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Abstract

The aim of this work is twofold: proving the existence of solution $(\boldsymbol{u}, \pi) \in \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$ in bounded domains of $\mathbb{R}^2$ and the whole plane for the Oseen problem (O) for solenoidal vector fields $\boldsymbol{v}$ in $\mathbf{L}^2(\Omega)$, and analyzing the same problem in bounded domains of $\mathbb{R}^n$ for $n = 2, 3$ when $h = 0$, $\boldsymbol{g} = \mathbf{0}$ and the solenoidal field $\boldsymbol{v}$ belongs to $\mathbf{L}^s(\Omega)$ for $s < n$.

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1. Introduction

This work is dedicated to the study of some existence aspect related to the Oseen problem in bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$:

$$(\mathrm{O}) - \Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = \mathrm{h} \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma.$$

In the 3-dimensional case, the existence of weak solutions $(\boldsymbol{u}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$, regular solution in $\mathbf{H}^2(\Omega) \times H^1(\Omega)$ and $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ (and intermediate Sobolev spaces) together with the analysis of the existence of very weak solutions in $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$ have been analyzed by the authors in [1], assuming $\boldsymbol{v}$ a solenoidal field belonging to $\mathbf{L}^s(\Omega)$ for $s \geq 3$ (from now on, we will denote this solenoidal space by $\mathbf{L}^s_\sigma(\Omega)$). However, the existence of solution for the 2-dimensional Oseen system has not been attacked in [1] because the "logical" assumption of considering the solenoidal field $\boldsymbol{v} \in \mathbf{L}^2(\Omega)$ (in order to obtain weak solutions for (O)) poses some difficulties in the treatment of the convective term $(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}$: On the one hand, it is not clear if the bilinear form associated is coercive and continuous. Some related results can be found in [2] for the scalar case (instead of considering a vector field solution $\boldsymbol{u}$, one considers a scalar unknown θ) and for $\boldsymbol{g} = \mathbf{0}$. On the other hand, when $\Omega = \mathbb{R}^2$ an additional awkwardness appears because

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even if we can prove $\nabla \boldsymbol{u} \in \mathbf{L}^2(\mathbb{R}^2)$, it is not evident that $\boldsymbol{u} \in \mathbf{L}^p(\mathbb{R}^2)$ (for any p). Giving a successful answer to both previous problems is our first aim.

Our second aim is to give a first answer to the question of the existence of solution for the Oseen problem (O) when $\boldsymbol{v}$ only belongs to $\mathbf{L}^s_\sigma(\Omega)$, with $s < n$ and $n = 2, 3$.

2. Solutions for the Oseen problem in the 2-dimensional case

The existence of weak solutions in $\mathbf{H}^1(\Omega)$ for Problem (O) in 2-dimensional domains is not known when a solenoidal field $\boldsymbol{v}$ that only belongs to $\mathbf{L}^2(\Omega)$ is considered. In this case, the term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}$ belongs only to $\mathbf{L}^1(\Omega)$. It is then not clear neither if the bilinear form associated to the Problem (O), with $h = 0$ and $\boldsymbol{g} = \mathbf{0}$:

$$a(\boldsymbol{u}, \boldsymbol{w}) = \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{w}\, d\boldsymbol{x} + \int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{w}\, d\boldsymbol{x}$$

is coercive on the space $\mathbf{V}(\Omega) = \{\boldsymbol{w} \in \mathbf{H}^1_0(\Omega);\, \operatorname{div} \boldsymbol{w} = 0 \,\text{in}\, \Omega\}$ nor if it is continuous on $\mathbf{V}(\Omega)\times\mathbf{V}(\Omega)$. In order to overcome this difficulty, we use the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$. One equivalent definition of such a space (in the n-dimensional case) is ([3]):

$$\mathcal{H}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n), R_j f \in L^1(\mathbb{R}^n),\, 1 \le j \le n\} \quad \text{where } R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}.$$

