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On Minimum Dominating Sets in cubic and (*claw*, *H*) – free graphs

Valentin Bouquet* François Delbot[†] Christophe Picouleau*
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February 28, 2020

Abstract

Given a graph $G = (V, E)$, $S \subseteq V$ is a dominating set if every $v \in V \setminus S$ is adjacent to an element of S . The Minimum Dominating Set problem asks for a dominating set with minimum cardinality. It is well known that its decision version is *NP*-complete even when G is a claw-free graph. We give a complexity dichotomy for the Minimum Dominating Set problem for the class of (*claw*, *H*)-free graphs when H has at most six vertices. In an intermediate step we show that the Minimum Dominating Set problem is *NP*-complete for cubic graphs.

Keywords: Minimum Dominating Set, cubic graphs, claw-free graphs, NP-complete.

1 Introduction

We will only be concerned with simple undirected graphs. The reader is referred to [3] and [11] for, respectively, the definitions and notations on graph theory and on computational complexity.

Given a graph $G = (V, E)$, a set $S \subseteq V$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S . When S is a dominating set of G we say that S *dominates* G . The minimum cardinality of a dominating set in G is denoted by $\gamma(G)$. A dominating set $S \subseteq V$ with $|S| = \gamma(G)$ is called a *Minimum Dominating Set*, a *mds* for short. Following [12] a *mds* is also called a γ -set.

Our aim is to determine the computational complexity of the task consisting of computing a γ -set or the domination number, for some subclasses of graphs defined by a finite set of forbidden subgraphs.

The decision problem associated with the Minimum Dominating Set is defined as:

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MINIMUM DOMINATING SET (MDS problem)

Instance: a graph $G = (V, E)$ and an integer $d, |V| \geq d \geq 0$.

Question: is $\gamma(G) \leq d$?

In this paper we focus on the (\textit{claw}, H) -free graphs complexity. The decision problems we study are defined this way.

(\textit{claw}, H) -FREE MINIMUM DOMINATING SET

Instance: a (\textit{claw}, H) -free graph $G = (V, E)$ and an integer $d, |V| \geq d \geq 0$.

Question: is $\gamma(G) \leq d$?

The paper is organized as follows. The two next sections give the notations, the results of the literature and some basic properties that will be used in the sequel of the paper. Then we prove that the MDS problem is NP -complete for the class of cubic graphs, a result that is strangely missing in the literature. This result and its proof will be useful for several demonstrations later on. The sections 5 to 8 are concerned with the MDS problem in the class of claw-free graphs when at least one other graph is excluded. Among our different results, in the section 7 we give a complexity dichotomy for the class of (\textit{claw}, H) -free graphs for all the graphs H with no more than six vertices. We give a partial result for the (\textit{claw}, H) -free MDS problem when H has at least seven vertices in the section 8. We summarize our main results and the problems left open in the conclusion.

2 Definitions and notations

An element $ab \in E$ is called an *edge*, if $ab \notin E$ then ab is called a *non-edge*. For a vertex $v \in V$ let us denote by $N(v)$ its neighborhood, $N[v] = N(v) \cup \{v\}$ its closed neighborhood. The set of vertices at distance exactly k of a vertex v is denoted by $N_k(v)$. Hence $N(v) = N_1(v)$ and $N[v] = N_0(v) \cup N_1(v)$. A vertex v is *isolated* if $N(v) = \emptyset$. A vertex v is *universal* if $N[v] = V$. Two distinct vertices u, v are *twins* if $N(v) = N(u)$, are *false twins* if $N[v] = N[u]$. The graph \overline{G} is the complementary graph of G , that is $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \mid uv \notin E(G)\}$.

For a subset $S \subseteq V$, we let $G[S]$ denote the subgraph of G induced by S , which has vertex set S and edge set $\{uv \in E \mid u, v \in S\}$. Moreover, for a vertex $v \in V$, we write $G - v = G[V \setminus \{v\}]$ and for a subset $V' \subseteq V$ we write $G - V' = G[V \setminus V']$. For a set $\{H_1, \dots, H_p\}$ of graphs, G is (H_1, \dots, H_p) -free if G has no induced subgraph isomorphic to a graph in $\{H_1, \dots, H_p\}$; if $p = 1$ we may write H_1 -free instead of (H_1) -free. For two vertex disjoint induced subgraphs $G[A], G[B]$, $G[A]$ is *complete* to $G[B]$ if ab is an edge for any $a \in A$ and $b \in B$, $G[A]$ is *anti-complete* to $G[B]$ if ab is a non-edge for any $a \in A$ and $b \in B$. We denote by $G + H$ the disjoint union of the graphs G and H .

A set $S \subseteq V$ is called a *stable set* or an *independent set* if any pairwise distinct vertices $u, v \in S$ are non adjacent. The maximum cardinality of an independent set in G is denoted by $\alpha(G)$. A set $S \subseteq V$ is called a *clique* if any pairwise distinct vertices $u, v \in S$ are adjacent. When $G[V]$ is a clique then G is a *complete graph*. We denote by $K_p, p \geq 1$, the clique or the complete graph on p vertices and $k.K_p$ is the disjoint union of k cliques ($k \geq 0$).

For $n \geq 1$, the graph $P_n = u_1 - u_2 - \dots - u_n$ denotes the *cordless path* on n vertices, that is, $V(P_n) = \{u_1, \dots, u_n\}$ and $E(P_n) = \{u_i u_{i+1} \mid 1 \leq i \leq n - 1\}$. For $n \geq 3$, the graph C_n denotes the *cordless cycle* on n vertices, that is, $V(C_n) = \{u_1, \dots, u_n\}$ and $E(C_n) = \{u_i u_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{u_n u_1\}$. For $n \geq 4$, C_n is called a *hole*. The graph $C_3 = K_3$ is also called the *triangle*. The *claw* $K_{1,3}$ is the 4-vertex star, that is, the graph with vertices u, v_1, v_2, v_3 and edges uv_1, uv_2, uv_3 . The *diamond* is the 4-vertex complete graph K_4 minus an edge. The *net* is the graph with six vertices $u_1, u_2, u_3, v_1, v_2, v_3$ and edges $u_1 u_2, u_2 u_3, u_1 u_3, u_1 v_1, u_2 v_2, u_3 v_3$. The *bull* is the graph with five vertices u_1, u_2, u_3, v_1, v_2 and edges $u_1 u_2, u_2 u_3, u_1 u_3, u_1 v_1, u_2 v_2$. The *paw* is the graph with four vertices u_1, u_2, u_3, v_1 and edges $u_1 u_2, u_2 u_3, u_1 u_3, u_1 v_1$. The *butterfly* is the graph with five vertices u_1, u_2, v, v_1, v_2 and edges $u_1 u_2, u_1 v, u_2 v, v_1 v_2, v_1 v, v_2 v$. The *house* is the graph with five vertices u_1, u_2, u_3, u_4, v and edges $u_1 u_2, u_2 u_3, u_3 u_4, u_4 u_1, u_1 v, u_2 v$. The *gem* is the graph with five vertices u_1, u_2, u_3, u_4, v and edges $u_1 u_2, u_2 u_3, u_3 u_4, u_1 v, u_2 v, u_3 v, u_4 v$. The Figure 2 below exposed all these graphs.

