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A RIEMANNIAN NEWTON OPTIMIZATION FRAMEWORK FOR THE SYMMETRIC TENSOR RANK APPROXIMATION PROBLEM

RIMA KHOUJA^{†‡}, HOUSSAM KHALIL^{*}, AND BERNARD MOURRAIN[†]

Abstract. The symmetric tensor rank approximation problem (STA) consists in computing the best low rank approximation of a symmetric tensor. We describe a Riemannian Newton iteration with trust region scheme for the STA problem. We formulate this problem as a Riemannian optimization problem by parameterizing the constraint set as the Cartesian product of Veronese manifolds. We present an explicit and exact formula for the gradient vector and the Hessian matrix of the method, in terms of the weights and points of the low rank approximation and the symmetric tensor to approximate, by exploiting the properties of the apolar product. We introduce a retraction operator on the Veronese manifold. The Newton Riemannian iterations are performed for best low rank approximation over the real or complex numbers. Numerical experiments are implemented to show the numerical behavior of the new method first against perturbation, to compute the best rank-1 approximation and the spectral norm of a symmetric tensor, and to compare with some existing state-of-the-art methods.

Key words. symmetric tensor decomposition, homogeneous polynomials, Riemannian optimization, Newton method, retraction, complex optimization, trust region method, Veronese manifold.

AMS subject classifications. 15A69, 15A18, 53B20, 53B21, 14P10, 65K10, 65Y20, 90-08

1. Introduction. A symmetric tensor T of order d and dimension n in $\mathcal{T}^d(\mathbb{C}^n) = \mathbb{C}^n \otimes \dots \otimes \mathbb{C}^n := \mathcal{T}_n^d$, is a special case of tensors, where its entries do not change under any permutation of its d indices. We denote their set $\mathcal{S}^d(\mathbb{C}^n) := \mathcal{S}_n^d$. The Symmetric Tensor Decomposition problem (STD) consists in decomposing the symmetric tensor into linear combination of symmetric tensors of rank one i.e.

$$(STD) \quad T = \sum_{i=1}^r w_i \underbrace{v_i \otimes \dots \otimes v_i}_{d \text{ times}}, \quad w_i \in \mathbb{C}, \quad v_i \in \mathbb{C}^n$$

The smallest r such that this decomposition exists is by definition the symmetric rank of P . The STD appears in many applications in the areas of the mobile communications, in blind identification of under-determined mixtures, machine learning, factor analysis of k-way arrays, statistics, biomedical engineering, psychometrics, and chemometrics. See e.g. [14, 16, 17, 34] and references therein. The decomposition of the tensor is often used to recover structural information in the application problem.

The Symmetric Tensor Approximation problem (STA) consists in finding the closest symmetric tensor $\in \mathcal{S}_n^d$, which has a symmetric rank at most r for a given $r \in \mathbb{N}$. Using the correspondence between \mathcal{S}_n^d and the set of homogeneous polynomials of degree d in n variables denoted $\mathbb{C}[x_1, \dots, x_n]_d := \mathbb{C}[\mathbf{x}]_d$, it consists in approximating an homogeneous polynomial P associated to a symmetric tensor T by an element in σ_r , where $\sigma_r = \{Q \in \mathbb{C}[\mathbf{x}]_d \mid \text{rank}_s(Q) \leq r\}$, i.e.

$$(STA) \quad \min_{Q \in \sigma_r} \frac{1}{2} \|P - Q\|_d^2.$$

Since in many problems, the input tensors are often computed from measurements or statistics, they are known with some errors on their coefficients and computing an

^{*}Laboratory of Mathematics and its Applications LaMa-Lebanon, Lebanese University, Faculty of Sciences, Lebanon. (houssam.khalil@ul.edu.lb).

[†]Aromath, Inria Sophia Antipolis Méditerranée, Université Côte d'Azur, France. (rima.khouja@inria.fr, bernard.mourrain@inria.fr).

approximate decomposition of low rank often gives better structural information than the exact or accurate decomposition of the approximate tensor [7, 6, 21].

An approach which has been investigated to address the STA problem is to extend the Singular Value Decomposition (SVD) to this problem, since the truncated SVD of a matrix yields its best approximation of a given rank. This so-called Highed Order Singular Value Decomposition (HOSVD) method has been studied for tensor decomposition [17] or for multi-linear rank computation based on matrix SVD [18, 33], or using iterative truncation strategies [49]. However, when dealing with best low rank approximations of tensors, these techniques do not provide the closest low rank tensor, since the structure is not taken into account in the flattening operations involved in these methods.

Another classical approach for computing an approximate tensor decomposition of low rank is the so-called Alternate Least Square method (ALS). It consists in minimizing the distance between a given tensor and a multilinear tensor by alternately updating the different factors of the tensor decomposition, solving a quadratic minimization problem at each step. See e.g. [11, 12, 24, 28]. This approach is well-suited for tensor represented in \mathcal{T}_n^d but it loses the symmetry property in the internal steps of the algorithm. The space in which the linear operations are performed is of large dimension n^d compared to the dimension $\binom{n+d-1}{d}$ of \mathcal{S}_n^d when n and d grow. Moreover the convergence is slow [48, 20].

Other iterative methods such as quasi-Newton methods have been considered to improve the convergence speed. See for instance [25, 41, 40, 45, 47, 43]. In [10, 9], a Gauss-Newton iteration on a real Riemannian manifold associated to a sum of rank-1 multilinear tensors is presented and its numerical behavior is analysed. Optimization techniques based on quasi-Newton iterations for decomposition of multilinear tensors over the complex numbers have also been presented in [45, 44]. In [43] quasi-Newton and limited memory quasi-Newton methods for distance optimization on products of Grassmannian varieties are proposed and applied for best low multi-rank tensor approximation. In all these approaches, an approximation on the Hessian is used to compute the descent direction, and the local quadratic convergence cannot be guaranteed.

Specific investigations have been developed, in the case of best rank-1 approximation. The problem is equivalent to the optimisation of a polynomial on the product of unitary spheres (see e.g. [52, 18]). Global polynomial optimization methods can be employed over the real or complex numbers, using for instance convex relaxations and SemiDefinite Programming [38]. However, the approach is facing scalability issues in practice for large size tensors.

In relation with polynomial representation and multivariate Hankel matrix properties, another least square optimization problem is presented in [37], for low rank symmetric tensor approximation. Good approximation of the best low rank approximation are obtained for small enough perturbations of low rank tensors.

Contributions. In this paper, we present a new Riemannian Newton method with trust region scheme for the STA problem. Exploiting the dimension reduction of the problem by working in $\mathbb{C}[\mathbf{x}]_d$, considering a suitable representation of the points on the Riemannian manifold, and combining the properties of the apolar product with efficient tools from complex optimization, we give an explicit, exact and tractable formulation of the Newton iterations for the distance minimization on the Riemannian manifold, which is the Cartesian product of Veronese manifolds. The explicit formulation is provided for low rank symmetric tensor approximation over the real and complex numbers. We propose an approximation method for a given homogeneous

polynomial in $\mathbb{C}[\mathbf{x}]_d$ into linear form to the d^{th} power, based on the rank-1 truncation of the SVD of Hankel matrices associated to the homogeneous polynomial. From this approximation method, we present a retraction operator on the Veronese manifold. The starting point of the numerical method is chosen, beside the classical random choice, by computing an approximate rank- r decomposition of the tensor by an algebraic method, called SHD (Spectral Hankel Decomposition), based on SVD of Hankel matrices and eigen computation [23, 35]. Numerical experiments show the good numerical behavior of the new method against perturbations. The good performance of the approach appears clearly in particular, for the best rank-1 approximation of real-valued symmetric tensors. Comparisons with existing state-of-the-art methods corroborate this analysis.

The paper is structured as follows. In [section 2](#) we give the main notation and preliminaries. [Section 3](#) describes the different steps of the construction of the Riemannian Newton with trust region scheme algorithm that we develop for the STA problem, called TR-RNS. In [subsection 3.1](#), we formulate the STA problem as a Riemannian least square optimization problem. In [subsection 3.2](#), we introduce the parameterization of the points on the Riemannian manifold that we use in the computation of the gradient vector and the Hessian matrix in [subsection 3.3](#). In [subsection 3.4](#), we present a retraction operator on the Veronese manifold with analysis. In [subsection 3.5](#) we discuss the choice of the initial point in the iterative algorithm. Our new algorithm TR-RNS is presented in [subsection 3.6](#). Numerical experiments are featured in [section 4](#). The final section is for our conclusions and outlook.

