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Fourier transform of a class of radial and semi-radial functions

Victor Rabiet

29 février 2020

Abstract

We give here the key to reduce the problem of computing a three dimensional Fourier transform - for a certain class of functions - to a one dimensional Fourier transform computation, allowing in some way to “reverse” the main theorem of [1], in this particular case. This class of functions being of the form

$$t \mapsto \frac{R(\cos(\|t\|), \sin(\|t\|))}{\|t\|^m} P(t) \in L^1_{\text{loc}}(\mathbb{R}^3),$$

where $R \in \mathbb{C}[X, Y]$ and $P \in \mathbb{C}[X, Y, Z]$

Key words : Fourier Transform, radial and semi-radial functions

1 Introduction

1.1 “Universal” notation for the Fourier Transform

There is a lot of conventions for the Fourier transform, so we give here a convenient notation to take care of all the cases once for all (with $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$) :

$$\mathcal{F}_n^{a,b}(f)(k) := \left(\frac{|b|}{(2\pi)^{1-a}} \right)^{n/2} \int_{\mathbb{R}^n} f(t) e^{ib\langle k,t \rangle} dt \quad (1.1)$$

$$(\mathcal{F}_n^{a,b})^{-1}(g)(t) := \left(\frac{|b|}{(2\pi)^{1+a}} \right)^{n/2} \int_{\mathbb{R}^n} g(k) e^{-ib\langle t,k \rangle} dk \quad (1.2)$$

To jump from a convention to another, we have the simple following corresponding formula :

$$\mathcal{F}^{a,b}(f)(k) = \left(\frac{|b/b'|}{(2\pi)^{a'-a}} \right)^{n/2} \mathcal{F}^{a',b'}(f)\left(\frac{b}{b'}k\right). \quad (1.3)$$

Fourier Transform of a tempered distribution

Let us recall that if $T \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, we define de Fourier transform of $\mathcal{F}_n^{a,b}T$ by

$$\langle \mathcal{F}_n^{a,b}T, \varphi \rangle := \langle T, \mathcal{F}_n^{a,b}\varphi \rangle \quad (1.4)$$

In order to give the formula to jump from a convention to another, we have to define the following operator : for $a \in \mathbb{R}$, we set $m_a : f \mapsto (x \mapsto f(ax), x \in \mathbb{R}^n)$.

This operator can be extended to the distributions : for all $a \neq 0$,

$$\langle m_a T, \varphi \rangle := \frac{1}{|a|^n} \langle T, m_{\frac{1}{a}} \varphi \rangle \quad (1.5)$$

From (1.3) and (1.5) we have

$$\mathcal{F}_n^{a,b} T = \left(\frac{|b'/b|}{(2\pi)^{a'-a}} \right)^{n/2} \mathcal{F}_n^{a',b'} (m_{\frac{b'}{b}} T). \quad (1.6)$$

(please notice that $|b/b'|$ in (1.3) “becoming” $|b'/b|$ in (1.6) is not a typo!)

1.2 Radial distributions

We recall here the definitions and de properties given and demonstrated in [1].

We set (where $\mathcal{S}(\mathbb{R}^n)$ stands for the space of Schwartz functions on \mathbb{R}^n)

$$\mathcal{S}_{\text{rad}}(\mathbb{R}^n) = \{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \varphi = \varphi \circ A, \forall A \in \mathcal{O}(n) \} \quad (1.7)$$

$$\mathcal{S}_{\text{rad}}(\mathbb{R}) = \mathcal{S}_{\text{even}}(\mathbb{R}) = \{ \varphi \in \mathcal{S}(\mathbb{R}) : \varphi(x) = \varphi(-x) \} \quad (1.8)$$

where $\mathcal{O}(n)$ is the set of the orthogonal transformations of \mathbb{R}^n .