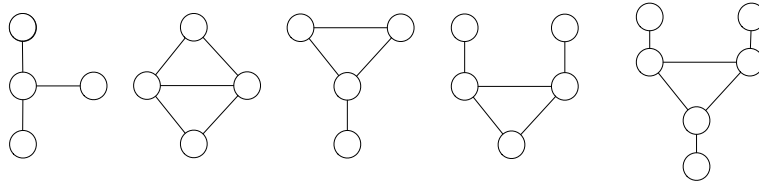


Figure 1: The claw, the diamond, the paw, the bull, the net.

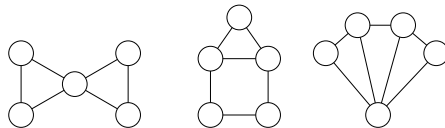


Figure 2: The butterfly, the house, the gem.

The (k_1, k_2, k_3) -*triangle* consists of a triangle $T = \{v_1, v_2, v_3\}$ and three vertex disjoint paths P_{k_i} connected to $v_i, 1 \leq i \leq 3$. Hence the net is the $(1, 1, 1)$ -triangle, the bull is the $(1, 1, 0)$ -triangle, the paw is the $(1, 0, 0)$ -triangle. The k -*double-triangle* consists of two vertex disjoint triangles $T_1 = \{v_1, v_2, v_3\}, T_2 = \{v_4, v_5, v_6\}$ and P_k between v_1 and v_4 . For simplicity the 0-double-triangle is called the double-triangle (see Figure 3).

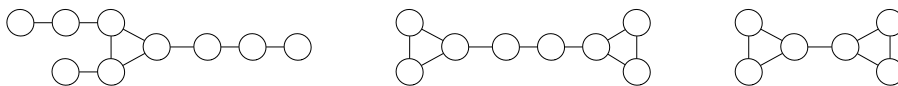


Figure 3: The $(3, 2, 1)$ -triangle, the 2-double-triangle, the double-triangle.

We denote $V^+ \subseteq V$ the subset of vertices v of G such that $\gamma(G - v) > \gamma(G)$. A mds which is also an independent set is an *independent dominating set* and the

minimum cardinality of an independent dominating set is denoted by $i(G)$. Clearly $\gamma(G) \leq i(G) \leq \alpha(G)$. Note also that a minimum independent dominating set is a *minimum maximal independent set*.

3 Preliminary results

We know from [1] that if the graph G is *claw*-free then $\gamma(G) = i(G)$. From [14] the MDS problem is *NP*-complete for the class of *claw*-free graphs.

The minimum edge dominating set problem consists of finding $F \subseteq E$ a minimum set of edges such that for each edge $e \notin F$, e is incident to an edge $f, f \in F$. Taking $L(G)$, the line graph of G , a minimum edge dominating set in G is a minimum dominating set in $L(G)$. In [14] Yannakakis et al. showed that the minimum edge dominating set problem is *NP*-complete for bipartite subcubic graphs. Moreover, one can check that the graphs built in the transformation (from a variant of the 3-SAT problem) are C_4 -free. Also, line graphs of bipartite graphs are perfect, so they have no odd holes. Thus, for any of these graphs, the corresponding line graphs are $(claw, diamond, K_4, C_4)$ -free and perfect (i.e. *odd hole, odd antihole*-free). It follows that the minimum dominating set problem is *NP*-complete for $(claw, diamond, K_4, C_4)$ -free perfect graphs.

The minimum dominating set problem is polynomial for \overline{paw} -free graphs. The class of $(claw, \overline{claw})$ -free graphs has bounded clique-width [7], so from [9] a γ -set can be computed in linear time. It is also polynomial for $(claw, net)$ -free graphs [6]. A minimum dominating set can be computed in polynomial time for $2K_2$ -free graphs so for $(claw, 2K_2)$ -free graphs [2]. In [4], it is shown that computing a minimum dominating set is polynomial for the class of $(claw, P_8)$ -free graphs. Since $K_2 + 2K_1 \subseteq_i P_8$ and $4K_1 \subseteq_i P_8$ computing a mds is polynomial for the classes of $(claw, K_2 + 2K_1)$ -free and $(claw, 4K_1)$ -free graphs.

Note that $(claw, H)$ -free graphs is equivalent to *claw*-free graphs when H contains a *claw*. Hence, the remaining H 's we consider are *claw*-free.

We give some preliminary easy properties that will be useful for many proofs given later.

Property 3.1 *Let H, H' two fixed graphs with $H' \subseteq_i H$. If H' -FREE MINIMUM DOMINATING SET is *NP* – complete then H -FREE MINIMUM DOMINATING SET is *NP* – complete.*

Property 3.2 *Let H, H' two fixed graphs with $H \subseteq_i H'$. If H' -FREE MINIMUM DOMINATING SET is polynomial then H -FREE MINIMUM DOMINATING SET is polynomial.*

Property 3.3 *Let $k > 0$ be a fixed positive integer and $G = (V, E)$ a graph. If there exists $T \subseteq_i G$ of size $|T| \leq k$ such that $V = N[T]$ then computing a minimum dominating set for G is polynomial.*

Proof: Since $|T| \leq k$ and $V = N[T]$, we have that $\gamma(G) \leq k$. So a minimum dominating set can be computed in $O(n^k)$ which is polynomial. \square

Property 3.4 *Let $k > 0$ be any fixed positive integer. If a graph G is $(\text{claw}, k.K_1)$ -free then computing a minimum dominating set for G is polynomial.*

Proof: Since G is $k.K_1$ -free we have $\alpha(G) < k$. Since G is claw-free $\gamma(G) = i(G) \leq \alpha(G)$. Now k is fixed, so enumerating all the independent sets of size less than k can be done in polynomial time. It follows that for any fixed $k \geq 1$, $(\text{claw}, k.K_1)$ -FREE MINIMUM DOMINATING SET is polynomial. \square

Property 3.5 *Let $G = (V, E)$ be a connected claw-free graph with $uv \in E$ such that u is a leaf. The MINIMUM DOMINATING SET problem is polynomial for $G' = G - N[v]$ if and only if it is polynomial for G .*

