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# Weak solutions for the Oseen system in 2D and when the given velocity is not sufficiently regular

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## Abstract

The aim of this work is twofold: proving the existence of solution  $(\mathbf{u}, \pi) \in \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$  in bounded domains of  $\mathbb{R}^2$  and the whole plane for the Oseen problem (O) for solenoidal vector fields  $\mathbf{v}$  in  $\mathbf{L}^2(\Omega)$ , and analyzing the same problem in bounded domains of  $\mathbb{R}^n$  for  $n = 2, 3$  when  $h = 0$ ,  $\mathbf{g} = \mathbf{0}$  and the solenoidal field  $\mathbf{v}$  belongs to  $\mathbf{L}^s(\Omega)$  for  $s < n$ .

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## 1. Introduction

This work is dedicated to the study of some existence aspect related to the Oseen problem in bounded domain  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$ :

$$(O) - \Delta \mathbf{u} + \mathbf{v} \cdot \nabla \mathbf{u} + \nabla \pi = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = h \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma.$$

In the 3-dimensional case, the existence of weak solutions  $(\mathbf{u}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ , regular solution in  $\mathbf{H}^2(\Omega) \times H^1(\Omega)$  and  $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$  (and intermediate Sobolev spaces) together with the analysis of the existence of very weak solutions in  $\mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$  have been analyzed by the authors in [1], assuming  $\mathbf{v}$  a solenoidal field belonging to  $\mathbf{L}^s(\Omega)$  for  $s \geq 3$  (from now on, we will denote this solenoidal space by  $\mathbf{L}_\sigma^s(\Omega)$ ). However, the existence of solution for the 2-dimensional Oseen system has not been attacked in [1] because the “logical” assumption of considering the solenoidal field  $\mathbf{v} \in \mathbf{L}^2(\Omega)$  (in order to obtain weak solutions for (O)) poses some difficulties in the treatment of the convective term  $(\mathbf{v} \cdot \nabla) \mathbf{u}$ : On the one hand, it is not clear if the bilinear form associated is coercive and continuous. Some related results can be found in [2] for the scalar case (instead of considering a vector field solution  $\mathbf{u}$ , one considers a scalar unknown  $\theta$ ) and for  $\mathbf{g} = \mathbf{0}$ . On the other hand, when  $\Omega = \mathbb{R}^2$  an additional awkwardness appears because

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even if we can prove  $\nabla \mathbf{u} \in \mathbf{L}^2(\mathbb{R}^2)$ , it is not evident that  $\mathbf{u} \in \mathbf{L}^p(\mathbb{R}^2)$  (for any  $p$ ). Giving  
 15 a successful answer to both previous problems is our first aim.

Our second aim is to give a first answer to the question of the existence of solution for the Oseen problem (O) when  $\mathbf{v}$  only belongs to  $\mathbf{L}_\sigma^s(\Omega)$ , with  $s < n$  and  $n = 2, 3$ .

## 2. Solutions for the Oseen problem in the 2-dimensional case

The existence of weak solutions in  $\mathbf{H}^1(\Omega)$  for Problem (O) in 2-dimensional domains is not known when a solenoidal field  $\mathbf{v}$  that only belongs to  $\mathbf{L}^2(\Omega)$  is considered. In this case, the term  $(\mathbf{v} \cdot \nabla)\mathbf{u}$  belongs only to  $\mathbf{L}^1(\Omega)$ . It is then not clear neither if the bilinear form associated to the Problem (O), with  $h = 0$  and  $\mathbf{g} = \mathbf{0}$  :

$$a(\mathbf{u}, \mathbf{w}) = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{w} \, dx + \int_{\Omega} (\mathbf{v} \cdot \nabla)\mathbf{u} \cdot \mathbf{w} \, dx$$

is coercive on the space  $\mathbf{V}(\Omega) = \{\mathbf{w} \in \mathbf{H}_0^1(\Omega); \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega\}$  nor if it is continuous on  $\mathbf{V}(\Omega) \times \mathbf{V}(\Omega)$ . In order to overcome this difficulty, we use the Hardy space  $\mathcal{H}^1(\mathbb{R}^2)$ . One equivalent definition of such a space (in the  $n$ -dimensional case) is ([3]):

$$\mathcal{H}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n), R_j f \in L^1(\mathbb{R}^n), 1 \leq j \leq n\} \quad \text{where } R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-1/2}.$$