We define then the following functions :

$$\begin{cases} \mathcal{S}(\mathbb{R}^n) & \rightarrow \mathcal{S}_{\text{rad}}(\mathbb{R}) \\ \varphi & \mapsto \left(r \mapsto \varphi^o(r) := \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \varphi(r\theta) d\theta \right) \end{cases} \quad (1.9)$$

$$\begin{cases} \mathcal{S}_{\text{rad}}(\mathbb{R}) & \rightarrow \mathcal{S}_{\text{rad}}(\mathbb{R}^n) \\ \varphi & \mapsto \left(x \mapsto \varphi^O(x) := \varphi(|x|) \right) \end{cases} \quad (1.10)$$

(where \mathbb{S}^{n-1} is the unit sphere on \mathbb{R}^n and ω_{n-1} its surface area; with the convention $\omega_0 = 2$ and $\varphi^o(x) = \frac{1}{2}(\varphi(x) + \varphi(-x))$, for $\varphi \in \mathcal{S}(\mathbb{R})$).

Definition 1.1 A distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ (with $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions on \mathbb{R}^n) is called radial if for all $A \in \mathcal{O}(n)$,

$$u = u \circ A,$$

that is,

$$\langle u, \varphi \rangle = \langle u, \varphi \circ A \rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The set of all radial tempered distributions is denoted by $\mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$.

Proposition 1.2 For $u \in \mathcal{S}'_{\text{rad}}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle u, \varphi \rangle = \langle u, \varphi^{\text{rad}} \rangle \quad (1.11)$$

where $\varphi^{\text{rad}} := (\varphi^o)^O$ (i.e. $\varphi^{\text{rad}}(x) = \varphi^o(|x|)$).

Let us define the space

$$\mathcal{R}_n := r^{n-1} \mathcal{S}_{\text{rad}}(\mathbb{R}) = \{(r \mapsto \psi(r)r^{n-1}), \psi \in \mathcal{S}_{\text{rad}}(\mathbb{R})\}. \quad (1.12)$$

Remark 1.3 \mathcal{R}_n is a subspace $\mathcal{S}(\mathbb{R})$ on which we can use the same topology; we denote its dual (set of the linear continuous functions defined over \mathcal{R}_n) by \mathcal{R}'_n .

We switch from \mathcal{R}'_n to radials distributions of $\mathcal{S}'(\mathbb{R}^n)$ as follows :

- if u is radial distribution, we define $u_\diamond \in \mathcal{R}'_n$ by

$$\langle u_\diamond, \psi(r)r^{n-1} \rangle := \frac{2}{\omega_{n-1}} \langle u, \psi^O \rangle, \quad \psi \in \mathcal{S}_{\text{rad}}(\mathbb{R}) \quad (1.13)$$

- if $u_\diamond \in \mathcal{R}'_n$, we define a radial distribution u by

$$\langle u, \varphi \rangle := \frac{\omega_{n-1}}{2} \langle u_\diamond, \varphi^o(r)r^{n-1} \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n) \quad (1.14)$$

1.3 Grafakos-Teschl's Theorem

Theorem 1.4 (Grafakos-Teschl (2013)) Given v_1 in $\mathcal{S}'(\mathbb{R})$, we define a radial distribution v_k on \mathbb{R}^k ($k \in \mathbb{N}^*$) by

$$\langle v_k, \varphi \rangle := \frac{\omega_{k-1}}{2} \langle v_1, \varphi^o(r)r^{k-1} \rangle, \quad \varphi \in \mathcal{S}_{\text{rad}}(\mathbb{R}^k) \quad (1.15)$$

(if $\varphi \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$, then $\varphi(x) = \varphi^o(|x|)$).

Let $u^k = \mathcal{F}_k^{a,b}(v_k)$. We have then

$$-\frac{(2\pi)^a}{|b|r} \frac{d}{dr} u_\diamond^n = u_\diamond^{n+2} \quad (1.16)$$

Remark 1.5 • Dans l'article de base, on partait d'une fonction $v_0 \in \mathcal{S}'(\mathbb{R})$, et on définissait $\langle v_k, \varphi \rangle := \frac{\omega_{k-1}}{2} \langle v_0, \varphi^o(r)r^{k-1} \rangle$, mais alors, on vérifie très rapidement que $v_0 = v_1$; cela n'a pas d'intérêt de commencer par v_0 .

