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# TITS ALTERNATIVE AND HIGHLY TRANSITIVE ACTIONS ON TORIC VARIETIES

I. ARZHANTSEV, M. ZAIDENBERG

ABSTRACT. Given a toric affine algebraic variety  $X$  and a collection of one-parameter unipotent subgroups  $U_1, \dots, U_s$  of  $\text{Aut}(X)$  which are normalized by the torus acting on  $X$ , we show that the group  $G$  generated by  $U_1, \dots, U_s$  verifies the Tits alternative, and, moreover, either is a unipotent algebraic group, or contains a nonabelian free subgroup. We deduce that if  $G$  is  $m$ -transitive for any positive integer  $m$ , then  $G$  contains a nonabelian free subgroup, and so, is of exponential growth.

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## INTRODUCTION

We fix an algebraically closed field  $\mathbf{k}$  of characteristic zero. Let  $\mathbb{A}^n$  stand for the affine space of dimension  $n$  over  $\mathbf{k}$  and  $\mathbb{G}_a$  ( $\mathbb{G}_m$ ) for the additive (multiplicative, respectively) group of  $\mathbf{k}$  viewed as an algebraic group. For an algebraic variety  $X$  over  $\mathbf{k}$ , a subgroup of  $\text{Aut}(X)$  isomorphic to  $\mathbb{G}_a$  and acting regularly on  $X$  is called a *one-parameter unipotent subgroup*, or  $\mathbb{G}_a$ -subgroup, for short. Any  $\mathbb{G}_a$ -subgroup has the form  $U = \{\exp(t\partial) \mid t \in \mathbf{k}\} \subset \text{SAut}(X)$ , where  $\partial$  is a locally nilpotent derivation of the structure ring  $\mathcal{O}(X)$ .

Let  $X$  be a toric affine variety over  $\mathbf{k}$  of dimension at least two with no torus factor, and let  $\text{SAut}(X) \subset \text{Aut}(X)$  be the subgroup generated by all the  $\mathbb{G}_a$ -subgroups of  $\text{Aut}(X)$ . It is known [AKZ12, Thm. 2.1] that  $\text{SAut}(X)$  acts highly transitively<sup>1</sup> on the smooth locus  $\text{reg}(X)$ , that is,  $m$ -transitively for any  $m \geq 1$ . A variety  $X$  with the latter property is called *flexible*; see [AFK+13, Thm. 1.1] for a criterion of flexibility.

A  $\mathbb{G}_a$ -subgroup acting on  $X$  is called a *root subgroup* if it is normalized by the acting torus. Such a subgroup is associated with a *Demazure root*, see Definition 1.2 for the terminology.

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<sup>1</sup>Or infinitely transitively, in the terminology of [AFK+13].

Assuming in addition that  $X$  is smooth in codimension two, one can find a finite number of root subgroups  $U_1, \dots, U_s$  of  $\text{Aut}(X)$  such that the group  $G = \langle U_1, \dots, U_s \rangle$  generated by these subgroups still acts highly transitively on  $\text{reg}(X)$  [AKZ19, Thm. 1.1].<sup>2</sup> If  $X = \mathbb{A}^n$ ,  $n \geq 2$ , then just three  $\mathbb{G}_a$ -subgroups (which are not root subgroups, in general) are enough [AKZ19, Thm. 1.3]; such subgroups are found explicitly in [And19]. For instance, for  $n = 2$  the group  $G$  generated by two root subgroups

$$U_1 = \{(x, y) \mapsto (x + t_1 y^2, y)\} \quad \text{and} \quad U_2 = \{(x, y) \mapsto (x, y + t_2 x)\}, \quad t_1, t_2 \in \mathbf{k}$$

acts highly transitively on  $\mathbb{A}^2 \setminus \{0\}$  equipped with the standard action of the 2-torus, see [LPS18, Cor. 21]. Adding one more root subgroup

$$U_3 = \{(x, y) \mapsto (x + t_3, y)\}, \quad t_3 \in \mathbf{k},$$

one gets the group  $\langle U_1, U_2, U_3 \rangle$  acting highly transitively on  $\mathbb{A}^2$  (cf. [Chis18]).

The following question arises: *What can one say about a group acting highly transitively on an algebraic variety?* More specifically, let us formulate the following conjecture.

**Conjecture 0.1.** *Let  $X$  be an affine variety over  $\mathbf{k}$  of dimension  $\geq 2$ . Consider the group*

$$G = \langle U_1, \dots, U_s \rangle$$

*generated by  $\mathbb{G}_a$ -subgroups  $U_1, \dots, U_s$  of  $\text{Aut}(X)$ . Suppose  $G$  is highly transitive on a  $G$ -orbit  $\mathcal{O}$ . Then  $G$  contains a non-abelian free subgroup.*

This conjecture is inspired by the following question proposed by J.-P. Demailly:

**Question 0.2.** *What can one say about the growth of the group*

$$G = \langle U_1, \dots, U_s \rangle,$$

*meaning by “growth” the maximal growth of its finitely generated subgroups?*

If our conjecture is true then the group  $G$  as in 0.1 has an exponential growth. It cannot have a polynomial growth, see Proposition 4.2. However, we do not know the answer to the following general question.

**Question 0.3.** *Let  $G$  be a finitely generated group. Assume  $G$  acts highly transitively on a set  $X$ . Can  $G$  be of intermediate growth?*

Notice that an algebraic subgroup  $G \subset \text{Aut}(X)$  cannot act highly transitively on its orbit, by a dimension count argument.

The following result confirms Conjecture 0.1 in our particular setting, cf. Theorem 4.4.

**Theorem 0.4.** *Consider a toric affine variety  $X$  with no torus factor. Let  $G = \langle U_1, \dots, U_s \rangle$  be a subgroup of  $\text{Aut}(X)$  generated by a finite collection of root subgroups. Assume  $G$  is highly transitive on a  $G$ -orbit. Then  $G$  contains a free subgroup of rank two.*

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<sup>2</sup>It is conjectured [AKZ19, Conj. 1.1] that an analogous result holds for any flexible affine variety.

The proof exploits an analog of the Tits alternative for the groups in question. We say that a group  $G$  satisfies the (restricted) Tits alternative if every (finitely generated) subgroup of  $G$  is either virtually solvable or contains a non-abelian free subgroup.<sup>3</sup> The restricted Tits alternative is known to hold for the linear algebraic groups over any field, while the Tits alternative holds for the linear groups in zero characteristic [Tit72], see also [Ben97, Thm. 3.10].

Abusing the language, in this paper we apply the term *Tits alternative* to address a class of groups such that any group in this class either is virtually solvable (resp., nilpotent, abelian, etc.), or contains a non-abelian free subgroup, disregarding whether or not the alternative remains true when passing to a subgroup.

Tits' alternative is known to hold for different classes of groups. Let us mention some results from the literature, especially interesting from the viewpoint of algebraic geometry.

**Theorem 0.5.** *The group  $\text{Bir}(V)$  of birational transformations of a compact complex Kähler variety  $V$  verifies the Tits alternative*

- if  $\dim(V) = 2$  [Can11, Ur17];
- if  $V$  is a hyperkähler variety [Og06].

The result of Cantat and Urech [Can11, Ur17] extends the earlier result of Lamy [Lam01] which says that  $\text{Aut}(\mathbb{A}^2)$  verifies the Tits alternative. The Tits alternative holds as well for the tame automorphism group of  $\text{SL}_2(\mathbb{C})$  viewed as an affine quadric threefold [BFL14, Thm. C].

In this paper we prove the following version of the Tits alternative, see Corollary 3.7.

**Theorem 0.6.** *The group  $G = \langle U_1, \dots, U_s \rangle$  as in Theorem 0.4 either is a unipotent algebraic group, or contains a free subgroup of rank two.*

In particular,  $G$  is of polynomial growth in the former case, and of exponential growth in the latter one; cf. Question 0.3.

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## 1. PRELIMINARIES

We start by recalling the standard definitions, see, e.g., [CLS11] or [AKZ19, Def. 4.2].