2. Notation and preliminaries. We denote by \mathcal{T}_n^d the set of outer product d times of \mathbb{C}^n . The set of symmetric tensors in \mathcal{T}_n^d is denoted \mathcal{S}_n^d . We have a correspondence between \mathcal{S}_n^d and the set of the homogeneous polynomials of degree d in n variables $\mathbb{C}[x_1, \dots, x_n]_d := \mathbb{C}[\mathbf{x}]_d$ (see e.g. [15]). This allows to reduce the dimension of the ambient space of the problem from n^d (dimension of \mathcal{T}_n^d) to $\binom{n+d-1}{d}$ (dimension of $\mathcal{S}_n^d \sim \mathbb{C}[\mathbf{x}]_d$). The capital letters P, Q and T denote the homogeneous polynomials in $\mathbb{C}[\mathbf{x}]_d$ or equivalently elements in \mathcal{S}_n^d . The multilinear tensor associated to the symmetric tensor or homogeneous polynomials P in is denoted $P^\tau \in \mathcal{T}_n^d$. An homogeneous polynomial P in $\mathbb{C}[\mathbf{x}]_d$ can be written as: $P = \sum_{|\alpha|=d} \binom{d}{\alpha} p_\alpha x^\alpha$, where $x = (x_1, \dots, x_n)$ is the vector of the variables x_1, \dots, x_n , $\alpha = (\alpha_1, \dots, \alpha_n)$ is a vector of the multi-indices in \mathbb{N}^n , $|\alpha| = \alpha_1 + \dots + \alpha_n$, $p_\alpha \in \mathbb{C}$, $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\binom{d}{\alpha} := \frac{d!}{\alpha_1! \dots \alpha_n!}$. Thus we can write the multilinear tensor P^τ as $P^\tau = \sum_{1 \leq i_1, \dots, i_d \leq n} (\sum_{\alpha | e_{i_1} + \dots + e_{i_d} = \sum_{i=1}^n \alpha_i e_i} p_\alpha) e_{i_1} \otimes \dots \otimes e_{i_d}$, where $(e_i)_{1 \leq i \leq n}$ denotes the canonical basis of \mathbb{C}^n . The superscripts $.^T$, $.^*$ and $.^{-1}$ are used respectively for the transpose, Hermitian conjugate, and the inverse matrix. The complex conjugate is denoted by an overbar, e.g., \bar{w} . We use parentheses to denote vectors e.g. $W = (w_i)_{1 \leq i \leq r}$, and the square brackets to denote matrices e.g. $V = [v_i]_{1 \leq i \leq r}$.

DEFINITION 2.1. For $P = \sum_{|\alpha|=d} \binom{d}{\alpha} p_\alpha x^\alpha$ and $Q = \sum_{|\alpha|=d} \binom{d}{\alpha} q_\alpha x^\alpha$ in $\mathbb{C}[\mathbf{x}]_d$, their apolar product is

$$\langle P, Q \rangle_d := \sum_{|\alpha|=d} \binom{d}{\alpha} \bar{p}_\alpha q_\alpha.$$

The apolar norm of P is $\|P\|_d = \sqrt{\langle P, P \rangle_d} = \sqrt{\sum_{|\alpha|=d} \binom{d}{\alpha} \bar{p}_\alpha p_\alpha}$.

For $T \in \mathcal{T}_n^d$, $\text{vec}(T)$ denotes the vectorization of T i.e. $\text{vec}(T) \in \mathbb{C}^{n^d}$. We define the Frobenius norm of T by: $\|T\|_F = \|\text{vec}(T)\|_2 = \sqrt{\text{vec}(T)^* \text{vec}(T)}$. From the definition

of P^τ , we have the following property:

LEMMA 2.2. *For $P \in \mathcal{S}_n^d$, we have $\|P\|_d = \|P^\tau\|_F$.*

The following properties of the apolar product can also be verified by direct calculus:

LEMMA 2.3. *Let $L = (v_1x_1 + \dots + v_nx_n)^d := (v^t x)^d \in \mathbb{C}[\mathbf{x}]_d$ where $v = (v_i)_{1 \leq i \leq n}$ is a vector in \mathbb{C}^n , $P \in \mathbb{C}[\mathbf{x}]_d$, $Q \in \mathbb{C}[\mathbf{x}]_{(d-1)}$, we have the following two properties:*

1. $\langle L, P \rangle_d = P(\bar{v})$,
2. $\langle P, x_i Q \rangle_d = \frac{1}{d} \langle \partial_{x_i} P, Q \rangle_{(d-1)}$, $\forall 1 \leq i \leq n$.

3. Riemannian Newton optimization for the STA problem. Riemannian optimization methods are solving optimization problems over a Riemannian manifold \mathcal{M} [2]. In our problem, we will consider the following least square minimization problem

$$(3.1) \quad \min_{y \in \mathcal{M}} \frac{1}{2} \|F(y)\|^2$$

where $F : \mathcal{M} \mapsto \mathbb{R}^N$ is a smooth objective function with $N \geq \dim \mathcal{M}$ and \mathcal{M} is a Riemannian manifold associated to rank- r symmetric tensors. A Riemannian Newton method (See Algorithm 3.1) for solving (3.1) requires a Riemannian metric, and a retraction operator R_y from the tangent space $T_y \mathcal{M}$ at $y \in \mathcal{M}$ to \mathcal{M} [2, Chapter 6]. Since we will assume that \mathcal{M} is embedded in some space \mathbb{R}^M , we will take the metric induced by the Euclidean space \mathbb{R}^M .

Algorithm 3.1 Riemannian Newton method

Data: Riemannian manifold \mathcal{M} ; retraction R on \mathcal{M} ; function $F : \mathcal{M} \rightarrow \mathbb{R}^N$; objective function $f = \frac{1}{2} \|F\|^2$.

Input: A starting point $y_0 \in \mathcal{M}$.

Output: A sequence of iterates y_k .

for $k = 0, 1, 2, \dots$ **do**

1. Solve the Newton equation $\text{Hess } f(y_k)[\eta_k] = -\text{grad } f(y_k)$ for the unknown vector η_k in the tangent space $T_{y_k} \mathcal{M}$ of \mathcal{M} at y_k ;
2. Set $y_{k+1} \leftarrow R_{y_k}(\eta_k)$;

end for

3.1. Formulation of the Riemannian least square problem. We describe now the Riemannian manifold that we use for the Riemannian Newton method.

DEFINITION 3.1. *Let $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}[\mathbf{x}]_d$, $v \mapsto (v^t x)^d = \sum_{|\alpha|=d} \binom{d}{\alpha} v^\alpha x^\alpha$. The Veronese manifold in $\mathbb{C}[\mathbf{x}]_d$ denoted by \mathcal{V}_n^d [26, 51] is the set of linear forms in $\mathbb{C}[\mathbf{x}]_d - \{0\}$ to the d^{th} power. It is the image of Φ after removing the zero polynomial i.e. $\mathcal{V}_n^d := \Phi(\mathbb{C}^n) - \{0\}$.*

Let $\sigma_r \subset \mathcal{S}_n^d$ be the set of the symmetric tensors of symmetric rank bounded by r . It is a subset of the image of the following map:

$$\begin{aligned} \Sigma_r : \mathcal{V}_n^{d \times r} &:= \mathcal{V}_n^d \times \dots \times \mathcal{V}_n^d \longrightarrow \mathbb{C}[\mathbf{x}]_d \\ ((v_i^t x)^d)_{1 \leq i \leq r} &\longmapsto \Sigma_r((v_i^t x)^d)_{1 \leq i \leq r} = \sum_{i=1}^r (v_i^t x)^d \end{aligned}$$

The closure of the image of Σ_r is called the r^{th} secant of the Veronese variety \mathcal{V}_n^d . We consider the case of r strictly subgeneric symmetric rank i.e. $r < r_g$, where

r_g denotes the generic symmetric rank, which given by the "Alexander-Hirschowitz" theorem [5] as follows: $r_g = \frac{1}{n} \binom{n+d-1}{d}$ for all $n, d \in \mathbb{N}$, except for the following cases: $(d, n) \in \{(3, 5), (4, 3), (4, 4), (4, 5)\}$, where it should be increased by 1. We are interested in such cases because then Σ_r has a differential map which is generically an embedding and provides a regular parametrization of σ_r (see Terracini's lemma, e.g. in [32, section 5.3]).

The Riemannian manifold that we will use is the r^{th} cartesian product $\mathcal{M} = \mathcal{V}_n^{d \times r}$ of the Veronese variety \mathcal{V}_n^d . We reformulate the Riemannian least square problem for the symmetric tensor approximation problem as follows:

$$(STA^*) \quad \min_{y \in \mathcal{M}} f(y)$$

where $\mathcal{M} = \mathcal{V}_n^{d \times r}$, $y = ((v_i^t x)^d)_{1 \leq i \leq r}$, $f(y) = \frac{1}{2} \|F(y)\|_d^2$ and $F(y) = \Sigma_r(y) - P$.

3.2. Parameterization. For a symmetric tensor P in the image of Σ_r , its decomposition can be rewritten as $P = \sum_{i=1}^r w_i (v_i^t x)^d$ with $w_i \in \mathbb{R}_+^*$ and $\|v_i\| = 1$, for $1 \leq i \leq r$; since we have $\sum_{i=1}^r (\tilde{v}_i^t x)^d = \sum_{i=1}^r \|\tilde{v}_i\|^d (\frac{\tilde{v}_i^t}{\|\tilde{v}_i\|} x)^d = \sum_{i=1}^r w_i (v_i^t x)^d$, with $w_i := \|\tilde{v}_i\|^d$ and $v_i := \frac{\tilde{v}_i^t}{\|\tilde{v}_i\|}$ such that $\|v_i\| = 1$.

The vector $(w_i)_{1 \leq i \leq r} \in \mathbb{C}^{\times r}$ in this decomposition is called "the weight vector", and is denoted by W . The coefficient vectors of the linear polynomials $(v_i^t x)$ such that $\|v_i\| = 1$ form a matrix denoted $V = [v_i]_{1 \leq i \leq r} \in \mathbb{C}^{n \times r}$.

The objective function f in subsection 3.1 is a real valued function of complex variables; such function is non-analytic, because it cannot verify the Cauchy-Riemann conditions [42]. To apply the Riemannian Newton method, we need the second order Taylor series expansion of f . As discussed in [44], we overcome this problem by converting the optimization problem to the real domain, regarding f as a function of the real and the imaginary parts of its complex variables.