A partial study of the BMO spaces (Bounded Mean Oscillation) will be also necessary taking into account the duality between $\mathcal{H}^1$ and the BMO (see [4]). Moreover, the VMO-space (Vanishing Mean Oscillator) is a subspace of the BMO: a function f in $BMO(\boldsymbol{R}^n)$ is said to be in $VMO(\boldsymbol{R}^n)$ if

$$\lim_{r\to 0} \sup_{\boldsymbol{x}_0\in\mathbb{R}^n} \frac{1}{r^n} \int\limits_{B(\boldsymbol{x}_0,r)} |f - \overline{f}\,|\, \mathrm{d}\boldsymbol{x} = 0, \quad \text{where } \overline{f} = \frac{1}{|B(\boldsymbol{x}_0, r)|} \int\limits_{B(\boldsymbol{x}_0,r)} f.$$

It is also crucial the fact that $H^1(\mathbb{R}^2) \hookrightarrow VMO(\mathbb{R}^2)$ (see [5]).

With these ingredients, we will prove one of the two main results of this work, namely Theorem 2.2 in bounded domains and Theorem 2.5 if $\Omega = \mathbb{R}^2$. In order to prove them, we use the following result:

Lemma 2.1. *Assume $\boldsymbol{v} \in \mathbf{L}^2_\sigma(\Omega)$ and $y \in H^1_0(\Omega)$. Then $(\boldsymbol{v}\cdot\nabla)y \in H^{-1}(\Omega)$ and*

$$\|(\boldsymbol{v}\cdot\nabla)y\|_{H^{-1}(\Omega)} \le C\, \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \|\nabla y\|_{\mathbf{L}^2(\Omega)}. \tag{1}$$

Moreover, we have that

$$\langle \boldsymbol{v}\cdot\nabla \boldsymbol{z}, \boldsymbol{z}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}^1_0(\Omega)} = 0 \quad \textit{for all } \boldsymbol{z} \in \mathbf{H}^1_0(\Omega). \tag{2}$$

Proof. Indeed, considering $\boldsymbol{w} \in \mathbf{L}^2(\mathbb{R}^2)$ the extension of $\boldsymbol{v}$ to $\mathbb{R}^2$ given by: $\boldsymbol{w} = \boldsymbol{v}$ in Ω, and $\boldsymbol{w} = \nabla\theta$ in $\Omega' = \mathbb{R}^2\backslash\overline{\Omega}$ where θ is the solution of the following problem:

$$\Delta\theta = 0 \ \text{ in } \Omega', \quad \frac{\partial\theta}{\partial \boldsymbol{n}} = -\boldsymbol{v}\cdot\boldsymbol{n} \ \text{ on } \Gamma,$$

with $\nabla\theta \in \mathbf{L}^2(\Omega')$ that satisfies

$$\|\nabla\theta\|_{\mathbf{L}^2(\Omega')} \leq C\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{H^{-1/2}(\Gamma)} \leq C\,\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}$$

because $\nabla\cdot\boldsymbol{v} = 0$ in Ω. Moreover, $\nabla\cdot\boldsymbol{v} = 0$ in Ω implies $\langle\boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_\Gamma = 0$ and the existence of θ is ensured by Theorem 3.1 [6]. Observe that $\nabla\cdot\boldsymbol{w} = 0$ in $\mathbb{R}^2$ because for $\varphi \in \mathcal{D}(\mathbb{R}^2)$:

$$\langle\nabla\cdot\boldsymbol{w}, \varphi\rangle = -\int_\Omega \boldsymbol{v}\cdot\nabla\varphi\,d\boldsymbol{x} - \int_{\Omega'}\nabla\theta\cdot\nabla\varphi\,d\boldsymbol{x} = \langle\boldsymbol{v}\cdot\boldsymbol{n},\varphi\rangle_\Gamma - \langle\boldsymbol{v}\cdot\boldsymbol{n},\varphi\rangle_\Gamma = 0,$$