A partial study of the *BMO* spaces (Bounded Mean Oscillation) will be also necessary taking into account the duality between  $\mathcal{H}^1$  and the *BMO* (see [4]). Moreover, the *VMO*-space (Vanishing Mean Oscillator) is a subspace of the *BMO*: a function  $f$  in *BMO*( $\mathbf{R}^n$ ) is said to be in *VMO*( $\mathbf{R}^n$ ) if

$$\lim_{r \rightarrow 0} \sup_{\mathbf{x}_0 \in \mathbb{R}^n} \frac{1}{r^n} \int_{B(\mathbf{x}_0, r)} |f - \bar{f}| \, d\mathbf{x} = 0, \quad \text{where } \bar{f} = \frac{1}{|B(\mathbf{x}_0, r)|} \int_{B(\mathbf{x}_0, r)} f.$$

It is also crucial the fact that  $H^1(\mathbb{R}^2) \hookrightarrow VMO(\mathbb{R}^2)$  (see [5]).

20 With these ingredients, we will prove one of the two main results of this work, namely Theorem 2.2 in bounded domains and Theorem 2.5 if  $\Omega = \mathbb{R}^2$ . In order to prove them, we use the following result:

**Lemma 2.1.** *Assume  $\mathbf{v} \in \mathbf{L}_\sigma^2(\Omega)$  and  $y \in H_0^1(\Omega)$ . Then  $(\mathbf{v} \cdot \nabla)y \in H^{-1}(\Omega)$  and*

$$\|(\mathbf{v} \cdot \nabla)y\|_{H^{-1}(\Omega)} \leq C \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \|\nabla y\|_{\mathbf{L}^2(\Omega)}. \quad (1)$$

Moreover, we have that

$$\langle \mathbf{v} \cdot \nabla z, z \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)} = 0 \quad \text{for all } z \in \mathbf{H}_0^1(\Omega). \quad (2)$$

PROOF. Indeed, considering  $\mathbf{w} \in \mathbf{L}^2(\mathbb{R}^2)$  the extension of  $\mathbf{v}$  to  $\mathbb{R}^2$  given by:  $\mathbf{w} = \mathbf{v}$  in  $\Omega$ , and  $\mathbf{w} = \nabla \theta$  in  $\Omega' = \mathbb{R}^2 \setminus \bar{\Omega}$  where  $\theta$  is the solution of the following problem:

$$\Delta \theta = 0 \text{ in } \Omega', \quad \frac{\partial \theta}{\partial \mathbf{n}} = -\mathbf{v} \cdot \mathbf{n} \text{ on } \Gamma,$$

with  $\nabla\theta \in \mathbf{L}^2(\Omega')$  that satisfies

$$\|\nabla\theta\|_{\mathbf{L}^2(\Omega')} \leq C \|\mathbf{v} \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} \leq C \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}$$

because  $\nabla \cdot \mathbf{v} = 0$  in  $\Omega$ . Moreover,  $\nabla \cdot \mathbf{v} = 0$  in  $\Omega$  implies  $\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma} = 0$  and the existence of  $\theta$  is ensured by Theorem 3.1 [6]. Observe that  $\nabla \cdot \mathbf{w} = 0$  in  $\mathbb{R}^2$  because for  $\varphi \in \mathcal{D}(\mathbb{R}^2)$ :

$$\langle \nabla \cdot \mathbf{w}, \varphi \rangle = - \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, d\mathbf{x} - \int_{\Omega'} \nabla \theta \cdot \nabla \varphi \, d\mathbf{x} = \langle \mathbf{v} \cdot \mathbf{n}, \varphi \rangle_{\Gamma} - \langle \mathbf{v} \cdot \mathbf{n}, \varphi \rangle_{\Gamma} = 0,$$