Let $\mathcal{N} = \{(W, \Re(V), \Im(V)) \mid W \in \mathbb{R}_+^{\times r}, V \in \mathbb{C}^{n \times r}, (\Re(v_i), \Im(v_i)) \in \mathbb{S}^{2n-1}, \forall 1 \leq i \leq r\}$, where \mathbb{S}^{2n-1} is the unit sphere in \mathbb{R}^{2n} . Routine calculation shows that \mathcal{N} is a Riemannian submanifold of \mathbb{R}^{r+2nr} of dimension $r + r(2n - 1) = 2nr$. This is the manifold that we will use for the parametrization of \mathcal{M} . We will consider f as a function of \mathcal{N} , in order to compute effectively its gradient vector and Hessian matrix.

3.3. Computation of the gradient vector and the Hessian matrix. Transforming the pair $(\Re(z), \Im(z))$ of the real and imaginary parts of a given complex variable z into the pair (z, \bar{z}) is a simple linear transformation, which will allow us to have obtain explicit and simple computation of the gradient and Hessian of f .

Let $R_r = \{(W, \Re(V), \Im(V)) \mid W \in \mathbb{R}^r, V \in \mathbb{C}^{n \times r}\}$, $C_r = \{(W, V, \bar{V}) \mid W \in \mathbb{R}^r, V \in \mathbb{C}^{n \times r}\}$ and

$$(3.2) \quad K = \begin{bmatrix} I_r & 0_{r \times 2nr} \\ 0_{2nr \times r} & J \end{bmatrix}$$

where $J = \begin{bmatrix} I_{nr} & iI_{nr} \\ I_{nr} & -iI_{nr} \end{bmatrix}$. The linear map K is an isomorphism from R_r to C_r and

its inverse is given by $K^{-1} = \begin{bmatrix} I_r & 0_{r \times 2nr} \\ 0_{2nr \times r} & \frac{1}{2} J^* \end{bmatrix}$.

The computation of the gradient and the Hessian of f as a function of C_r yields more elegant expressions than considering f as a function of R_r . For this reason, we

consider f as a function of C_r to compute them, and then we use the isomorphism K in (3.2) to define the gradient and the Hessian of f as a function of R_r .

Denote the gradient and the Hessian of f as a function of R_r (resp. C_r) at a point $(W, \Re(V), \Im(V))$ (resp. (W, V, \bar{V})) by G^R and H^R (resp. by G^C and H^C).

Recall that f is a function on \mathcal{N} . We can define the gradient and the Hessian of f at a point $p \in \mathcal{N}$, denoted respectively by G and H , by:

$$G = Q^T G^R, H = Q^T H^R Q$$

where the columns of Q form an orthonormal basis of $T_p \mathcal{N}$ [2, Ch.5].

Hereafter, we detail the computation of Q , G^R and H^R .

PROPOSITION 3.2. *Let $p = (W, \Re(V), \Im(V)) \in \mathcal{N}$. For all i in $\{1, \dots, r\}$, we denote by u_i the vector $(\Re(v_i), \Im(v_i)) \in \mathbb{R}^{2n}$, and by $Q_{i, re}$ (resp. $Q_{i, im}$) the matrix given by the first n rows (resp. the last n rows) and the first $2n - 1$ columns of the factor Q_i of the QR decomposition of $I_{2n} - u_i u_i^t$: $(I_{2n} - u_i u_i^t) P_i = Q_i R_i$ such that $Q_i Q_i^T = I_{2n}$, R_i is upper triangular, and P_i is a permutation matrix. Let $M = \begin{bmatrix} Q_{re} \\ Q_{im} \end{bmatrix} \in \mathbb{R}^{2nr \times (2n-1)r}$, where $Q_{re} = \text{diag}(Q_{i, re})_{1 \leq i \leq r}$ and $Q_{im} = \text{diag}(Q_{i, im})_{1 \leq i \leq r}$. Then the columns of $Q = \text{diag}(I_r, M)$ form an orthonormal basis of $T_p \mathcal{N}$.*

Proof. Denote $(\Re(V), \Im(V))$ by Z . Using that $T_p \mathcal{N} \simeq T_W(\mathbb{R}_+^*)^r \times T_Z \mathcal{V}$, where $\mathcal{V} = \{(\Re(V), \Im(V)) \mid V \in \mathbb{C}^{n \times r}, \|v_i\|^2 = 1, \forall 1 \leq i \leq r\}$. We have $T_W(\mathbb{R}_+^*)^r = \mathbb{R}^r$ and I_r is an orthonormal basis of $T_W(\mathbb{R}_+^*)^r$.

We verify that M is an orthonormal basis of $T_Z \mathcal{V}$. Firstly, for each $i \in \{1, \dots, r\}$, $u_i \in \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$, thus the first $(2n - 1)$ columns of the factor Q_i of the QR decomposition of $I_{2n} - u_i u_i^t$ give an orthonormal basis of the image of $(I_{2n} - u_i u_i^t)$, which is $T_{u_i} \mathbb{S}^{2n-1}$.

Secondly, $T_Z \mathcal{V}$ is a vector space of dimension $r(2n - 1)$ which is the cartesian product of the tangent spaces $T_{u_i} \mathbb{S}^{2n-1}$. Therefore, by construction, the columns of M form an orthonormal basis of $T_Z \mathcal{V}$.

Finally $Q = \text{diag}(I_r, M)$ is an orthonormal basis of $T_p \mathcal{N}$. \square

PROPOSITION 3.3. *The gradient G^R of f on R_r is the vector*

$$G^R = \begin{pmatrix} \Re(G_1) \\ 2 \Re(G_2) \\ -2 \Im(G_2) \end{pmatrix} \in \mathbb{R}^{r+2nr},$$

where

- $G_1 = (\sum_{i=1}^r w_i (v_j^* v_i)^d - P(\bar{v}_j))_{1 \leq j \leq r} \in \mathbb{C}^r$
- $G_2 = \frac{1}{2} (d \sum_{i=1}^r w_i w_j (v_i^* v_j)^{(d-1)} \bar{v}_i - w_j \nabla \bar{P}(v_j))_{1 \leq j \leq r} \in \mathbb{C}^{nr}$.

Proof. By using the apolar identities in Lemma 2.3, we can write f as:

$$(3.3) \quad \frac{1}{2} (f_1 - f_2 - f_3 + f_4)$$

where $f_1 = \|\sum_{i=1}^r w_i (v_i^t x)^d\|_d^2$, $f_2 = \langle \sum_{i=1}^r w_i (v_i^t x)^d, P \rangle_d = \sum_{i=1}^r w_i P(\bar{v}_i)$, $f_3 = \bar{f}_2 = \sum_{i=1}^r w_i \bar{P}(v_i)$, $f_4 = \|P\|_d^2$.

Let us decompose G^C as $G^C = \begin{pmatrix} G_1^C \\ G_2^C \\ G_3^C \end{pmatrix}$ with $G_1^C = (\frac{\partial f}{\partial w_i})_{1 \leq i \leq r}$, $G_2^C = [\frac{\partial f}{\partial v_i}]_{1 \leq i \leq r}$

and $G_3^C = [\frac{\partial f}{\partial \bar{v}_i}]_{1 \leq i \leq r}$. As f is a real valued function, we have that $\frac{\partial f}{\partial v_i} = \overline{\frac{\partial f}{\partial \bar{v}_i}}$ [36, 42], thus $G_3 = \overline{G_2}$. Using the apolar identities (e.g. $\frac{\partial f_2}{\partial w_i} = \langle (v_i^t x)^d, P \rangle_d = P(\bar{v}_i)$, $\frac{\partial f_2}{\partial v_{i,l}} =$

$dw_i \langle x_l (v_i^t x)^{d-1}, P \rangle_d = w_i \frac{\partial P}{\partial x_l}(\bar{v}_i)$, $1 \leq l \leq n$, we find by direct calculus the formula of G_1 and G_2 given in the proposition above. Now by using (3.2) and the relation $G^R = K^T G^C$, we obtain the formula of G^R in terms of G_1 , and the real and the imaginary parts of G_2 . \square

PROPOSITION 3.4. *The Hessian H^R of f on R_r is given by the following block matrix:*

$$H^R = \begin{bmatrix} \Re(A) & 2\Re(B)^T & -2\Im(B)^T \\ 2\Re(B) & 2(\Re(C) + \Re(D)) & -2(\Im(C) + \Im(D)) \\ -2\Im(B) & 2(\Im(D) - \Im(C)) & 2(\Re(D) - \Re(C)) \end{bmatrix} \in \mathbb{R}^{(r+2nr) \times (r+2nr)}$$

with

- $A = [(v_i^* v_j)^d]_{1 \leq i, j \leq r} \in \mathbb{C}^{r \times r}$,
- $B = \frac{1}{2} [dw_i (v_j^* v_i)^{d-1} \bar{v}_j + \delta_{i,j} (d \sum_{l=1}^r w_l (v_l^* v_i)^{d-1} \bar{v}_l - \nabla \bar{P}(v_j))]_{1 \leq i, j \leq r} \in \mathbb{C}^{nr \times r}$,
- $C = \frac{1}{2} \text{diag} [d(d-1) [\sum_{i=1}^r w_i w_j \bar{v}_{i,k} v_{i,l} (v_i^* v_j)^{d-2}]_{1 \leq k, l \leq n} - w_j \Delta \bar{P}(v_j)]_{1 \leq j \leq r} \in \mathbb{C}^{nr \times nr}$,
- $D = \frac{1}{2} [dw_i w_j (v_i^* v_j)^{d-2} ((v_i^* v_j) I_n + (d-1) v_j v_i^*)]_{1 \leq i, j \leq r} \in \mathbb{C}^{nr \times nr}$.