Proof: Since G is claw-free $K = N(v) - u$ is a clique. Let $s \in K$, we show that $s \notin V^+$. For contradiction we assume that $s \in V^+$. As shown in [5], s is in every γ -set of G . Let Γ be a γ -set with $v \in \Gamma$. Let $W = N(s) \setminus N(v)$. If $W = \emptyset$ then $\Gamma - s$ is a dominating set, a contradiction. It follows $W \neq \emptyset$ and since G is claw-free, W is a clique. Let $w \in W$. $\Gamma' = (\Gamma - \{s\}) \cup \{w\}$ is another γ -set, a contradiction. Hence $\gamma(G - K) \leq \gamma(G)$. Since $G - K$ consists of G' and the component uv , we have that $\gamma(G - K) = \gamma(G') + 1$. Then from Γ' a γ -set of G' we obtain $\Gamma' \cup \{v\}$ a γ -set of G in polynomial time. Reciprocally, let Γ be a γ -set of G . Since u is a leaf we assume that $v \in \Gamma$. Then $\Gamma - \{v\}$ is a γ -set for G' . Trivially it can be done in polynomial time from Γ . \square

4 Cubic and k -regular graphs for $k \geq 3$ when k is odd

In [10] Demange et al. it is proved that, for any fixed $k \geq 3$, the Minimum Edge Dominating Set problem is NP -complete for bipartite k -regular graphs. Transferring this result into the line graph we have that the Minimum Dominating Set problem is NP -complete for $2(k - 1)$ -regular graphs, $k \geq 3$, that is regular graphs with even degree at least four.

Theorem 4.1 *The Minimum Dominating Set problem is NP -complete for cubic graphs.*

Proof: We give a polynomial reduction from the Minimum Dominating Set problem which is NP -complete for 4-regular graphs.

Let G be a 4-regular graph and d be an instance of Minimum Dominating Set. We build G' and d' another instance of Minimum Dominating Set where G' is cubic. Let n be the number of vertices of G , we take $d' = 11n + d$. Each 4-vertex v of G is

replaced with the gadget shown in Figure 4. We denote by H_v the gadget associated with v .

The gadget H_v has four *corners*, that is the vertices a, b, c, d in Figure 4. Each of them corresponds to an edge incident to v in G . To the edge $a = uv$ in G corresponds an edge connecting the two corners a of H_v and H_u . Hence G' is cubic.

For each H_v , the subgraph of G' induced by the 44 vertices of H_v , satisfies the following properties. Note that by symmetry the four corners play the same role. Clearly $\gamma(H_v) = 12$ and there exists a minimum dominating set that contains the four corners, see Figure 5. For each corner, says a , $H_v - a$ has a unique minimum dominating set of size $\gamma(H_v - a) = 11$, see Figure 6. None of the three other corners b, c, d are in this minimum dominating set. With ten vertices it is not possible to dominate $H_v - \{a, b, c, d\}$.

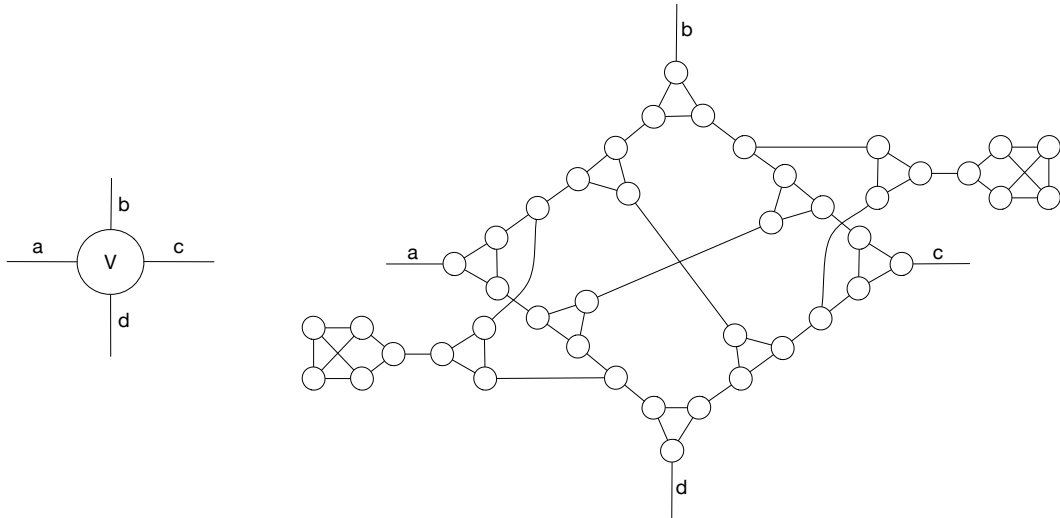


Figure 4: A 4-vertex v and its associated gadget H_v .

Let Γ be a dominating set of G such that $|\Gamma| \leq d$. We give Γ' a dominating set of G' as follows: if $v \in \Gamma$ then the twelve vertices of the minimum dominating set of H_v (the 12 black vertices of Figure 5), are put in Γ' . If $v \notin \Gamma$ then v is dominated by a vertex $w, w \in \Gamma$. Let $a = vw$. The eleven vertices of the minimum dominating set of $H_v - a$ (the 11 black vertices of Figure 6), are put in Γ' . Hence the corner a of H_v is dominated by the corner a of H_w . So Γ' is a dominating set of size $|\Gamma'| = 11n + d = d'$.

Let Γ' be a dominating set of G' such that $|\Gamma'| \leq 11n + d = d'$. We show that $\gamma(G) \leq d$. From the last property, we gave at least eleven vertices to dominate each H_v (note that it is necessary to dominate $H_v - \{a, b, c, d\}$). Since $d' = 11n + d'$ at most d gadgets H_v must contain at least 12 vertices. Since twelve vertices are enough for one H_v , we can assume there are d gadgets with $|H_v \cap \Gamma'| = 12$. Taking the d corresponding vertices v in Γ , Γ is a dominating set of G . Hence we have that $\gamma(G) \leq |\Gamma| \leq d$. \square

The graphs built in the proof above are (*butterfly, house, gem*)-free, so we have the following.