and  $\|\mathbf{w}\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq C \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}$ . On the other hand, we consider  $\tilde{y}$  the extension by zero of  $y$  that satisfies  $\tilde{y} \in H^1(\mathbb{R}^2)$ . Using Theorem II.2 point 2) or Theorem II.1 point 2) in [3], we can deduce that  $\mathbf{w} \cdot \nabla \tilde{y} \in \mathcal{H}^1(\mathbb{R}^2)$  and the bound

$$\|\mathbf{w} \cdot \nabla \tilde{y}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C \|\mathbf{w}\|_{\mathbf{L}^2(\mathbb{R}^2)} \|\nabla \tilde{y}\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq C \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \|\nabla y\|_{\mathbf{L}^2(\Omega)}.$$

Now, we have to prove that  $\mathbf{v} \cdot \nabla y \in H^{-1}(\Omega)$  and  $\langle \mathbf{v} \cdot \nabla y, y \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = 0$ . Indeed, for  $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned} \left| \int_{\Omega} \varphi \mathbf{v} \cdot \nabla y \, d\mathbf{x} \right| &= \left| \int_{\mathbb{R}^2} \tilde{\varphi} \mathbf{w} \cdot \nabla \tilde{y} \, d\mathbf{x} \right| \leq \|\mathbf{w} \cdot \nabla \tilde{y}\|_{\mathcal{H}^1(\mathbb{R}^2)} \|\tilde{\varphi}\|_{BMO(\mathbb{R}^2)} \\ &\leq C \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \|\nabla y\|_{\mathbf{L}^2(\Omega)} \|\tilde{\varphi}\|_{H^1(\mathbb{R}^2)} \\ &\leq C \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \|\nabla y\|_{\mathbf{L}^2(\Omega)} \|\varphi\|_{H^1(\Omega)} \end{aligned}$$

because  $H^1(\mathbb{R}^2) \hookrightarrow VMO(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^2)$ . In that way, as  $\mathcal{D}(\Omega)$  is dense in  $H_0^1(\Omega)$ , we can deduce that  $\mathbf{v} \cdot \nabla y \in H^{-1}(\Omega)$  and estimate (1).

For the proof of (2), let us consider  $\mathbf{z}_k \in \mathcal{D}(\Omega)$  be such that  $\mathbf{z}_k \rightarrow \mathbf{z}$  in  $\mathbf{H}_0^1(\Omega)$ . Then,

$$\begin{aligned} &|\langle \mathbf{v} \cdot \nabla \mathbf{z}, \mathbf{z} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)} - \langle \mathbf{v} \cdot \nabla \mathbf{z}_k, \mathbf{z}_k \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)}| \\ &\leq |\langle \mathbf{v} \cdot \nabla (\mathbf{z} - \mathbf{z}_k), \mathbf{z} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)}| + |\langle \mathbf{v} \cdot \nabla \mathbf{z}_k, (\mathbf{z}_k - \mathbf{z}) \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)}| \end{aligned}$$

25 Using (1) and the convergence of  $\mathbf{z}_k$  to  $\mathbf{z}$  in  $\mathbf{H}_0^1(\Omega)$ , both duality terms on the right-hand-side of the previous inequality tend to 0 when  $k \rightarrow +\infty$ .

Finally, from  $\langle \mathbf{v} \cdot \nabla \mathbf{z}_k, \mathbf{z}_k \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)} = 0$ , we can deduce (2).  $\square$

**Theorem 2.2 (Existence of weak solution for (O)).** *Let  $\Omega$  be a Lipschitz bounded domain in  $\mathbb{R}^2$ . Let*

$$\mathbf{f} \in \mathbf{H}^{-1}(\Omega), \quad \mathbf{v} \in \mathbf{L}_\sigma^2(\Omega), \quad h \in L^2(\Omega) \quad \text{and} \quad \mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$$

*satisfy the compatibility condition*

$$\int_{\Omega} h(\mathbf{x}) \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, d\sigma. \quad (3)$$