Proof. H^C is given by the following block matrix:

$$H^C = \begin{bmatrix} \left[\frac{\partial^2 f}{\partial w_i \partial w_j} \right]_{1 \leq i, j \leq r} & \left[\frac{\partial^2 f}{\partial w_i \partial v_j^t} \right]_{1 \leq i, j \leq r} & \left[\frac{\partial^2 f}{\partial w_i \partial \bar{v}_j^t} \right]_{1 \leq i, j \leq r} \\ \left[\frac{\partial^2 f}{\partial v_i \partial w_j} \right]_{1 \leq i, j \leq r} & \left[\frac{\partial^2 f}{\partial v_i \partial v_j^t} \right]_{1 \leq i, j \leq r} & \left[\frac{\partial^2 f}{\partial v_i \partial \bar{v}_j^t} \right]_{1 \leq i, j \leq r} \\ \left[\frac{\partial^2 f}{\partial \bar{v}_i \partial w_j} \right]_{1 \leq i, j \leq r} & \left[\frac{\partial^2 f}{\partial \bar{v}_i \partial v_j^t} \right]_{1 \leq i, j \leq r} & \left[\frac{\partial^2 f}{\partial \bar{v}_i \partial \bar{v}_j^t} \right]_{1 \leq i, j \leq r} \end{bmatrix}.$$

Let $A = \left[\frac{\partial^2 f}{\partial w_i \partial w_j} \right]_{1 \leq i, j \leq r}$, $B = \left[\frac{\partial^2 f}{\partial v_i \partial w_j} \right]_{1 \leq i, j \leq r}$, $C = \left[\frac{\partial^2 f}{\partial v_i \partial \bar{v}_j^t} \right]_{1 \leq i, j \leq r}$, and $D = \left[\frac{\partial^2 f}{\partial \bar{v}_i \partial v_j^t} \right]_{1 \leq i, j \leq r}$. Using the relations $\frac{\partial^2 f}{\partial \bar{z} \partial \bar{z}^t} = \overline{\frac{\partial^2 f}{\partial z \partial z^t}}$, and $\frac{\partial^2 f}{\partial z \partial \bar{z}^t} = \frac{\partial^2 f}{\partial \bar{z} \partial z^t}$, for a complex variable z , we can decompose H^C as:

$$H^C = \begin{bmatrix} A & B^T & \overline{B^T} \\ B & C & D^T \\ \overline{B} & D & \overline{C} \end{bmatrix}.$$

Now, as in the previous proof, by writing f as in (3.3), and using the apolar identities, we obtain the formula of A, B, C and D given in the above proposition. Using (3.2) we have that $H^R = K^T H^C K$, which gives the formula for H^R . \square

3.4. Retraction. The retraction is an important ingredient of a Riemannian optimization and choosing an efficient retraction is crucial [2, 4, 30].

DEFINITION 3.5. *Let \mathcal{M} be a manifold and $p \in \mathcal{M}$. A retraction R_p is a map $T_p \mathcal{M} \rightarrow \mathcal{M}$, which satisfies the following properties:*

1. $R_p(0_p) = p$;
2. there exists an open neighborhood $\mathcal{U}_p \subset T_p \mathcal{M}$ of 0_p such that the restriction on \mathcal{U}_p is well-defined and a smooth map;
3. R_p satisfies the local rigidity condition

$$DR_p(0_p) = id_{T_p \mathcal{M}}$$

where $id_{T_p \mathcal{M}}$ denotes the identity map on $T_p \mathcal{M}$.

We will use the following well-known lemma to construct a retraction on $\mathcal{M} = \mathcal{V}_n^{d \times r}$ [3, 10].

LEMMA 3.6. *Let $\mathcal{M}_1, \dots, \mathcal{M}_r$ be manifolds, $p_i \in \mathcal{M}_i$ and $\mathcal{M} = \mathcal{M}_1 \times \dots \times \mathcal{M}_r$ and $p = (p_1, \dots, p_r) \in \mathcal{M}$. Let $R_i : T_{p_i} \mathcal{M}_i \rightarrow \mathcal{M}_i$ be retractions. Then $R_p : T_p \mathcal{M} \rightarrow \mathcal{M}$ defined as follows: $R_p(\xi_1, \dots, \xi_r) = (R_{p_1}(\xi_1), \dots, R_{p_r}(\xi_r))$ for $\xi_i \in T_{p_i} \mathcal{M}_i$, $1 \leq i \leq r$, is a retraction on \mathcal{M} .*

Our construction of a retraction on \mathcal{V}_n^d is described in the following definitions.

DEFINITION 3.7. *The Hankel matrix of degree $k, d-k$ associated to a polynomial P in $\mathbb{C}[\mathbf{x}]_d$ is given by:*

$$H_P^{k, d-k} = (\langle P, x^{\alpha+\beta} \rangle_d)_{|\alpha|=k, |\beta|=d-k}$$

DEFINITION 3.8. *Let $\Pi : \mathbb{C}[\mathbf{x}]_d \rightarrow \mathcal{V}_n^d$, $P \mapsto \Pi(P) = P(\bar{q})(q^t x)^d$; where q is the first left singular vector from the SVD decomposition of $H_P^{1, d-1}$. For $P \in \mathcal{V}_n^d$, let $R_P : T\mathcal{V}_n^d \rightarrow \mathcal{V}_n^d$, such that $R_P : T_P \mathcal{V}_n^d \rightarrow \mathcal{V}_n^d$, $Q \mapsto R_P(Q) = \Pi(P + Q)$.*

We verify that R_P is a retraction on \mathcal{V}_n^d .

LEMMA 3.9. *Let $P \in \mathcal{V}_n^d$, P is a fixed point by Π .*

Proof. Let $P \in \mathcal{V}_n^d$, we have to show that $\Pi(P) = P$. In fact, as $P \in \mathcal{V}_n^d$, we can write $P = w(v^t x)^d$ with $w \in \mathbb{R}_+^*$, $v \in \mathbb{C}^n$, $\|v\|^2 = 1$. We have $\Pi(P) = P(\bar{q})(q^t x)^d$, where q is as in Definition 3.8. Note that $H_P^{1, d-1}$ is of rank one since $P \in \mathcal{V}_n^d$ and its image is spanned by v : $\text{span}(H_P^{1, d-1}) = \langle v \rangle$. As $q \in \text{span}(H_P^{1, d-1})$ and $\|q\| = 1$, we have that $q = e^{i\theta} v$ with $\theta \in [0, 2\pi[$. By substituting q by $e^{i\theta} v$ in $\Pi(P)$, we find that $\Pi(P) = P$, which ends the proof. \square

PROPOSITION 3.10. *Let $P \in \mathcal{V}_n^d$. There exists a neighborhood $\mathcal{U}_P \subset \mathbb{C}[\mathbf{x}]_d$ of P such that the restriction $\Pi|_{\mathcal{U}_P}$ is well-defined and C^∞ smooth.*

Proof. Let $P \in \mathcal{V}_n^d$ and $\theta : \mathbb{C}[\mathbf{x}]_d \rightarrow \mathbb{C}^n$, $Q \mapsto q$ where q is the first left singular vector of the SVD decomposition of $H_Q^{1, d-1}$. Let $\delta : \mathbb{C}^n \rightarrow \mathcal{V}_n^d$, $a \mapsto (a^t x)^d$ and let $\gamma : \mathbb{C}[\mathbf{x}]_d \rightarrow \mathcal{V}_n^d = \delta \circ \theta$ be the composition map. By construction, we have $\Pi : P \mapsto \langle P, \gamma(P) \rangle_d \gamma(P)$. Let \mathcal{O} denotes the open set of homogeneous polynomials $Q \in \mathbb{C}[\mathbf{x}]_d$ such that the Hankel matrix $H_Q^{1, d-1}$ has a nonzero gap between the first and the second singular values. It follows from [13] that the map θ is well-defined and smooth on \mathcal{O} . As P is in \mathcal{V}_n^d and $H_P^{1, d-1}$ is of rank 1, $P \in \mathcal{O}$. Let \mathcal{U}_P be a neighborhood of P in $\mathbb{C}[\mathbf{x}]_d$ such that $\delta|_{\mathcal{U}_P}$ is well-defined and smooth. As the apolar product $\langle \cdot, \cdot \rangle_d$ and the multiplication are well-defined and smooth on $\mathbb{C}[\mathbf{x}]_d \times \mathbb{C}[\mathbf{x}]_d$, Π is well-defined and smooth on \mathcal{U}_P , which ends the proof. \square

DEFINITION 3.11. *Let $T \in \mathbb{C}^{n_1 \times \dots \times n_d}$ be a tensor of dimension d and size (n_1, \dots, n_d) . For $i \in \{1, \dots, d\}$, the i^{th} mode matricization of T*

$$T_{(i)} \in \mathbb{C}^{n_i \times \pi_{j \neq i} n_j}$$

is a rearrangement of the entries of T into the matrix $T_{(i)}$, such that the i^{th} mode index the row of $T_{(i)}$ and the other $(d-1)$ modes index the columns of $T_{(i)}$.

The following result similar to the case of multilinear tensors [10][22, Ch.10] [31], shows that Π gives a good rank-1 approximation:

PROPOSITION 3.12. *Let $P \in \mathbb{C}[\mathbf{x}]_d$. We have the following inequality:*

$$\|P - \Pi(P)\|_d \leq \sqrt{d} \|P - P_{\text{best}}\|_d$$

where $P_{best} \in \mathcal{V}_n^d$ is a best approximation of P in \mathcal{V}_n^d for the apolar norm $\|\cdot\|_d$.