**1.1.** Consider a normal toric affine variety  $X$  over  $\mathbf{k}$  of dimension  $n$  with the torus  $\mathbb{T} = (\mathbb{G}_m)^n$  acting faithfully on  $X$ . Let  $N$  be the lattice of one-parameter subgroups of  $\mathbb{T}$ ,  $N^\vee = \text{Hom}(\mathbb{T}, \mathbb{G}_m)$  the dual lattice of characters, and  $\langle \cdot, \cdot \rangle: N \times N^\vee \rightarrow \mathbb{Z}$  the natural pairing. Let  $\chi^m$  stand for the character of  $\mathbb{T}$  which corresponds to  $m \in N^\vee$ , so that  $\chi^m \chi^{m'} = \chi^{m+m'}$ . The group algebra  $\mathbf{k}[N^\vee] = \bigoplus_{m \in N^\vee} \mathbf{k} \chi^m$  can be identified with the structure algebra  $\mathcal{O}(\mathbb{T})$ . The toric affine variety  $X$  is associated with a polyhedral lattice cone  $\Delta \subset N$  in such a way that

$$X = \text{Spec} \left( \bigoplus_{m \in \Delta^\vee} \mathbf{k} \chi^m \right),$$

where  $\Delta^\vee \subset N^\vee$  is the dual lattice cone. By abuse of language, by a *lattice cone*  $\Delta$  ( $\Delta^\vee$ , respectively) we mean here the intersection of the lattice  $N$  ( $N^\vee$ , respectively) with a cone

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<sup>3</sup>The restricted Tits alternative in our terminology corresponds to the Tits alternative in the usual meaning.

$\Delta_{\mathbb{Q}}$  ( $\Delta_{\mathbb{Q}}^{\vee}$ , respectively) in the vector space  $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$  ( $N_{\mathbb{Q}}^{\vee}$ , respectively). The  $\mathbb{T}$ -action on  $X$  arises from the  $\Delta^{\vee}$ -grading on the structure algebra  $\mathcal{O}(X)$ . By Gordon's Lemma, the cones  $\Delta$  and  $\Delta^{\vee}$  are both finitely generated monoids. The lattice vectors  $(\rho_j)_{j=1,\dots,k}$  on the extremal rays of  $\Delta_{\mathbb{Q}}$ , which are elements of the minimal system of generators of  $\Delta$ , are called *ray generators*. The variety  $X$  has no torus factor if and only if  $\Delta_{\mathbb{Q}}$  is full dimensional, if and only if  $\Delta_{\mathbb{Q}}^{\vee} \subset N_{\mathbb{Q}}^{\vee}$  is pointed, that is, contains no line.

**Definition 1.2** (*Root derivations and root subgroups*). The cone  $\Delta^{\vee}$  is surrounded in  $N^{\vee}$  by the *Demazure facets*

$$S_j = \{e \in N^{\vee} \mid \langle \rho_j, e \rangle = -1, \quad \langle \rho_i, e \rangle \geq 0, \quad i = 1, \dots, k, \quad i \neq j\}, \quad j = 1, \dots, k.$$

The lattice vectors  $e \in \bigcup_{j=1}^k S_j$  are called *Demazure roots*. To a Demazure root  $e \in S_j$  one associates the *root derivation*  $\partial_{\rho_j, e} \in \text{Der}(\mathcal{O}(X))$ , which acts on the character  $\chi^m$  via

$$\partial_{\rho_j, e}(\chi^m) = \langle \rho_j, m \rangle \chi^{m+e}.$$

The root derivations are homogeneous locally nilpotent derivations of the graded algebra  $\mathcal{O}(X) = \bigoplus_{m \in \Delta^{\vee}} \mathbf{k} \chi^m$ . The  $\mathbb{G}_a$ -subgroup  $\exp(t\partial_{\rho_j, e})$  is called a *root subgroup*. The root subgroups are the  $\mathbb{G}_a$ -subgroups of  $\text{Aut}(X)$  normalized by  $\mathbb{T}$ .

**1.3.** Let  $X$  be a normal toric affine variety  $X$  with no torus factor. The divisor class group  $\text{Cl}(X)$  is the abelian group generated by the classes of the prime  $\mathbb{T}$ -invariant divisors  $D_1, \dots, D_k$  on  $X$ . The *Cox ring* of  $X$  is the polynomial ring  $\mathcal{O}(\mathbb{A}^k) = \mathbf{k}[x_1, \dots, x_k]$  on a distinguished set of variables called the *total coordinates*. It is equipped with a  $\text{Cl}(X)$ -grading defined by  $\deg(x_i) = [D_i]$ ,  $i = 1, \dots, k$ . This grading corresponds to a diagonal action on  $\mathbb{A}^k = \text{Spec}(\mathbf{k}[x_1, \dots, x_k])$  of the *Cox quasitorus*  $F_{\text{Cox}} = \text{Hom}(\text{Cl}(X), \mathbb{G}_m)$ . Recall that a *quasitorus* is a direct product of an algebraic torus and a finite abelian group. One has [ADHL15, Thm. 2.1.3.2]

$$X \cong \text{Spec}(\mathcal{O}(\mathbb{A}^k)^{F_{\text{Cox}}}) = \mathbb{A}^k // F_{\text{Cox}}.$$

**Lemma 1.4.** ([AKZ19, Lem. 4.20]) *Let  $e \in S_j$  be a Demazure root, and let  $\hat{e} = (c_1, \dots, c_k) \in \mathbb{Z}^k$ , where  $c_i = \langle \rho_i, e \rangle$ . Then the following hold.*

- (a) *The integer lattice vector  $\hat{e}$  is a Demazure root of  $\mathbb{A}^k$  (viewed as a toric variety with the standard action of the  $k$ -torus) which belongs to the  $j$ th Demazure facet  $\hat{S}_j$  of the first octant  $\mathbb{Z}_{\geq 0}^k \subset \mathbb{Z}^k$ .*
- (b) *Let  $(\varepsilon_i)_{i=1,\dots,k}$  be the ray generators of the lattice cone  $\mathbb{Z}_{\geq 0}^k$ . Then one has [AKZ19, (12)]*

$$(1) \quad \hat{\partial} := \partial_{\varepsilon_j, \hat{e}} = M_j \frac{\partial}{\partial x_j}, \quad \text{where} \quad M_j = x_1^{c_1} \cdots x_{j-1}^{c_{j-1}} x_{j+1}^{c_{j+1}} \cdots x_k^{c_k} \in \mathbf{k}[x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k].$$

*The associate  $\mathbb{G}_a$ -subgroup consists of elementary transformations*

$$(2) \quad \exp(t\hat{\partial}) : (x_1, \dots, x_k) \mapsto (x_1, \dots, x_{j-1}, x_j + tM_j, x_{j+1}, \dots, x_k).$$

*This is a subgroup of the tame automorphism group  $\text{Tame}(\mathbb{A}^k)$ .*

- (c) *The Cox quasitorus  $F_{\text{Cox}}$  and the  $\mathbb{G}_a$ -subgroup  $\exp(t\hat{\partial})$  commute in  $\text{Aut}(\mathcal{O}(\mathbb{A}^k))$ , and*

$$\exp(t\hat{\partial})|_{\mathcal{O}(\mathbb{A}^k)^{F_{\text{Cox}}}} = \exp(t\partial_{\rho_j, e}).$$

In the sequel we need the following technical results.

**Proposition 1.5.** *Given a collection of Demazure roots  $(e_{j(i),i} \in S_{j(i)})_{i=1,\dots,s}$ , let*

$$G = \langle U_i \mid i = 1, \dots, s \rangle \subset \text{Aut}(X) \quad \text{where} \quad U_i = \exp(t\partial_{\rho_{j(i),e_{j(i),i}}}).$$

*Consider the root derivations  $\hat{\partial}_i = \hat{\partial}_{\varepsilon_{j(i),e_{j(i),i}}}$  and the root subgroups  $\hat{U}_i = \exp(t\hat{\partial}_i)$  acting on  $\mathbb{A}^k$ ,  $i = 1, \dots, s$ . Let*

$$\hat{G} = \langle \hat{U}_i \mid i = 1, \dots, s \rangle \subset \text{Aut}(\mathbb{A}^k).$$

*Then the following holds.*

- (a) *If  $\hat{G}$  is a unipotent algebraic group then  $G$  is.*
- (b) *If  $\hat{G}$  contains a free subgroup  $F_m$  of rank  $m \geq 2$  then  $G$  does.*