*Then, the problem (O) has a unique solution  $(\mathbf{u}, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ . Moreover, there exist some constants  $C_1 > 0$  and  $C_2 > 0$  such that:*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq C_1 \left( \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + (1 + \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}) (\|h\|_{L^2(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{1/2}(\Gamma)}) \right), \quad (4)$$

$$\|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C_2 \left( \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + (1 + \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}) (\|h\|_{L^2(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{1/2}(\Gamma)}) \right), \quad (5)$$

30 where  $C_1 = C(\Omega)$  and  $C_2 = C_1 (1 + \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)})$ .

PROOF. Although some parts of this proof are identical to the proof made in [1], we include the whole argument here for completeness. In order to prove the existence of solution, first (using Lemma 3.3 in [7], for instance) we lift the boundary and the divergence data. Then, there exists  $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$  such that  $\nabla \cdot \mathbf{u}_0 = h$  in  $\Omega$ ,  $\mathbf{u}_0 = \mathbf{g}$  on  $\Gamma$  and:

$$\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \leq C (\|h\|_{L^2(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{1/2}(\Gamma)}). \quad (6)$$

Therefore, it remains to find  $(\mathbf{z}, \pi) = (\mathbf{u} - \mathbf{u}_0, \pi)$  in  $\mathbf{H}_0^1(\Omega) \times L^2(\Omega)$  such that:

$$-\Delta \mathbf{z} + \mathbf{v} \cdot \nabla \mathbf{z} + \nabla \pi = \mathbf{F} \quad \text{and} \quad \nabla \cdot \mathbf{z} = 0 \quad \text{in } \Omega, \quad \mathbf{z} = \mathbf{0} \quad \text{on } \Gamma. \quad (7)$$

being  $\mathbf{F} = \mathbf{f} + \Delta \mathbf{u}_0 - (\mathbf{v} \cdot \nabla) \mathbf{u}_0$ . From Lemma 2.1, we deduce that  $(\mathbf{v} \cdot \nabla) \mathbf{u}_0 \in \mathbf{H}^{-1}(\Omega)$ , then  $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ . Since the space  $\mathcal{D}_\sigma(\Omega) = \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega); \nabla \cdot \boldsymbol{\varphi} = 0\}$  is dense in the space  $\mathbf{V}(\Omega)$ , the previous problem is equivalent to:

$$\begin{aligned} &\text{Find } \mathbf{z} \in \mathbf{V}(\Omega) \text{ such that: } \forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega) \\ &\int_{\Omega} \nabla \mathbf{z} \cdot \nabla \boldsymbol{\varphi} \, d\mathbf{x} + \langle (\mathbf{v} \cdot \nabla) \mathbf{z}, \boldsymbol{\varphi} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)} = \langle \mathbf{F}, \boldsymbol{\varphi} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)}. \end{aligned}$$

Now, using (2) by Lax-Milgram's Theorem, if we assume that  $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$ , then we can deduce the existence of a unique  $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$  solution of (7) verifying:

$$\begin{aligned} \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} &\leq C \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)} \\ &\leq C \left( \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + (1 + \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}) (\|h\|_{L^2(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{1/2}(\Gamma)}) \right), \end{aligned} \quad (8)$$

which added to estimate (6) makes (4). We can recover the pressure  $\pi$  thanks to the De Rham's Lemma (see Lemma 6 in [1] and Corollary III.5.1 in [8]). Now,  $-\Delta \mathbf{z} + \mathbf{v} \cdot \nabla \mathbf{z} - \mathbf{F} \in \mathbf{H}^{-1}(\Omega)$  and:

$$\forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega), \quad \langle -\Delta \mathbf{z} + \mathbf{v} \cdot \nabla \mathbf{z} - \mathbf{F}, \boldsymbol{\varphi} \rangle_{\mathbf{H}^{-1}(\Omega) \times \mathbf{H}_0^1(\Omega)} = 0.$$