Proof. Let $\theta_i^\tau : \mathcal{T}_n^d \rightarrow \mathbb{C}^n$, such that for $T \in \mathcal{T}_n^d$ $\theta_i^\tau(T)$ is the first left singular vector of $T_{(i)}$. Note that $H_P^{1,d-1}$ and the d -mode of matricization of P^τ are equals all together up to a right projection N : $H_P^{1,d-1} = P_{[i]}^\tau N$. This implies that the first left singular vectors $\theta_i^\tau(P^\tau)$ for all i in $\{1, \dots, d\}$ are equal to the left singular vector $q = \theta(P)$ of $H_P^{1,d-1}$. By direct calculus, we verify that $\Pi(P) = P(\bar{q})(q^t x)^d$ is equal to $(q, \dots, q)(q^*, \dots, q^*)P^\tau$.

Let $\mathcal{V}_n^{d\tau} = \{a \otimes \dots \otimes a \mid a \in \mathbb{C}^n - \{0\}\}$. We denote by P_{best}^τ any best approximation of P^τ in $\mathcal{V}_n^{d\tau}$ in the norm $\|\cdot\|_F$. For i in $\{1, \dots, d\}$. Let $\text{Trunc}_i : \mathcal{T}_n^d \rightarrow \mathcal{T}_n^d$, $T \mapsto (Q_{i,1}, \dots, Q_{i,d})(Q_{i,1}^*, \dots, Q_{i,d}^*)T$ where

$$Q_j = \begin{cases} \text{the square matrix made of first left singular vector of } T_{(i)} & \text{if } i = j, \\ I_n & \text{otherwise.} \end{cases}$$

Thus $(q, \dots, q)(q^*, \dots, q^*)P^\tau$ is equal to $\prod_{i=1}^d \text{Trunc}_i(P^\tau)$. Notice that $\text{Trunc}_i(P^\tau)_{(i)}$ is the rank-1 SVD truncation of $P_{(i)}^\tau$, and that it is the best rank-1 matrix approximation of $P_{(i)}^\tau$. Then we have

$$\|P^\tau - \text{Trunc}_i(P^\tau)\|_F = \|P_{(i)}^\tau - \text{Trunc}_i(P^\tau)_{(i)}\|_F \leq \|P_{(i)}^\tau - P_{best(i)}^\tau\|_F = \|P^\tau - P_{best}^\tau\|_F.$$

Using the inequality $\|T - \prod_{i=1}^d \text{Trunc}_i(T)\|_F^2 \leq \sum_{i=1}^d \|T - \text{Trunc}_i(T)\|_F^2$ for $T \in \mathcal{T}_n^d$ proved in [22, Lemma 4.123], we have

$$\begin{aligned} \|P - \Pi(P)\|_d^2 &= \|P^\tau - (q, \dots, q)(q^*, \dots, q^*)P^\tau\|_F^2 = \|P^\tau - \prod_{i=1}^d \text{Trunc}_i(P^\tau)\|_F^2 \\ &\leq \sum_{i=1}^d \|P^\tau - \text{Trunc}_i(P^\tau)\|_F^2 \leq d \|P^\tau - P_{best}^\tau\|_F^2 = d \|P - P_{best}\|_d^2, \end{aligned}$$

which ends the proof. \square

PROPOSITION 3.13. *Let $P \in \mathcal{V}_n^d$. The map $R_P : T_P \mathcal{V}_n^d \rightarrow \mathcal{V}_n^d$, $Q \mapsto R_P(Q) = \Pi(P + Q)$ is a retraction operator on the Veronese manifold \mathcal{V}_n^d .*

Proof. We have to prove that R_P verify the three properties in Definition 3.5.

1. $R_P(0_P) = \Pi(P + 0_P) = \Pi(P) = P$, by using Lemma 3.9.
2. Let $S_P : T_P \mathcal{V}_n^d \rightarrow \mathbb{C}[\mathbf{x}]_d$, $Q \mapsto P + Q$. The map S_P is well-defined and smooth on $T_P \mathcal{V}_n^d$. By Proposition 3.10, Π is well-defined and smooth in a neighborhood \mathcal{U}_P of $P \in \mathcal{V}_n^d$. Thus $R_P = \Pi \circ S_P$ is well-defined and smooth in a neighborhood $\mathcal{U}'_P \subset T_P \mathcal{V}_n^d$ of 0_P .
3. Let $\rho_{\mathcal{V}_n^d} : \mathcal{U}_P \rightarrow \mathcal{V}_n^d$, $P \in \mathcal{U}_P \mapsto P_{best}$ be the projection map onto \mathcal{V}_n^d . As $T_P \mathcal{V}_n^d$ is the first order approximation of \mathcal{V}_n^d around $P \in T_P \mathcal{V}_n^d$, it follows that $\|(P + tQ) - \rho_{\mathcal{V}_n^d}(P + tQ)\|_d = O(t^2)$ for $t \rightarrow 0$. Using Proposition 3.12 we have

$$\|(P + tQ) - R_P(tQ)\|_d \leq \sqrt{d} \|(P + tQ) - \rho_{\mathcal{V}_n^d}(P + tQ)\|_d = O(t^2),$$

which implies that $R_P(tQ) = P + tQ + O(t^2)$. Thus $\frac{d}{dt} R_P(tQ) |_{t=0} = Q$, which is equivalent to $DR_P(0_P).(Q) = Q$. Therefore we have $DR_P(0_P) = id_{T_P \mathcal{V}_n^d}$. \square

3.5. Choice of the initial point. The choice of the initial point is a crucial step in Riemannian Newton iterative methods. We consider two cases.

In the first case, the initial point (W_0, V_0) is chosen such that $W_0 = (w_i^0)_{1 \leq i \leq r} \in \mathbb{R}^r$ and $V_0 = [v_i^0]_{1 \leq i \leq r} \in \mathbb{K}^{n \times r}$, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , randomly according to an uniform distribution.

In the second case, we use the direct algorithm of [23], based on SVD decomposition of Hankel matrices and eigen computation (see also [35]), to compute an initial rank- r approximation. This algorithm, denoted SHD, works only with r such that $r < r_g$ and $\iota \leq \frac{d-1}{2}$, where ι denotes the interpolation degree of the points in the rank- r decomposition [19, Ch.4]. The rationale behind choosing the initial point with this method is when the symmetric tensor is already of symmetric rank r with $r < r_g$ and $\iota \leq \frac{d-1}{2}$, then this computation gives a good numerical approximation of the exact decomposition, so that the Riemannian Newton algorithm needs few iterations to converge numerically. We will see in the second numerical experiment in subsection 4.2 that this initial point is an efficient choice to get a good real rank-1 approximation of a real symmetric tensor.

After we have chosen the initial point (W_0, V_0) by one of the two methods above, we solve the quadratic least square problem

$$(3.4) \quad \min_{(w_i)_{1 \leq i \leq r} \in \mathbb{K}^{\times r}} \left\| \sum_{i=1}^r w_i w_i^0 (v_i^{0t} x)^d - P \right\|_d;$$

in order to get the weight vector $W_0 = (w_i w_i^0)_{1 \leq i \leq r}$. The aim of this step is to get the optimal combination of the directions v_i^0 , which yields a reduction in the number of iterations for the numerical convergence of Riemannian Newton iterations [10]. This initial points (W_0, V_0) are normalized so that $w_i^0 \in \mathbb{R}_*^+$ and $\|v_i^0\| = 1$ as in subsection 3.2.

3.6. Riemannian Newton algorithm with trust region scheme for the STA problem. In the previous sections we described all the ingredients of a Riemannian Newton algorithm for the STA problem. Unfortunately, the convergence of this algorithm may not occur from the beginning, that is because Newton method converges if the initial point is close enough to a fix point solution. By adding a trust region scheme to the Riemannian Newton algorithm, we enhance the algorithm, with the desirable global properties of convergence to a local minimum, with a local superlinear rate of convergence [2, Chapter 7], [1].

Let $p_k = (W_k, \mathfrak{R}(V_k), \mathfrak{S}(V_k)) \in \mathcal{N}$. The idea is to approximate the objective function f to its second order Taylor series expansion in a ball of center $0_{p_k} \in T_{p_k} \mathcal{N}$ and radius Δ_k denoted by $B_{\Delta_k} := \{u \in T_{p_k} \mathcal{N} \mid \|u\| \leq \Delta_k\}$, and to solve the subproblem

$$(3.5) \quad \min_{u \in B_{\Delta_k}} m_{p_k}(u)$$

where $m_{p_k}(u) := f(p_k) + G_k^T u + \frac{1}{2} u^T H_k u$ and G_k is the gradient at p_k (Proposition 3.3) and H_k is the Hessian of f at p_k (Proposition 3.4).

By solving (3.5), we obtain a solution $u_k \in T_{p_k} \mathcal{N}$. Accepting or rejecting the candidate new point $p_{k+1} = R_{p_k}(u_k)$ is based on the quotient $\rho_k = \frac{f(p_k) - f(p_{k+1})}{m_{p_k}(0) - m_{p_k}(u_k)}$. If ρ_k exceeds 0.2 then the current point p_k is updated, otherwise the current point p_k remains unchanged.

The radius of the trust region Δ_k is also updated based on ρ_k . We choose to update the trust region as in [10] with a few changes.