*Proof.* We start with the proof of (b). Since any subgroup  $\hat{U}_i$ ,  $i = 1, \dots, s$  commutes with the quasitorus  $F_{\text{Cox}}$  in  $\text{Aut}(\mathbb{A}^k)$  one has

$$\hat{G} \subset \text{Centr}_{\text{Aut}(\mathbb{A}^k)}(F_{\text{Cox}}) \subset \text{Norm}_{\text{Aut}(\mathbb{A}^k)}(F_{\text{Cox}}),$$

where  $\text{Centr}_{\text{Aut}(\mathbb{A}^k)}(F_{\text{Cox}})$  and  $\text{Norm}_{\text{Aut}(\mathbb{A}^k)}(F_{\text{Cox}})$  are the centralizer and the normalizer of  $F_{\text{Cox}}$  in  $\text{Aut}(\mathbb{A}^k)$ , respectively. There is the exact sequence [AG10, Thm. 5.1]

$$1 \rightarrow F_{\text{Cox}} \rightarrow \text{Norm}_{\text{Aut}(\mathbb{A}^k)}(F_{\text{Cox}}) \xrightarrow{\tau} \text{Aut}(X) \rightarrow 1.$$

Assume  $\hat{G}$  contains a free subgroup  $F_m$  of rank  $m \geq 2$ . We claim that the restriction

$$\tau|_{F_m}: F_m \rightarrow F_m / (F_m \cap F_{\text{Cox}}) \subset \text{Aut}(X)$$

is an isomorphism, that is,  $F_m \cap F_{\text{Cox}} = 1$ . Indeed, the latter intersection is a normal abelian subgroup of  $F_m$ , hence the trivial group, see, e.g., Corollary 5.5.

- (a) If  $\hat{G}$  is a unipotent algebraic group, then its image  $G = \tau(\hat{G})$  in  $\text{Aut}(X)$  is as well.  $\square$

## 2. TITS' ALTERNATIVE FOR A PAIR OF ROOT SUBGROUPS

In this section we prove the following partial result, cf. Theorem 0.6. We still deal with a toric affine variety  $X$  over  $\mathbf{k}$  with no torus factor. We freely use the notation from 1.1–1.3.

**Theorem 2.1.** *Consider the group  $H = \langle U_1, U_2 \rangle \subset \text{Aut}(X)$  generated by the root subgroups  $U_i = \exp(t\partial_i)$ ,  $i = 1, 2$ , associated with two different ray generators, say,  $\rho_1$  and  $\rho_2$ , respectively. Then either  $H$  is a unipotent algebraic group, or  $H$  contains a free subgroup of rank 2.*

*Proof.* Introducing the total coordinates  $(x_1, \dots, x_k)$ , we let  $U_1$  and  $U_2$  act on  $\mathbb{A}^k$  as  $\mathbb{G}_a$ -subgroups  $\hat{U}_1$  and  $\hat{U}_2$  of the tame automorphism group  $\text{Tame}(\mathbb{A}^k)$  commuting with the Cox quasitorus  $F_{\text{Cox}}$ , see Lemma 1.4. We let  $\hat{H} = \langle \hat{U}_1, \hat{U}_2 \rangle$ . By Proposition 1.5 it suffices to prove the above alternative for  $\hat{H}$  instead of  $H$ . To simplify the notation, we write in the sequel  $e_i, U_i, H$  instead of  $\hat{e}_i, \hat{U}_i, \hat{H}$ .

Let in these coordinates  $e_i = (c_{ij})$  where  $c_{ii} = -1$  and  $c_{ij} \geq 0$  for  $j \neq i$ ,  $i \in \{1, 2\}$ . One can write

$$e_1 = (-1, c, *, \dots, *) \quad \text{and} \quad e_2 = (d, -1, *, \dots, *),$$

where the stars stand for nonnegative integers. The elements  $u_i \in U_i$ ,  $i = 1, 2$  can be written as

$$(3) \quad u_1 = (x_1 + sx_2^c N_1, x_2, \dots, x_k) \quad \text{and} \quad u_2 = (x_1, x_2 + tx_1^d N_2, x_3, \dots, x_k),$$

where  $s, t \in \mathbf{k}$  and  $N_1, N_2 \in \mathbf{k}[x_3, \dots, x_k]$  are monomials, cf. (1)–(2). Recall [Li10, Thm. 2.4] that any homogeneous derivation of  $\mathcal{O}(X)$  is proportional to a one of the form  $\partial_{\rho, e}$  for some  $\rho \in N$  and  $e \in N^\vee$  acting via

$$\partial_{\rho, e}(\chi^m) = \langle \rho, m \rangle \chi^{m+e},$$

where  $e$  is called the *degree* of  $\partial_{\rho, e}$ . One has [Rom14, Sect. 3]

$$(4) \quad [\partial_1, \partial_2] = \partial_{\hat{\rho}, e_1+e_2} \quad \text{with} \quad \hat{\rho} = d\rho_2 - c\rho_1.$$

Thus,  $H$  is abelian (and then  $H \cong \mathbb{G}_a \times \mathbb{G}_a$ ) if and only if  $c = d = 0$ . More generally, the following holds.

**Claim 1.** *Assume  $c = 0$  and  $d > 0$ . Then  $H = \langle U_1, U_2 \rangle$  is a unipotent algebraic group.*

*Proof of Claim 1.* Under our assumption one obtains by (4),

$$(5) \quad [\partial_1, \partial_2] = d\partial_{\rho_2, e_1+e_2}.$$

Letting  $L = \text{Lie}(\partial_1, \partial_2)$  and

$$\partial_3 := [\partial_1, \partial_2], \quad \partial_4 := [\partial_1, \partial_3], \quad \dots \quad \partial_{d+2} := [\partial_1, \partial_{d+1}] \neq 0,$$

for the lower central series  $L^i = [L, L^{i-1}]$  of  $L$  one gets

$$L = \text{span}(\partial_1, \partial_2, \dots, \partial_{d+2}) \supset \dots \supset L^i = \text{span}(\partial_{i+1}, \dots, \partial_{d+2}) \supset \dots \supset L^{d+1} = \text{span}(\partial_{d+2}) \supset L^{d+2} = 0.$$

Hence,  $L$  is nilpotent of dimension  $d + 2$ . It follows that  $H$  is a unipotent algebraic group with Lie algebra  $\text{Lie}(H) = L$ , see Proposition 3.6 below.  $\square$

Suppose further that  $c \geq 1$  and  $d \geq 1$ . In this case we show, using ping-pong type arguments, that  $H$  contains a free subgroup of rank two, see Claims 2–4.

By (3), any  $h \in H$  can be written as

$$(6) \quad h = (p, q, x_3, \dots, x_k) \quad \text{with} \quad p, q \in \mathbf{k}[x_1, \dots, x_k] \setminus \mathbf{k}.$$

We distinguish between several cases.

**Claim 2.** *Assume  $c, d \geq 2$ .<sup>4</sup> Then one has  $H = U_1 * U_2 \cong \mathbb{G}_a * \mathbb{G}_a$ . Consequently, any two non-unit elements  $u_i \in U_i$ ,  $i = 1, 2$ , generate a free subgroup of rank two.*

*Proof of Claim 2.* Fixing  $u_i \in U_i$ ,  $i = 1, 2$  as in (3) with nonzero  $s, t \in \mathbf{k}$ , for  $h$  as in (6) one has

$$u_1 h = (p_1, q, x_3, \dots, x_k) \quad \text{and} \quad u_2 h = (p, q_2, x_3, \dots, x_k),$$

where by (3),

$$(7) \quad p_1 = p + sq^c N_1 \quad \text{and} \quad q_2 = q + tp^d N_2.$$

For  $\deg(p) \leq \deg(q)$  one gets

$$(8) \quad \deg(p_1) = c \deg(q) + \deg(N_1) > \deg(q),$$

and, similarly, for  $\deg(p) \geq \deg(q)$  one deduces

$$(9) \quad \deg(p) < \deg(q_2).$$

Consider a nontrivial reduced word  $w \in F_2$ , and let  $h = w(u_1, u_2) \in H$ , where  $u_1, u_2 \neq 1$ . Using (8)–(9) one concludes by recursion on the length of  $w$  that  $\deg(p) > \deg(q)$  if  $w$  starts

<sup>4</sup>The assertion remains valid under a weaker assumption  $c + \deg(N_1) \geq 2$ ,  $d + \deg(N_2) \geq 2$ .

on the left with  $u_1$ , and  $\deg(p) < \deg(q)$  if  $w$  starts with  $u_2$ . Anyway,  $\deg(p) \neq \deg(q)$ , hence  $h \neq 1$ .  $\square$

**Claim 3.** *Assume  $c \geq 2$  and  $d = 1$ . Then  $\langle u_1, u_2 \rangle$  is a free subgroup of rank two for the general  $(u_1, u_2) \in U_1 \times U_2 = \mathbb{A}^2$ .*

*Proof of Claim 3.* The Jung-van der Kulk Theorem [Jun42, vdK53] implies the presentation

$$\text{Aut}(\mathbb{A}^2) = A *_C B,$$

where  $A = \text{Aff}(\mathbb{A}^2)$  is the affine group,  $B$  is the de Jonquières subgroup of  $\text{Aut}(\mathbb{A}^2)$ , and  $C = A \cap B$ ; see [Nag72, Wr75, Dic83], [Kam75, Thm. 2], and [Kam79, Lem. 4.1].