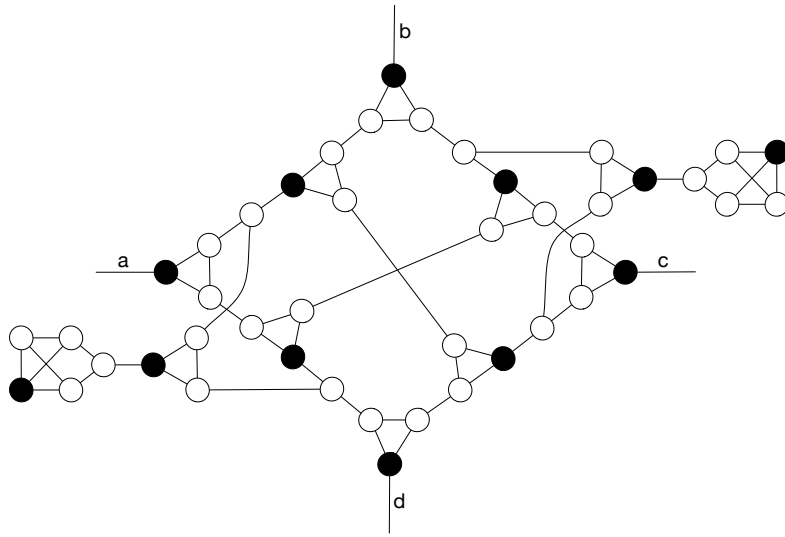


Figure 5: A minimum dominating set for the gadget.

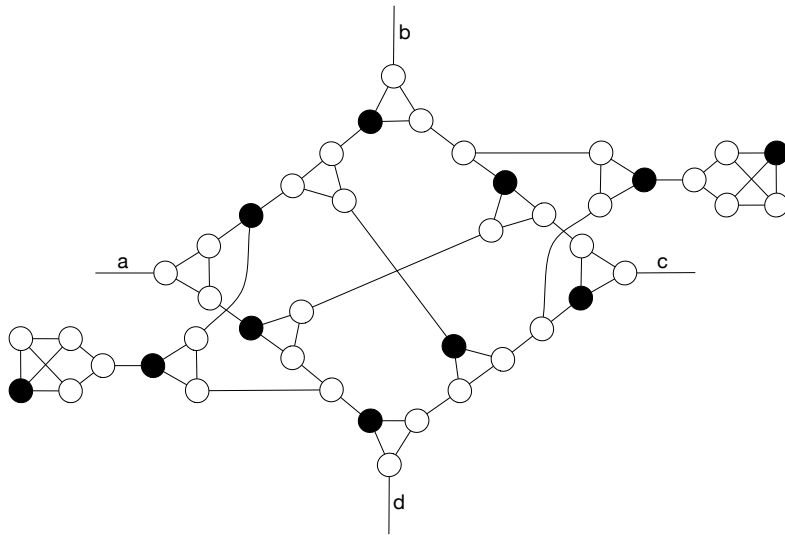


Figure 6: A partial dominating set of size 11 for the gadget.

Corollary 4.2 *The Minimum Dominating Set problem is NP-complete for $(K_4, \text{butterfly}, \text{house}, \text{gem})$ -free cubic graphs.*

Theorem 4.3 *For any odd $k \geq 3$, the Minimum Dominating Set problem is NP-complete for k -regular graphs.*

Proof: The case $k = 3$ is proved by Theorem 4.1. Now let $k \geq 5$ be an odd integer.

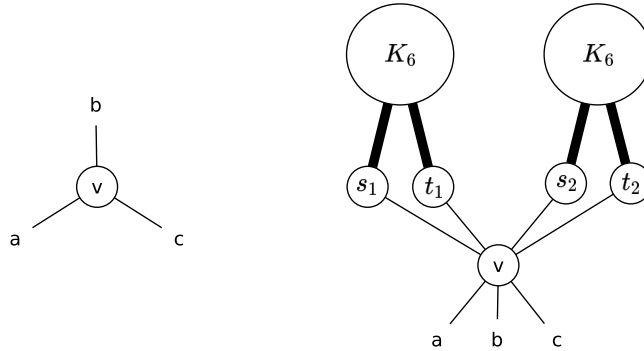


Figure 7: A 3-vertex v and its replacement in the case of 7-regular graphs. The bold edge s_1K_6 means that s_1 is connected to the 6 vertices of K_6 .

Given a cubic graph $G = (V, E)$, we construct a graph $G' = (V', E')$ as follows. Each vertex $v, v \in V'$, is connected to $\frac{(k-3)}{2}$ components $K_{k+1} - e$ (the complete graph with $k + 1$ vertices minus an edge). For each component $K_{k+1} - e$, let $e = st$. The vertex v is connected to s and t . Figure 7 shows the transformation for $k = 7$. For each component, one vertex $w, w \neq s, t$ is necessary in a minimum dominating set. Since v is not dominated by w , the sequel of the proof is easy. \square

5 $(\text{claw}, C_4, \dots, C_k)$ -free graphs

With similar reduction as the one used for proving Theorem 4.1, we show the following theorem.

Theorem 5.1 *For any fixed integer $k, k \geq 4$, the Minimum Dominating Set problem is NP-complete for $(\text{claw}, C_4, \dots, C_k)$ -free subcubic graphs.*

Proof: The arguments are similar of those given in the proof of Theorem 4.1. We give a polynomial reduction from the Minimum Dominating Set problem which is NP-complete for 4-regular graphs. To each 4-vertex v is associated the gadget depicted in Figure 8. In this gadget each dashed box corresponds to an induced path of $3p$ vertices, $p \geq 0$.

We have the following: $\gamma(H_v) = 8 + 4p$ and there exists a minimum dominating set that contains the four corners. For each corner, says a , $H_v - a$ has a unique minimum dominating set of size $\gamma(H_v - a) = 7 + 4p$. None of the three other corners

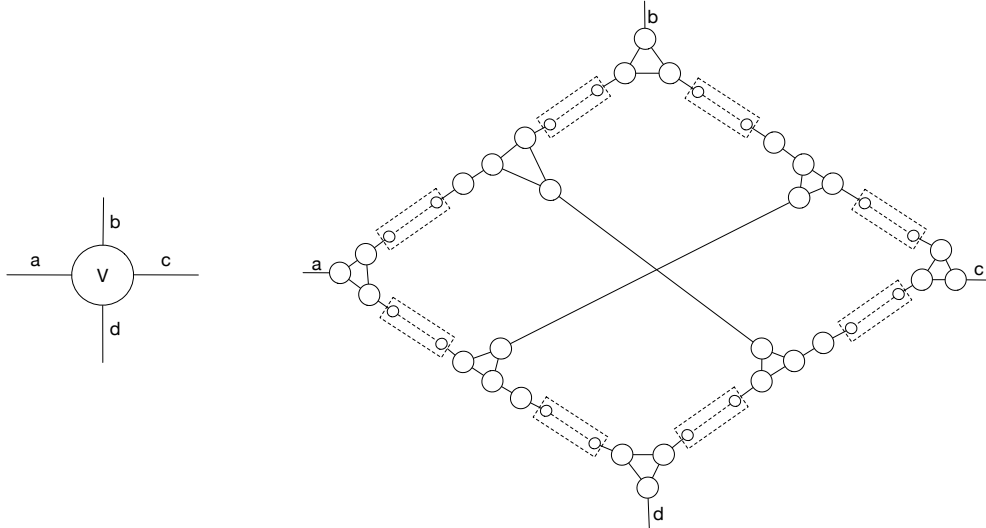


Figure 8: A 4-vertex v and its associated gadget H_v .

b, c, d are in this minimum dominating set. With $6 + 4p$ vertices it is not possible to dominate $H_v - \{a, b, c, d\}$.