Let $\Delta_{p_0} := 10^{-1} \sqrt{\frac{d}{r} \sum_{i=1}^r \|w_i^0\|^2}$, $\Delta_{max} := \frac{1}{2} \|P\|_d$. We take the initial radius as $\Delta_0 = \min\{\Delta_{p_0}, \Delta_{max}\}$, if $\rho_k > 0.6$ then the trust region is enlarged as follow: $\Delta_{k+1} = \min\{2\|u_k\|, \Delta_{max}\}$. Otherwise the trust region is shrunked by taking $\Delta_{k+1} = \min\left\{\left(\frac{1}{3} + \frac{2}{3} \cdot (1 + e^{-14 \cdot (\rho_k - \frac{1}{3})})^{-1}\right) \Delta_k, \Delta_{max}\right\}$.

We choose the so-called dogleg method to solve the subproblem (3.5) [39]. Let p_N be the Newton direction given by the Newton equation $H p_N = -G$, and let p_c denotes the Cauchy point given by $p_c = -\frac{G^T G}{G^T H G} G$. Then the optimal solution p^* of (3.5) by the dogleg method is given as follows:

$$p^* = \begin{cases} p_N & \text{if } \|p_N\| \leq \Delta \\ -\frac{\Delta}{\|G\|} G & \text{if } \|p_N\| > \Delta \text{ and } \|p_c\| \geq \Delta \\ p_I & \text{otherwise} \end{cases}$$

where p_I is the intersection of the boundary of the sphere B_Δ and the vector pointing from p_c to p_N .

The algorithm of the Riemannian Newton method with trust region scheme for the STA problem is denoted by TR-RNS, and it is given by:

Algorithm 3.2 Riemannian Newton algorithm with trust region scheme for the STA problem (TR-RNS)

Input: The homogeneous polynomial $P \in \mathbb{C}[\mathbf{x}]_d$ associated to the symmetric tensor to approximate, $r < r_g$

Choose initial point (W_0, V_0)

while the method has not converged **do**

1. Compute the vectors G_1 and G_2 and the matrices A, B, C and D
2. Compute the vector G^R and the matrix H^R
3. Compute the basis matrix Q
4. Solve the subproblem (3.5) for the search direction $u_k \in B_{\Delta_k}$ by using the dogleg method
5. Compute the candidate next new point $p_{k+1} = R_{p_k}(u_k)$
6. Compute the quotient ρ_k
7. Accept or reject p_{k+1} based on the quotient ρ_k
8. Update the trust region radius Δ_k

end while

Output: (\tilde{W}, \tilde{V})

The Algorithm 3.2 is stopped when $\Delta_k \leq 10^{-3}$, or when the maximum number of iterations exceeds N_{max} .

REMARK 3.14. For the implementation of the retraction in the step 5 in Algorithm 3.2; let us assume that the subproblem (3.5) is solved at a point $p = (W, \Re(V), \Im(V)) \in \mathcal{N}$, such that $W = (w_i)_{1 \leq i \leq r}$, and $V = [v_i]_{1 \leq i \leq r}$, in local coordinates with respect to the basis Q as in Proposition 3.2. It yields a solution vector $\hat{p} \in \mathbb{R}^{r+2r(2n-1)}$. The tangent vector $p^* \in T_p \mathcal{N}$ is given by: $p^* = Q \hat{p}$, of size $r + 2nr$. For each $j \in \{1, \dots, r\}$, Now, let denote $w_j^* = p^*[j]$, and $v_j^* = p^*[r + (j-1)n + 1 : r + jn] + p^*[r + rn + (j-1)n + 1 : r + rn + jn]i$.

Using Lemma 3.6 for each j in $\{1, \dots, r\}$, we denote $P_j := w_j(v_j^t x)^d \in \mathcal{V}_n^d$, $tg_j := w_j^*(v_j^t x)^d + dw_j(v_j^t x)^{d-1}(v_j^{*t} x) \in T_{P_j} \mathcal{V}_n^d$.

Using the retraction in [Proposition 3.13](#), we have $R_{P_j}(tg_j) = (P_j + tg_j)(\bar{q}_j) (q_j^t x)^d$, where q_j is the first left singular vector of $H_{P_j + tg_j}^{1,d-1}$.

Finally, we define the new point $(\tilde{W}, \mathfrak{R}(\tilde{V}), \mathfrak{S}(\tilde{V})) \in \mathcal{N}$ with $\tilde{W} = (\tilde{w}_j)_{1 \leq j \leq r}$, $\tilde{V} = [\tilde{v}_j]_{1 \leq j \leq r}$, such that $\tilde{w}_j = |z_j|$, $\tilde{v}_j = e^{i \frac{\theta_j}{d}} q_j$, where $z_j := (P_j + tg_j)(\bar{q}_j) \in \mathbb{C}$ and $\theta_j := \text{argument of } z_j$.

REMARK 3.15. *The TR-RNS algorithm can handle ill-conditioned Hessian matrices. This can happen if the symmetric tensor P has infinitely many decompositions or if its border symmetric rank is not equal to its symmetric rank [32, section 2.4], [15], [10]. In practice we use the Moore-Penrose pseudoinverse ill-conditioned matrices [8, 46, 29].*

4. Numerical experiments. In this section we present three numerical experiments of using the TR-RNS algorithm. In [subsection 4.1](#) we choose randomly some real and complex examples of symmetric decompositions uniformly distributed, that we perturb. Then we apply the TR-RNS algorithm in order to explore its practical behavior, choosing for the initial point the non-perturbed rank- r tensor. In [subsection 4.2](#) we explore the performance of the RNS algorithm (which is the TR-RNS algorithm after removing the trust region scheme) to find a best real rank-1 approximation of a real symmetric tensor. In [subsection 4.3](#) we compare with some examples of real and complex valued symmetric tensors, the performance of the TR-RNS algorithm with state-of-the-art nonlinear least-square solvers CCPD-NLS and SDF-NLS from Tensorlab v3 [50] in MATLAB 7.10. The TR-RNS algorithm is implemented in Julia version 1.1.1 in the package `TensorDec.jl`¹ The experimentation have been done on a Dell Window desktop with 8 GB memory and Intel 2.3 GHz CPU. We fix a maximum of $N_{max} = 500$ iterations in the iterative algorithms.

4.1. The TR-RNS algorithm against perturbation. Here is the setting for this experimentation with $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , $r \leq r_g$.

1. Let $(W_0, V_0) \in \mathbb{K}^n \times \mathbb{K}^{n \times r}$ be a random symmetric decomposition uniformly distributed.
2. Let $P_0 = \sum_{i=1}^r w_i^0 (v_i^0)^t x^d$. As P_0 is a homogeneous polynomial in $\mathbb{K}[\mathbf{x}]_d$, we can write $P_0 = \sum_{|\alpha|=d} \binom{d}{\alpha} a_\alpha x^\alpha$.
3. Create a perturbation of P_0 : $\tilde{P} = \sum_{|\alpha|=d} \left(\binom{d}{\alpha} a_\alpha + 10^{-e} b_\alpha \right) x^\alpha$, where $b_\alpha \in \mathbb{K}$ is a random number chosen according to the normal distribution, and $e \in \{0, 1, 2, 3\}$.
4. Apply the TR-RNS algorithm on \tilde{P} with (W_0, V_0) as initial point for finding rank- r approximation.

The analysis of this numerical experiment is as follow: First the symmetric tensor P_0 is of symmetric rank r . In the step 3 of the experimental setup above, the perturbed symmetric tensor \tilde{P} has a generic symmetric rank r , and then we can consider (W_0, V_0) as a rank- r approximation of \tilde{P} . In order to improve this rank- r approximation, we can apply the TR-RNS algorithm, since it will converge to a local minimum in the neighborhood of (W_0, V_0) .

We apply this numerical experiment on two examples of real (resp. of complex) symmetric tensor. We denote by d_0 the norm between \tilde{P} and P_0 i.e. d_0 is the initial error, and by d_* the norm between \tilde{P} and P_* , where P_* denotes the rank- r symmetric

¹It can be obtained from "<https://gitlab.inria.fr/AlgebraicGeometricModeling/TensorDec.jl>". See functions RNS and TR-RNS.

tensor approximation that we find by applying the TR-RNS algorithm; N and t denote respectively the number of iterations and the consumed time in seconds(s).

A. Real symmetric tensor examples.

EXAMPLE 4.1. *In this example we choose $n = 2, d = 3, r = 2$*

TABLE 1
Computational results of Example 4.1

e	3	2	1	0
d_0	0.00134307	0.03194333	0.17372914	1.53017120
d_*	0.00118456	0.01805463	0.08153017	0.36553821
N	2	2	5	10
$t(s)$	0.000593	0.000596	0.00149	0.002978

As expected the TR-RNS algorithm converges to a local minimum, and the algorithm needs up to 10 iterations to numerically converge, and the time consumed for one iteration is around 3ms.

EXAMPLE 4.2. *In this example we choose $n = 7, d = 5, r = 15$.*

TABLE 2
Computational results of Example 4.2

e	3	2	1	0
d_0	0.02010075	0.2134006	2.0286789	22.68697293
d_*	0.01861737	0.19530094	1.86973115	20.99543927
N	4	8	10	12
$t(s)$	0.940281	1.880589	2.344324	2.761686

In this example, we chose to enlarge n, d and r . We notice that as in the previous examples the method improves the initial rank- r approximation, but with a higher number of iterations, and that the computation time per iteration increases. This is normal since the complexity of the algorithm increases with n, d and r . Despite that, the time is still low around 0.23 seconds per iteration.