- On a $u_\diamond^1 = u^1$: en effet

$$\langle u_\diamond^1, \psi(r)r^{1-1} \rangle := \frac{2}{\omega_0} \langle u, \psi^O \rangle = \langle u, \psi \rangle$$

soit $\langle u_\diamond^1, \psi \rangle = \langle u, \psi \rangle$ pour tout $\psi \in \mathcal{S}_{\text{rad}}(\mathbb{R})$.

Corollary 1.6 Given v_1 in $\mathcal{S}'(\mathbb{R})$, we define the radial distribution v_3 on \mathbb{R}^3 by

$$\langle v_3, \varphi \rangle := \frac{\omega_2}{2} \langle v_1, \varphi^o(r)r^2 \rangle, \quad \varphi \in \mathcal{S}_{\text{rad}}(\mathbb{R}^3) \quad (1.17)$$

(if $\varphi \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$, then $\varphi(x) = \varphi^o(|x|)$). We have then, for all $\varphi \in \mathcal{S}_{\text{rad}}(\mathbb{R}^3)$

$$\langle \mathcal{F}_3^{a,b}(v_3), \varphi \rangle = -\frac{(2\pi)^{a+1}}{|b|} \left\langle r \frac{d}{dr} (\mathcal{F}_1^{a,b}(v_1)), \varphi^o(r) \right\rangle \quad (1.18)$$

Proof: With $u^3 := \mathcal{F}_3^{a,b}(v_3)$ and $u^1 := \mathcal{F}_1^{a,b}(v_3)$ we have

$$\begin{aligned} \langle u^3, \varphi \rangle &= \frac{\omega_2}{2} \langle u_\diamond^3, \varphi^o(r)r^2 \rangle && \text{(cf. (1.14))} \\ &= \frac{\omega_2}{2} \left\langle -\frac{(2\pi)^a}{|b|r} \frac{d}{dr} u^1, \varphi^o(r)r^2 \right\rangle && \text{(cf. (1.16) and Remark 1.5)} \\ &= -\frac{(2\pi)^{a+1}}{|b|} \left\langle r \frac{d}{dr} (u^1), \varphi^o(r) \right\rangle \end{aligned}$$

since $\omega_2 = 4\pi$. •

2 Fourier transform of $t \mapsto \frac{R(\cos(\|t\|), \sin(\|t\|))}{\|t\|^m} \in L_{\text{loc}}^1(\mathbb{R}^3)$

Lemma 2.1 *Let f be a smooth function such that*

$$t \mapsto \frac{f(\|t\|)}{\|t\|^m} \in L_{\text{loc}}^1(\mathbb{R}^3). \quad (2.19)$$

If we set $T_1 = f_e T_{\frac{1}{r^m}}$ and $T_2 = f_o T_{\frac{\text{sgn}(r)}{r^m}}$, (with $f_e = \frac{f(x)+f(-x)}{2}$ and $f_o = \frac{f(x)-f(-x)}{2}$) if m is even, and $T_1 = f_o T_{\frac{1}{r^m}}$ and $T_2 = f_e T_{\frac{\text{sgn}(r)}{r^m}}$ otherwise, the radial distribution T defined by

$$\langle T, \varphi \rangle := \frac{\omega_2}{2} \langle T_1 + T_2, r^2 \varphi^o \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^3)$$

verifies

$$\frac{f(\|t\|)}{\|t\|^m} = T.$$

Proof:

We set

$$j := \inf \left(\left(\inf_{u \in \mathbb{N}} \{u \mid f^{(u)}(0) \neq 0\}, m-2 \right) \right)$$

and $n := m - j - 2$. Let us notice that, from condition (2.19) and since either $f(r) \sim Cr^j$ (Taylor's Theorem) or $j = m - 2$, it becomes that

$$0 \geq n. \quad (2.20)$$

1) First, we will show the result for $T := T_2 = f(r) T_{\frac{\text{sgn}(r)}{r^m}}$ (f even or odd, according to m).