and $\|\boldsymbol{w}\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq C\,\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}$. On the other hand, we consider $\widetilde{y}$ the extension by zero of y that satisfies $\widetilde{y} \in H^1(\mathbb{R}^2)$. Using Theorem II.2 point 2) or Theorem II.1 point 2) in [3], we can deduce that $\boldsymbol{w}\cdot\nabla\widetilde{y} \in \mathcal{H}^1(\mathbb{R}^2)$ and the bound

$$\|\boldsymbol{w}\cdot\nabla\widetilde{y}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C\|\boldsymbol{w}\|_{\mathbf{L}^2(\mathbb{R}^2)}\|\nabla\widetilde{y}\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq C\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\|\nabla y\|_{\mathbf{L}^2(\Omega)}.$$

Now, we have to prove that $\boldsymbol{v}\cdot\nabla y \in H^{-1}(\Omega)$ and $\langle\boldsymbol{v}\cdot\nabla y, y\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0$. Indeed, for $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned}\left|\int_\Omega \varphi\boldsymbol{v}\cdot\nabla y\,d\boldsymbol{x}\right| = \left|\int_{\mathbb{R}^2}\widetilde{\varphi}\boldsymbol{w}\cdot\nabla\widetilde{y}\,d\boldsymbol{x}\right| &\leq \|\boldsymbol{w}\cdot\nabla\widetilde{y}\|_{\mathcal{H}^1(\mathbb{R}^2)}\|\widetilde{\varphi}\|_{BMO(\mathbb{R}^2)}\\ &\leq C\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\|\nabla y\|_{\mathbf{L}^2(\Omega)}\|\widetilde{\varphi}\|_{H^1(\mathbb{R}^2)}\\ &\leq C\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\|\nabla y\|_{\mathbf{L}^2(\Omega)}\|\varphi\|_{H^1(\Omega)}\end{aligned}$$

because $H^1(\mathbb{R}^2) \hookrightarrow VMO(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2)$. In that way, as $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, we can deduce that $\boldsymbol{v}\cdot\nabla y \in H^{-1}(\Omega)$ and estimate (1).

For the proof of (2), let us consider $\boldsymbol{z}_k \in \boldsymbol{\mathcal{D}}(\Omega)$ be such that $\boldsymbol{z}_k \to \boldsymbol{z}$ in $\mathbf{H}_0^1(\Omega)$. Then,

$$\begin{aligned}&|\langle\boldsymbol{v}\cdot\nabla\boldsymbol{z},\boldsymbol{z}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} - \langle\boldsymbol{v}\cdot\nabla\boldsymbol{z}_k,\boldsymbol{z}_k\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)}|\\ &\quad\leq |\langle\boldsymbol{v}\cdot\nabla(\boldsymbol{z}-\boldsymbol{z}_k),\boldsymbol{z}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)}| + |\langle\boldsymbol{v}\cdot\nabla\boldsymbol{z}_k,(\boldsymbol{z}_k - z)\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)}|\end{aligned}$$

Using (1) and the convergence of $\boldsymbol{z}_k$ to $\boldsymbol{z}$ in $\mathbf{H}_0^1(\Omega)$, both duality terms on the right-hand-side of the previous inequality tend to 0 when $k \to +\infty$.

Finally, from $\langle\boldsymbol{v}\cdot\nabla\boldsymbol{z}_k,\boldsymbol{z}_k\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)} = 0$, we can deduce (2). □

Theorem 2.2 (Existence of weak solution for (O)**).** *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^2$. Let*

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega), \quad \boldsymbol{v} \in \mathbf{L}^2_\sigma(\Omega), \quad h \in L^2(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{H}^{1/2}(\Gamma)$$

satisfy the compatibility condition

$$\int_\Omega h(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\partial\Omega}\boldsymbol{g}\cdot\boldsymbol{n}\,d\sigma. \tag{3}$$

Then, the problem (O) *has a unique solution $(\boldsymbol{u},\pi) \in \mathbf{H}^1(\Omega)\times L^2(\Omega)/\mathbb{R}$. Moreover, there exist some constants $C_1 > 0$ and $C_2 > 0$ such that:*

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leq C_1\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\big)\big(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\big)\Big), \tag{4}$$

$$\|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C_2\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \big(1 + \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\big)\big(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\big)\Big), \tag{5}$$

where $C_1 = C(\Omega)$ and $C_2 = C_1\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\right)$.