B. Complex symmetric tensor examples.

EXAMPLE 4.3. *In this example we choose $n = 3, d = 4, r = 10$*

TABLE 3
Computational results of Example 4.3

e	3	2	1	0
d_0	0.00282422	0.03660158	0.42268605	3.09545345
d_*	0.00131745	0.02277818	0.27910713	2.23842866
N	2	2	16	6
t	0.020744	0.020744	0.165978	0.062222

In this example the TR-RNS method consumed around 0.01037 seconds by iteration.

EXAMPLE 4.4. *In this example we choose $n = 8, d = 3, r = 5$.*

TABLE 4
Computational results of Example 4.4

e	3	2	1	0
d_0	0.01069129	0.10194451	1.0410976	11.45692044
d_1	0.00884072	0.08534489	0.8858317	9.91119889
N	4	4	12	20
t	0.133396	0.126917	0.397085	0.664668

In this example TR-RNS method consumed around 0.033 seconds by iteration.

This numerical experiment shows that for real or complex valued symmetric tensor the TR-RNS algorithm has a good numerical behavior in term of consumed time per iteration, and of the number of iterations needed to the convergence.

4.2. Best rank-1 approximation and spectral norm. Let $P \in \mathcal{S}^d(\mathbb{R}^n)$, a best real rank-1 approximation of P is a minimizer of the optimization problem

$$(4.1) \quad \text{dist}_1(P) := \min_{X \in \mathcal{S}^d(\mathbb{R}^n), \text{rank} X=1} \|P - X\|_d^2 = \min_{(w,v) \in \mathbb{R} \times \mathbb{S}^{n-1}} \|P - w(v^t x)^d\|_d.$$

where $\mathbb{S}^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$ is the unit sphere. This problem is equivalent to $\min_{X \in \mathcal{T}^d(\mathbb{R}^n), \text{rank} X=1} \|P^T - X\|_F^2$ since at least one global minimizer is a symmetric rank-1 tensor [52].

The real spectral norm of $P \in \mathcal{S}^d(\mathbb{R}^n)$, denoted by $\|P\|_{\sigma, \mathbb{R}}$ or equivalently $\|P\|_{\sigma, \mathbb{R}}$, is by definition:

$$(4.2) \quad \|P\|_{\sigma, \mathbb{R}}^2 := \max_{v \in \mathbb{S}^{n-1}} |P(v)|$$

The two problems (4.1) and (4.2) are related by the following equality:

$$\text{dist}_1(P)^2 = \|P\|_d^2 - \|P\|_{\sigma, \mathbb{R}}^2$$

which we deduce by simple calculus and properties of the apolar norm (see also [52, 18]):

$$\begin{aligned} \text{dist}_1(P)^2 &= \min_{(w,v) \in \mathbb{R} \times \mathbb{S}^{n-1}} \|P - w(v^t x)^d\|_d^2 \\ &= \min_{(w,v) \in \mathbb{R} \times \mathbb{S}^{n-1}} \|P\|_d^2 - 2\langle P, w(v^t x)^d \rangle_d + \|w(v^t x)^d\|_d^2 \\ &= \min_{(w,v) \in \mathbb{R} \times \mathbb{S}^{n-1}} \|P\|_d^2 - 2wP(v) + w^2 \\ &= \min_{v \in \mathbb{S}^{n-1}} \|P\|_d^2 - |P(v)|^2 = \|P\|_d^2 - \max_{v \in \mathbb{S}^{n-1}} |P(v)|^2 = \|P\|_d^2 - \|P\|_{\sigma, \mathbb{R}}^2. \end{aligned}$$

Therefore if v is a global maximizer of (4.2) such that $w = P(v)$, then $w v^{\otimes d}$ is a best rank-1 approximation of P . Herein, a rank-1 approximation $w v^{\otimes d}$, such that $w = P(v)$ and $\|v\| = 1$, is better when $|w|$ is higher. Therefore, in the following experimentation, we report the weight w obtained by the different methods.

In [38] the authors present an algorithm called hereafter "SDP" based on semidefinite relaxations to find a best real rank-1 approximation of a real symmetric tensor by finding the global optimum of P on \mathbb{S}^{n-1} . In our second numerical experiment we choose to apply, on some of examples from [38] our method RNS with real initial point chosen according to the SHD algorithm as in subsection 3.5. Recall that

the RNS algorithm is an exact Riemannian Newton method, and then it has a local quadratic rate of convergence [2, Theorem 6.3.2]. Note that the computations in [38] are implemented in MATLAB 7.10 on a Dell Linux desktop with 8 GB memory and Intel 2.8 GHz CPU, using double precision arithmetic (64 bits).

We denote by $w_{RNS}(v_{RNS}^t x)^d$ (resp. $w_{sdp}(v_{sdp}^t x)^d$) the rank-1 approximation of $P \in \mathbb{R}[\mathbf{x}]_d$ obtained by RNS method (resp. SDP method). Note that $|w_{sdp}|$ is the spectral norm of P , since SDP gives a best rank-1 approximation. We denote by N the number of iterations needed for the convergence of RNS method. We report the time spent of RNS method in the seconds (s) and we denote it by t_{RNS} , while the computation time in SDP method, in the format hours:minutes:seconds, is denoted by t_{sdp} . We denote by d_0 the norm between P and the initial point of RNS method, and by d_* the norm between P and the local minimizer obtained after the convergence of the RNS method.

EXAMPLE 4.5. [13, Example 3.5]. Consider the tensor $P \in \mathcal{S}^3(\mathbb{R}^n)$ with entries: $(P)_{i_1, i_2, i_3} = \frac{(-1)^{i_1}}{i_1} + \frac{(-1)^{i_2}}{i_2} + \frac{(-1)^{i_3}}{i_3}$.

TABLE 5
Computational results for Example 4.5

n	$ w_{RNS} $	d_0	d_*	N	t_{RNS}	$ w_{sdp} $	t_{sdp}
10	17.80	32.3787	28.4489	5	0.0988	17.80	0:00:02
20	34.16	80.7752	64.884	6	0.7817	34.16	0:00:06
30	50.14	133.5443	102.1653	8	3.2093	50.14	0:00:30
40	65.93	187.9941	139.8421	7	7.1714	65.93	0:04:05
50	81.59	290.2738	177.7605	6	14.8683	81.59	0:32:45

In this example the RNS algorithm finds a best rank-1 approximation, which implies that the RNS algorithm found a minimizer in a neighborhood of the initial point chosen by the SHD algorithm, and it is in fact a global minimizer, and it converged quadratically to this point with very reduced time compared to the SDP algorithm especially when n grows.

EXAMPLE 4.6. [13, Example 3.7]. Consider the tensor $P \in \mathcal{S}^5(\mathbb{R}^n)$ given as: $(P)_{i_1, \dots, i_5} = (-1)^{i_1} \ln(i_1) + \dots + (-1)^{i_5} \ln(i_5)$.

TABLE 6
Computational results for 4.6

n	$ w_{RNS} $	d_0	d_*	N	t_{RNS}	$ w_{sdp} $	t_{sdp}
5	1.100e+2	526.1339	477.4742	5	0.0582	1.100e+2	0:00:01
10	8.833e+2	6558.5283	6095.9062	5	0.50071	8.833e+2	0:00:22
15	2.697e+3	26317.7883	24642.9612	6	3.8200	2.697e+3	0:01:18
20	6.237e+3	64267.4747	60435.2823	6	18.2790	6.237e+3	0:22:30
25	11.504e+3	132213.3759	121892.3211	6	34.7682		

In this example, we notice that t_{sdp} increases a lot with n . For example, for $n = 20$, SDP method needs 22.5 minutes to find a best rank-1 approximation. We added one row to the Table 6, for $n = 25$, with the RNS method, such case is not implemented with the SDP method because of its high complexity. Nevertheless, we can notice that for $n = 25$, the RNS algorithm needs only 6 iterations to converge with 34.7682 seconds, but we can't know if it is a best rank-1 approximation or not since we don't have the rank-1 approximation by the SDP method.

EXAMPLE 4.7. [13, corresponds to Example 3.10]. Let $P \in \mathcal{S}^d(\mathbb{R}^n)$ such that $(P)_{i_1, \dots, i_d} = \sin(i_1 + \dots + i_d)$. In this example we compare the performance of RNS, SDF-NLS, CCPD-NLS, SPD and SHOPM [27] all together, for $d = 3$ with $n = 10, 15, 20, 25$, and for $d = 4$ with $n = 10, 15$. The numerical results of SPD and SHOPM (resp. SDF-NLS and CCPD-NLS) are taken from [38] (resp. from our implementation in Tensorlab V3.0 in MATLAB 7.10).

We denote by $w_{sdf}(v_{sdf}^t x)^d$ (resp. $w_{ccpd}(v_{ccpd}^t x)^d$, $w_{shopm}(v_{shopm}^t x)^d$) the rank-1 approximation obtained by SDF-NLS (resp. CCPD-NLS, SHOPM). The time spent by SHOPM and SDP methods is not available in [38]. We mention the time spent by the other methods in this comparison. We denote by t_{sdf} (resp. t_{ccpd}) the time in seconds(s) spent by SDF-NLS (resp. CCPD-NLS). We denote by N_{RNS} (resp. N_{sdf} , N_{ccpd}) the number of iterations needed in RNS method (resp. SDF-NLS, CCPD-NLS).