Let  $u_1 = u_1(s)$  and  $u_2 = u_2(t)$  be as in (3). Evaluating the nonzero monomials  $N_1, N_2 \in \mathbf{k}[x_3, \dots, x_k]$  at the general point  $P_0 = (x_3^0, \dots, x_k^0) \in \mathbb{A}^{k-2}$  and letting

$$s = N_1(P_0)^{-1}, \quad t = N_2(P_0)^{-1}, \quad u_1^{(0)} = u_1(s, P_0), \quad u_2^{(0)} = u_2(t, P_0)$$

one gets

$$\langle u_1^{(0)}, u_2^{(0)} \rangle = \langle (x_1 + x_2^c, x_2), (x_1, x_2 + x_1) \rangle \subset \text{Aut}(\mathbb{A}^2).$$

Since  $c > 1$ , for any  $m \in \mathbb{Z}$ ,  $m \neq 0$ , one has  $u_1^m \in B \setminus C$  and  $u_2^m \in A \setminus C$ . Write  $h \in \langle u_1^{(0)}, u_2^{(0)} \rangle$  as a reduced word

$$h = w(u_1^{(0)}, u_2^{(0)}) \quad \text{with} \quad w \in F_2.$$

Applying the Jung-van der Kulk Theorem to  $w(u_1^{(0)}, u_2^{(0)})$  we conclude that  $h = 1$  if and only if  $w = 1 \in F_2$ . Thus, one has  $\langle u_1^{(0)}, u_2^{(0)} \rangle \cong F_2$ . The specialization  $(u_1, u_2) \mapsto (u_1^{(0)}, u_2^{(0)})$  defines an isomorphism  $\langle u_1, u_2 \rangle \cong F_2$ . The same argument works for the general  $(s, t) \in \mathbb{A}^2$ .  $\square$

The next claim ends the proof of the theorem.

**Claim 4.** *Assume  $c = d = 1$ . Then for a suitable  $(u_1, u_2) \in U_1 \times U_2 = \mathbb{A}^2$ , the group  $\langle u_1, u_2 \rangle$  surjects onto  $\text{SL}_2(\mathbb{Z})$  and so, contains a free subgroup of rank two.*

*Proof of Claim 4.* Repeating the argument from the proof of Claim 3 gives

$$\langle u_1^{(0)}, u_2^{(0)} \rangle = \langle (x_1 + x_2, x_2), (x_1, x_2 + x_1) \rangle = \text{SL}_2(\mathbb{Z}).$$

This yields the desired surjection  $\langle u_1, u_2 \rangle \rightarrow \text{SL}_2(\mathbb{Z})$ . It remains to recall [Wiki, 3.1] that  $\text{SL}_2(\mathbb{Z})$  is virtually free with  $\langle (x_1 + 2x_2, x_2), (x_1, x_2 + 2x_1) \rangle \cong F_2$ .  $\square$

Inspecting our proof we come to the following conclusion.

**Corollary 2.2.** *In the notation as before,  $H = \langle U_1, U_2 \rangle$  is a unipotent algebraic group if and only if*

$$(10) \quad \min\{\langle \rho_1, e_2 \rangle, \langle \rho_2, e_1 \rangle\} = \min\{c, d\} = 0.$$

So, if the group  $G = \langle U_1, \dots, U_s \rangle$  generated by root subgroups does not contain any non-abelian free subgroup, then for any  $i \neq j$  either  $U_i$  and  $U_j$  belong to the same ray generator (and then commute), or they belong to two different ray generators  $\rho$  and  $\rho'$  and for the corresponding roots  $e, e'$  one has  $\min\{\langle \rho, e' \rangle, \langle \rho', e \rangle\} = 0$ . In Theorem 3.2 we establish that such a group  $G$  is a unipotent algebraic group. Let us give an example.



**Example 2.3.** Consider the group  $G = \langle U_1, U_2, U_3, U_4 \rangle$  acting on  $\mathbb{A}^3 = \text{Spec } \mathbf{k}[x, y, z]$ , where  $U_i = \exp(t\partial_i)$ ,  $i = 1, \dots, 4$  with

$$\partial_1 = yz \frac{\partial}{\partial x}, \quad \partial_2 = z \frac{\partial}{\partial y}, \quad \partial_3 = z^2 \frac{\partial}{\partial y}, \quad \partial_4 = \frac{\partial}{\partial z}.$$

We have

$$\partial_1 = \partial_{\rho_1, e_1}, \quad \partial_2 = \partial_{\rho_2, e_2}, \quad \partial_3 = \partial_{\rho_2, e_3}, \quad \partial_4 = \partial_{\rho_3, e_4},$$

where the ray generators  $\rho_1, \rho_2, \rho_3$  are the vectors of the standard basis in  $\mathbb{A}^3$ , and

$$e_1 = (-1, 1, 1), \quad e_2 = (0, -1, 1), \quad e_3 = (0, -1, 2), \quad e_4 = (0, 0, -1).$$

Any pair of these root derivations verify (10). They generate the Lie algebra

$$L = \text{span} \left( \frac{\partial}{\partial x}, y \frac{\partial}{\partial x}, yz \frac{\partial}{\partial x}, z \frac{\partial}{\partial x}, z^2 \frac{\partial}{\partial x}, z^3 \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, z \frac{\partial}{\partial y}, z^2 \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Consider the abelian Lie subalgebras

$$L_1 = \text{span} \left( \frac{\partial}{\partial x}, y \frac{\partial}{\partial x}, yz \frac{\partial}{\partial x}, z \frac{\partial}{\partial x}, z^2 \frac{\partial}{\partial x}, z^3 \frac{\partial}{\partial x} \right),$$

$$L_2 = \text{span} \left( \frac{\partial}{\partial y}, z \frac{\partial}{\partial y}, z^2 \frac{\partial}{\partial y} \right), \quad \text{and} \quad L_3 = \text{span} \left( \frac{\partial}{\partial z} \right).$$

We have

$$L = L_1 \oplus L_2 \oplus L_3, \quad \text{where} \quad [L_1, L_i] \subset L_1, \quad i = 2, 3, \quad [L_2, L_3] \subset L_2,$$

and, furthermore,

$$\text{ad}(L_i)(L_i) = 0, \quad i = 1, 2, 3, \quad \text{ad}(L_3)^4(L_1) = 0, \quad \text{ad}(L_2)^2(L_1) = 0, \quad \text{ad}(L_3)^3(L_2) = 0.$$

For the lower central series  $L^i = [L, L^{i-1}]$  of  $L$  we obtain  $L^5 = 0$ . Thus,  $L$  is nilpotent, and so, by Proposition 3.6,  $G$  is a unipotent algebraic group.

### 3. TITS' ALTERNATIVE FOR A SEQUENCE OF ROOT SUBGROUPS

**3.1.** Let as before  $X$  be a toric affine variety with no torus factor, and let

$$G = \langle U_1, \dots, U_s \rangle$$

be the group generated by the root subgroups  $U_j = \exp(t\partial_j) \subset \text{SAut}(X)$ ,  $j = 1, \dots, s$ . The Lie algebra  $L$  generated by the root derivations  $\partial_j$ ,  $j = 1, \dots, s$ , might contain extra root derivations, cf. Example 2.3. Let  $R_i$  be the set of Demazure roots  $e_{ij} \in S_i$  of  $X$  such that  $\partial_{\rho_i, e_{ij}} \in L$ ,  $j = 1, \dots, \#(R_i)$ . We may suppose that

$$R_i \neq \emptyset \quad \forall i = 1, \dots, r \quad \text{and} \quad R_i = \emptyset \quad \forall i = r+1, \dots, k.$$

Let  $R = \bigcup_{i=1}^r R_i$ . For  $e \in R_i$  we let  $U_e = \exp(t\partial_{\rho_i, e})$ .