From (5.33), we have (with $g(r) := \frac{f(r)}{r^j}$) and $s \in \mathbb{N}$ such that $m + s = j + 2$),

$$\begin{aligned} \frac{\omega_2}{2} \langle f(r) T_{\frac{\text{sgn}(r)}{r^m}}, r^2 \varphi^o \rangle &= \frac{\omega_2}{2} \langle g(r) r^s \text{sgn}(r), \varphi^o \rangle \\ &= \frac{\omega_2}{2} \left(- \int_{-\infty}^0 g(u) u^s \varphi^o(u) du + \int_0^{+\infty} g(u) u^s \varphi^o(u) du \right) \\ &= \frac{\omega_2}{2} \left(- \int_{-\infty}^0 \frac{f(u)}{u^m} \varphi^o(u) u^2 du + \int_0^{+\infty} \frac{f(u)}{u^m} \varphi^o(u) u^2 du \right) \\ &= \omega_2 \int_0^{+\infty} \frac{f(u)}{u^m} \varphi^o(u) u^2 du \\ &= \int_{\mathbb{R}^3} \frac{f(\|t\|)}{\|t\|^m} \varphi^o(\|t\|) dt = \int_{\mathbb{R}^3} \frac{f(\|t\|)}{\|t\|^m} \varphi(t) dt. \end{aligned}$$

2) Similarly, for $T := T_1 = f(r)T_{\frac{1}{r^m}}$ (f even or odd, according to m), From (5.32), we have (with $g(r) := \frac{f(r)}{r^j}$) and $s \in \mathbb{N}$ such that $m + s = j + 2$),

$$\begin{aligned}
\frac{\omega_2}{2} \langle f(r)T_{\frac{1}{r^m}}, r^2 \varphi^o \rangle &= \frac{\omega_2}{2} \langle g(r)r^s, \varphi^o \rangle \\
&= \frac{\omega_2}{2} \left(\int_{-\infty}^0 g(u)u^s \varphi^o(u) du + \int_0^{+\infty} g(u)u^s \varphi^o(u) du \right) \\
&= \frac{\omega_2}{2} \left(\int_{-\infty}^0 \frac{f(u)}{u^m} \varphi^o(u) u^2 du + \int_0^{+\infty} \frac{f(u)}{u^m} \varphi^o(u) u^2 du \right) \\
&= \omega_2 \int_0^{+\infty} \frac{f(u)}{u^m} \varphi^o(u) u^2 du \\
&= \int_{\mathbb{R}^3} \frac{f(\|t\|)}{\|t\|^m} \varphi^o(\|t\|) dt = \int_{\mathbb{R}^3} \frac{f(\|t\|)}{\|t\|^m} \varphi(t) dt.
\end{aligned}$$

•

3 Fourier transform of $t \mapsto P(t) \frac{R(\cos(\|t\|), \sin(\|t\|))}{\|t\|^m} \in L^1_{\text{loc}}(\mathbb{R}^3)$

Lemma 3.1 *Let f be a smooth function and $P \in \mathbb{C}[X, Y, Z]$, such that*

$$t \mapsto P(t) \frac{f(\|t\|)}{\|t\|^m} \in L^1_{\text{loc}}(\mathbb{R}^3). \quad (3.21)$$

If we set $T_1 = f_e T_{\frac{1}{r^m}}$ and $T_2 = f_o T_{\frac{\text{sgn}(r)}{r^m}}$, (with $f_e = \frac{f(x)+f(-x)}{2}$ and $f_o = \frac{f(x)-f(-x)}{2}$) if m is even, and $T_1 = f_o T_{\frac{1}{r^m}}$ and $T_2 = f_e T_{\frac{\text{sgn}(r)}{r^m}}$ otherwise, the radial distribution T defined by

$$\langle T, \varphi \rangle := \frac{\omega_2}{2} \langle T_1 + T_2, r^2 \varphi^o \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^3)$$

verifies

$$P(t) \frac{f(\|t\|)}{\|t\|^m} = P(t)T.$$

Proof: If $P(0) \neq 0$, it is a straightforward consequence of Lemma 2.1; hence, we will consider that $P(0) = 0$ in the sequel of this proof and, using the spherical coordinates, we then define the quantity $d \in \mathbb{N}^*$ by

$$P(rX(\varphi, \theta)) = r^d h(X(\varphi, \theta)). \quad (3.22)$$

We set

$$j = \inf \left(\left(\inf_{u \in \mathbb{N}} \{u \mid f^{(u)}(0) \neq 0\}, m - 2 \right) \right)$$

and $n := m - j - 2$. Let us notice that, from condition (3.21) and since either $f(r) \sim Cr^j$ (Taylor's Theorem) or $j = m - 2$, it becomes that

$$d \geq n. \quad (3.23)$$

If $n \leq 0$, we will have $t \mapsto \frac{f(\|t\|)}{\|t\|^m} \in L^1_{\text{loc}}(\mathbb{R}^3)$, so that Lemma 2.1 can be, again, directly applied.