TABLE 7
Computational results for Example 4.7 with $d=3, 4$.

n	10	15	20	25	10	15
$ w_{RNS} $	12.12	22.07	32.98	44.46	27.27	58.8483
N_{RNS}	6	8	8	8	6	6
t_{RNS}	0.0706	0.2783	0.8599	1.7010	0.2208	0.8496
$ w_{sdf} $	11.1384	19.3699	29.0885	43.7844	22.6844	51.2141
N_{sdf}	20	21	28	17	19	21
t_{sdf}	0.1477	0.1647	0.2163	0.2036	1.4939	0.2474
$ w_{ccpd} $	11.1384	19.3699	29.0885	44.3332	22.6844	53.3712
N_{ccpd}	20	21	20	20	23	23
t_{ccpd}	0.5294	0.1388	0.1341	0.1758	0.2170	0.1717
$ w_{sdp} $	12.12	22.07	32.98	44.46	27.27	61.42
$ w_{shopm} $	3.01	13.74	26.99	33.33	1.18	50.43

The computational results presented in Table 7 show that SHOPM, SDF-NLS and CCPD-NLS didn't find best rank-1 approximation, but SDF-NLS and CCPD-NLS found a good rank-1 approximation in comparison with SHOPM. On the other side, RNS method found a best rank-1 approximation in all cases except when $d = 4$ and $n = 15$; but even with this case the rank-1 approximation given by the RNS method is better than the SDF-NLS, CCPD-NLS and SHOPM methods.

These numerical experiments show that RNS algorithm with initial point chosen according to the SHD algorithm gives a best real rank-1 approximation for real symmetric tensor in all examples except Example 4.7 with $d = 4$, $n = 15$. We noticed that the combination between the RNS algorithm and this method to choose the initial point respectively outperforms the SDP method in term of the consumed time, and the SDF-NLS, CCPD-NLS and SHOPM methods in term of the rank-1 approximation given by each of these algorithms. Nevertheless, we have no certification that TR-RNS method converges to a best rank-1 approximation. All we can say is that RNS method with initial point chosen by using the SHD algorithm is an efficient method to get a good real rank-1 approximation of a real symmetric tensor.

4.3. TR-RNS, CCPD-NLS, and SDF-NLS for symmetric rank- r approximation. In this numerical experiment we choose to compare the performance of TR-RNS method with state-of-the-art nonlinear least-square solvers CCPD-NLS and SDF-NLS designed for the symmetric tensor decomposition from Tensorlab v3 [50]. These solvers employ factor matrices as parameterization and use a trust region method with dogleg steps called "nls-gndl" which means that they use the Gauss-

Newton approximation of the Hessian matrix. The difference in general between TR-RNS and the other methods is, first in the parameterization since the TR-RNS method use the parameterization on the Veronese manifold, second in the use of the exact Hessian matrix in the TR-RNS instead of the Gauss-Newton approximation involved in the other methods. We apply this numerical experiment on some examples of real and complex symmetric tensor. We denote by d_0 the initial error, and by d_* the distance between P and the symmetric tensor, which we find by solving the linear least square problem in (3.4). We denote respectively by d_{tr} , d_{ccpd} , d_{sdf} the distance between P and the symmetric tensor that we find respectively by TR-RNS, CCPD-NLS and SDF-NLS method, N_{tr} (resp. N_{ccpd} , N_{sdf}) denotes the number of iterations of the TR-RNS (resp. CCPD-NLS, SDF-NLS) method and we denote by t_{tr} (resp. t_{ccpd} , t_{sdf}) the time in seconds consumed by the TR-RNS (resp. CCPD-NLS, SDF-NLS) method.

EXAMPLE 4.8. Let $P \in \mathcal{S}^5(\mathbb{R}^5)$ such that:

$$(P)_{i_1, i_2, i_3, i_4, i_5} = \arcsin\left((-1)^{i_1} \frac{i_1}{n}\right) + \dots + \arcsin\left((-1)^{i_5} \frac{i_5}{n}\right).$$

TABLE 8

Computational results for Example 4.8 with random real initial point uniformly distributed.

	r=3	r=5	r=10
d_0	685.0378	742.1689	708.2716
d_1	589.3293	336.8402	277.3143
d_{tr}	0.5574	0.5292	0.2451
d_{ccpd}	1.4066	1.1008	0.93769762
d_{sdf}	8.4312	1.1654	0.59840579
N_{tr}	41	30	292
N_{ccpd}	500	200	200
N_{sdf}	500	200	200
t_{tr}	1.131837	0.727448	15.236333
t_{ccpd}	1.823980	1.043719	0.976040
t_{sdf}	8.431258	3.766470	3.997201

In this example we see that starting with the same initial point, TR-RNS methods gives a better symmetric rank- r approximation than the other methods, but on the other hand the time consumed by one iteration is higher in the TR-RNS method than in the other methods.

TABLE 9

Computational results for Example 4.8 with random complex initial point uniformly distributed.

	r=3	r=5	r=10
d_0	734.3344	766.3735	773.7966
d_1	565.7515	434.8503	249.2298
d_{tr}	0.7845907	0.41194819	0.00016571
d_{ccpd}	2.3364	2.7222	2.2161
d_{sdf}	0.96876578	2015.9299	0.21967377
N_{tr}	50	500	246
N_{ccpd}	500	500	243
N_{sdf}	500	231	298
t_{tr}	1.092889	10.289029	10.805855
t_{ccpd}	2.176546	2.412442	1.576053
t_{sdf}	8.277246	5.874064	8.596538

We find a complex symmetric decomposition for this example of real symmetric tensor.

Nevertheless, as in the previous case, we find that with the same initial point, the TR-RNS succeed to find a better rank- r approximation than the other two methods but with more time consumed per iteration, which is normal since the use of the exact Hessian in the TR-RNS method increases the complexity of the algorithm.

EXAMPLE 4.9. Let $P \in \mathcal{S}^4(\mathbb{C}^7)$ with entries: $(P)_{i_1, i_2, i_3, i_4} = \frac{(i)^{i_1}}{i_1} + \dots + \frac{(i)^{i_4}}{i_4}$.

TABLE 10

Computational results for [Example 4.9](#) with random complex initial point uniformly distributed.

	r=3	r=5	r=10
d_0	211.7385	487.5316	301.8507
d_1	128.1921	149.6419	84.6598
d_{tr}	0.09780803	8.3285e-5	2.5257e-5
d_{ccpd}	0.8689672	0.66874246	2.444e8
d_{sdf}	0.21156654	0.03242244	2.4439e8
N_{tr}	130	40	89
N_{ccpd}	500	466	200
N_{sdf}	64	13	200
t_{tr}	3.420527	1.882588	9.254812
t_{ccpd}	1.766338	2.209960	1.319475
t_{sdf}	1.986222	0.777425	4.819483

In this example of complex symmetric tensor, we find that the TR-RNS method gives a better symmetric rank-3 approximation than CCPD-NLS and SDF-NLS algorithms. Also the TR-RNS algorithm gives an exact symmetric rank-5 and symmetric rank-10 decompositions, which is not the case for the other two algorithms, especially for $r = 10$ where they experience numerical troubles. We can notice also that the step of solving the linear least square problem (3.4) in the TR-RNS algorithm yields an important decrease in the initial error, which reduce later the number of iterations and then the consumed time of the method.

EXAMPLE 4.10. Consider the tensor $P \in \mathcal{S}^3(\mathbb{R}^n)$ with entries: $(P)_{i_1, i_2, i_3} = \frac{(-1)^{i_1}}{i_1} + \frac{(-1)^{i_2}}{i_2} + \frac{(-1)^{i_3}}{i_3}$. We use the SHD algorithm to choose an initial point for the TR-RNS algorithm, and a random real initial point uniformly distributed for the other two algorithms.

TABLE 11

Computational results for [Example 4.10](#) with $r = 2$.

	$n = 5$	$n = 10$	$n = 15$
d_{tr}	4.5169e-7	4.59e-7	1.0048e-6
d_{ccpd}	0.01348249	67.5542	0.20721736
d_{sdf}	0.0230521	36.2859	0.20246552
N_{tr}	1	1	1
N_{ccpd}	200	200	500
N_{sdf}	200	200	500
t_{tr}	0.009038	0.011488	0.062949
t_{ccpd}	2.654101	2.877974	5.126160
t_{sdf}	2.667050	3.452365	6.435196

It is clear that the TR-RNS algorithm gives an exact rank-2 symmetric decomposition with very low time in one iteration. In this example, it outperforms the other two methods.

These numerical experiments show that the TR-RNS method has a good numerical behavior compared to the existing state-of-the-art solvers in Tensorlab v3 for the

symmetric tensor rank approximation problem.

5. Conclusions. We presented the first Riemannian Newton optimization framework for approximating a given complex-valued symmetric tensor by a low rank symmetric tensor. We parametrized in [subsection 3.1](#) the constraint set as the Cartesian product of Veronese manifolds. Since we search a complex decomposition, we obtained a non-analytic real-valued objective function with complex variables. We have developed an exact Riemannian Newton iteration without approximating the Hessian matrix. We proposed in [subsection 3.2](#) a suitable representation of the points in the constraint set, and we exploited in [subsection 3.3](#) the properties of the apolar product and of partial complex derivatives, to deduce a simplified and explicit computation of the gradient and Hessian of the square distance function in terms of the points, weights of the decomposition and the tensor to approximate. In [subsection 3.4](#), we presented a retraction operator on the Veronese manifold. We proposed in [subsection 3.5](#), beside the classical random method, the first non-random method for choosing the initial point by the SHD algorithm, and we showed the impact of this choice in the numerical experiment in [subsection 4.2](#) to compute a best real rank-1 approximation of a real symmetric tensor, or in [subsection 4.3](#) to compute a close decomposition to a perturbation of symmetric tensors of low rank. In future works, we plan to investigate the computation of better initial points for the Riemannian Newton iterations and the STA problem for other families of tensors, such as multi-symmetric or skew symmetric tensors.

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