PROOF. Although some parts of this proof are identical to the proof made in [1], we include the whole argument here for completeness. In order to prove the existence of solution, first (using Lemma 3.3 in [7], for instance) we lift the boundary and the divergence data. Then, there exists $\boldsymbol{u}_0 \in \mathbf{H}^1(\Omega)$ such that $\nabla \cdot \boldsymbol{u}_0 = h$ in Ω, $\boldsymbol{u}_0 = \boldsymbol{g}$ on Γ and:

$$\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} \leq C\, \left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right). \tag{6}$$

Therefore, it remains to find $(\boldsymbol{z}, \pi) = (\boldsymbol{u} - \boldsymbol{u}_0, \pi)$ in $\mathbf{H}^1_0(\Omega) \times L^2(\Omega)$ such that:

$$-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \pi = \mathbf{F} \quad \text{and} \quad \nabla \cdot \boldsymbol{z} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{z} = \mathbf{0} \quad \text{on } \Gamma. \tag{7}$$

being $\boldsymbol{F} = \boldsymbol{f} + \Delta \boldsymbol{u}_0 - (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}_0$. From Lemma 2.1, we deduce that $(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}_0 \in \mathbf{H}^{-1}(\Omega)$, then $\boldsymbol{F} \in \mathbf{H}^{-1}(\Omega)$. Since the space $\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \nabla \cdot \boldsymbol{\varphi} = 0\}$ is dense in the space $\mathbf{V}(\Omega)$, the previous problem is equivalent to:

$$\begin{aligned}&\text{Find } \boldsymbol{z} \in \mathbf{V}(\Omega) \text{ such that:} \quad \forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega)\\ &\int_\Omega \nabla \boldsymbol{z} \cdot \nabla \boldsymbol{\varphi}\, d\boldsymbol{x} + \langle (\boldsymbol{v} \cdot \nabla)\boldsymbol{z}, \boldsymbol{\varphi}\rangle_{\mathbf{H}^{-1}(\Omega)\times \mathbf{H}^1_0(\Omega)} = \langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\mathbf{H}^{-1}(\Omega)\times \mathbf{H}^1_0(\Omega)}.\end{aligned}$$

Now, using (2) by Lax-Milgram's Theorem, if we assume that $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$, then we can deduce the existence of a unique $\boldsymbol{z} \in \mathbf{H}^1_0(\Omega)$ solution of (7) verifying:

$$\begin{aligned}\|\boldsymbol{z}\|_{\mathbf{H}^1(\Omega)} &\leq C\, \|\boldsymbol{F}\|_{\mathbf{H}^{-1}(\Omega)}\\ &\leq C\Big(\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\right)\left(\|h\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{H}^{1/2}(\Gamma)}\right)\Big),\end{aligned} \tag{8}$$

which added to estimate (6) makes (4). We can recover the pressure π thanks to the De Rham's Lemma (see Lemma 6 in [1] and Corollary III.5.1 in [8]). Now, $-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} - \boldsymbol{F} \in \mathbf{H}^{-1}(\Omega)$ and:

$$\forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega), \qquad \langle -\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} - \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\mathbf{H}^{-1}(\Omega)\times \mathbf{H}^1_0(\Omega)} = 0.$$

Thanks to De Rham's Lemma, there exists a unique $\pi \in L^2(\Omega)/\mathbb{R}$ such that

$$-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \pi = \mathbf{F}$$

with $\|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C\, \|\nabla \pi\|_{\mathbf{H}^{-1}(\Omega)}$. Finally, estimate (5) follows from the previous equation and estimate (8) for $\boldsymbol{z}$. □

With the same procedure than in [1], we can prove strong and weak-$W^{1,p}(\Omega)$ regularity for (O) in the 2-dimensional bounded case. These results can be stated as follows:

Theorem 2.3 (Existence of strong solution for (O)). *Let $p > 1$,*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad h \in W^{1,p}(\Omega), \quad \boldsymbol{v} \in \mathbf{L}^s_\sigma(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$$

satisfying the compatibility condition (3) with $s = 2$ if $p < 2$, $s = p$ if $p > 2$ and $s = 2 + \varepsilon$ ($\varepsilon > 0$) if $p = 2$. Then, the unique solution of (O) given by Theorem 2.2 $(\boldsymbol{u}, \pi)$ belongs to $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$, and there exists a constant $C > 0$ such that:

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} \;+\;& \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\\ \times\;& \left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^s(\Omega)}\right)\left(\|h\|_{W^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}\right)\right)\end{aligned}$$

Theorem 2.4. *Let*

$$p > 1, \quad \boldsymbol{f} \in \mathbf{W}^{-1,p}(\Omega), \quad h \in L^p(\Omega), \quad \boldsymbol{v} \in \mathbf{L}^3_\sigma(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$$

satisfying the compatibility condition (3). Then, the problem (*O*) *has a unique solution* $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$*, and there exists a constant* $C > 0$ *such that:*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} &+ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)^\gamma \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C\left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right) \\ &\times \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^3(\Omega)}\right)\left(\|h\|_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\right)\right) \end{aligned}$$

with $\gamma = 0$ *if* $p \geq 2$ *and* $\gamma = -1$ *if* $p < 2$.

If we treat the case of $\Omega = \mathbb{R}^2$, we have to introduce the Sobolev spaces:

$$W_0^{1,2}(\mathbb{R}^2) = \left\{\varphi \in \mathcal{D}'(\mathbb{R}^2);\ \frac{\varphi}{w_1} \in L^2(\mathbb{R}^2),\ \nabla\varphi \in L^2(\mathbb{R}^2)\right\},$$

$$W_0^{2,2}(\mathbb{R}^2) = \left\{\varphi \in \mathcal{D}'(\mathbb{R}^2);\ \frac{\varphi}{w_2} \in L^2(\mathbb{R}^2),\ \frac{\nabla\varphi}{w_1} \in L^2(\mathbb{R}^2,\ \nabla^2\varphi \in L^2(\mathbb{R}^2)\right\},$$

where $w_1 = (1+|\boldsymbol{x}|)\ \ln(2+|\boldsymbol{x}|)$ and $w_2 = (1+|\boldsymbol{x}|)^2\ \ln(2+|\boldsymbol{x}|)$ (see Definition (7.1),p. 593 in [9]). We denote by $W_0^{-1,2}(\mathbb{R}^2)$ the dual space of $W_0^{1,2}(\mathbb{R}^2)$. Recall ([5]) that the space $W_0^{1,2}(\mathbb{R}^2)$ is densely embedded in $VMO(\mathbb{R}^2)$, and therefore $\mathcal{H}^1(\mathbb{R}^2) = \left[VMO(\mathbb{R}^2)\right]' \hookrightarrow W_0^{-1,2}(\mathbb{R}^2)$.

Theorem 2.5 (Case $\Omega = \mathbb{R}^2$). *i) Let*

$$\boldsymbol{f} = \operatorname{div}\mathbb{F} \quad \text{with} \quad \mathbb{F} \in \mathbf{L}^2(\mathbb{R}^2) \quad \text{and} \quad \mathrm{h} \in \mathrm{L}^2(\mathbb{R}^2).$$