Hence, in the sequel, we will assume always that $n \geq 1$.

1) First, we will show the result for $T := T_2 = f(r)T_{\frac{\text{sgn}(r)}{r^m}}$ (f even or odd, according to m).

From (5.35), we have (with $g(r) := \frac{f(r)}{r^j}$)

$$\frac{\omega_2}{2} \langle f(r) T_{\frac{\text{sgn}(r)}{r^m}}, r^2 \varphi^o \rangle = \frac{\omega_2}{2} \langle g(r) T_{\frac{\text{sgn}(r)}{r^m}}, \varphi^o \rangle + (-1)^{n-1} \frac{\omega_2}{2} C(j+2, n) \langle \delta_0^{(n-1)}, \varphi^o \rangle \quad (3.24)$$

and (with (5.37)),

$$\begin{aligned} & \langle T_{\frac{\text{sgn}(r)}{r^m}}, g(r) \varphi^o \rangle \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{r > \varepsilon} \frac{g(r) \varphi^o(r) + (-1)^{n+1} g(-r) \varphi^o(-r)}{r^n} dr \\ & \quad + \frac{1}{(n-1)!} \left(\ln \varepsilon \left((g\varphi^o)^{(n-1)}(\varepsilon) + (g\varphi^o)^{(n-1)}(-\varepsilon) \right) - \sum_{k=2}^n \frac{(k-2)!}{\varepsilon^{k-1}} \left((g\varphi^o)^{(n-k)}(\varepsilon) + (-1)^{(k+1)} (g\varphi^o)^{(n-k)}(-\varepsilon) \right) \right) \end{aligned}$$

Since we have the following different cases :

	m even and f odd		m odd and f even	
	j even	j odd	j even	j odd
g	odd	even	even	odd
n	even	odd	odd	even

and since $\varphi^o(r) = \varphi^o(-r)$, we always have

$$\frac{g(r) \varphi^o(r) + (-1)^{n+1} g(-r) \varphi^o(-r)}{r^n} = \varphi^o(r) \frac{g(r) + (-1)^{n+1} g(-r)}{r^n} = \frac{2\varphi^o(r)g(r)}{r^n} = 2 \frac{\varphi^o(r)f(r)}{r^m} r^2.$$

So

$$\begin{aligned} & \langle T_{\frac{\text{sgn}(r)}{r^m}}, g(r) \varphi^o \rangle \\ &= \frac{2}{\omega_2} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi^o(\|x\|) dx \\ & \quad + \frac{1}{(n-1)!} \left(\ln \varepsilon \left((g\varphi^o)^{(n-1)}(\varepsilon) + (g\varphi^o)^{(n-1)}(-\varepsilon) \right) - \sum_{k=2}^n \frac{(k-2)!}{\varepsilon^{k-1}} \left((g\varphi^o)^{(n-k)}(\varepsilon) + (-1)^{(k+1)} (g\varphi^o)^{(n-k)}(-\varepsilon) \right) \right) \\ &= \frac{2}{\omega_2} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi^o(\|x\|) dx + F_\varepsilon(g\varphi^o) \\ &= \frac{2}{\omega_2} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi^o(\|x\|) dx + F_\varepsilon(g\varphi^o) \end{aligned}$$

But if we define the tempered distribution u by

$$\langle u, \varphi \rangle := \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi(x) dx + \frac{\omega_2}{2} F_\varepsilon(g\varphi^o)$$

it is straightforward that u is radial distribution (clearly $u \circ A = u$ for any $A \in \mathcal{O}(n)$, and from Proposition 1.2,

$$\langle u, \varphi \rangle = \langle u, \varphi^{\text{rad}} \rangle = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi^o(\|x\|) dx + \frac{\omega_2}{2} F_\varepsilon(g\varphi^o) = \frac{\omega_2}{2} \langle T_{\frac{\text{sgn}(r)}{r^m}}, g(r) \varphi^o \rangle. \quad (3.25)$$

