m_n} = \frac{1}{m_{n-1}} - \frac{1}{N} = \frac{1}{m_0} - \frac{n}{N} \quad \text{for } n \in \mathbb{N}^*.$$

If n^* is the largest integer smallest than $\frac{N}{m_0}$, then $v \in W^{1,m_{n^*+1}-\tau}(\Omega)$ for any $\tau > 0$, hence $v \in W^{1,m_{n^*+1}-\tau}(\Omega) \subset W^{s,\infty}(\Omega)$ for some $s \in (0,1)$. By Proposition 3.6, $v \in W^{1+s,\infty}(\Omega)$. Iterating this method we obtain the claim. □

Remark. The sign assumption on u may look unusual, but it must be noticed that the problem is by essence non-local. The only local aspect is the one dealing with the local properties of nonnegative harmonic functions and the solutions of elliptic equations with measure data. If we want to get rid of it, we need $\nabla u \in L^{\frac{N}{N-1},\infty}(K)$ for all compact set $K \subset \overline{\Omega} \setminus N$ as a starting point of the proof of Proposition 3.7.

An important application deals with nonlinear boundary value such as

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} + g(u) &= 0 && \text{in } \partial\Omega, \end{aligned} \tag{3.39}$$

where $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a C^{k+1} function. Putting $\mu = g(u)$ and iterating Proposition 3.7 we obtain

$$\|u\|_{W^{k+1+s,\infty}(\Omega\setminus N_{2\epsilon})} \leq c\,\|u\|_{L^\infty(\partial\Omega\setminus N_\epsilon)} + b_a\,\|u\|_{L^1(\Omega\setminus N_\epsilon)}\,. \tag{3.40}$$

Lemma 3.8 *Let $1 < p \leq \frac{N}{N-1}$ and u be a nonnegative solution of (1.12) such that $|x|^{\frac{1}{p-1}}u(x)$ is bounded, then $|x|^{\frac{1}{p-1}+\ell}|D^\ell u(x)|$ is also bounded for $\ell = 1, 2, 3$.*

Proof. For $k \in (0, 1]$ we set $u_k(x) = T_k[u](x) = k^{\frac{1}{p-1}}u(kx)$ where T_k is already defined in (1.4). Then u_k satisfies (1.12) in $\Omega^k := k^{-1}\Omega$. Since $u(x) \leq c|x|^{-\frac{1}{p-1}}$ in Ω, u_k satisfies the same estimate with the same constant in Ω^k. Let $r > 0$ such that $\frac{k}{4} \leq r \leq 8k$. By Proposition 3.7 we have

$$\|u_k\|_{W^{3,\infty}(\Omega^k\cap\Gamma^{4r/k}_{2r/k})} \leq c\,\|u_k\|_{L^\infty(\partial\Omega^k\cap\Gamma^{6r/k}_{r/k})} + b_a\,\|u_k\|_{L^1(\Omega^k\cap\Gamma^{6r/k}_{r/k})}\,,$$

where $\Gamma^b_a = \{x \in \mathbb{R}^N : a \leq |x| \leq b\}$. Since the curvature of Ω^k is bounded independently of k, the constant c is independent of k too. Furthermore

$$\begin{aligned}\int_{\Omega^k\cap\Gamma^{6r/k}_{r/k}} u_k(x)dx &= k^{1-N+\frac{1}{p-1}}\int_{\Omega\cap\Gamma^{6r}_r} u(y)dy\\ &\leq ck^{1-N+\frac{1}{p-1}}\int_{\Omega\cap\Gamma^{6r}_r} |y|^{-\frac{1}{p-1}}dy\\ &= cc_N k^{1-N+\frac{1}{p-1}}\int_r^{6r} s^{N-2-\frac{1}{p-1}}ds.\end{aligned}$$

This last term is bounded as we have chosen $\frac{k}{4} \leq r \leq 8k$. Since $D^\ell u_k(x) = k^{\frac{1}{p-1}+\ell}D^k u(kx)$, we take $k = r$ and deduce

$$|D^\ell u(x)| \leq c'|x|^{-\frac{1}{p-1}-\ell} + c'',$$

which ends to proof. □

3.3 Proof of Theorem B

We denote by $(x, z) \mapsto \mathcal{N}_\Omega(x, z)$ be the kernel function defined in $\Omega \times \partial\Omega$ with Neumann boundary data δ_z, that is the solution of $v = v_z$ of

$$\begin{aligned}-\Delta v + v &= 0 \qquad &&\text{in } \Omega\\ \frac{\partial v}{\partial \mathbf{n}} &= \delta_z \qquad &&\text{in } \partial\Omega\end{aligned} \tag{3.41}$$

It is known that

$$\mathcal{N}_\Omega(x, z) \approx \begin{cases} |x - z|^{2-N} & \text{if } N \geq 3\\ -\ln|x - z| & \text{if } N = 2.\end{cases} \tag{3.42}$$

Furthermore, if $\mu \in \mathfrak{M}(\partial\Omega)$ the solution of

$$\begin{aligned}-\Delta v + v &= 0 \qquad &&\text{in } \Omega\\ \frac{\partial v}{\partial \mathbf{n}} &= \mu \qquad &&\text{in } \partial\Omega\end{aligned} \tag{3.43}$$

is expressed by

$$v(x) = \int_{\partial\Omega} \mathcal{N}_\Omega(x,y) d\mu(y). \tag{3.44}$$

Let $j : \mathbb{R} \mapsto \mathbb{R}_+$ be a C^2 nondecreasing convex function, vanishing on $(-\infty, 0]$, such that $0 < j'(r) \leq 1$ on $(0,\infty)$. For $\epsilon > 0$ set $w_\epsilon = j(u - \epsilon\mathcal{N}_\Omega(.,0))$, then

$$-\Delta w_\epsilon = -\epsilon\mathcal{N}_\Omega(.,0) j'(u - \epsilon\mathcal{N}_\Omega(.,0)) - j''(u - \epsilon\mathcal{N}_\Omega(.,0))|\nabla(u - \epsilon\mathcal{N}_\Omega(.,0))|^2 \leq 0.$$

Since w_ϵ vanishes in a neighborhood of 0,

$$\int_\Omega |\nabla w_\epsilon|^2 + \int_{\partial\Omega} w_\epsilon g(u) dS \leq 0.$$

As $g(u)$ has the sign of u, it is nonnegative on the support of w_ϵ. Hence $\nabla w_\epsilon = 0$. This implies that $j(u - \epsilon\mathcal{N}_\Omega(.,0))$ is equal to some consatnt c_ϵ which is decreasing with ϵ. Letting $\epsilon \to 0$ we infer that u_+ is constant. Similarly u_- is constant and such is u. Notice that for this constant u, $g(u) = 0$. □

3.4 Proof of Theorem C

3.4.1 Straightening the boundary

If $p > \frac{N-1}{N-2}$, then $u(x) = O(|x|^{-\frac{1}{p-1}}) = o(|x|^{2-N})$ and $u = 0$ by Theorem B. Therefore we can assume $1 < p \leq \frac{N-1}{N-2}$ in the sequel. The basic technique is to straighten the boundary and transform the study near the singular point into a problem in a infinite cylinder. We abridge the proof since the details of the method (initialy introduced in [20]) can be found in [16]. We assume that the orthonormal basis $\mathbf{e}_1, ..., \mathbf{e}_N$ is $\mathbb{R}^N$ is such that at 0, $\mathbf{n} = -\mathbf{e}_N$ and that $\partial\Omega$ is locally the graph of a C^2 function θ defined in $B_{R'} = B_R \cap \{x : x_N = 0\}$ and satisfying $\theta(0) = 0$, $D\theta(0) = 0$. Putting

$$y_j = x_j = \Theta_j(x) \text{ if } j = 1, ..., N-1 \text{ and } y_N = x_N - \theta(x') = \Theta_N(x),$$

then $\Theta = (\Theta_1, ..., \Theta_N)$ is a local diffeomorphism near 0. We set $u(x) = \tilde{u}(y) = \tilde{u}(r,\sigma) = r^{-\frac{1}{p-1}} v(t,\sigma)$ with $t = \ln r$. Performing a lengthy computation we derive that v satisfies

$$\begin{aligned}(1+\epsilon_1)v_{tt} + \left(N - \frac{2p}{p-1} + \epsilon_2\right) v_t + (\ell_{N,p} + \epsilon_3)\, v + \Delta' v + \langle \nabla' v, \epsilon_4\rangle \\ + \langle \nabla' v_t, \epsilon_5\rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N\rangle, \epsilon_6\rangle = 0\end{aligned} \tag{3.45}$$