Then, the problem (O) *has a unique solution* $(\boldsymbol{u}, \pi)$ *satisfying* $\boldsymbol{u} \in \mathbf{W}_0^{1,2}(\mathbb{R}^2)$ *and* $\pi \in L^2(\mathbb{R}^2)$*, where* π *is unique and* $\boldsymbol{u}$ *is unique up to an additive constant vector field.*
ii) Moreover, if

$$\boldsymbol{f} \in \mathcal{H}^1(\mathbb{R}^2) \quad \text{and} \quad \nabla h \in \mathcal{H}^1(\mathbb{R}^2),$$

then

$$\nabla^2\boldsymbol{u} \in \mathcal{H}^1(\mathbb{R}^2), \quad \nabla\pi \in \mathcal{H}^1(\mathbb{R}^2), \quad \nabla\boldsymbol{u} \in \boldsymbol{L}^{2,1}(\mathbb{R}^2) \quad \textit{and} \quad \boldsymbol{u} \in L^\infty(\mathbb{R}^2), \tag{9}$$

being $L^{2,1}(\mathbb{R}^2)$ *is the Lorentz space of all measurable functions* f *satisfying*

$$\int_0^\infty t^{-1/2} f^*(t)\, dt < +\infty,$$

where the rearrangement function f^* *is defined by* $f^*(t) = \sup\{s \in (0,\infty);\ \mu(\{\boldsymbol{x} \in \mathbb{R}^2;\ |f(\boldsymbol{x})| > s\}) > t\}$*, for* μ *the Lebesgue measure on* $\mathbb{R}^2$.

Proof. *i) Existence:* Let $\chi \in W_0^{2,2}(\mathbb{R}^2)$ be the unique solution, up to a polynomial function of degree one, of $\Delta\chi = h$ in $\mathbb{R}^2$ (see Theorem 9.6 in [9]). Then, we take $\boldsymbol{u}_h = \nabla\chi \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^2)$. Problem (O) is then written as:

$$-\Delta\boldsymbol{z} + \boldsymbol{v}\cdot\nabla\boldsymbol{z} + \nabla\pi = \mathbf{k}, \quad \nabla\cdot\boldsymbol{z} = 0 \quad \text{in } \mathbb{R}^2,$$

with $\mathbf{k} = \boldsymbol{f} + \Delta \boldsymbol{u}_h - \boldsymbol{v} \cdot \nabla \boldsymbol{u}_h$. Because of $(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}_h \in \mathcal{H}^1(\mathbb{R}^2) \hookrightarrow W_0^{-1,2}(\mathbb{R}^2)$, by using Lax-Milgram's Lemma (as in the bounded case) we can deduce the existence of a solution $\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^2)$ with $\nabla \cdot \boldsymbol{z} = 0$, unique up to a constant vector of $\mathbb{R}^2$. Lax-Milgram's Lemma hypotheses are satisfied because, on the one hand, we know that the quotient norm $\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^2)/\mathbb{R}^2}$ is equivalent to that one defined as $\|\nabla \boldsymbol{z}\|_{\boldsymbol{L}^2(\mathbb{R}^2)}$, and, on the other hand, $(\boldsymbol{v} \cdot \nabla)\boldsymbol{z} \in \mathcal{H}^1(\mathbb{R}^2)$ for any $\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^2)$. The pressure can be recovered by using Theorem 1 in [10].

ii) Regularity: Assume that $\boldsymbol{f} \in \mathcal{H}^1(\mathbb{R}^2)$ and $\nabla h \in \mathcal{H}^1(\mathbb{R}^2)$ (which, in particular, implies that $h \in L^{2,1}(\mathbb{R}^2)$). Therefore,

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} - \boldsymbol{v} \cdot \nabla \boldsymbol{u} \in \mathcal{H}^1(\mathbb{R}^2) \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = h.$$

By using Theorem 3.14 in [10], one deduces (9).