**Theorem 3.2.** *In the notation of 3.1, suppose for any  $e, e' \in R$  the group  $\langle U_e, U_{e'} \rangle$  does not contain any free subgroup of rank two, that is, (10) holds, see Corollary 2.2. Then  $G$  is a unipotent algebraic group.*

The proof exploits [ALS19, Prop. 5.3] adopted to our particular setting, see Proposition 3.5. Let us recall the terminology of [ALS19] and introduce the necessary notation.

**Definition 3.3.** Consider a finite sequence of root derivations

$$\mathcal{D} = (D_1, \dots, D_t, D_{t+1}) \text{ where } D_i = \partial_{\rho_{j(i)}, e_{j(i), i}} \text{ with } e_{j(i), i} \in R_{j(i)}, j(i) \in \{1, \dots, r\}.$$

One says that  $\mathcal{D}$  is a *cycle* (more precisely, a *t-cycle*) if  $D_{t+1} = D_1$  and

$$(11) \quad \langle \rho_{j(i+1)}, e_{j(i), i} \rangle > 0 \quad \forall i = 1, \dots, t.$$

For instance,  $(D_1, D_2, D_1)$  forms a 2-cycle if and only if (10) fails, that is,

$$\langle \rho_{j(2)}, e_{j(1), 1} \rangle > 0 \quad \text{and} \quad \langle \rho_{j(1)}, e_{j(2), 2} \rangle > 0.$$

We say that  $\mathcal{D}$  is a *pseudo-cycle* if (11) holds and  $j(t+1) = j(1)$ , but not necessarily  $e_{j(t+1), t+1} = e_{j(1), 1}$ ; that is,  $\rho_{j(t+1)} = \rho_{j(1)}$  but possibly  $D_{t+1} \neq D_1$ .

**Lemma 3.4.** *The following are equivalent:*

- (i) *L contains no pseudo-cycle;*
- (ii) *L contains no cycle;*
- (iii) *L contains no 2-cycle.*

*Proof.* It suffices to prove (iii) $\Rightarrow$ (i), the two other implications being immediate. Assume  $L$  contains no 2-cycle, that is, cf. (10),

$$(12) \quad \min\{\langle \rho_i, e_j \rangle, \langle \rho_j, e_i \rangle\} = 0 \quad \forall e_i \in R_i, \quad \forall e_j \in R_j \quad \text{with} \quad 1 \leq i \neq j \leq r.$$

Notice that  $\langle \rho_j, e_i \rangle = c > 0$  for some  $e_i \in R_i$  implies by (12)

$$(13) \quad \langle \rho_i, e_j \rangle = 0 \quad \forall e_j \in R_j.$$

It follows that  $L$  has no 2-pseudo-cycle.

For any  $j = 1, \dots, r$  consider the abelian subalgebra

$$L_j = \text{Lie}(\partial_{\rho_j, e_j} \mid e_j \in R_j) \subset L.$$

From (5) and (13) one deduces

$$(14) \quad [\partial_{\rho_j, e_j}, \partial_{\rho_i, e_i}] = c \partial_{\rho_i, e_i + e_j} \in L_i \quad \text{where} \quad e_i + e_j \in R_i.$$

It follows that

$$(15) \quad 0 \neq [L_j, L_i] \subset L_i,$$

that is,  $L_i$  is a proper ideal of the solvable nonabelian Lie algebra  $\text{Lie}(L_i, L_j)$ .

We express (15) by writing  $[L_j \rightarrow L_i]$ . This defines a directed graph  $\Gamma_r$  on  $r$  vertices  $\{L_i\}_{i=1, \dots, r}$ . The vertices  $L_i$  and  $L_j$  are not joint by an edge in  $\Gamma_r$  if and only if  $[L_i, L_j] = 0$ , that is,  $\text{Lie}(L_i, L_j)$  is abelian.

Suppose to the contrary that  $\mathcal{D} = \{D_1, \dots, D_N, D_{N+1}\}$  is a pseudo-cycle in  $L$  with  $N \geq 3$ . Then  $\Gamma_r$  has the oriented cycle

$$L_{\rho_{j(1)}} \rightarrow L_{\rho_{j(N)}} \rightarrow \dots \rightarrow L_{\rho_{j(2)}} \rightarrow L_{\rho_{j(1)}}.$$

The sequence  $\rho_{j(1)}, \dots, \rho_{j(N)}$  of the corresponding ray generators can eventually contain repetitions. However, it is possible to subtract a subsequence  $\rho_{j(1)}, \dots, \rho_{j(t)}$  without repetition, where  $2 \leq t \leq N$ , such that  $\rho_{j(t+1)} = \rho_{j(1)}$ . Then  $\mathcal{D}' = \{D_1, \dots, D_t, D_{t+1}\}$  is again a pseudo-cycle.

To any  $e \in R$  we associate the integer vector of length  $t$ ,

$$v(e) = (\langle \rho_{j(1)}, e \rangle, \dots, \langle \rho_{j(t)}, e \rangle) \in \mathbb{Z}^t.$$

One has

$$\begin{aligned}
(16) \quad & v(e_{j(1),1}) = (-1, \bullet, *, \dots, *, *) \\
& v(e_{j(2),2}) = (0, -1, \bullet, *, \dots, *) \\
& v(e_{j(3),3}) = (*, 0, -1, \bullet, \dots, *) \\
& \quad \quad \quad \vdots \\
& v(e_{j(t-1),t-1}) = (*, *, \dots, 0, -1, \bullet) \\
& v(e_{j(t),t}) = (\bullet, *, *, \dots, 0, -1)
\end{aligned}$$

where the stars stand for nonnegative integers, the bullets stand for positive integers, and the zeros on the lower subdiagonal are due to (11) and (12). In fact, (11) and (12) imply

$$(17) \quad \langle \rho_{j(i)}, e \rangle = 0 \quad \forall e \in R_{j(i+1)}.$$

From (5) and (17) one deduces

$$\langle \rho_{j(i+2)}, e'_{j(i),i} \rangle > 0 \quad \text{where} \quad e'_{j(i),i} := e_{j(i),i} + e_{j(i+1),i+1} \in R_{j(i)}, \quad i = 1, \dots, t-2.$$

Then (12) gives

$$\langle \rho_{j(i)}, e \rangle = 0 \quad \forall e \in R_{j(i+2)}.$$

This means that the second lower subdiagonal in (16) consists of zeros. Continuing in this fashion we show finally that the matrix in (16) is upper triangular. Moreover, one has

$$\langle \rho_{j(t+1)}, e \rangle = \langle \rho_{j(1)}, e \rangle = 0 \quad \forall e \in R_{j(t)}.$$

The latter contradicts (11) for  $i = t$  and  $e = e_{j(t),t}$ .  $\square$

Due to Lemma 3.4, the following statement is equivalent to Proposition 5.3 in [ALS19]. For the convenience of the reader we provide a proof. It is slightly different from the one in [ALS19], while it is based on the same ideas.