in $(-\infty, T_0] \times S^{N-1}_+$ and

$$\left(v_t - \frac{1}{p-1} v\right)(\epsilon_8 - \langle \mathbf{c}, \mathbf{e}_N\rangle \epsilon_7) - \langle \nabla' v, \mathbf{e}_N\rangle \epsilon_7 + \langle \nabla' v, \nu\rangle + v^p = 0 \tag{3.46}$$

in $(-\infty, T_0] \times \partial S^{N-1}_+$, where $\mathbf{c} = \frac{y}{|y|}$ and the ϵ_j satisfy

$$|\epsilon_j(t,.)| + |\epsilon_{j\,t}(t,.)| + |\nabla' \epsilon_j(t,.)| \leq c e^t, \tag{3.47}$$

as a consequence of the fact that $|\theta(x')| = 0(|x'|^2)$ near 0. Furthermore the quantities v_t, v_{tt}, v_{ttt} and $\nabla^\alpha \partial_{t^\beta} v$ are uniformly bounded on $(-\infty, T_1] \times S^{N-1}_+$ if $|\alpha| + \beta \leq 3$ and $T_1 < T_0$ by Lemma 3.8.

Lemma 3.9 *There holds*

$$\int_{-\infty}^{T_1}\int_{S_+^{N-1}} \left(v_t^2 + v_{tt}^2 + |\nabla' v_t|^2\right) dSdt < \infty. \tag{3.48}$$

Proof. We multiply the first equation (3.45) by v_t, integrate on S_+^{N-1} and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\int_{S_+^{N-1}} \left(v_t^2 + \ell_{N,p} v^2 - |\nabla' v|^2\right) dS - \frac{2}{p+1}\int_{\partial S_+^{N-1}} |v|^{p+1} dS'\right] \\ + \left(N - \frac{2p}{p-1} + \epsilon_2\right)\int_{S_+^{N-1}} v_t^2 dS + \eta_1(t) + \eta_2(t) = 0,\end{aligned} \tag{3.49}$$

where

$$\eta_1(t) = \int_{S_+^{N-1}} \left(\epsilon_1 v_{tt} + \epsilon_3 v + \langle \nabla' v, \epsilon_4\rangle + \langle \nabla' v_t, \epsilon_5\rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_{_N}\rangle, \epsilon_6\rangle\right) v_t dS,$$

$$\eta_2(t) = \int_{\partial S_+^{N-1}} \left(\left(v_t - \frac{1}{p-1} v\right)\left(\langle \mathbf{c}, \mathbf{e}_{_N}\rangle \epsilon_7 - \epsilon_8\right) + \langle \nabla' v, \mathbf{e}_{_N}\rangle \epsilon_7\right) v_t dS'.$$

By (3.47), $|\eta_j(t)| \le ce^t$. The fact that v_t and $\nabla' v$ are uniformly bounded and $N - \frac{2p}{p-1} \neq 0$ as $p \neq \frac{N}{N-2}$, we infer

$$\int_{-\infty}^{T_1}\int_{S_+^{N-1}} v_t^2 dSdt < \infty. \tag{3.50}$$

Since $vv_{tt} = (vv_t)_t - v_t^2$ and $\langle \nabla' v, \nabla' v_{tt}\rangle = (\langle \nabla' v, \nabla' v_t\rangle)_t - |\nabla' v_t|^2$, we obtain by multiplying (3.45) by v_{tt} and integrating on S_+^{N-1},

$$\begin{aligned}\frac{d}{dt}\left[\int_{S_+^{N-1}} \left(\left(\frac{N}{2} - \frac{p}{p-1}\right) v_t^2 + \ell_{N,p} v v_t - \nabla' v.\nabla' v_t\right) dS - \int_{\partial S_+^{N-1}} |v|^{p-1} v v_t dS'\right] \\ + \int_{S_+^{N-1}} (1+\epsilon_1)(v_{tt}^2 - \ell_{N,p} v_t^2 + |\nabla' v_t|^2) dS + p\int_{\partial S_+^{N-1}} |v|^{p-1} v_t^2 dS' \\ + \gamma_1(t) + \gamma_2(t) = 0,\end{aligned} \tag{3.51}$$

where

$$\gamma_1(t) = \int_{S_+^{N-1}} \left(\epsilon_2 v_t + \epsilon_3 v + \langle \nabla' v, \epsilon_4\rangle + \langle \nabla' v_t, \epsilon_5\rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_{_N}\rangle, \epsilon_6\rangle\right) v_{tt} dS,$$

$$\gamma_2(t) = \int_{\partial S_+^{N-1}} \left(v_t - \frac{1}{p-1} v\right)\left(\langle \mathbf{c}, \nabla' v\rangle \epsilon_7 - \epsilon_8\right) v_{tt} dS'.$$

Again $|\gamma_j(t)| \le ce^t$ by (3.47). Since v_t, vv_t, $\nabla' v.\nabla' v_t$ and $|v|^{p-1} v_t^2$ are uniformly bounded on $(-\infty, T_1] \times S_+^{N-1}$ we infer

$$\int_{-\infty}^{T_1}\int_{S_+^{N-1}} \left(v_{tt}^2 + |\nabla' v_t|^2\right) dSdt < \infty,$$

which ends the proof. □

3.4.2 Strong singularities

Because the functions v_t, v_{tt} and $\nabla' v_t$ are uniformly continuous on $(-\infty, T_1] \times S_+^{N-1}$ we deduce easily from (3.48) that

$$\lim_{t\to\infty} (v_t^2 + v_{tt}^2 + |\nabla' v_t|^2)(t,.) = 0 \quad \text{uniformly on } S_+^{N-1}. \tag{3.52}$$

The negative trajectory of $t \mapsto v(t,.)$ in $C^2(S_+^{N-1})$ is $\mathcal{T}_-[v] := \bigcup_{t\le T_1}\{v(t,.)\}$. By Lemma 3.8, $\mathcal{T}_-[v]$ is bounded in $C^3(S_+^{N-1})$, hence it is relatively compact in $C^2(S_+^{N-1})$ by the Arzela-Ascoli theorem. Therefore, the alpha-limit set of $\mathcal{T}_-[v]$ defined by

$$\mathcal{A}[\mathcal{T}_-[v]] := \bigcap_{t\le T_1} \mathrm{clo}_{C^2(S_+^{N-1})} \left(\bigcup_{\tau\le t} \{v(\tau,.)\} \right) \tag{3.53}$$

is a non-empty compact connected set in $C^2(S_+^{N-1})$. Using (3.50) and letting $t \to -\infty$ in (3.45), we conclude that if $\omega \in \mathcal{A}[\mathcal{T}_-[v]]$, then

$$\begin{aligned} \Delta'\omega + \ell_{N,p}\omega &= 0 \qquad && \text{in } S_+^{N-1} \\ \langle \nabla'\omega, \nu\rangle + \omega^p &= 0 && \text{in } \partial S_+^{N-1}, \end{aligned} \tag{3.54}$$

hence $\mathcal{A}[\mathcal{T}_-[v]] \subset \mathcal{E}_+$.
If $p = \frac{N-1}{N-2}$, then $\frac{1}{p-1} = N-2$ and $\mathcal{E} = \{0\}$. Hence $v(t,.) \to 0$ when $t \to -\infty$, equivalently

$$\lim_{x\to 0} |x|^{N-2} u(x) = 0. \tag{3.55}$$

By Theorem B it implies that $u = 0$.
If $\frac{N}{N-1} < p < \frac{N-1}{N-2}$, $\mathcal{E}_+$ is discrete. Then either $v(t,.) \to \omega_s$ of $v(t,.) \to 0$ when $t \to -\infty$. In the first case it is equivalent to

$$|x|^{\frac{1}{p-1}} u(x) = \omega_s(\tfrac{x}{|x|})(1 + o(1)) \quad \text{as } x \to 0, \tag{3.56}$$

and in the second case

$$\lim_{x\to 0} |x|^{\frac{1}{p-1}} u(x) = 0. \tag{3.57}$$

3.4.3 Weak singularities

In the sequel, we assume $N > 2$, the proof in the case $N = 2$ can be carried out by the same techniques with minor technical modifications.

Proposition 3.10 *If* (3.57) *holds we claim that there exists* $\delta > 0$ *such that*

$$u(x) \le c|x|^{\delta - \frac{1}{p-1}}, \tag{3.58}$$

near $x = 0$ *for some* $c > 0$.