The graph we obtain has no claw. Its maximum degree is three. The smallest cycle that is not a triangle has length more than $12p$. As p can be chosen arbitrary large taking $k = 12p$ we obtain the result. \square

6 (*claw, k – double – triangle*)-free graphs

Using the Theorem 4.1, we show the NP -completeness of the MDS problem for the class of (*claw, k – double – triangle*)-free subcubic graphs. In a first step we prove the following lemma.

Lemma 6.1 *The (*claw, butterfly, diamond, C_4, C_5, K_4*)-free Minimum Dominating Set problem is NP -complete for cubic perfect graphs.*

Proof: We give a polynomial reduction from the Minimum Dominating Set problem for cubic graphs which is proved NP -complete in Theorem 4.1.

Let G be a cubic graph and d be an instance of Minimum Dominating Set. We build G' and d' another instance of Minimum Dominating Set where G' is (*claw, butterfly, diamond, C_4, C_5, K_4*)-free and perfect. Let n be the number of vertices in G , we take $d' = 2n + d$. Each 3-vertex v of G is replaced with the gadget shown in Figure 9. We denote by H_v the gadget associated with v .

The gadget H_v has three *corners* - the vertices a, b, c in Figure 9 - each of them corresponding to an edge incident to v in G . To the edge $a = uv$ in G corresponds an edge connecting the two corners a of H_v and H_u . Hence G' is (*claw, butterfly, diamond, C_4, C_5, K_4*)-free. Moreover G' has no odd hole. Since C_5 is isomorphic to $\overline{C_5}$ we have $C_5 \not\subseteq_i G'$. One can remark that for any $k, k \geq 6$, we have $C_4 \subseteq_i \overline{C_k}$. Thus G' being C_4 -free, G' has no odd anti-hole. Then from the perfect graph theorem [8] G' is perfect.

For each H_v , the subgraph of G' induced by the nine vertices of H_v , satisfies the following properties. By symmetry the three corners play the same role. Clearly $\gamma(H_v) = 3$ and there exists a minimum dominating set that contains the three corners, see Figure 10. For each corner, says a , $H_v - a$ has a unique minimum dominating set of size $\gamma(H_v - a) = 2$, see Figure 10. None of the two other corners b, c are in this minimum dominating set. With one vertex it is not possible to dominate $H_v - \{a, b, c\}$.

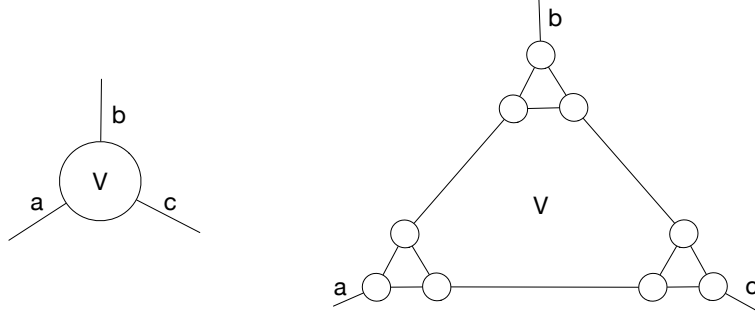


Figure 9: A 3-vertex v and its associated gadget H_v .

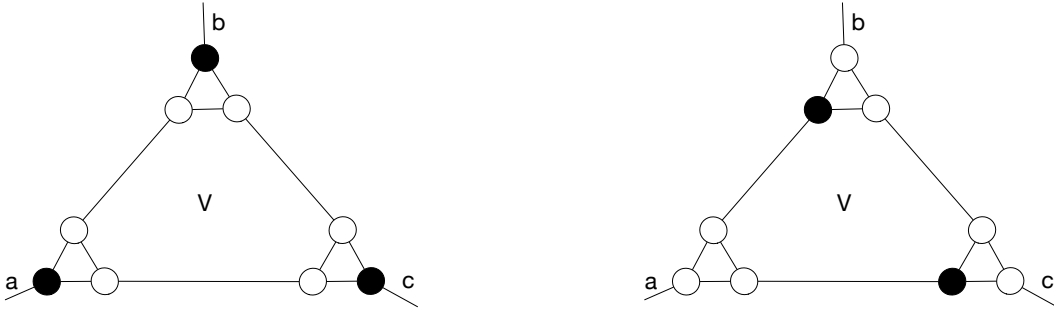


Figure 10: A minimum dominating set and partial dominating set of size 2 for the gadget

Let Γ be a dominating set of G such that $|\Gamma| \leq d$. We give Γ' a dominating set of G' as follows: If $v \in \Gamma$ then the three vertices of the minimum dominating set of H_v (the three black vertices of Figure 10), are put in Γ' . If $v \notin \Gamma$ then v is dominated by a vertex $w, w \in \Gamma$. Let $a = vw$. The two vertices of the minimum dominating set of $H_v - a$ (the two black vertices of Figure 10), are put in Γ' . Hence the corner a of H_v is dominated by the corner a of H_w . So Γ' is a dominating set of size $|\Gamma'| = 2n + d = d'$.

Let Γ' be a dominating set of G' such that $|\Gamma'| \leq 2n + d = d'$. We show that $\gamma(G) \leq d$. From the last property we gave for H_v at least two vertices are necessary to dominate $H_v - \{a, b, c\}$. Since $d' = 2n + d'$ at most d gadgets H_v must contain at least three vertices. Since three vertices are enough for one H_v , we can assume there are d gadgets with $|H_v \cap \Gamma'| = 3$. Taking the d corresponding vertices v in Γ , Γ is a dominating set of G . Hence we have that $\gamma(G) \leq |\Gamma| \leq d$. \square

So we can prove the following theorem.