3. The Oseen problem in bounded domains for a less regular $\boldsymbol{v}$

The aim of this section is the analysis of the existence of solutions of (O) in a bounded domain ($n = 2$ or 3) when $\boldsymbol{v} \in \mathbf{L}^s_\sigma(\Omega)$ for $s < n$. We analyze the case for $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$, $h = 0$ and $\boldsymbol{g} = \mathbf{0}$. Observe that the term $(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}$ can also be written as $\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{v})$. The proof of Theorem 3.2 ($n = 2$) applies directly from that one of Theorem 3.1 ($n = 3$).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ a Lipschitz bounded domain,*

$$\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega), \quad h = 0, \quad \boldsymbol{g} = \mathbf{0} \quad \text{and} \quad \boldsymbol{v} \in \boldsymbol{L}^{6/5+\alpha}_\sigma(\Omega)$$

for any $0 < \alpha \leq 9/5$. Then, there exists a solution of (O) *such that $(\boldsymbol{u}, \pi) \in \mathbf{H}^1_0(\Omega) \times L^{q(\alpha)}(\Omega)/\mathbb{R}$ for $q(\alpha) = (6(6+5\alpha))/(36+5\alpha)$ with the estimate:*

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^{q(\alpha)}(\Omega)/\mathbb{R}} \leq C\ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^{6/5+\alpha}(\Omega)}\right)\ \|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} \tag{10}$$

Proof. We approximate $\boldsymbol{v}$ by $\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ in the $\mathbf{L}^{6/5+\alpha}(\Omega)$-norm and look for the solution of the problem:

$$(\mathrm{O}_\lambda) \qquad -\Delta \boldsymbol{u}_\lambda + \nabla \cdot (\boldsymbol{u}_\lambda \otimes \boldsymbol{v}_\lambda) + \nabla \pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u}_\lambda = 0 \quad \text{in } \Omega, \quad \boldsymbol{u}_\lambda = \mathbf{0} \quad \text{on } \Gamma$$

Taking $\boldsymbol{u}_\lambda$ as test function in (O_λ), we get the estimate:

$$\|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1_0(\Omega)} \leq C(\Omega)\,\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)}. \tag{11}$$

By De Rham Theorem, there exists $\pi_\lambda \in L^2(\Omega)$ (unique up to a constant) such that:

$$\nabla \pi_\lambda = \boldsymbol{f} + \Delta \boldsymbol{u}_\lambda - \nabla \cdot (\boldsymbol{u}_\lambda \otimes \boldsymbol{v}_\lambda).$$

Moreover, $\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda$ belongs to a bounded set of $\mathbb{L}^{q(\alpha)}(\Omega)$ with $q(\alpha) = (6(6+5\alpha))/(36+5\alpha)$ and which implies that $\nabla \cdot (\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda)$ belongs to a bounded set of $\mathbf{W}^{-1,q(\alpha)}(\Omega)$. Note that if $0 < \alpha \leq 9/5$ then $1 < q(\alpha) \leq 2$. Using (11),

$$\begin{aligned}
\|\nabla \pi_\lambda\|_{W^{-1,q(\alpha)}(\Omega)} &\leq C_1\,(1 + C(\Omega))\,\|f\|_{\mathbf{H}^{-1}(\Omega)} + \|\boldsymbol{u}_\lambda \otimes \boldsymbol{v}_\lambda\|_{\mathbb{L}^{q(\alpha)}(\Omega)} \\
&\leq C_1\,(1 + C(\Omega))\,\|f\|_{\mathbf{H}^{-1}(\Omega)} + C_2\,\|\boldsymbol{v}_\lambda\|_{\mathbf{L}^{6/5+\alpha}(\Omega)}\|\boldsymbol{u}_\lambda\|_{\mathbf{H}^1_0(\Omega)} \\
&\leq C(\Omega)\ \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^{6/5+\alpha}(\Omega)}\right)\ \|f\|_{\mathbf{H}^{-1}(\Omega)}
\end{aligned} \tag{12}$$

where C_1 and C_2 are the constant of the Sobolev embeddings $\mathbf{H}^{-1}(\Omega) \hookrightarrow \mathbf{W}^{-1,q(\alpha)}(\Omega)$ and $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$, respectively. Therefore, from (12) we obtain:

$$\inf_{K\in\mathbb{R}} \|\pi_\lambda + K\|_{L^{q(\alpha)}(\Omega)} \leq C(\Omega) \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^{6/5+\alpha}(\Omega)}\right) \|f\|_{\mathbf{H}^{-1}(\Omega)}$$