Thanks to De Rham's Lemma, there exists a unique  $\pi \in L^2(\Omega)/\mathbb{R}$  such that

$$-\Delta \mathbf{z} + \mathbf{v} \cdot \nabla \mathbf{z} + \nabla \pi = \mathbf{F}$$

with  $\|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq C \|\nabla \pi\|_{\mathbf{H}^{-1}(\Omega)}$ . Finally, estimate (5) follows from the previous equation and estimate (8) for  $\mathbf{z}$ .  $\square$

With the same procedure than in [1], we can prove strong and weak- $W^{1,p}(\Omega)$  regularity for (O) in the 2-dimensional bounded case. These results can be stated as follows:

**Theorem 2.3 (Existence of strong solution for (O)).** *Let  $p > 1$ ,*

$$\mathbf{f} \in \mathbf{L}^p(\Omega), \quad h \in W^{1,p}(\Omega), \quad \mathbf{v} \in \mathbf{L}_\sigma^s(\Omega) \quad \text{and} \quad \mathbf{g} \in \mathbf{W}^{2-1/p,p}(\Gamma)$$

*satisfying the compatibility condition (3) with  $s = 2$  if  $p < 2$ ,  $s = p$  if  $p > 2$  and  $s = 2 + \varepsilon$  ( $\varepsilon > 0$ ) if  $p = 2$ . Then, the unique solution of (O) given by Theorem 2.2  $(\mathbf{u}, \pi)$  belongs to  $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ , and there exists a constant  $C > 0$  such that:*

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} &+ \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C (1 + \|\mathbf{v}\|_{\mathbf{L}^s(\Omega)}) \\ &\times \left( \|\mathbf{f}\|_{\mathbf{L}^p(\Omega)} + (1 + \|\mathbf{v}\|_{\mathbf{L}^s(\Omega)}) (\|h\|_{W^{1,p}(\Omega)} + \|\mathbf{g}\|_{\mathbf{W}^{2-1/p,p}(\Gamma)}) \right) \end{aligned}$$

**Theorem 2.4.** *Let*

$$p > 1, \quad \mathbf{f} \in \mathbf{W}^{-1,p}(\Omega), \quad h \in L^p(\Omega), \quad \mathbf{v} \in \mathbf{L}_\sigma^3(\Omega) \quad \text{and} \quad \mathbf{g} \in \mathbf{W}^{1-1/p,p}(\Gamma)$$

*satisfying the compatibility condition (3). Then, the problem (O) has a unique solution  $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ , and there exists a constant  $C > 0$  such that:*

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} &+ (1 + \|\mathbf{v}\|_{\mathbf{L}^3(\Omega)})^\gamma \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C (1 + \|\mathbf{v}\|_{\mathbf{L}^3(\Omega)}) \\ &\times (\|\mathbf{f}\|_{\mathbf{W}^{-1,p}(\Omega)} + (1 + \|\mathbf{v}\|_{\mathbf{L}^3(\Omega)}) (\|h\|_{L^p(\Omega)} + \|\mathbf{g}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)})) \end{aligned}$$

35 *with  $\gamma = 0$  if  $p \geq 2$  and  $\gamma = -1$  if  $p < 2$ .*

If we treat the case of  $\Omega = \mathbb{R}^2$ , we have to introduce the Sobolev spaces:

$$W_0^{1,2}(\mathbb{R}^2) = \left\{ \varphi \in \mathcal{D}'(\mathbb{R}^2); \frac{\varphi}{w_1} \in L^2(\mathbb{R}^2), \nabla \varphi \in L^2(\mathbb{R}^2) \right\},$$

$$W_0^{2,2}(\mathbb{R}^2) = \left\{ \varphi \in \mathcal{D}'(\mathbb{R}^2); \frac{\varphi}{w_2} \in L^2(\mathbb{R}^2), \frac{\nabla \varphi}{w_1} \in L^2(\mathbb{R}^2), \nabla^2 \varphi \in L^2(\mathbb{R}^2) \right\},$$