Theorem 6.2 *For any fixed integer $k, k \geq 0$, the Minimum Dominating Set problem is NP-complete for (claw, k – double – triangle)-free subcubic graphs.*

Proof: The arguments are similar of those given in the proof of Lemma 6.1. We give a polynomial reduction from the Minimum Dominating Set problem which is NP-complete for cubic graphs. Let $G = (V, E)$ be a cubic graph. To each 3-vertex $v, v \in V$, is associated the gadget depicted by Figure 11. In this gadget each dashed box corresponds to an induced path of $3p$ vertices (we depicted the case $p = 1$). To each edge $uv \in E$ is associated a P_{3p} (the same dashed box in the picture). We take $d' = d + (2 + 3p)|V| + p|E|$.

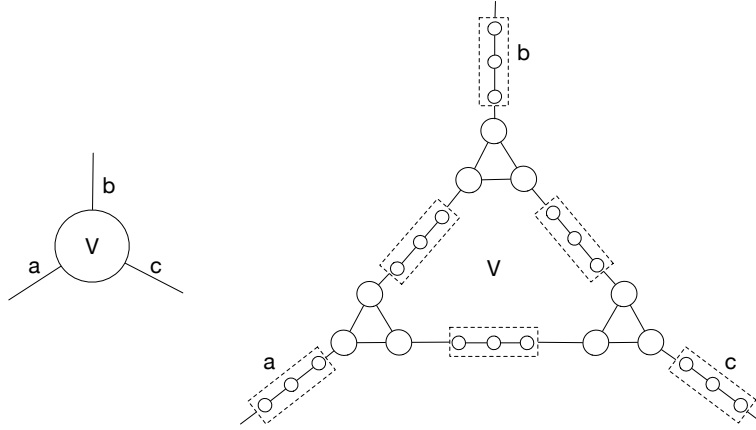


Figure 11: A 3-vertex v and its associated gadget H_v .



Figure 12: On the left : the vertex v is in the dominating set; on the right: the vertex v is not in the dominating set.

The graph we obtain is (claw, $(3p - 1)$ – double – triangle)-free and its maximum degree is three. As shown by Figure 12 the remaining arguments are the same as in the proof of Lemma 6.1.

As p is a positive integer that can be chosen arbitrary large we obtain the result.

□

7 (claw, H) – free graphs

For any fixed graph H with no more than six vertices we determine the complexity of the MDS problem for the class of (claw, H) -free graphs. Trivially the graphs G we consider are assumed to be connected. When H has no more than two vertices the MDS problem is trivially polynomial, so we prove the results for all the graphs H with three vertices up to six vertices.

Property 7.1 *For any fixed graph H with three vertices computing a minimum dominating set for G a connected (claw, H) -free graph is polynomial.*

Proof: Since a connected (claw, C_3) -free graph is either a path or a cycle, the MDS problem is polynomial when G is (claw, H) -free with $H = C_3$. When $H \neq C_3$ we have $H \subseteq_i P_8$. From [4] we know that computing a mds is polynomial for (claw, P_8) -free graphs. Then the result holds from Property 3.2. \square

The next three sections deal with the case where H contains four, five, six vertices.

7.1 (claw, H) – free graphs when H has four vertices

When H has exactly four vertices the state of the art is the following. In [14], the minimum edge dominating set problem is shown to be NP -complete for bipartite subcubic graphs. Moreover, one can check that the graphs G built in the transformation (from a variant of the 3-SAT problem) are C_4 -free. Also, line graphs of bipartite graphs are perfect, so they have no odd holes. Thus, $L(G)$ the line graph of such a G is $(\text{claw}, \text{diamond}, K_4, C_4)$ -free and perfect (i.e., $(\text{odd hole}, \text{odd antihole})$ -free). It follows that the minimum dominating set problem is NP -complete for $(\text{claw}, \text{diamond}, K_4, C_4)$ -free perfect graphs. From [4] computing a mds is polynomial for (claw, P_k) -free graphs when $k \leq 8$. It is also polynomial when $H \in \{P_4, P_3 + K_1, 2K_2, K_2 + 2K_1, 4K_1\}$ since $H \subseteq_i P_8$. The class of $(\text{claw}, \overline{\text{claw}})$ -free graphs has bounded clique-width [7], so from [9] a γ -set can be computed in linear time (remark that $\overline{\text{claw}} \equiv K_3 + K_1$).

Taking these results together we obtain the following dichotomy.

Property 7.2 *Let H be a fixed graph with four vertices. Computing a minimum dominating set for G a connected (claw, H) -free graph is NP -complete when $H \in \{\text{diamond}, K_4, C_4\}$, otherwise it is polynomial.*

Remark 7.1 *In [13] Lin et al. remark that the minimum dominating set problem is NP -complete for $(\text{claw}, \text{diamond}, K_4, C_4)$ -free graphs, here we restrict this result for the class of perfect graphs.*

7.2 (claw, H) – free graphs when H has five vertices

In this section, we focus on H with exactly five vertices. From [14], we know that the MDS problem is NP -complete when G is claw -free and perfect, hence for (claw, C_5) -free graphs. Also, from Lemmas 3.1 and 6.1, we know that the MDS

problem is NP -complete for (claw, H) -free graphs when H contains a C_4 , a K_4 , a *diamond*, or a *butterfly*.

Using Property 3.1 the MDS problem is NP -complete when $H \in \{K_5, K_5 - e, \overline{P_3 + 2K_1}, W_4, \overline{\text{claw} + K_1}, \overline{P_2 + P_3}, \text{gem}, \overline{K_3 + 2K_1}, K_4 + K_1, C_4 + K_1, \text{dart}, \text{house}, \text{diamond} + K_1\}$. From Property 3.2, it is polynomial for (claw, H) -free graphs when $H \subseteq_i \text{net}$ or $H \subseteq_i P_7$, and from Property 3.4 it is polynomial for $(\text{claw}, k.K_1)$ -free graphs. Hence, the MDS problem is polynomial for $H \in \{\text{bull}, P_5, P_3 + 2K_1, 2K_2 + K_1, K_2 + 3K_1, K_2 + P_3, P_4 + K_1, 5K_1\}$.

It remains to treat the cases where $H \in \{\text{paw} + K_1, \{2, 0, 0\}\text{-triangle}, K_3 + 2K_1, K_3 + K_2\}$.