Putting together (3.24) and (3.25), we have

$$\langle T, \varphi \rangle = \langle u, \varphi \rangle + \frac{\omega_2}{2} C(j+2, n) (-1)^{n-1} \langle \delta_0^{(n-1)}, \varphi^o \rangle$$

so,

$$\langle PT, \varphi \rangle = \langle u, P\varphi \rangle + \frac{\omega_2}{2} C(j+2, n) (-1)^{n-1} \underbrace{\langle \delta_0^{(n-1)}, (P\varphi)^o \rangle}_{= ((P\varphi)^o)^{(n-1)}(0)=0} = \langle u, P\varphi \rangle.$$

The fact that $((P\varphi)^o)^{(n-1)}(0) = 0$ is a straightforward consequence of (3.23) and the following Lemma 3.2 (with $g = 1$).

Now, since we have

$$\lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(gP\varphi^o) = 0$$

which is a consequence from the easy following lemma

* * *

Lemma 3.2 *For all $\ell \in \mathbb{N}$ such that $\ell < d$, there exists a smooth function h_ℓ such that*

$$(g(P\varphi)^o)^{(\ell)}(\varepsilon) = h_\ell(\varepsilon)\varepsilon^{d-\ell}.$$

Proof:

By induction : for $\ell = 0$, since g is a smooth function (cf. Lemma 6.1), and

$$\begin{aligned} (P\varphi)^o &= \frac{1}{\omega_2} \int_{\mathbb{S}^2} P\varphi(\varepsilon\sigma) d\sigma \\ &= \frac{1}{\omega_2} \int_0^{2\pi} \int_0^\pi P\varphi(\varepsilon\Theta(\theta, \psi)) \sin(\varphi) d\sigma \\ &= \varepsilon^d \frac{1}{\omega_2} \int_{\mathbb{S}^2} P(\sigma)\varphi(\varepsilon\sigma) d\sigma \end{aligned}$$

we have (it is straightforward that $\varepsilon \mapsto \frac{1}{\omega_2} \int_{\mathbb{S}^2} P(\sigma)\varphi(\varepsilon\sigma) d\sigma$ is a smooth function)

$$g(P\varphi)^o(\varepsilon) = h_0(\varepsilon)\varepsilon^d$$

with $h_0(\varepsilon) := g(\varepsilon) \frac{1}{\omega_2} \int_{\mathbb{S}^2} P(\sigma)\varphi(\varepsilon\sigma) d\sigma$.

For $\ell + 1$:

$$\begin{aligned} \left((g(P\varphi)^o)^\ell \right)' &= (h_\ell(\varepsilon)\varepsilon^{d-\ell})' \\ &= h'_\ell(\varepsilon)\varepsilon^{d-\ell} + h_\ell(\varepsilon)\varepsilon^{d-\ell-1} \\ &= h_{\ell+1}(\varepsilon)\varepsilon^{d-(\ell+1)} \end{aligned}$$

with $h_{\ell+1}(\varepsilon) := h'_\ell(\varepsilon)\varepsilon + h_\ell(\varepsilon)$.

* * *

we have

$$\langle PT, \varphi \rangle = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} P\varphi(x) dx = \int_{\mathbb{R}^3} \frac{f(\|x\|)}{\|x\|^m} P\varphi(x) dx.$$

2) Similarly, we will show the result for $T_1 = f(r)T_{\frac{1}{r^m}}$ (f even or odd, according to m).