**Proposition 3.5.** *Assume  $L$  contains no 2-cycle of root derivations. Then  $L$  is finite-dimensional and nilpotent.*

*Proof.* We freely use the notation from the proof of Lemma 3.4. Consider the one-dimensional Lie subalgebras  $l_{\rho_i, e_i}$  of  $L_i$  generated by the root derivations, where

$$l_{\rho_i, e_i} = \text{span}(\partial_{\rho_i, e_i}) = \mathbf{k}\partial_{\rho_i, e_i} \quad \text{where} \quad e_i \in R_i.$$

Since  $L$  has no 2-cycle then (12) holds. Hence, for  $i \neq j$  one has the alternative:

$$\text{either } [l_{\rho_i, e_i}, l_{\rho_j, e_j}] = 0, \quad \text{or} \quad [l_{\rho_i, e_i}, l_{\rho_j, e_j}] \in \{l_{\rho_i, e_i + e_j}, l_{\rho_j, e_i + e_j}\}.$$

Recall, cf. (14) and (15), that  $[l_{\rho_i, e_i}, l_{\rho_j, e_j}] = l_{\rho_i, e_i + e_j}$  if and only if  $\langle \rho_j, e_i \rangle > 0$ . In the latter case  $\Gamma_r$  has the directed edge  $[L_j \rightarrow L_i]$ . It is clear that

$$L_i = \bigoplus_{e \in R_i} l_{\rho_i, e} \quad \text{and} \quad L = \bigoplus_{i=1}^r L_i.$$

Therefore, one has

$$(18) \quad \dim(L) = \sum_{i=1}^r \dim(L_i) = \sum_{i=1}^r \text{card}(R_i) = \text{card}(R).$$

We claim that  $\Gamma_r$  is acyclic, that is, does not contain any oriented cycle. Indeed, given an oriented cycle in  $\Gamma_r$ ,

$$L_{j(1)} \rightarrow L_{j(2)} \rightarrow \dots \rightarrow L_{j(t)} \rightarrow L_{j(t+1)} = L_{j(1)},$$

one can find a sequence of roots  $e_{j(i),i} \in R_{j(i)}$  such that, with the usual convention  $\rho_{j(t+1)} = \rho_{j(1)}$ , one has

$$\langle \rho_{j(i+1)}, e_{j(i),i} \rangle > 0, \quad i = 1, \dots, t.$$

Thus,  $\mathcal{D} = (D_i = \partial_{\rho_{j(i)}, e_{j(i),i}})_{i=1, \dots, t}$  is a pseudo-cycle of root derivations in  $L$ . By Lemma 3.4, the latter contradicts our assumption on absence of 2-cycles in  $L$ .

A vertex  $L_i$  is called a *sink* if either  $L_i$  is isolated in  $\Gamma_r$ , or all the incident edges of  $\Gamma_r$  at  $L_i$  have the incoming direction. A sink  $L_i$  of  $\Gamma_r$  is an ideal of the Lie algebra  $L$ .

The end vertex of any maximal oriented path in  $\Gamma_r$  is a sink. Since  $\Gamma_r$  is acyclic it has at least one sink. Moreover, any connected component of  $\Gamma_r$  contains a sink.

We can choose a new enumeration of the vertices of  $\Gamma_r$  taking for  $L_1$  a vertex which is a sink of  $\Gamma_r$ . Deleting  $L_1$  from  $\Gamma_r$  along with its incident edges yields a directed graph  $\Gamma_{r-1}$ . The corresponding Lie subalgebra of  $L$  still has no pseudo-cycle of root derivations. Hence,  $\Gamma_{r-1}$  has at least one sink. We choose a sink of  $\Gamma_{r-1}$  to be  $L_2$ , etc.

With this new enumeration one has (cf. Example 2.3)

$$(19) \quad \begin{aligned} [L_i, L_1] &\subset L_1, & i = 2, \dots, r, \\ [L_i, L_2] &\subset L_2, & i = 3, \dots, r, \\ &\vdots \\ [L_r, L_{r-1}] &\subset L_{r-1}. \end{aligned}$$

Let us show next that  $L$  is of finite dimension. Recall (see 3.1) that  $L$  is generated by the finite set of root derivations  $\partial_i$ ,  $i = 1, \dots, s$ . Let  $R_j^{(0)} \subset R_j$  be the set of roots which serve as roots for some of the  $\partial_i$ . It follows from (19) that  $R_r = R_r^{(0)}$ , and so,  $R_r$  is finite. Moreover, by (13), (15), and (19), for any  $e \in R_r$  one has

$$\langle \rho_i, e \rangle = 0 \quad \forall i = 1, \dots, r-1 \quad \text{and} \quad \langle \rho_r, e \rangle = -1.$$

Any root  $e \in R_{r-1}$  is of the form

$$e = e_{r-1}^{(0)} + e_{r,1} + \dots + e_{r,m} \quad \text{with} \quad e_{r-1}^{(0)} \in R_{r-1}^{(0)} \quad \text{and} \quad e_{r,i} \in R_r, \quad i = 1, \dots, m,$$

where  $0 \leq m \leq \langle \rho_r, e_{r-1}^{(0)} \rangle$  because  $\langle \rho_r, e_{r,i} \rangle = -1$ ,  $i = 1, \dots, m$ , and  $\langle \rho_r, e \rangle \geq 0$ . Since both  $R_{r-1}^{(0)}$  and  $R_r$  are finite, we conclude that  $R_{r-1}$  is as well.

Suppose by induction that  $\bigcup_{i=2}^r R_i$  is finite. Any root  $e \in R_1$  is of the form

$$e = e_1^{(0)} + \sum_{i=2}^r \sum_{j=1}^{m_i} e_{i,j} \quad \text{with} \quad e_1^{(0)} \in R_1^{(0)} \quad \text{and} \quad e_{i,j} \in R_i.$$

Again by (13), (15), and (19), the  $(\sum_{i=2}^r m_i) \times r$  matrix of coordinates  $\langle \rho_l, e_{i,j} \rangle$  of the vectors  $e_{i,j}$  is triangular-like with  $(-1)$ 's on the "diagonal", that is,

$$(20) \quad \langle \rho_l, e_{i,j} \rangle = 0 \quad \forall l = 1, \dots, i-1 \quad \text{and} \quad \langle \rho_i, e_{i,j} \rangle = -1, \quad j = 1, \dots, m_i.$$

Since

$$\langle \rho_2, e \rangle = \langle \rho_2, e_1^{(0)} \rangle - m_2 \geq 0$$

one has  $m_2 \leq \langle \rho_2, e_1^{(0)} \rangle =: \tilde{m}_2$ . Letting

$$\tilde{m}_3 = \langle \rho_3, e_1^{(0)} \rangle + \max_{m_2 \leq \tilde{m}_2} \left\{ \sum_{j=1}^{m_2} \langle \rho_3, e_{2,j} \rangle \mid (e_{2,1}, \dots, e_{2,m_2}) \in R_2^{m_2} \right\} < +\infty$$

and applying the same argument as before yields  $m_3 \leq \tilde{m}_3$ , etc. This implies, by recursion, the finiteness of  $R_1$  and the one of  $R$ . Thus, the dimension  $\dim(L) = \text{card}(R)$  is finite, see (18).

It follows as well that for  $N \gg 1$  one has

$$(21) \quad \text{ad}(L_j)^N(L_i) = 0 \quad \forall j \geq i, \quad i, j \in \{1, \dots, r\}.$$

Taking into account (19) the latter implies

$$\text{ad}(L)^{Nr}(L) = 0,$$

and so,  $L$  is nilpotent.  $\square$

It is well known, see, e.g., [Hoch81, Ch. XVI, Thm. 4.2], that the Lie functor realizes the equivalence of categories of the unipotent algebraic groups over fields of zero characteristic and of the nilpotent Lie algebras. In our particular case, this correspondence is quite explicite.

**Proposition 3.6.** *Suppose  $L$  as in 3.1 has no 2-cycle of root derivations, and so, is nilpotent. Then the following hold.*

- (a) Any  $\partial \in L$  is a locally nilpotent derivation of  $\mathcal{O}(X)$ ;
- (b)  $\exp(L) = \{\exp(\partial) \mid \partial \in L\}$  is a subgroup of the group of unitriangular automorphisms of  $\mathbb{A}^k$ ;
- (c) the map  $\exp: L \rightarrow \exp(L)$  is bijective;
- (d) the degrees of  $\exp(\partial)$  are uniformly bounded for  $\partial \in L$ ;
- (e)  $G = \langle U_1, \dots, U_s \rangle$  as in 3.1 is a unipotent algebraic group acting morphically on  $X$ ;
- (f) one has  $G = \exp(L)$  and so,  $L = \text{Lie}(G)$ .

*Proof.* According to Proposition 3.5,  $L$  is a finite-dimensional nilpotent Lie algebra. We use the enumeration of the subalgebras  $L_i \subset L$ ,  $i = 1, \dots, r$  introduced in the proof of this proposition, so that (19) and (21) hold.