Property 7.3 *Let $G = (V, E)$ be a connected (claw, H) -free graph. If $H = K_3 + 2K_1$ then computing a minimum dominating set for G is polynomial.*

Proof: When G is a (claw, K_3) -free graph computing a mds is polynomial. Now we suppose that $K_3 \subseteq_i G$. Let $G[\{v_1, v_2, v_3\}] = K_3$. From Property 3.3 we can assume that $N[K_3] \neq V$. Let $W = V - N[T] \neq \emptyset$. If W is not complete then there are $u, v \in W$ such that $uv \notin E$, but $K_3 \cup \{u, v\} = H \subseteq_i G$. Thus W is complete and $\gamma(G) \leq 4$ (taking $\{v_1, v_2, v_3\}$ and $w \in W$). Hence a mds can be computed in time $O(n^4)$. \square

Property 7.4 *Let $G = (V, E)$ be a connected (claw, H) -free graph. If $H \in \{K_3 + K_2, \{2, 0, 0\}\text{-triangle}, \text{paw} + K_1\}$ then computing a minimum dominating set for G is polynomial.*

Proof: When G is a (claw, P_8) -free graph computing a mds is polynomial (see [4]). Now we suppose that $P_8 \subseteq_i G$. Let $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 = P$. From Property 3.3 we can assume that $N[P] \neq V$. Let $W = V \setminus N[P]$. Since G is connected and claw -free, there are $w \in W, v \in N(P)$ such that v is a neighbor of w and v has exactly two neighbors v_i, v_{i+1} in P . By symmetry we can assume that $i \leq 4$. Clearly $G[\{v, v_i, v_{i+1}, v_{i+3}\}] = \text{paw} + K_1$, $G[\{v, v_i, v_{i+1}, v_{i+2}, v_{i+3}\}] = \{2, 0, 0\}\text{-triangle}$ and $G[\{v, v_i, v_{i+1}, v_{i+3}, v_{i+4}\}] = K_3 + K_2$. \square

Taking the results together we obtain the following dichotomy.

Theorem 7.1 *Let H be a fixed graph with five vertices. Computing a minimum dominating set for G a connected (claw, H) -free graph is NP -complete when $H \in \{C_5, K_5, K_5 - e, \overline{P_3 + 2K_1}, W_4, \overline{\text{claw} + K_1}, \overline{P_2 + P_3}, \text{gem}, \overline{K_3 + 2K_1}, K_4 + K_1, C_4 + K_1, \text{dart}, \text{house}, \text{diamond} + K_1, \text{butterfly}\}$ and polynomial otherwise.*

7.3 (claw, H) – free graphs when H has six vertices

We consider H with exactly six vertices. From Theorems 5.1, 6.2, the MDS problem is NP -complete when G is $(\text{claw}, C_4, \dots, C_k)$ -free (for any fixed $k \geq 4$) and when G is $(\text{claw}, k\text{-double-triangle})$ -free (for any fixed $k \geq 0$). Hence, the MDS problem is NP -complete for (claw, C_6) -free graphs and for $(\text{claw}, \text{double-triangle})$ -free graphs.

Furthermore, the MDS problem is NP -complete for (claw, H) -free graphs when H contains at least one of the following graphs: *diamond, butterfly, claw, K_4* .

Also from lemma 3.2, the MDS problem is *polynomial* for (claw, H) -free graphs when $H \subseteq_i H'$ and the MDS problem is *polynomial* for (claw, H') -free graphs. Hence, if H is a net or is an induced subgraph of P_8 , then the MDS problem is *polynomial* for (claw, H) -free graphs.

So, we focus on (claw, H) -free graphs where the complexity of the MDS problem cannot be deduced from the previous arguments.

Property 7.5 *Let $G = (V, E)$ a connected (claw, H) -free graph. Computing a minimum dominating set for G is polynomial if $H \in \{K_3 + P_3, \{3, 0, 0\}\text{-triangle}, \{2, 0, 0\}\text{-triangle} + K_1, 2K_3, P_3 + 3K_1, 2K_2 + 2K_1, \text{paw} + 2K_1, \text{bull} + K_1, K_3 + K_2 + K_1, \text{paw} + K_2, \{2, 1, 0\}\text{-triangle}, K_2 + 4K_1, K_3 + 3K_1\}$.*

Proof: Let $P = v_1 - \dots - v_k$ a maximum induced path of G . Since computing a minimum dominating set is polynomial for (claw, P_8) -free graphs [4], we can assume that $k \geq 8$. Since G is *claw*-free, any vertex $s \in N(P)$ is such that $|N_P(s)| \leq 4$. Also, from Property 3.3, we can assume that $N[P] \neq V$, so $W = V - N[P] \neq \emptyset$. Let $w \in W$ with a neighbor $v \in V$ such that $N_P(v) \neq \emptyset$. Since G is *claw*-free v has exactly two neighbors that are consecutive in P , that is $N_P(v) = \{v_i, v_{i+1}\}$, $1 \leq i \leq k - 1$. Since P is a maximal path, these two neighbors cannot be v_1, v_2 or v_{k-1}, v_k . Hence $N_P(v) = \{v_i, v_{i+1}\}$, $2 \leq i \leq k - 2$.

From Property 3.5 we can assume that both v_1 and v_k are not leaves. Let $u_1, u_k \in V \setminus P$ be two neighbors of v_1 and v_k respectively. We show how the vertices u_1 and u_k are connected to P . First, we deal with $u_1 = u_k$. From above u_1 has no neighbor in W . If $N_P(u_1) \supset \{v_1, v_2, v_3, v_k\}$ or $N_P(u_1) \supseteq \{v_1, v_{k-2}, v_{k-1}, v_k\}$ then G has a *claw*. Hence, when $|N_P(u_1)| = 4$, $N_P(u_1) = \{v_1, v_2, v_{k-1}, v_k\}$ and when $|N_P(u_1)| = 3$, $N_P(u_1) = \{v_1, v_2, v_k\}$ or $N_P(u_1) = \{v_1, v_{k-1}, v_k\}$. Last, if $|N_P(u_1)| = 2$, i.e. $N_P(u_1) = \{v_1, v_k\}$ then $w - v - v_i - \dots - v_1 - u_1 - v_k - \dots - v_{i+1} = P_{k+2}$, a contradiction. Hence there is no vertex $u \in V$ such that $N_P(u) = \{v_1, v_k\}$. Now when $u_1 \neq u_k$ we have $N_P(u_1) = \{v_1, v_2\}$ or $N_P(u_1) = \{v_1, v_2, v_3\}$ or $N_P(u_1) = \{v_1, v_2, v_3, v_4\}$, and $N_P(u_k) = \{v_{k-1}, v_k\}$ or $N_P(u_k) = \{v_{k-2}, v_{k-1}, v_k\}$ or $N_P(u_k) = \{v_{k-3}, v_{k-2}, v_{k-1}, v_k\}$. Hence, up to symmetry, in any case $G[\{v_1, v_2, u_1\}]$ is a triangle, and, u_1 and u_k have no neighbor in W .

In the following we take $k = 8$, the arguments being the same for $k > 8$. Thus $N_P(v) = \{v_i, v_{i+1}\}$, $2 \leq i \leq 6$.