From (5.34), we have (with $g(r) := \frac{f(r)}{r^j}$)

$$\frac{\omega_2}{2} \langle f(r)T_{\frac{1}{r^m}}, r^2\varphi^o \rangle = \frac{\omega_2}{2} \langle g(r)T_{\frac{1}{r^m}}, \varphi^o \rangle \tag{3.26}$$

and (with (5.36)),

$$\begin{aligned} \langle T_{\frac{\text{sgn}(r)}{r^n}}, g(r)\varphi^o \rangle &= \lim_{\varepsilon \rightarrow 0^+} \int_{r > \varepsilon} \frac{g(r)\varphi^o(r) - (-1)^{n+1}g(-r)\varphi^o(-r)}{r^n} dr \\ &\quad - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \frac{(k-1)!}{\varepsilon^k} \left((g\varphi^o)^{(n-k-1)}(\varepsilon) - (-1)^k (g\varphi^o)^{(n-k-1)}(-\varepsilon) \right) \end{aligned}$$

Since we have the following different cases :

	m even and f even		m odd and f odd	
	j even	j odd	j even	j odd
g	even	odd	odd	even
n	even	odd	odd	even

and since $\varphi^o(r) = \varphi^o(-r)$, we always have

$$\frac{g(r)\varphi^o(r) - (-1)^{n+1}g(-r)\varphi^o(-r)}{r^n} = \varphi^o(r) \frac{g(r) - (-1)^{n+1}g(-r)}{r^n} = \frac{2\varphi^o(r)g(r)}{r^n} = 2 \frac{\varphi^o(r)f(r)}{r^m} r^2.$$

So

$$\begin{aligned} \langle T_{\frac{1}{r^n}}, g(r)\varphi^o \rangle &= \frac{2}{\omega_2} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi^o(\|x\|) dx \\ &\quad - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \frac{(k-1)!}{\varepsilon^k} \left((g\varphi^o)^{(n-k-1)}(\varepsilon) - (-1)^k (g\varphi^o)^{(n-k-1)}(-\varepsilon) \right) \\ &= \frac{2}{\omega_2} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi^o(\|x\|) dx + F'_\varepsilon(g\varphi^o) \\ &= \frac{2}{\omega_2} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi^o(\|x\|) dx + F'_\varepsilon(g\varphi^o) \end{aligned}$$

But if we define the tempered distribution u by

$$\langle u, \varphi \rangle := \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi(x) dx + \frac{\omega_2}{2} F'_\varepsilon(g\varphi^o)$$

it is straightforward that u is radial distribution (clearly $u \circ A = u$ for any $A \in \mathcal{O}(n)$, and from Proposition 1.2,

$$\langle u, \varphi \rangle = \langle u, \varphi^{\text{rad}} \rangle = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} \varphi^o(\|x\|) dx + \frac{\omega_2}{2} F'_\varepsilon(g\varphi^o) = \frac{\omega_2}{2} \langle T_{\frac{1}{r^n}}, g(r)\varphi^o \rangle. \quad (3.27)$$

With (3.26) and (3.27), we have directly

$$\langle T, \varphi \rangle = \langle u, \varphi \rangle$$

so,

$$\langle PT, \varphi \rangle = \langle u, P\varphi \rangle.$$

Now, since we have (using Lemma 3.2),

$$\lim_{\varepsilon \rightarrow 0^+} F'_\varepsilon(gP\varphi^o) = 0,$$

then

$$\langle PT, \varphi \rangle = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^3 \setminus B(0, \varepsilon)} \frac{f(\|x\|)}{\|x\|^m} P\varphi(x) dx = \int_{\mathbb{R}^3} \frac{f(\|x\|)}{\|x\|^m} P\varphi(x) dx.$$

•

4 Appendix

4.1 Distributions associated to $\frac{1}{x^n}$ and $\frac{\text{sgn}(x)}{x^n}$

$$T_{\frac{1}{x^n}} := \frac{(-1)^{n-1}}{(n-1)!} (\ln(|x|))^{(n)} \quad (4.28)$$

$$T_{\frac{\text{sgn}(x)}{x^n}} := \frac{(-1)^{n-1}}{(n-1)!} (\text{sgn}(x) \ln(|x|))^{(n)} \quad (4.29)$$

Remark 4.1 1. We then have directly de followings inductive properties

$$T_{\frac{1}{x^n}} = \frac{-1}{n-1} T'_{\frac{1}{x^{n-1}}} \quad (4.30)$$

$$T_{\frac{\text{sgn}(x)}{x^n}} = \frac{-1}{n-1} T'_{\frac{\text{sgn}(x)}{x^{n-1}}}. \quad (4.31)$$

2. $T_{\frac{1}{x^n}}$ and $T_{\frac{\text{sgn}(x)}{x^n}}$ sont des distributions tempérés car elles sont, à constantes près, issues respectivement des dérivations successives (au sens des distributions) des fonctions $x \mapsto x \ln(|x|) - x$ et $x \mapsto |x| \ln(|x|) - x$ qui sont continues et à croissance lente, donc elles-mêmes des distributions tempérées.