We proceed by contradiction, set $\rho(t) = \|v(t,.)\|_{C^0(S_+^{N-1})}$ and assume that for any $\epsilon > 0$,

$$\limsup_{t\to-\infty} e^{-\epsilon t}\rho(t) = \infty. \tag{3.59}$$

The following lemma proved in [14] is the key for starting the proof of the decay of the solution.

Lemma 3.11 *There exists a function $\eta \in C^\infty((-\infty, T_1])$ satisfying*

$$\begin{aligned}
&(i) && \eta > 0\,,\ \eta_t > 0\,,\ \lim_{t\to-\infty} \eta(t) = 0,\\
&(ii) && 0 < \limsup_{t\to-\infty} \frac{\rho(t)}{\eta(t)} < \infty,\\
&(iii) && \lim_{t\to-\infty} e^{-\epsilon t}\eta(t) = \infty \quad \textit{for all } \epsilon > 0,\\
&(iv) && \frac{\eta_t}{\eta} \textit{ and } \left(\frac{\eta_t}{\eta}\right)_t \textit{ are bounded and integrable on } (-\infty, T_1],\\
&(v) && \lim_{t\to-\infty} \frac{\eta_t}{\eta}(t) = \lim_{t\to-\infty} \left(\frac{\eta_t}{\eta}\right)_t(t) = 0.
\end{aligned} \tag{3.60}$$

Proof of Proposition 3.10. Define $w(t,.) = \eta^{-1}(t)v(t,.)$. Then w is bounded and satisfies

$$\begin{aligned}
(1+\epsilon_1)w_{tt} &+ \left(N - \frac{2p}{p-1} + \epsilon_2 + 2(1+\epsilon_1)\frac{\eta_t}{\eta}\right) w_t + \Delta' w\\
&+ \left(\ell_{N,p} + \epsilon_3 + (1+\epsilon_1)\frac{\eta_{tt}}{\eta} + \left(N - \frac{2p}{p-1} + \epsilon_2\right)\frac{\eta_t}{\eta}\right) w\\
&+ \langle \nabla' w, \epsilon_4 + \frac{\eta_t}{\eta}\epsilon_5\rangle + \langle \nabla' w_t, \epsilon_5\rangle + \langle \nabla'\langle \nabla' w, \mathbf{e}_{_N}\rangle, \epsilon_6\rangle = 0
\end{aligned} \tag{3.61}$$

in $(-\infty, T_0] \times S^{N-1}_+$ and

$$\begin{aligned}
-\Bigg(\left(w_t + \left(\frac{\eta_t}{\eta} - \frac{1}{p-1}\right) w\right) \langle \mathbf{c}, \mathbf{e}_{_N}\rangle + \langle \nabla' w, \mathbf{e}_{_N}\rangle\Bigg)\epsilon_7\\
+ \left(w_t + \left(\frac{\eta_t}{\eta} - \frac{1}{p-1}\right) w\right)\epsilon_8 + \langle \nabla' w, \nu\rangle + \eta^{p-1} w^p = 0
\end{aligned} \tag{3.62}$$

in $(-\infty, T_0] \times \partial S^{N-1}_+$. Since w is bounded, a standard adaptation of Lemma 3.8 shows that w_t, w_{tt}, w_{ttt} and $\nabla^\alpha \partial_{t^\beta} w$ are uniformly bounded on $(-\infty, T_1] \times S^{N-1}_+$ whenever $|\alpha| + \beta \le 3$ and $T_1 < T_0$. The negative trajectory of $t \mapsto w(t,.)$ in $C^2(S^{N-1}_+)$ is defined by $\mathcal{T}_-[w] := \bigcup_{t\le T_1}\{w(t,.)\}$. By the previous statements, $\mathcal{T}_-[w]$ is bounded in $C^3(S^{N-1}_+)$, hence it is relatively compact in $C^2(S^{N-1}_+)$ by the Arzela-Ascoli theorem. Therefore, the alpha-limit set of $\mathcal{T}_-[w]$ defined by

$$\mathcal{A}[\mathcal{T}_-[w]] := \bigcap_{t\le T_1} \mathrm{clo}_{C^2(S^{N-1}_+)}\left(\bigcup_{\tau\le t}\{w(\tau,.)\}\right) \tag{3.63}$$

is a non-empty compact connected set in $C^2(S^{N-1}_+)$. The integrability assumptions on η allows us to prove

Lemma 3.12 *There holds*

$$\int_{-\infty}^{T_1}\int_{S^{N-1}_+} \left(w_t^2 + w_{tt}^2 + |\nabla' w_t|^2\right) dS dt < \infty. \tag{3.64}$$

Proof. We multiply equation (3.61) by w_t and integrate over S^{N-1}_+.

$$\frac{1}{2}\frac{d}{dt}\left[\int_{S^{N-1}_+}\left(w_t^2+\ell_{N,p}w^2-|\nabla' w|^2\right)dS-\frac{2\eta^{p-1}}{p+1}\int_{\partial S^{N-1}_+}|w|^{p+1}dS'\right]$$
$$+\left(N-\frac{2p}{p-1}+\epsilon_2+2(1+\epsilon_1)\frac{\eta_t}{\eta}\right)\int_{S^{N-1}_+}w_t^2dS+\alpha_1(t)+\alpha_2(t)=0, \tag{3.65}$$

where α_1 and α_2 are defined by

$$\alpha_1(t)=\int_{S^{N-1}_+}\left[\epsilon_1 w_{tt}+\epsilon_2 w_t+\left(\epsilon_3+(1+\epsilon_1)\frac{\eta_{tt}}{\eta}+\left(N-\frac{2p}{p-1}+\epsilon_2\right)\frac{\eta_t}{\eta}\right)w\right.$$
$$\left.+\langle\nabla' w,\epsilon_4+\frac{\eta_t}{\eta}\epsilon_5\rangle+\langle\nabla' w_t,\epsilon_5\rangle+\langle\nabla'\langle\nabla' w,\mathbf{e}_{_N}\rangle,\epsilon_6\rangle\right]w_t dS,$$

$$\alpha_2(t)=\int_{\partial S^{N-1}_+}\left[\left(w_t+\left(\frac{\eta_t}{\eta}-\frac{1}{p-1}\right)w\right)(\langle\mathbf{c},\mathbf{e}_{_N}\rangle)\epsilon_7+\epsilon_8)\right.$$
$$\left.-\langle\nabla' w,\mathbf{e}_{_N}\rangle\epsilon_7-\frac{p-1}{p+1}w^{p+1}\eta^{p-1}\frac{\eta_t}{\eta}\right]w_t dS'.$$

Using the estimates on ϵ_j and (3.60), we obtain that

$$\int_{-\infty}^{T_1}\int_{S^{N-1}_+}w_t^2 dSdt<\infty.$$

Multiplying equation (3.61) by w_{tt}, integrating over S^{N-1}_+ and using (3.60) yield

$$\int_{-\infty}^{T_1}\int_{S^{N-1}_+}\left(w_{tt}^2+|\nabla' w_t|^2\right)dSdt<\infty,$$

which ends the proof. □

End of the proof of Proposition 3.10. Since w_t and w_{tt} are uniformly continuous on $(-\infty,T_1]\times S^{N-1}_+$ we infer from (3.64) that

$$\lim_{t\to-\infty}w_t(t,.)=\lim_{t\to-\infty}w_{tt}(t,.)=0, \tag{3.66}$$

uniformly on S^{N-1}_+. Therefore $\mathcal{A}[\mathcal{T}_-[w]]$ is a subset of the set of nonnegative solutions of

$$\begin{aligned}\Delta'\phi+\ell_{N,p}\phi&=0 \qquad \text{in } S^{N-1}_+\\ \langle\nabla'\phi,\nu\rangle&=0 \qquad \text{in } \partial S^{N-1}_+,\end{aligned} \tag{3.67}$$

and by (3.60)-(ii) it contains a positive element. Since $\ell_{N,p}>0$ this is a contradiction and (3.59) does not hold. This ends the proof of Proposition 3.10. □