where  $w_1 = (1 + |\mathbf{x}|) \ln(2 + |\mathbf{x}|)$  and  $w_2 = (1 + |\mathbf{x}|)^2 \ln(2 + |\mathbf{x}|)$  (see Definition (7.1), p. 593 in [9]). We denote by  $W_0^{-1,2}(\mathbb{R}^2)$  the dual space of  $W_0^{1,2}(\mathbb{R}^2)$ . Recall ([5]) that the space  $W_0^{1,2}(\mathbb{R}^2)$  is densely embedded in  $VMO(\mathbb{R}^2)$ , and therefore  $\mathcal{H}^1(\mathbb{R}^2) = [VMO(\mathbb{R}^2)]' \hookrightarrow W_0^{-1,2}(\mathbb{R}^2)$ .

**Theorem 2.5 (Case  $\Omega = \mathbb{R}^2$ ).** *i) Let*

$$\mathbf{f} = \operatorname{div} \mathbb{F} \quad \text{with} \quad \mathbb{F} \in \mathbf{L}^2(\mathbb{R}^2) \quad \text{and} \quad h \in L^2(\mathbb{R}^2).$$

*Then, the problem (O) has a unique solution  $(\mathbf{u}, \pi)$  satisfying  $\mathbf{u} \in \mathbf{W}_0^{1,2}(\mathbb{R}^2)$  and  $\pi \in L^2(\mathbb{R}^2)$ , where  $\pi$  is unique and  $\mathbf{u}$  is unique up to an additive constant vector field.*

*ii) Moreover, if*

$$\mathbf{f} \in \mathcal{H}^1(\mathbb{R}^2) \quad \text{and} \quad \nabla h \in \mathcal{H}^1(\mathbb{R}^2),$$

*then*

$$\nabla^2 \mathbf{u} \in \mathcal{H}^1(\mathbb{R}^2), \quad \nabla \pi \in \mathcal{H}^1(\mathbb{R}^2), \quad \nabla \mathbf{u} \in \mathbf{L}^{2,1}(\mathbb{R}^2) \quad \text{and} \quad \mathbf{u} \in L^\infty(\mathbb{R}^2), \quad (9)$$

*being  $L^{2,1}(\mathbb{R}^2)$  is the Lorentz space of all measurable functions  $f$  satisfying*

$$\int_0^\infty t^{-1/2} f^*(t) dt < +\infty,$$

40 *where the rearrangement function  $f^*$  is defined by  $f^*(t) = \sup\{s \in (0, \infty); \mu(\{\mathbf{x} \in \mathbb{R}^2; |f(\mathbf{x})| > s\}) > t\}$ , for  $\mu$  the Lebesgue measure on  $\mathbb{R}^2$ .*

**PROOF.** *i) Existence:* Let  $\chi \in W_0^{2,2}(\mathbb{R}^2)$  be the unique solution, up to a polynomial function of degree one, of  $\Delta \chi = h$  in  $\mathbb{R}^2$  (see Theorem 9.6 in [9]). Then, we take  $\mathbf{u}_h = \nabla \chi \in \mathbf{W}_0^{1,2}(\mathbb{R}^2)$ . Problem (O) is then written as:

$$-\Delta \mathbf{z} + \mathbf{v} \cdot \nabla \mathbf{z} + \nabla \pi = \mathbf{k}, \quad \nabla \cdot \mathbf{z} = 0 \quad \text{in } \mathbb{R}^2,$$