Proposition 4.2 1.

$$x^n T_{\frac{1}{x^n}} = 1 \quad (4.32)$$

$$x^n T_{\frac{\text{sgn}(x)}{x^n}} = \text{sgn}(x) \quad (4.33)$$

2.

$$x^n T_{\frac{1}{x^{m+n}}} = T_{\frac{1}{x^m}} \quad (4.34)$$

$$x^n T_{\frac{\text{sgn}(x)}{x^{m+n}}} = T_{\frac{\text{sgn}(x)}{x^m}} - (-1)^{m-1} C(m, n) \delta_0^{(m-1)} \quad (4.35)$$

where δ_0 is the Dirac distribution in 0, with the convention $\delta_0^{(0)} := \delta_0$ and with¹

$$C(m, n) = \frac{2}{(m-1)!} \sum_{k=0}^{n-1} \frac{1}{m+k} = \frac{2(H_{m+n-1} - H_{m-1})}{(m-1)!}.$$

Proof:

•

1. Where H_n is the n -th Harmonic number :

$$H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}.$$

Proposition 4.3 For all $\varphi \in \mathcal{S}(\mathbb{R})$, we have (with the conventions $\varphi^{(0)} = \varphi$ and $\sum_{k=a}^b f(k) = 0$ if $a > b$)

1.

$$\begin{aligned} \langle T_{\frac{1}{x^n}}, \varphi \rangle &= \lim_{\varepsilon \rightarrow 0^+} \int_{x > \varepsilon} \frac{\varphi(x) - (-1)^{n-1} \varphi(-x)}{x^n} dx \\ &\quad - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \frac{(k-1)!}{\varepsilon^k} \left(\varphi^{(n-1-k)}(\varepsilon) - (-1)^k \varphi^{(n-1-k)}(-\varepsilon) \right) \end{aligned} \quad (4.36)$$

2.

$$\begin{aligned} \langle T_{\frac{\text{sgn}(x)}{x^n}}, \varphi \rangle &= \lim_{\varepsilon \rightarrow 0^+} \int_{x > \varepsilon} \frac{\varphi(x) - (-1)^n \varphi(-x)}{x^n} dx \\ &\quad + \frac{1}{(n-1)!} \left(\ln \varepsilon \left(\varphi^{(n-1)}(\varepsilon) + \varphi^{(n-1)}(-\varepsilon) \right) - \sum_{k=2}^n \frac{(k-2)!}{\varepsilon^{k-1}} \left(\varphi^{(n-k)}(\varepsilon) + (-1)^{(k-1)} \varphi^{(n-k)}(-\varepsilon) \right) \right) \end{aligned} \quad (4.37)$$

Proof:

1. By induction (or direct calculation).
2. By induction.

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5 Micellaneous

Lemma 5.1 Let f be a smooth function and $j \in \mathbb{N}^*$ such that

$$f(0) = f'(0) = \dots = f^{(j-1)}(0) = 0$$

then the function

$$g : x \mapsto \frac{f(x)}{x^j}$$

is also smooth.

Proof:

The Taylor's Theorem with integral remainder gives

$$f(x) = \frac{x^j}{j!} \int_0^1 (1-t)^j f^{(j)}(tx) dt$$

so

$$g(x) = \frac{1}{j!} \int_0^1 (1-t)^j f^{(j)}(tx) dt$$

and the Leibniz integral rule, allows us to conclude the smoothness of g .

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Références

- [1] Loukas Grafakos and Gerald Teschl. On fourier transforms of radial functions and distributions. *Journal of Fourier Analysis and Applications*, 19(1) :167–179, 2013.