Step 1. We claim that

$$u(x)\le c|x|^{2-N} \quad \text{in a neighborhood of 0.} \tag{3.68}$$

If $\delta \geq \frac{1}{p-1}+2-N$, (3.68) is a consequence of (3.58). In what follows we assume that

$$0 < \delta < \frac{1}{p-1} + 2 - N. \tag{3.69}$$

We set $v_\delta = e^{-\delta t}v$. Then v_δ is bounded in $(-\infty, T_1]\times S^{N-1}_+$ and, as in the proof of Proposition 3.10, the quantities $\partial_t v_\delta$, $\partial_{tt} v_\delta$, $\partial_{ttt} v_\delta$ and $\nabla^\alpha \partial_{t^\beta} v_\delta$ are uniformly bounded on $(-\infty, T_1]\times S^{N-1}_+$. Furthermore there holds

$$\begin{aligned}(1+\epsilon_1)v_{\delta\,tt} + \left(N - \frac{2p}{p-1} + \epsilon_2 + 2(1+\epsilon_1)\delta\right) v_{\delta\,t} + \Delta' v_\delta \\ + \left(\ell_{N,p} + \epsilon_3 + (1+\epsilon_1)\delta^2 + \left(N - \frac{2p}{p-1} + \epsilon_2\right)\delta\right) v_\delta \\ + \langle \nabla' v_\delta, \epsilon_4 + \delta\epsilon_5\rangle + \langle \nabla' v_{\delta\,t}, \epsilon_5\rangle + \langle \nabla'\langle \nabla' v_\delta, \mathbf{e}_{_N}\rangle, \epsilon_6\rangle = 0\end{aligned} \tag{3.70}$$

in $(-\infty, T_0]\times S^{N-1}_+$ and

$$\begin{aligned}-\left(\left(v_{\delta\,t} + \left(\delta - \frac{1}{p-1}\right) v_\delta\right)\langle \mathbf{c}, \mathbf{e}_{_N}\rangle + \langle \nabla' v_\delta, \mathbf{e}_{_N}\rangle\right)\epsilon_7 \\ + \left(v_{\delta\,t} + \left(\delta - \frac{1}{p-1}\right) v_\delta\right)\epsilon_8 + \langle \nabla' v_\delta, \nu\rangle + e^{(p-1)t} v_\delta^p = 0\end{aligned} \tag{3.71}$$

in $(-\infty, T_0]\times \partial S^{N-1}_+$. We denote by $\tilde v_\delta$ the projection of v onto $H := [\ker(-\Delta')]^\perp$, the operator being defined in $H^1(S^{N-1}_+)$, and by P_H the corresponding projection operator. Then

$$\begin{aligned}\tilde v_{\delta\,tt} + \left(N - \frac{2p}{p-1} + 2\delta\right)\tilde v_{\delta\,t} + \Delta'\tilde v_\delta \\ + \left(\ell_{N,p} + \delta^2 + \left(N - \frac{2p}{p-1}\right)\delta\right)\tilde v_\delta + \tilde F = 0\end{aligned} \tag{3.72}$$

where

$$\begin{aligned}\tilde F := P\big[\epsilon_1 v_{\delta\,tt} + (\epsilon_2 + 2\epsilon_1\delta)v_{\delta\,t} + (\epsilon_3 + \epsilon_1\delta^2 + \epsilon_2\delta)v_\delta \\ + \langle \nabla' v, \epsilon_4\rangle + \langle \nabla' v_t, \epsilon_5\rangle + \langle \nabla'\langle \nabla' v, \mathbf{e}_{_N}\rangle, \epsilon_6\rangle\big] = O(e^t).\end{aligned}$$

We multiply (3.71) by $\tilde v_\delta$, integrate over S^{N-1}_+ and use the boundary condition and the fact that $N-1$ is the first eigenvalue of $-\Delta'$ in H. We deduce that $\tilde X_\delta(t) := \|\tilde v_\delta(t,.)\|_{L^2(S^{N-1}_+)}$ satisfies in the sense of distributions in $(-\infty, T_1)$,

$$\begin{aligned}\tilde X''_\delta + \left(N - \frac{2p}{p-1} + 2\delta\right)\tilde X'_\delta \\ + \left(\ell_{N,p} + \delta^2 + \left(N - \frac{2p}{p-1}\right)\delta + 1 - N\right)\tilde X_\delta \geq -c^* e^{mt}.\end{aligned} \tag{3.73}$$

where $m = \inf\{1, p-1\}$, for some constant $c^* > 0$. Note that the nonlinear term on ∂S^{N-1}_+ is at the origin of the term $e^{(p-1)t}$. The characteristic polynomial of (3.73) is

$$P_\delta(\xi) = \xi^2 + \left(N - \frac{2p}{p-1} + 2\delta\right)\xi + \ell_{N,p} + \delta^2 + \left(N - \frac{2p}{p-1}\right)\delta + 1 - N.$$

It is noticeable that its discriminant is N^2, independent of δ, and as a consequence its roots are expressed easily by

$$\xi_{1,\delta} = \frac{p}{p-1} - \delta > m \quad \text{and} \quad \xi_{2,\delta} = \frac{p}{p-1} - N - \delta < 0, \tag{3.74}$$

since (3.69) holds. Therefore $P_\delta(m) < 0$. For $a, \gamma, \epsilon > 0$ set

$$X_\epsilon(t) = ae^{\xi_{1,\delta} t} + \epsilon e^{\xi_{1,\delta} t} + \gamma e^{mt}.$$

Then

$$X'' + \left(N - \frac{2p}{p-1} + 2\delta\right) X' \\ + \left(\ell_{N,p} + \delta^2 + \left(N - \frac{2p}{p-1}\right)\delta + 1 - N\right) X = \gamma P_\delta(m) e^{mt}.$$

We can choose γ such that $\gamma P_\delta(1) \geq -c^*$ and $a = \|\tilde{v}_\delta(T_1, .)\|_{L^2(S_+^{N-1})} e^{-\xi_{1,\delta} T_1}$. By the maximum principle $\tilde{X}_\delta(t) \leq X_\epsilon(t)$ for $t \leq T_1$ and all $\epsilon > 0$. This implies

$$\|\tilde{v}_\delta(t, .)\|_{L^2(S_+^{N-1})} \leq \|\tilde{v}_\delta(T_1, .)\|_{L^2(S_+^{N-1})} e^{\xi_{1,\delta}(t-T_1)} + \gamma e^{mt} \quad \text{for } t \leq T_1. \tag{3.75}$$

Using standard regularizing effect for elliptic equations, we can improve (3.75) and obtain a uniform estimate

$$\|\tilde{v}_\delta(t, .)\|_{L^\infty(S_+^{N-1})} \leq A \|\tilde{v}_\delta(T_1, .)\|_{L^2(S_+^{N-1})} e^{\xi_{1,\delta}(t-T_1)} + \gamma e^{mt} \quad \text{for } t \leq T_1 - 1. \tag{3.76}$$

Next we denote by X_δ the projection of v_δ onto $\ker(-\Delta')$ (i.e. the average on S_+^{N-1}), then

$$X_\delta'' + \left(N - \frac{2p}{p-1} + 2\delta\right) X_\delta' + \left(\ell_{N,p} + \delta^2 + \left(N - \frac{2p}{p-1}\right)\delta\right) X_\delta + F = 0 \tag{3.77}$$

where

$$F := \frac{1}{|S_+^{N-1}|} \int_{S_+^{N-1}} \left[\epsilon_1 v_{\delta\, tt} + (\epsilon_2 + 2\epsilon_1\delta) v_{\delta\, t} + (\epsilon_3 + \epsilon_1\delta^2 + \epsilon_2\delta) v_\delta \right. \\ \left. + \langle \nabla' v, \epsilon_4 \rangle + \langle \nabla' v_t, \epsilon_5 \rangle + \langle \nabla' \langle \nabla' v, \mathbf{e}_N \rangle, \epsilon_6 \rangle \right] dS = O(e^{mt}).$$

The characteristic roots of the equation

$$y'' + \left(N - \frac{2p}{p-1} + 2\delta\right) y' + \left(\ell_{N,p} + \delta^2 + \left(N - \frac{2p}{p-1}\right)\delta\right) y = 0$$

are $\theta_{1,\delta}$, $\theta_{2,\delta}$. They can easily be computed and for $\delta > 0$ small enough

$$\theta_{1,\delta} = \frac{1}{p-1} - \delta > N - 2 \geq 1 > \theta_{2,\delta} = \frac{1}{p-1} + 2 - N - \delta. \tag{3.78}$$

The solution of (3.77) admits the general expression

$$X_\delta(t) = ae^{t\theta_{1,\delta}} + be^{t\theta_{2,\delta}} - \frac{1}{\theta_{1,\delta} - \theta_{2,\delta}} \int_t^{T_1} F(s) \left(e^{(t-s)\theta_{1,\delta}} - e^{(t-s)\theta_{2,\delta}}\right) ds \tag{3.79}$$

Since $m < \theta_{1,\delta}$, it is easy to see that there exists $c \geq 0$ such that, when $t \to -\infty$, there holds

$$X_\delta(t) = e^{t \inf\{p-1, \theta_{2,\delta}\}}(c + o(1)), \tag{3.80}$$

if $p - 1 \neq \theta_{2,\delta}$ and

$$X_\delta(t) = (-t)e^{t(p-1)}(c + o(1)), \tag{3.81}$$

if $p - 1 = \theta_{2,\delta}$. We consider only the case $p - 1 \neq \theta_{2,\delta}$, the case of equality requiring only some technical modifications of the proof.