with  $\mathbf{k} = \mathbf{f} + \Delta \mathbf{u}_h - \mathbf{v} \cdot \nabla \mathbf{u}_h$ . Because of  $(\mathbf{v} \cdot \nabla) \mathbf{u}_h \in \mathcal{H}^1(\mathbb{R}^2) \hookrightarrow W_0^{-1,2}(\mathbb{R}^2)$ , by using Lax-Milgram's Lemma (as in the bounded case) we can deduce the existence of a solution  $\mathbf{z} \in \mathbf{W}_0^{1,2}(\mathbb{R}^2)$  with  $\nabla \cdot \mathbf{z} = 0$ , unique up to a constant vector of  $\mathbb{R}^2$ . Lax-Milgram's  
45 Lemma hypotheses are satisfied because, on the one hand, we know that the quotient norm  $\|\mathbf{z}\|_{\mathbf{W}_0^{1,2}(\mathbb{R}^2)/\mathbb{R}^2}$  is equivalent to that one defined as  $\|\nabla \mathbf{z}\|_{\mathbf{L}^2(\mathbb{R}^2)}$ , and, on the other hand,  $(\mathbf{v} \cdot \nabla) \mathbf{z} \in \mathcal{H}^1(\mathbb{R}^2)$  for any  $\mathbf{z} \in \mathbf{W}_0^{1,2}(\mathbb{R}^2)$ . The pressure can be recovered by using Theorem 1 in [10].

ii) *Regularity*: Assume that  $\mathbf{f} \in \mathcal{H}^1(\mathbb{R}^2)$  and  $\nabla h \in \mathcal{H}^1(\mathbb{R}^2)$  (which, in particular, implies that  $h \in L^{2,1}(\mathbb{R}^2)$ ). Therefore,

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{f} - \mathbf{v} \cdot \nabla \mathbf{u} \in \mathcal{H}^1(\mathbb{R}^2) \quad \text{and} \quad \nabla \cdot \mathbf{u} = h.$$

By using Theorem 3.14 in [10], one deduces (9).

### 50 3. The Oseen problem in bounded domains for a less regular $\mathbf{v}$

The aim of this section is the analysis of the existence of solutions of (O) in a bounded domain ( $n = 2$  or  $3$ ) when  $\mathbf{v} \in \mathbf{L}_\sigma^s(\Omega)$  for  $s < n$ . We analyze the case for  $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ ,  $h = 0$  and  $\mathbf{g} = \mathbf{0}$ . Observe that the term  $(\mathbf{v} \cdot \nabla) \mathbf{u}$  can also be written as  $\nabla \cdot (\mathbf{u} \otimes \mathbf{v})$ . The proof of Theorem 3.2 ( $n = 2$ ) applies directly from that one of Theorem 3.1 ( $n = 3$ ).

**Theorem 3.1.** *Let  $\Omega \subset \mathbb{R}^3$  a Lipschitz bounded domain,*

$$\mathbf{f} \in \mathbf{H}^{-1}(\Omega), \quad h = 0, \quad \mathbf{g} = \mathbf{0} \quad \text{and} \quad \mathbf{v} \in \mathbf{L}_\sigma^{6/5+\alpha}(\Omega)$$

for any  $0 < \alpha \leq 9/5$ . Then, there exists a solution of (O) such that  $(\mathbf{u}, \pi) \in \mathbf{H}_0^1(\Omega) \times L^{q(\alpha)}(\Omega)/\mathbb{R}$  for  $q(\alpha) = (6(6+5\alpha))/(36+5\alpha)$  with the estimate:

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^{q(\alpha)}(\Omega)/\mathbb{R}} \leq C \left(1 + \|\mathbf{v}\|_{\mathbf{L}^{6/5+\alpha}(\Omega)}\right) \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \quad (10)$$

PROOF. We approximate  $\mathbf{v}$  by  $\mathbf{v}_\lambda \in \mathcal{D}_\sigma(\bar{\Omega})$  in the  $\mathbf{L}^{6/5+\alpha}(\Omega)$ -norm and look for the solution of the problem:

$$(O_\lambda) \quad -\Delta \mathbf{u}_\lambda + \nabla \cdot (\mathbf{u}_\lambda \otimes \mathbf{v}_\lambda) + \nabla \pi_\lambda = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u}_\lambda = 0 \quad \text{in } \Omega, \quad \mathbf{u}_\lambda = \mathbf{0} \quad \text{on } \Gamma$$

Taking  $\mathbf{u}_\lambda$  as test function in  $(O_\lambda)$ , we get the estimate:

$$\|\mathbf{u}_\lambda\|_{\mathbf{H}_0^1(\Omega)} \leq C(\Omega) \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}. \quad (11)$$