Due to (1), for  $i \in \{1, \dots, r\}$  in the total coordinates  $(x_1, \dots, x_k)$  on  $\mathbb{A}^k$  any derivation  $\partial_i \in L_i$  acts via

$$(22) \quad \partial_i = p_i \partial / \partial x_i \quad \text{where} \quad p_i \in \mathbf{k}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k].$$

The set of monomials  $M_i$  (see (1)), taken up to proportionality, of all possible polynomials  $p_i$  in (22) is in one-to-one correspondence with the set of roots in  $R_i$ . Since  $R_i$  is finite, it follows that

$$(23) \quad \max_{\partial_i \in L_i} \{\deg(p_i)\} \leq d_i \quad \text{for some} \quad d_i \in \mathbb{N}.$$

The total coordinates of these roots form a triangular-like matrix, see (20). The latter means that in (22) one has  $p_i \in \mathbf{k}[x_{i+1}, \dots, x_k]$ . Since  $\partial_i^2(x_i) = 0$ ,  $i = 1, \dots, k$ , the  $\mathbb{G}_a$ -subgroup  $\exp(t\partial_i)$  of  $\text{Aut}(\mathbb{A}^k)$  generated by  $\partial_i$  from (22) acts on  $\mathbb{A}^k$  via the unipotent triangular (i.e., unitriangular) transformations

$$(24) \quad \exp(t\partial_i): (x_1, \dots, x_k) \mapsto (x_1, \dots, x_{i-1}, x_i + tp_i(x_{i+1}, \dots, x_k), x_{i+1}, \dots, x_k), \quad t \in \mathbf{k},$$

cf. (2).

Any derivation  $\partial \in L$  is triangular of the form

$$(25) \quad \partial = \sum_{i=1}^r \partial_i = \sum_{i=1}^r p_i \partial / \partial x_i, \quad \text{where } \partial_i \in L_i, \quad \text{and so, } p_i \in \mathbf{k}[x_{i+1}, \dots, x_k], \quad i = 1, \dots, r.$$

According to [Fre17, Prop. 3.29],  $\partial$  is locally nilpotent on  $\mathcal{O}(\mathbb{A}^k)$ , and also on  $\mathcal{O}(X) = \mathcal{O}(\mathbb{A}^k)^{F_{\text{Cox}}}$ , as stated in (a). Notice that for  $r < k$  the variables  $x_{r+1}, \dots, x_k$  belong to the kernel of any  $\partial \in L$ .

Since  $R$  is finite, the degrees of the polynomials  $p_i$  in (25) are uniformly bounded from above by, say,  $d \in \mathbb{N}$ . Any automorphism  $\exp(\partial) \in \exp(L) \subset \text{Aut}(\mathbb{A}^k)$  is unitriangular of the form

$$(26) \quad \exp(\partial)(x_1, \dots, x_k) = (x_1 + f_1, \dots, x_r + f_r, x_{r+1}, \dots, x_k) \text{ with } f_i \in \mathbf{k}[x_{i+1}, \dots, x_k].$$

Any unitriangular automorphism  $\alpha \in \text{Aut}(\mathbb{A}^k)$  is the exponent  $\alpha = \exp(c)$  of the triangular derivation

$$c = \log(\alpha) = \log(\text{id} + (\alpha - \text{id})) \in \text{Der}(\mathcal{O}(\mathbb{A}^k)),$$

see [Fre17, Prop. 3.30] and its proof. The derivation  $c$  being locally nilpotent, the exponential series for

$$(27) \quad \alpha(x_i) = \exp(c)(x_i) = x_i + f_i^{(\alpha)}(x_{i+1}, \dots, x_k)$$

has a finite number of nonzero terms for every  $i \in \{1, \dots, k\}$ .

Consider a pair  $(a, b)$  of triangular derivations of  $\mathcal{O}(\mathbb{A}^k)$ . The product  $\exp(a)\exp(b)$  of the corresponding unitriangular automorphisms is again unitriangular. In more detail,  $\exp(a)\exp(b) = \exp(c)$  with a triangular derivation

$$c = \log(\exp(a)\exp(b)) \in \text{Der}(\mathcal{O}(\mathbb{A}^k)),$$

see [Fre17, Cor. 3.31]. According to the Baker-Campbell-Hausdorff formula (see, e.g., [Man12]) one has

$$(28) \quad c = a + b + \frac{1}{2}[a, b] + \frac{1}{12}[a, [a, b]] + \frac{1}{12}[b, [b, a]] + \dots,$$

where each term is a rational multiple of a Lie product  $[z_1, [z_2, [\dots, z_m] \dots]]$  of length  $m$  such that each  $z_i$  is equal either to  $a$  or to  $b$ .

Let  $n$  be the nilpotency index of  $L$ , that is,  $n > 0$  is minimal with  $L^n = 0$ . For  $a, b \in L$  the series in (28) is truncated, namely, all the terms of length  $m \geq n$  vanish, and so, one has  $c \in L$ . Thus,  $\exp(L)$  is a subgroup of the group of unitriangular automorphisms of  $\mathbb{A}^k$ . This proves (b). Since  $\log$  and  $\exp$  are mutually inverse, the map  $\exp : L \rightarrow \exp(L)$  is a bijection, as stated in (c).

To show (d) we apply the multivariate Zassenhaus formula [WGJ19]

$$(29) \quad \exp(a_1 + \dots + a_\nu) = \exp(a_1) \cdots \exp(a_\nu) \prod_{m=2}^{\infty} \exp(\psi_m(a_1, \dots, a_\nu)),$$

where  $\psi_m(a_1, \dots, a_\nu)$  is a homogeneous Lie polynomial in  $a_1, \dots, a_\nu \in L$  of degree  $m$ . Once again, we have the vanishing  $\psi_m(a_1, \dots, a_\nu) = 0$  for  $m \geq n$ . Hence the product in (29) is finite.

Take  $a_i = p_i \partial / \partial x_i \in L_i$ , where  $p_i \in \mathbf{k}[x_{i+1}, \dots, x_k]$  with  $\deg(p_i) \leq d_i$ , see (23). Then

$$a_1 + \dots + a_\nu \in \mathcal{L}_\nu := L_1 \oplus \dots \oplus L_\nu, \quad \text{where } L_1 = \mathcal{L}_1 \subset \mathcal{L}_2 \subset \dots \subset \mathcal{L}_r = L$$

is an increasing chain of ideals of  $L$ , see (19). Henceforth, one has  $\psi_m(a_1, \dots, a_\nu) \in \mathcal{L}_{\nu-1}$  for any  $m \geq 2$  and any  $\nu \in \{2, \dots, r\}$ .

We proceed by induction on  $\nu$ . For  $\nu = 1$ , the assertion of (d) follows from (24). Suppose it holds for some  $\nu - 1 < r$ . Letting  $\partial = \sum_{i=1}^\nu p_i \partial / \partial x_i \in \mathcal{L}_\nu$ , by the preceding observation (29) can be written as

$$(30) \quad \exp(\partial) = g_1 \exp(p_\nu \partial / \partial x_\nu) g_2 \quad \text{with} \quad g_1, g_2 \in \exp(\mathcal{L}_{\nu-1}).$$

Since (d) holds for  $\exp(\mathcal{L}_{\nu-1})$  by the induction hypothesis, and for  $\exp(L_\nu)$  by (23)–(24), the degrees of all the automorphisms in (30) are uniformly bounded above. This concludes the induction.

Consider the finite dimensional affine spaces  $P = \text{vect}(L)$  and  $F = \text{span}(\exp(L)) \subset \text{End}(\mathbb{A}^k)$ . One can take for the coordinates in  $P$  and  $F$  the coefficients of the polynomials  $p_1, \dots, p_r$  and  $f_1, \dots, f_r$  as in (25) and (26), respectively, with  $\partial \in L$ . The map

$$L \ni \partial \mapsto \exp(\partial) \in \exp(L), \quad (p_1, \dots, p_r) \mapsto (f_1, \dots, f_r),$$

defines a morphism  $P \rightarrow F$ . The image  $\exp(P) = \exp(L)$  is a constructible set. Being a connected group,  $\exp(L)$  is an irreducible, smooth, locally closed subvariety of  $F$ . By Zariski's Main Theorem, the bijection  $\exp : L \rightarrow \exp(L)$  is an isomorphism of affine varieties. Hence,  $\exp(L)$  is an affine algebraic group, and an algebraic subgroup of the ind-group  $\text{Aut}(\mathbb{A}^k)$ . So,  $\exp(L)$  acts morphically on  $\mathbb{A}^k$ . This group is unipotent since all its elements are.

Recall (see 3.1), that  $G = \langle U_1, \dots, U_s \rangle$ , where  $U_j = \exp(t\partial_j)$  with  $\partial_j \in L$ . So,  $U_j \subset \exp(L)$  for any  $j = 1, \dots, s$ . It follows that  $G \subset \exp(L)$ . Since the Lie subalgebras  $\text{Lie}(U_j)$ ,  $j = 1, \dots, s$ , generate  $L$  we have  $G = \exp(L)$ , as claimed in (f).