Case 1: Assume $\theta_{2,\delta} < p - 1$. Then $X_\delta(t) = e^{t\theta_{2,\delta}\}}(c + o(1))$. Since $\xi_{1,\delta} > 1$ and $m > \theta_{2,\delta}$, we infer from (3.76) and (3.78) and the definition of v_δ that

$$0 < v(t, \sigma) = e^{t(\delta+\theta_{2,\delta})}(c + o(1)) \quad \text{when } t \to -\infty, \text{ uniformly on } S^{N-1}_+. \tag{3.82}$$

for some constant $c > 0$. This implies not only (3.68) but also (1.14).

Case 2: Assume $1 > \theta_{2,\delta} > p - 1 = m$. Then

$$0 < v(t, \sigma) \leq ce^{(\delta+p-1)t} \quad \text{for all } t \in (-\infty, T_1] \times S^{N-1}_+. \tag{3.83}$$

Then we restart the previous construction, replacing δ by $\delta_1 := \delta + p - 1$. After a finite number j of iterations of this construction and setting $\delta_j := \delta + j(p-1)$ we finally obtain

$$0 < v(t, \sigma) = e^{t(\delta_j+\theta_{2,\delta_j})}(c + o(1)) \quad \text{when } t \to -\infty, \text{ uniformly on } S^{N-1}_+. \tag{3.84}$$

which again implies not only (3.68) but also (1.14). □

Remark. The results of Theorem C can be extended to signed solutions u of (1.12) provided they satisfy not only the same a priori estimates $|u(x)| \leq c|x|^{-\frac{1}{p-1}}$ but also $|D^\alpha u(x)| \leq c|x|^{-\frac{1}{p-1}-|\alpha|}$ for $|\alpha| = 1, 2, 3$. If this holds, the energy method applies and we infer that the limit set of the trajectory $\mathcal{A}[\mathcal{T}_-[v]]$ is a connected subset of the set $\mathcal{E}$. In particular, if $p \geq \frac{N-1}{N-2}$ then $\mathcal{A}[\mathcal{T}_-[v]] = \{0\}$ and by Theorem B it implies that $u = 0$. If $\frac{N}{N-1} < p < \frac{N-1}{N-2}$ with $N > 2$, then $\mathcal{A}[\mathcal{T}_-[v]] \subset \{\omega_s, -\omega_s, 0\}$. If $N = 2$ and $\frac{1}{p-1}$ is an integer, then $\mathcal{A}[\mathcal{T}_-[v]] = \{0\}$ and $u = 0$ by Theorem B, while if $\frac{1}{p-1}$ is not an integer then $\mathcal{A}[\mathcal{T}_-[v]] \subset \{\omega_s, -\omega_s, 0\}$. Furthermore, if $\mathcal{A}[\mathcal{T}_-[v]] = \{0\}$ and $\ell_{N,p}$ is not an eigenvalue of $-\Delta'$ in $H^1(S^{N-1}_+)$ it is possible to adapt the method developed in the proof of Proposition 3.10 and obtain that $r^{N-2+k}u(r, .)$ converges to a nonzero eigenfunction of $-\Delta'$ in $H^1(S^{N-1}_+)$ for some $k \in \mathbb{N}$ such that $N - 2 + k < \frac{1}{p-1}$. The method for such a task is an adaptation of the ideas introduced in [14, Theorem 2.1] and [20, Theorem 5.1]. Note that the assumption $\ell_{N,p} \notin \sigma\left(-\Delta', H^1(S^{N-1}_+)\right)$ is fundamental to prove and estimate of type (3.58), which is the starting point of the proof.

4 Measure boundary data

Let u and v two solutions of (1.1) with the same data μ. By Lemma 3.3

$$\int_\Omega |u - v| \left(-\Delta\zeta + \zeta\right) dx + \int_{\partial\Omega} \text{sign}_0(u - v)(g(u) - g(v)\zeta dS \leq 0 \tag{4.1}$$

for all $\zeta \in \mathcal{C}(\Omega)$, $\zeta \geq 0$. Since g is nondecreasing, we take $\zeta = 1$ and get $u = v$.

4.1 Proof of Theorem D

In this section we assume $N \geq 3$. Let $\{\mu_k\}$ be a sequence of smooth functions on $\partial\Omega$ and u_k the solution of

$$\begin{array}{ll} -\Delta u_k + u_k = 0 & \text{in } \Omega \\ \dfrac{\partial u_k}{\partial \mathbf{n}} + g(u_k) = \mu_k & \text{in } \partial\Omega, \end{array} \tag{4.2}$$

obtained by minimization. By Lemma 3.3

$$\int_\Omega |u_k| dx + \int_{\partial\Omega} |g(u_k)| dS \leq \int_{\partial\Omega} |\mu_k| dS. \tag{4.3}$$

Hence

$$\|u_k\|_{L^{\frac{N}{N-2},\infty}(\Omega)} + \|\nabla u_k\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \leq c' \int_{\partial\Omega} |\mu_k| dS \leq c \|\mu\|_{\mathfrak{M}(\partial\Omega)}, \tag{4.4}$$

by Proposition 3.2.

Therefore there exist a function $u \in L^{\frac{N}{N-2},\infty}(\Omega)$ verifying $\nabla u \in L^{\frac{N}{N-1},\infty}(\Omega)$ and a subsequence $\{u_{k_j}\}$ such that $u_{k_j} \to u$ a.e. in Ω and in $L^1(\Omega)$. By [31] the boundary trace of a function $v \in L^{\frac{N}{N-1},\infty}(\Omega)$ such that $\nabla v \in L^{\frac{N}{N-1},\infty}(\Omega)$ belongs to the fractional Besov-Lorentz space $B^{\frac{1}{N},\frac{N}{N-1},\infty}(\partial\Omega)$ and there holds

$$\|v\lfloor_{\partial\Omega}\|_{B^{\frac{1}{N},\frac{N}{N-1},\infty}(\partial\Omega)} \leq c \left(\|v\|_{L^{\frac{N}{N-1},\infty}(\Omega)} + \|\nabla v\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \right). \tag{4.5}$$

Using Sobolev imbedding theorem for Besov-Lorentz spaces, classicaly obtained by the real interpolation method [23] from the same indexed Sobolev spaces [31], we obtain

$$\|v\lfloor_{\partial\Omega}\|_{L^{\frac{N-1}{N-2},\infty}(\partial\Omega)} \leq c \|v\lfloor_{\partial\Omega}\|_{B^{\frac{1}{N},\frac{N}{N-1},\infty}(\partial\Omega)}. \tag{4.6}$$

Therefore

$$\|u_k\lfloor_{\partial\Omega}\|_{L^{\frac{N-1}{N-2},\infty}(\partial\Omega)} \leq c \|\mu\|_{\mathfrak{M}(\partial\Omega)}, \tag{4.7}$$

and $u_{k_j}\lfloor_{\partial\Omega} \to u\lfloor_{\partial\Omega}$ a.e. in $\partial\Omega$. In order to prove the convergence of $\{g(u_{k_j})\lfloor_{\partial\Omega}\}$ to $\{g(u)\lfloor_{\partial\Omega}\}$ we use Vitali's theorem. Let $E \subset \partial\Omega$ be a Borel set and $|E|_{N-1} = \mathcal{H}_{\partial\mathbb{R}^N_+}(E)$ is its the (N-1)-Hausdorff measure on $\partial\Omega$; for any $\lambda > 0$,

$$\begin{aligned} \int_E |g(u_{k_j})| dS &= \int_{E\cap\{|u_{k_j}|\leq\lambda\}} |g(u_{k_j})| dS + \int_{E\cap\{|u_{k_j}|>\lambda\}} |g(u_{k_j})| dS \\ &\leq |E|_{N-1} (g(\lambda) - g(-\lambda)) + \int_{\{|u_{k_j}\lfloor_{\partial\Omega}|>\lambda\}} |g(u_{k_j})| dS. \end{aligned} \tag{4.8}$$