Now, it is necessary to take the limit when $\lambda \to 0$: We can extract a subsequence of $(\boldsymbol{u}_\lambda)$ and $(\pi_\lambda + C_\lambda)$ (that will be called in the same way that the original one) such that:

$$\boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{u} \quad \text{in } \mathbf{H}_0^1(\Omega), \qquad \pi_\lambda + C_\lambda \rightharpoonup \pi \quad \text{in } L^{q(\alpha)}(\Omega),$$

where $(\boldsymbol{u}, \pi)$ is solution of (O) and satisfies (10). □

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^2$ a Lipschitz bounded domain, $\boldsymbol{f} \in \mathbf{H}^{-1}(\Omega)$, $h = 0$, $\boldsymbol{g} = \mathbf{0}$ and $\boldsymbol{v} \in \mathbf{L}_\sigma^{1+\alpha}(\Omega)$ with $0 < \alpha \leq 1$. Then, there exists a solution of* (O) *such that $(\boldsymbol{u}, \pi) \in \mathbf{H}_0^1(\Omega) \times L^{q(\beta)}(\Omega)/\mathbb{R}$ for $q(\beta) = 1 + \beta$, for any $0 < \beta < \alpha$, with the estimate:*

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^{q(\beta)}(\Omega)/\mathbb{R}} \leq C \left(1 + \|\boldsymbol{v}\|_{\mathbf{L}^{1+\alpha}(\Omega)}\right) \|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)}$$

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References

[1] C. Amrouche, M. Á. Rodríguez-Bellido, Stationary Stokes, Oseen and Navier-Stokes equations with singular data, Arch. Ration. Mech. Anal. 199 (2) (2011) 597–651. doi:10.1007/s00205-010-0340-8. URL http://0-dx.doi.org.fama.us.es/10.1007/s00205-010-0340-8

[2] H. Kim, Y.-H. Kim, On weak solutions of elliptic equations with singular drifts, SIAM J. Math. Anal. 47 (2) (2015) 1271–1290. doi:10.1137/14096270X. URL https://doi.org/10.1137/14096270X

[3] R. Coifman, P.-L. Lions, Y. Meyer, S. Semmes, Compensated compactness and Hardy spaces, J. Math. Pures Appl. (9) 72 (3) (1993) 247–286.

[4] C. Fefferman, E. M. Stein, H^p spaces of several variables, Acta Math. 129 (3-4) (1972) 137–193. doi:10.1007/BF02392215. URL https://doi.org/10.1007/BF02392215

[5] H. Brezis, L. Nirenberg, Degree theory and BMO. II. Compact manifolds with boundaries, Selecta Math. (N.S.) 2 (3) (1996) 309–368, with an appendix by the authors and Petru Mironescu. doi:10.1007/BF01587948. URL https://doi.org/10.1007/BF01587948

[6] C. Amrouche, V. Girault, J. Giroire, Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces, J. Math. Pures Appl. (9) 76 (1) (1997) 55–81. doi:10.1016/S0021-7824(97)89945-X. URL https://doi.org/10.1016/S0021-7824(97)89945-X

[7] C. Amrouche, V. Girault, Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension, Czechoslovak Math. J. 44(119) (1) (1994) 109–140.

[8] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, 2nd Edition, Springer Monographs in Mathematics, Springer, New York, 2011, steady-state problems. doi:10.1007/978-0-387-09620-9. URL https://doi.org/10.1007/978-0-387-09620-9

[9] C. Amrouche, V. Girault, J. Giroire, Weighted Sobolev spaces for Laplace's equation in $\mathbf{R}^n$, J. Math. Pures Appl. (9) 73 (6) (1994) 579–606.

[10] F. Alliot, C. Amrouche, The Stokes problem in $\mathbf{R}^n$: an approach in weighted Sobolev spaces, Math. Models Methods Appl. Sci. 9 (5) (1999) 723–754. doi:10.1142/S0218202599000361. URL https://doi.org/10.1142/S0218202599000361