Finally,  $G = \exp(L)$  is a unipotent affine algebraic group acting morphically on  $\mathbb{A}^k$  and centralized by the quasitorus  $F_{\text{Cox}}$ . Hence, it acts morphically on  $X = \text{Spec}(\mathcal{O}(\mathbb{A}^k)^{F_{\text{Cox}}})$ . We have  $\text{Lie}(G) = L$ . This ends the proof of (e) and (f).  $\square$

*Proof of Theorem 3.2.* Due to Corollary 2.2, under the assumption of Theorem 3.2 the group  $U_{e,e'} = \langle U_e, U_{e'} \rangle$  is nilpotent and (12) holds for any  $e \in R_i$ ,  $e' \in R_j$ . Then  $L$  has no 2-cycle of root derivations. Now the assertion follows from Proposition 3.6.  $\square$

Let us finally state an analog of the Tits alternative for a group acting on a toric affine variety and generated by a finite sequence of  $\mathbb{G}_a$ -subgroups.

**Corollary 3.7.** *Any group  $G = \langle U_1, \dots, U_s \rangle$  as in 3.1 either is a unipotent algebraic group, or contains a free subgroup of rank two.*

*Proof.* The assertion is immediate from Theorems 2.1 and 3.2.  $\square$

#### 4. HIGHLY TRANSITIVE GROUPS ACTING ON TORIC AFFINE VARIETIES

In this section we apply the Tits alternative to answer Question 0.2 under the assumption of high transitivity of the group in question. Recall the following definition.

**Definition 4.1.** Let  $G$  be a group. We say that  $G$  is *highly transitive* if  $G$  admits an effective action on a set  $X$  which is  $m$ -transitive for any  $m \in \mathbb{N}$ .

Attention: one can find in the literature another definition of high transitivity, which does not require effectiveness.

Recall that a subgroup  $N$  of a group  $G$  is called *subnormal* if there exists a descending normal series

$$(31) \quad G \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \dots \trianglerighteq N_k = N.$$

The following fact is a direct consequence of [DM96, Cor. 7.2A]. For the reader's convenience, in the next section we provide a proof.

**Proposition 4.2.** *Assume that a group  $G$  acts faithfully and highly transitively on an infinite set  $X$ . Then any nontrivial subnormal subgroup  $N$  of  $G$  is also highly transitive on  $X$ . In particular,  $N$  cannot be virtually solvable.*

The next corollary is immediate (cf. Question 0.3).

**Corollary 4.3.** *Let  $G$  be a highly transitive group which satisfies the Tits alternative. Then  $G$  contains a free subgroup of rank two, hence is of exponential growth.*

For the groups generated by a finite collection of root subgroups, we have the following

**Theorem 4.4.** *Let  $G = \langle U_1, \dots, U_s \rangle$  be as in 3.1. If  $G$  is highly transitive, then  $G$  contains a free subgroup of rank two.*

*Proof.* By Proposition 4.2, if  $G$  is nilpotent it cannot be highly transitive. Thus, the assertion follows from Corollary 3.7.  $\square$

## 5. APPENDIX: TRANSITIVITY OF A SUBNORMAL SUBGROUP

For the sake of completeness, we provide here a proof of Proposition 4.2, which imitates the one of [DM96, Cor. 7.2A]. The first assertion of 4.2 follows from Proposition 5.1 below by recursion on the length of the normal series (31), and the second follows from Corollary 5.4.

**Proposition 5.1.** *Assume that a group  $G$  acts effectively and highly transitively on a set  $X$ . Let  $H$  be a non-trivial normal subgroup in  $G$ . Then  $H$  acts on  $X$  highly transitively.*

We begin with an elementary lemma.

**Lemma 5.2.** *Assume that a group  $G$  acts effectively and 2-transitively on a set  $Y$ . Let  $H$  be a non-trivial normal subgroup in  $G$ . Then  $H$  is transitive on  $Y$ .*

*Proof.* It suffices to notice that  $G$  permutes the  $H$ -orbits on  $Y$ .  $\square$

*Proof of Proposition 5.1.* For any  $m$ -tuple  $\alpha = \{x_1, \dots, x_m\}$  of pairwise distinct points in  $X$  we consider the stabilizers

$$G_\alpha = G_{x_1} \cap \dots \cap G_{x_m} \quad \text{and} \quad H_\alpha = H_{x_1} \cap \dots \cap H_{x_m}.$$

Then  $H_\alpha$  is a normal subgroup in  $G_\alpha$ .

We have to show that for any positive integer  $m$  and for any  $m$ -tuple  $\alpha$  the group  $H_\alpha$  acts transitively on  $X \setminus \{x_1, \dots, x_m\}$ .

By assumption,  $G_\alpha$  acts highly transitively on  $X \setminus \{x_1, \dots, x_m\}$ . By Lemma 5.2, either  $H_\alpha$  is transitive on  $X \setminus \{x_1, \dots, x_m\}$ , or  $H_\alpha = \{e\}$ .

Assuming the latter, take the minimal  $m$  with this property, where  $m \geq 1$  by Lemma 5.2. Let  $\beta = \{x_1, \dots, x_{m-1}\}$ . By assumption, the stabilizer  $H_\beta$  is transitive on  $X \setminus \{x_1, \dots, x_{m-1}\}$  (for  $m = 1$  we have  $H_\beta = H$ , and the latter follows by Lemma 5.2). Moreover,  $H_\beta$  is simply



transitive on  $X \setminus \{x_1, \dots, x_{m-1}\}$ , and so, we can identify the set  $X \setminus \{x_1, \dots, x_m\}$  with  $H_\beta \setminus \{e\}$  via the bijection

$$X \setminus \{x_1, \dots, x_m\} \ni y \mapsto h \in H_\beta \setminus \{e\}, \quad \text{where } y = hx_m.$$

Under this identification, the action of  $G_\alpha$  on  $X \setminus \{x_1, \dots, x_m\}$  goes to the action of  $G_\alpha$  on  $H_\beta \setminus \{e\}$  by conjugation, due to the relation

$$ghx_m = ghg^{-1}gx_m = ghg^{-1}x_m \quad \forall g \in G_\alpha, \quad \forall h \in H_\beta \setminus \{e\}.$$

The action by conjugation sends a pair  $(h, h^{-1})$  to a pair of the same type. Since  $H_\beta$  is infinite, it follows that the action of  $G_x \subset \text{Aut}(N)$  on  $H_\beta \setminus \{e\}$  cannot be 2-transitive, unless  $H_\beta$  is a group of exponent two.

Suppose finally that  $H_\beta$  is a group of exponent two. It is well known that such  $H_\beta$  is a power of  $\mathbb{Z}/2\mathbb{Z}$ , or, in other words, the additive group of a vector space  $V$  over the field  $\mathbb{F}_2$ . However, the action of  $\text{Aut}(H_\beta) = \mathbf{GL}(V)$  is not 3-transitive on  $H_\beta \setminus \{e\} = V \setminus \{0\}$  contrary to our assumption, because it preserves the linear (in)dependence. This contradiction completes the proof.  $\square$

**Remark 5.3.** Notice that the affine group  $G = \text{Aff}(V)$  of the vector space  $V = \mathbb{A}_{\mathbb{F}_2}^n$ ,  $n \geq 3$ , acts 3-transitively on  $V$ , while the normal subgroup of translations acts just simply transitively on  $V$ , contrary to [DM96, Exercise 2.1.16].

**Corollary 5.4.** *A virtually solvable group  $G$  cannot be highly transitive.*

*Proof.* Any virtually solvable group  $G$  contains a normal solvable subgroup  $H$  of finite index. In turn,  $H$  contains a nontrivial normal abelian subgroup  $A$ , and  $A$  contains a nontrivial cyclic subgroup  $C$ . Clearly, a cyclic subgroup cannot be highly transitive.  $\square$

By Gromov's theorem, the same conclusion applies to any finitely generated group of polynomial growth. As a curiosity, we deduce also the following elementary fact.

**Corollary 5.5.** *A nonabelian free group contains no nontrivial subnormal abelian subgroup.*

*Proof.* Indeed, any nonabelian free group is highly transitive [Cam87, MD77], while an abelian group is not.  $\square$

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