We set $A_\lambda(u_{k_j}) = \{x \in \partial\Omega : |u_{k_j}\lfloor_{\partial\Omega}(x)| > \lambda\}$ and $\alpha_{k_j}(\lambda) = |A_\lambda(u_{k_j})|_{N-1}$. Since (4.7) holds,

$$\alpha_{k_j}(\lambda) \leq c \|\mu\|_{\mathfrak{M}(\partial\Omega)} \lambda^{-\frac{N-1}{N-2}}.$$

Using Cavalieri's formula [13],

$$\begin{aligned}\int_{\{|u_{k_j}|>\lambda\}} |g(u_{k_j})|dS &= -\int_\lambda^\infty g(s)d\alpha_{k_j}(s)\\ &\le c\tfrac{N-1}{N-2}\,\|\mu\|_{\mathfrak{M}(\partial\Omega)}\int_\lambda^\infty (g(s)-g(-s))\,s^{-\frac{2N-3}{N-2}}ds.\end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9), we can choose λ large enough and deduce that $\int_E |g(u_{k_j})|dS \to 0$ when $|E|_{N-1} \to 0$, uniformly with respect to k_j. Hence $g(u_{k_j})\lfloor_{\partial\Omega} \to g(u)\lfloor_{\partial\Omega}$ in $L^1(\partial\Omega)$. If $\xi \in \mathcal{C}(\Omega)$, there holds

$$\int_\Omega u_{k_j}\,(-\Delta\xi+\xi)\,dx + \int_{\partial\Omega} g(u_{k_j})\xi dS = \int_{\partial\Omega}\xi\mu_{k_j}dS. \tag{4.10}$$

Letting $k_j \to \infty$, we infer that (1.18) holds. Actually, the whole sequence $\{u_k\}$ converges and we denote by u_μ its limit. Notice also that by the monotonicity of g, $\mu \ge \mu'$ implies $u_\mu \ge u_{\mu'}$. □

Remark. If $g(r) = |r|^{p-1}r$ with $p > 0$, condition (1.20) is satisfied if and only if $p < \frac{N-1}{N-2}$.

4.2 Proof of Theorem E

In this section we assume $N = 2$.

Proof of assertion 1. As in the proof of Theorem D, we denote by u_k the solution of (4.2). Estimate (4.3) is valid and (4.11) is replaced by

$$\|u_k\|_{L^q(\Omega)} + \|\nabla u_k\|_{L^{2,\infty}(\Omega)} \le c'_q\int_{\partial\Omega}|\mu_k|dS \le c_q\,\|\mu\|_{\mathfrak{M}(\partial\Omega)} \quad \text{for all } q\in[1,\infty). \tag{4.11}$$

By an extension of Moser's inequality to Lorentz spaces [36, Theorem 3.1] , there exist constants $c^*, c' > 0$ depending on Ω such that for any function $v \in L^2(\partial\Omega)$ such that $(-\Delta)^{\frac12}v \in L^{2,\infty}(\partial\Omega)$ (equivalently $v \in B^{\frac12,2,\infty}(\partial\Omega)$), there holds

$$\sup_{\left\|(-\Delta)^{\frac12}v\right\|_{L^{2,\infty}}\le 1}\int_{\partial\Omega} e^{c^*|v(x)|}dS \le c'. \tag{4.12}$$

Using (4.11), (4.12) we deduce, with $c = \frac{c^*}{c_2}$,

$$\int_{\partial\Omega} e^{\frac{c|u_k(x)|}{\|\mu\|_{\mathfrak{M}}}}dS \le c'. \tag{4.13}$$

This implies that $\{(u_k, u_k\lfloor_{\partial\Omega})\}$ is compact in $L^q(\Omega)\times L^q(\partial\Omega)$ for any $q < \infty$ and up to a subsequence $\{u_{k_j}\}$ converges a. e. and in $L^q(\Omega)\times L^q(\partial\Omega)$ to some u such that $\nabla u \in L^{2,\infty}(\Omega)$ and therefore $u\lfloor_{\partial\Omega}\in L^q(\partial\Omega)$. Thus $u\lfloor_{\partial\Omega}$ satisfies (4.13). As a consequence problem (1.1) admits a solution if $|g(r)| \le c_1|r|^q + c_2$ for some $q \in (0,\infty)$ and $c_1, c_2 \ge 0$. We have actually a more general result if we assume that $a_+(g) = a_-(g) = 0$. From (4.13) there holds for $\lambda > 0$,

$$e^{\frac{c\lambda}{\|\mu\|_{\mathfrak{M}}}}\alpha_{k_j}(\lambda) \le c' \Longrightarrow \alpha_{k_j} \le c'e^{-\frac{c\lambda}{\|\mu\|_{\mathfrak{M}}}}, \tag{4.14}$$

where $A_\lambda(u_{k_j})$ and $\alpha_{k_j}(\lambda)$ are defined in the proof of Theorem D. If $E \subset \partial\Omega$ is a Borel set,

$$\begin{aligned}\int_E |g(u_{k_j})|dS &\le |E|_1(g(\lambda)-g(-\lambda)) + \int_{\{|u_{k_j}|\lfloor_{\partial\Omega}\}} |g(u_{k_j})|dS \\ &\le |E|_1(g(\lambda)-g(-\lambda)) - \int_\lambda^\infty (g(s)-g(-s))d\alpha_{k_j}(s) \\ &\le |E|_1(g(\lambda)-g(-\lambda)) + \frac{c'}{\|\mu\|_{\mathfrak{M}}}\int_\lambda^\infty (g(s)-g(-s))e^{-\frac{cs}{\|\mu\|_{\mathfrak{M}}}}ds\end{aligned} \tag{4.15}$$

Since

$$\int_\lambda^\infty (g(s)-g(-s))e^{-as}ds < \infty$$

for any $a > 0$ the result follows as in Theorem D. □

Remark. Actually we have a stronger result since we only use

$$\int_\lambda^\infty (g(s)-g(-s))e^{-\frac{cs}{\|\mu\|_{\mathfrak{M}}}}ds < \infty. \tag{4.16}$$

Therefore the assumption $a_+(g) = a_-(g) = 0$ can be replaced by $\|\mu\|_{\mathfrak{M}} \le c_g$ where

$$\int_\lambda^\infty (g(s)-g(-s))e^{-\frac{cs}{c_g}}ds < \infty. \tag{4.17}$$

However the constant c is not explicitely known.

Proof of assertion 2. Set $\mu = \sum_{j=1}^k \alpha_j\delta_{a_j}$. For $\ell > 0$ set $g_\ell(r) = \min\{g(\ell), \sup\{g(-\ell), g(r)\}\}$. Since $a_+(g_\ell) = a_-(g_\ell) = 0$, there exists a weak solution to

$$\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} + g_\ell(u) &= \mu && \text{on } \partial\Omega, \end{aligned} \tag{4.18}$$

and this solution denoted by $u_{\ell,\mu}$ is unique since g_ℓ is nonnecreasing. Put $J_+ := \{j = 1, ..., k : \alpha_j > 0\}$, $J_- := \{j = 1, ..., k : \alpha_j < 0\}$ and denote by u_{ℓ,μ_+} (resp. u_{ℓ,μ_-}) the solution of

$$\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} + g_\ell(u) &= \mu_+ \quad (\text{resp. } = \mu_-) && \text{on } \partial\Omega, \end{aligned} \tag{4.19}$$

with $\mu_+ = \sum_{j\in J_+} \alpha_j\delta_{a_j}$ (resp. $\mu_- = \sum_{j\in J_-} \alpha_j\delta_{a_j}$). Then $u_{\ell,\mu_+} \ge 0$ (resp. $u_{\ell,\mu_-} \le 0$) and

$$\sum_{j\in J_-} \alpha_j\mathcal{N}_\Omega(., a_j) \le u_{\ell,\mu_-} \le u_{\ell,\mu} \le u_{\ell,\mu_+} \le \sum_{j\in J_+} \alpha_j\mathcal{N}_\Omega(., a_j).$$

Thus

$$g\left(\sum_{j\in J_-}\alpha_j\mathcal{N}_\Omega(.,a_j)\right)\le g_\ell\left(u_{\ell,\mu_-}\right)\le g_\ell(u_{\ell,\mu})\le g_\ell\left(u_{\ell,\mu_+}\right)\le g\left(\sum_{j\in J_+}\alpha_j\mathcal{N}_\Omega(.,a_j)\right).$$