By De Rham Theorem, there exists  $\pi_\lambda \in L^2(\Omega)$  (unique up to a constant) such that:

$$\nabla \pi_\lambda = \mathbf{f} + \Delta \mathbf{u}_\lambda - \nabla \cdot (\mathbf{u}_\lambda \otimes \mathbf{v}_\lambda).$$

Moreover,  $\mathbf{v}_\lambda \otimes \mathbf{u}_\lambda$  belongs to a bounded set of  $L^{q(\alpha)}(\Omega)$  with  $q(\alpha) = (6(6+5\alpha))/(36+5\alpha)$  and which implies that  $\nabla \cdot (\mathbf{v}_\lambda \otimes \mathbf{u}_\lambda)$  belongs to a bounded set of  $\mathbf{W}^{-1,q(\alpha)}(\Omega)$ . Note that if  $0 < \alpha \leq 9/5$  then  $1 < q(\alpha) \leq 2$ . Using (11),

$$\begin{aligned} \|\nabla \pi_\lambda\|_{W^{-1,q(\alpha)}(\Omega)} &\leq C_1 (1 + C(\Omega)) \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{u}_\lambda \otimes \mathbf{v}_\lambda\|_{L^{q(\alpha)}(\Omega)} \\ &\leq C_1 (1 + C(\Omega)) \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + C_2 \|\mathbf{v}_\lambda\|_{\mathbf{L}^{6/5+\alpha}(\Omega)} \|\mathbf{u}_\lambda\|_{\mathbf{H}_0^1(\Omega)} \\ &\leq C(\Omega) \left(1 + \|\mathbf{v}\|_{\mathbf{L}^{6/5+\alpha}(\Omega)}\right) \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \end{aligned} \quad (12)$$

where  $C_1$  and  $C_2$  are the constant of the Sobolev embeddings  $\mathbf{H}^{-1}(\Omega) \hookrightarrow \mathbf{W}^{-1,q(\alpha)}(\Omega)$  and  $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^\beta(\Omega)$ , respectively. Therefore, from (12) we obtain:

$$\inf_{K \in \mathbb{R}} \|\pi_\lambda + K\|_{L^{q(\alpha)}(\Omega)} \leq C(\Omega) (1 + \|\mathbf{v}\|_{\mathbf{L}^{6/5+\alpha}(\Omega)}) \|f\|_{\mathbf{H}^{-1}(\Omega)}$$

Now, it is necessary to take the limit when  $\lambda \rightarrow 0$ : We can extract a subsequence of  $(\mathbf{u}_\lambda)$  and  $(\pi_\lambda + C_\lambda)$  (that will be called in the same way that the original one) such that:

$$\mathbf{u}_\lambda \rightharpoonup \mathbf{u} \quad \text{in } \mathbf{H}_0^1(\Omega), \quad \pi_\lambda + C_\lambda \rightharpoonup \pi \quad \text{in } L^{q(\alpha)}(\Omega),$$

55 where  $(\mathbf{u}, \pi)$  is solution of (O) and satisfies (10). □

**Theorem 3.2.** *Let  $\Omega \subset \mathbb{R}^2$  a Lipschitz bounded domain,  $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ ,  $h = 0$ ,  $\mathbf{g} = \mathbf{0}$  and  $\mathbf{v} \in \mathbf{L}_\sigma^{1+\alpha}(\Omega)$  with  $0 < \alpha \leq 1$ . Then, there exists a solution of (O) such that  $(\mathbf{u}, \pi) \in \mathbf{H}_0^1(\Omega) \times L^{q(\beta)}(\Omega)/\mathbb{R}$  for  $q(\beta) = 1 + \beta$ , for any  $0 < \beta < \alpha$ , with the estimate:*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\pi\|_{L^{q(\beta)}(\Omega)/\mathbb{R}} \leq C (1 + \|\mathbf{v}\|_{\mathbf{L}^{1+\alpha}(\Omega)}) \|f\|_{\mathbf{H}^{-1}(\Omega)}$$

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