Since

$$\alpha_j\mathcal{N}_\Omega(x,a_j)=\frac{\alpha_j}{\pi}\ln\left(\frac{1}{|x-a_j|}\right)(1+o(1))\text{as }\ x\to a_j,$$

for any $\epsilon>0$, there exists $K_\epsilon>0$ such that

$$\begin{aligned}\sum_{j\in J_-}g\left(\frac{\alpha_j-\epsilon}{\pi}\ln\left(\frac{1}{|x-a_j|}\right)\right)-K_\epsilon\le g_\ell\left(u_{\ell,\mu_-}\right)\le g_\ell(u_{\ell,\mu})\\ \le g_\ell\left(u_{\ell,\mu_+}\right)\le\sum_{j\in J_+}g\left(\frac{\alpha_j+\epsilon}{\pi}\ln\left(\frac{1}{|x-a_j|}\right)\right)+K_\epsilon.\end{aligned}$$

We take $\epsilon>0$ small enough such that

$$\frac{\pi}{a_-(g)}\inf_{j\in J_-}\alpha_j-\epsilon\le 0\le\sup_{j\in J_+}\alpha_j+\epsilon\le\frac{\pi}{a_+(g)}.$$

This implies that $\{g_\ell\left(u_{\ell,\mu_-}\right)\}_\ell$ and $\{g_\ell\left(u_{\ell,\mu_+}\right)\}_\ell$ are uniformly integrable in $L^1(\partial\Omega)$. Consequently $\{g_\ell\left(u_{\ell,\mu}\right)\}_\ell$ is also uniformly integrable in $L^1(\partial\Omega)$. Letting $\ell\to\infty$ we deduce that up to a subsequence, $u_{\ell_j,\mu}$ converges to the unique weak solution $u=u_\mu$ of (1.12). □

Remark. By adapting the construction in [32] (see also [33] for a slightly simpler proof), it can be proved that when $N=2$ the problem (1.12) can be solved with any measure on $\partial\Omega$ with Jordan decomposition $\mu=\mu_r+\mu_a$ where μ_r is the diffuse part and $\mu_a=\sum_{j=1}^k\alpha_j\delta_{a_j}$ is the atomic part, provided the α_j satisfy (1.23). In particular no assumption on μ_r is required.

4.3 The supercritical case: proof of Theorem F

Let $\mathbb{P}_\Omega$ be the Poisson operator for $-\Delta+I$ in Ω and $\mathbb{D}_\Omega$ the Dirichlet to Neumann operator for $-\Delta+I$. Thus if $\eta\in\mathfrak{M}(\partial\Omega)$, $v=\mathbb{P}_\Omega[\eta]$ if

$$\begin{aligned}-\Delta v+v&=0 &&\text{in }\ \Omega\\ v&=\eta &&\text{on }\ \partial\Omega,\end{aligned}\tag{4.20}$$

and

$$\mathbb{D}_\Omega[\eta]=\frac{\partial v}{\partial\mathbf{n}}=\frac{\partial}{\partial\mathbf{n}}\mathbb{P}_\Omega[\eta]\quad\text{on }\ \partial\Omega.\tag{4.21}$$

Let $\mathbb{N}_\Omega$ be the Neumann operator from $\partial\Omega$ to Ω defined by $v=\mathbb{N}_\Omega[\mu]$ where

$$\begin{aligned}-\Delta v+v&=0 &&\text{in }\ \Omega\\ \frac{\partial v}{\partial\mathbf{n}}&=\mu &&\text{on }\ \partial\Omega,\end{aligned}\tag{4.22}$$

and some results of regularity of $\mathbb{N}_\Omega$ are recalled in Proposition 3.4.

Lemma 4.1 *Let $1 < q < \infty$ and μ is a distribution in $\mathbb{R}^N$ with support included in $\partial\Omega$. Then the following assertions are equivalent:*
(i) $\mathbb{N}_\Omega[\mu] \in L^q(\partial\Omega)$,
(ii) $\mu \in W^{-1,q}(\partial\Omega)$.
Furthermore there exists $c > 0$ such that

$$c^{-1} \|\mu\|_{W^{-1,q}(\partial\Omega)} \leq \|\mathbb{N}_\Omega[\mu]\|_{L^q(\partial\Omega)} \leq c \|\mu\|_{W^{-1,q}(\partial\Omega)} \tag{4.23}$$

Proof. We recall that by Calderon's theorem the operator $\mathbb{D}_\Omega$ is an isomorphism from $L^q(\partial\Omega)$ to $W^{1,q}(\partial\Omega)$ (see e.g. [1, Theorem 1.2.3]) and in particular for any $q \in (1, \infty)$,

$$c^{-1} \|\eta\|_{W^{1,q}(\partial\Omega)} \leq \|\mathbb{D}_\Omega[\eta]\|_{L^q(\partial\Omega)} \leq c \|\eta\|_{W^{1,q}(\partial\Omega)}. \tag{4.24}$$

This follows from the fact the following identity holds if $v = \mathbb{N}_\Omega[\mu]$

$$v(x) = \int_{\partial\Omega} \mathcal{N}_\Omega(x,y) \frac{\partial v}{\partial \mathbf{n}}(y) dS(y) = \int_{\partial\Omega} \mathcal{N}_\Omega(x,y) d\mu(y)$$

and that $\mathcal{N}_\Omega$ satisfies (3.42). In the flat case it is exactely the Calderon-Zygmund theory as it is shown in [30, Theorem V-3].
Let $v = \mathbb{N}_\Omega[\mu]$, $\xi \in C^2(\Omega)$ and $\psi = \mathbb{P}_\Omega[\xi]$, then

$$\langle \mu, \xi \rangle = \int_{\partial\Omega} v\lfloor_{\partial\Omega} \frac{\partial \psi}{\partial \mathbf{n}} dS = \int_{\partial\Omega} \mathbb{N}_\Omega[\mu] \mathbb{D}_\Omega[\xi] dS.$$

Using (4.24), we see that if $\mathbb{N}_\Omega[\mu]$ belongs to $L^q(\partial\Omega)$, then

$$|\langle \mu, \xi \rangle| \leq \|\mathbb{N}_\Omega[\mu]\|_{L^q(\partial\Omega)} \|\mathbb{D}_\Omega[\xi]\|_{L^{q'}(\partial\Omega)} \leq c \|\mathbb{N}_\Omega[\mu]\|_{L^q(\partial\Omega)} \|\xi\|_{W^{1,q'}(\partial\Omega)},$$

which implies that $\xi \mapsto \langle \mu, \xi \rangle$ is a continuous linear map on $W^{1,q}(\partial\Omega)$, thus belongs to $W^{-1,q}(\partial\Omega)$ and there holds

$$\|\mu\|_{W^{-1,q}(\partial\Omega)} \leq c \|\mathbb{N}_\Omega[\mu]\|_{L^q(\partial\Omega)}. \tag{4.25}$$

Conversely, if $\mu \in W^{-1,q}(\partial\Omega)$ and $\xi \in C^2(\partial\Omega)$, we set $\phi = \mathbb{N}_\Omega[\mu]$ and $w = \mathbb{D}_\Omega[\xi]$. Then, using (4.24)

$$\begin{aligned} \left| \int_{\partial\Omega} \mathbb{N}_\Omega[\mu\lfloor_{\partial\Omega}] \mathbb{D}_\Omega[\xi] \right| &= |\langle \mu, \xi \rangle| \leq \|\mu\|_{W^{-1,q}(\partial\Omega)} \|\xi\|_{W^{1,q'}(\partial\Omega)} \\ &\leq c^{-1} \|\mu\|_{W^{-1,q}(\partial\Omega)} \|\mathbb{D}_\Omega[\xi]\|_{L^{q'}(\partial\Omega)}. \end{aligned}$$

By density, it implies that

$$\left| \int_{\partial\Omega} \mathbb{N}_\Omega[\mu\lfloor_{\partial\Omega}] h \right| \leq c^{-1} \|\mu\|_{W^{-1,q}(\partial\Omega)} \|h\|_{L^{q'}(\partial\Omega)},$$

Hence $\mathbb{N}_\Omega[\mu\lfloor_{\partial\Omega}] \in L^q(\partial\Omega)$ and

$$\|\mathbb{N}_\Omega[\mu]\|_{L^q(\partial\Omega)} \leq c^{-1} \|\mu\|_{W^{-1,q}(\partial\Omega)}. \tag{4.26}$$

□

Lemma 4.2 *Assume $\mu \in W^{-1,q}(\partial\Omega) \cap \mathfrak{M}_+(\partial\Omega)$, then problem* (1.24) *admits a weak solution.*

Proof. We denote by u_k the solution of

$$\begin{aligned} -\Delta u_k + u_k &= 0 \qquad && \text{in } \Omega \\ \frac{\partial u_k}{\partial \mathbf{n}} + \min\{u_k^p, k^p\} &= \mu \qquad && \text{on } \partial\Omega, \end{aligned} \tag{4.27}$$

the existence of which comes from Theorem D. Then $0 \le u_k \le \mathbb{N}[\mu]$, and the sequence $\{u_k\}$ is nonincreasing and bounded from above by $\mathbb{N}_\Omega[\mu]$. Then it converges to some nonnegative harmonic function u in Ω. Since $\min\{u_k^p, k^p\} \le (\mathbb{N}[\mu])^p$ and $\mathbb{N}[\mu] \in L^p(\partial\Omega)$, it implies that $\min\{u_k^p, k^p\}$ converges to u^p a.e. in $\partial\Omega$ and in $L^1(\partial\Omega)$. Then u satisfies (1.24). □

Corollary 4.3 *Let $\{\mu_m\}$ be an increasing sequence of nonnegative measures on $\partial\Omega$ belonging to $W^{-1,p}(\partial\Omega)$ and converging to a measure μ in $\mathfrak{M}(\Omega)$. Then problem* (1.24) *with boundary data μ admits a weak solution.*

Proof. Let u_m be the solution of

$$\begin{aligned} -\Delta u_m + u_m &= 0 \qquad && \text{in } \Omega \\ \frac{\partial u_m}{\partial \mathbf{n}} + u_m^p &= \mu_m \qquad && \text{on } \partial\Omega, \end{aligned} \tag{4.28}$$

The sequence $\{u_m\}$ is increasing. For any $\zeta \in \mathcal{C}(\Omega)$, there holds

$$\int_\Omega u_m \left(-\Delta\zeta + \zeta\right) dx + \int_{\partial\Omega} u_m^p \zeta dS = \int_{\partial\Omega} \zeta d\mu_m \tag{4.29}$$

If we take in particular $\zeta = 1$, then

$$\int_\Omega u_m dx + \int_{\partial\Omega} u_m^p dS = \int_{\partial\Omega} d\mu_m \le \int_{\partial\Omega} d\mu.$$

Then u_m is bounded in $L^{\frac{N}{N-2},\infty}(\Omega)$ (or any $L^q(\Omega)$ if $N = 2$) and ∇u_m is bounded in $L^{\frac{N}{N-1},\infty}(\Omega)$. By the monotone convergence theorem $\{u_m\}$ converges in $L^1(\Omega)$ to some u and $\{u_m \lfloor_{\partial\Omega}\}$ converges in $L^p(\partial\Omega)$ to $u\lfloor_{\partial\Omega}$. Letting $m \to \infty$ in (4.29) we obtain

$$\int_\Omega u \left(-\Delta\zeta + \zeta\right) dx + \int_{\partial\Omega} u^p \zeta dS = \int_{\partial\Omega} \zeta d\mu,$$

which ends the proof. □

In the next result we denote by $C^{1,p'}_{\mathbb{R}^{N-1}}$ the Bessel (or Sobolev) capacity on $\partial\Omega$ associated to $W^{1,p'}(\mathbb{R}^{N-1})$. The corresponding capacity $C^{1,p'}_{\partial\Omega}$ on the boundary is defined by local charts and the zero-capacity property does not depend on the charts.

Proposition 4.4 *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ such that problem* (1.24) *admits a weak solution, then μ vanishes on Borel set $E \subset \partial\Omega$ satisfying $C^{1,p'}_{\partial\Omega}(E) = 0$.*

Proof. Without loss of generality we can assume that $E \subset \partial\Omega$ is a compact set. Because of uniqueness, u is nonnegative. Let $\eta \in C_0^2(\partial\Omega)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in a neighborhood of E and $v_\eta = \mathbb{P}_\Omega[\eta]$. If $\zeta \in C^2(\overline{\Omega})$ we have

$$\int_\Omega u\left(-\Delta\zeta + \zeta\right) dx + \int_{\partial\Omega} u^p \zeta dS = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial \mathbf{n}} u dS + \int_{\partial\Omega} \zeta d\mu.$$

For $k > 1$ we take $\zeta = v_\eta^k$, then

$$\int_\Omega u\left(-k(k-1)v_\eta^{k-2}|\nabla v_\eta|^2 + (1-k)v_\eta^k\right) dx + \int_{\partial\Omega} u^p v_\eta^k dS \\ = -k\int_{\partial\Omega} \eta^{k-1}\mathbb{D}_\Omega[\eta] u dS + \int_{\partial\Omega} \eta^k d\mu.$$

Since $v_\eta \geq 0$, we obtain

$$\int_{\partial\Omega} u^p v_\eta^k dS + k\int_{\partial\Omega} \eta^{k-1}|\mathbb{D}_\Omega[\eta]| u dS \geq \mu(E).$$

Furthermore.

$$\int_{\partial\Omega} \eta^{k-1}|\mathbb{D}_\Omega[\eta]| u dS \leq \left(\int_{\partial\Omega} \eta^k u^p dS\right)^{\frac{1}{p}} \left(\int_{\partial\Omega} \eta^{k-p'}|\mathbb{D}_\Omega[\eta]|^{p'} dS\right)^{\frac{1}{p'}}.$$

Taking $k = p'$ and using (4.24), we infer

$$\int_{\partial\Omega} u^p v_\eta^k dS + cp'\left(\int_{\partial\Omega} \eta^k u^p dS\right)^{\frac{1}{p}} \|\eta\|_{W^{1,p'}(\partial\Omega)} \geq \mu(E). \tag{4.30}$$

If $C_{\partial\Omega}^{1,p'}(E) = 0$, there exists a sequence $\{\eta_m\} \subset C_0^2(\partial\Omega)$ such that $0 \leq \eta_m \leq 1$, $\eta = 1$ in a neighborhood of E and $\|\eta_m\|_{W^{1,p'}(\partial\Omega)} \to 0$ when $m \to \infty$. This implies that $v_{\eta_m} \to 0$ in $L^1(\Omega)$, hence the left-hand side of (4.30) tends to 0, and finally $\mu(E) = 0$. □

We end the proof of Theorem F with the sufficient condition which follows from a general result due to Feyel and de la Pradelle [18].

Proposition 4.5 *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ such that $\mu(E) = 0$ for any Borel set $E \subset \partial\Omega$ satisfying $C_{\partial\Omega}^{1,p'}(E) = 0$. Then there exists an increasing sequence $\{\mu_n\} \subset \mathfrak{M}_+(\partial\Omega) \cap W^{-1,p}(\partial\Omega)$ converging to μ.*

Remark. If $1 < p < \frac{N-1}{N-2}$, $W^{1,p'}(\partial\Omega) \subset C(\partial\Omega)$. Therefore the only set with zero $C_{\partial\Omega}^{1,p'}$-capacity is the empty set. If $p \geq \frac{N-1}{N-2}$, a single point has zero $C_{\partial\Omega}^{1,p'}$-capacity. Since $\delta_a(a) = 1$ for any $a \in \partial\Omega$ there is no solution of problem (1.24) with $\mu = \delta_a$.

As a consequence we have a non-removability result.

Corollary 4.6 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$. Then any compact subset $K \subset \partial\Omega$ with positive $C_{\partial\Omega}^{1,p'}$-capacity is non-removable in the sense that there exists a nonnegative non-trivial function $u_K \in C^1(\overline{\Omega} \setminus K)$ satisfying*

$$\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} + |u|^{p-1}u &= 0 && \text{in } \partial\Omega \setminus K. \end{aligned} \tag{4.31}$$

Proof. By [1, Theorem 2.5.3] there exists a positive measure, called the capacitary measure μ_K with support in K and such that $\mu_K \in W^{-1,p'}(\Omega)$. For such a measure there exists a positive solution to (1.24), hence u satisfies (4.31). □

Remark. We conjecture that the condition $C^{1,p'}_{\partial\Omega}(K) = 0$ is also a sufficient condition for a compact set $K \subset \partial\Omega$ to be removable. This is even not known if K is a singleton.

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