We demonstrate that $H \subseteq_i G$ when P is of length 8 thus for $k = 8$. In the following, i is the indice of the first neighbor of the vertex v in P .

- $H = K_3 + P_3$: $G[\{u_1, v_1, v_2, v_5, v_6, v_7\}] = H$;
- $H = \{3, 0, 0\}\text{-triangle}$: when $2 \leq i \leq 4$ then $G[\{v, v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}] = H$ (the case $5 \leq i \leq 6$ is symmetric);
- $H = \{2, 0, 0\}\text{-triangle} + K_1$: when $i \in \{2, 3\}$ then $G[\{v, v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_8\}] = H$ ($i \in \{5, 6\}$ is symmetric). When $i = 4$ then $G[\{v, v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_1\}] = H$;

- $2K_3$: when $2 \leq i \leq 4$ then $u_8v \notin E$ else G contains a claw. Thus $G[\{v, v_i, v_{i+1}, v_{k-1}, v_k, u_k\}] = H$ (the case $i \in \{5, 6\}$ is symmetric);
- $H = P_3 + 3K_1$: $G[\{v_1, v_2, v_3, v_5, v_7, w\}] = H$;
- $H = 2K_2 + 2K_1$: $G[\{v_1, v_2, v_4, v_5, v_7, w\}] = H$;
- $H = paw + 2K_1$: when $i \in \{2, 3\}$ then $G[\{v, v_i, v_{i+1}, w, v_6, v_8\}] = H$ (the case $i \in \{5, 6\}$ is symmetric). When $i = 4$ then $G[\{v, v_i, v_{i+1}, v_{i-1}, v_1, v_8\}] = H$;
- $H = bull + K_1$: when $2 \leq i \leq 4$ then $G[\{v, v_i, v_{i+1}, v_{i-1}, w, v_8\}] = H$ (the case $i \in \{5, 6\}$ is symmetric);
- $H = K_3 + K_2 + K_1$: $G[\{u_1, v_1, v_2, v_5, v_6, w\}] = H$;
- $H = paw + K_2$: when $2 \leq i \leq 4$ then $G[\{v, v_i, v_{i+1}, w, v_8\}] = H$ (the case $i \in \{5, 6\}$ is symmetric);
- $H = \{2, 1, 0\}$ -triangle: when $2 \leq i \leq 4$ then $G[\{v, v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_{i+3}, w\}] = H$ (the case $i \in \{5, 6\}$ is symmetric);
- $H = K_2 + 4K_1$: $G[\{v_1, v_2, v_4, v_6, v_8, w\}] = H$;
- $H = K_3 + 3K_1$: $G[\{u_1, v_1, v_2, v_4, v_6, w\}] = H$.

□

7.4 The dichotomy for the H 's with no more than six vertices

Taking together all the results proved in this section we can conclude with the following result.

Theorem 7.2 *Let G be a connected $(claw, H)$ -free graph such that H has at most six vertices. When $H' \subseteq_i H$ with $H' \in \{C_4, C_5, C_6, K_4, diamond, butterfly, double-triangle\}$, computing a minimum dominating set for G is NP-complete, otherwise it is polynomial.*

8 More results for $(claw, H)$ – free graphs

In the previous section we give a complexity dichotomy when H has no more than six vertices. Here we give a partial result for the H 's with more than six vertices.

Theorem 8.1 *Let $k > 0$ be a fixed integer and H be a fixed graph with k vertices such that H has a connected component that contains two distinct triangles. The Minimum Dominating Set problem is NP-complete for $(claw, H)$ -free graphs.*

Proof: Let $T_1 \subseteq_i H, T_2 \subseteq_i H$ be two distinct triangles in a same component of H . When T_1 and T_2 are not vertex disjoint then $C_4 \subseteq_i H$ or *diamond* $\subseteq_i H$ or *butterfly* $\subseteq_i H$. Now let T_1 and T_2 be vertex disjoint. When there is an edge from T_1 to T_2 then *double – triangle* $\subseteq_i H$ or $C_4 \subseteq_i H$ or $K_4 \subseteq_i H$ or *diamond* $\subseteq_i H$. If there is no edge between T_1 and T_2 then there exists $k' < k$ such that $k - \text{double – triangle} \subseteq_i H$. From our previous results we know that when $H \in \{C_4, K_4, \text{diamond}, \text{butterfly}, k' - \text{double – triangle}\}$ the MDS problem is *NP*-complete for (claw, H) -free graphs. Thus the result follows from Property 3.1. \square

9 Conclusion

We gave some complexity results for the Minimum Dominating Set problem for the class of *claw*-free graphs when another fixed graph H is forbidden as an induced subgraph. Especially we gave a complexity dichotomy for the class of (claw, H) -free graphs when H has less than seven vertices. When H has at least seven vertices, we gave a partial result for the cases where H contains a connected component with two triangles. For the other H 's we left the complexity problem open.

To reach our goal, as an intermediary step, we proved the *NP*-completeness of the Minimum Dominating Set problem for the class of cubic graphs, which was unknown.

References

- [1] B. Allan, R. Laskar (1978), *On domination and independent domination numbers of a graph*, Discrete Mathematics 23, 73-76.
- [2] B. Boliak, V. Lozin (2003) *Independent domination in finitely defined classes of graphs*, Theor. Comput. Scienc. 301, 271-284.
- [3] J.A. Bondy, U. S. R. Murty, *Graphs Theory*, Springer, (2008).
- [4] V. Bouquet, C. Picouleau (2019) *The Minimum Dominating Set problem is polynomial for (claw, P_8) -free graphs*, Manuscript.
- [5] D. Bauer, F. Harary, J. Nieminen, and C. L. Suffel (1983), *Domination alteration sets in graphs*. Discrete Mathematics, 47:153-161.
- [6] A. Brandstädt, F. Dragan (2003) *On linear and circular structure of $(\text{claw}, \text{net})$ -free graph*, Discr. Appl. Math. 129 (2-3), 285-303.
- [7] A. Brandstädt, S. Mahfud (2002) *Maximum weight stable set on graphs without claw and co-claw (and similar graph classes) can be solved in linear time.*, Inf. Proc. Let. 84, 251-259.
- [8] M. Chudnovsky, N. Robertson, P. D. Seymour and R. Thomas, (2006) *The strong perfect graph theorem*, Annals of Math. 164, 51-229.

- [9] B. Courcelle, J.A. Makowsky, U. Rotics (2000) *Linear time solvable optimization problems on graphs of bounded clique-width*, Theory Comput. Syst. 32 (2), 125-150.
- [10] M. Demange, T. Ekim, C. Tanasescu (2014) *Hardness and approximation of minimum maximal matchings*, International Journal of Computer Mathematics 91 (8), 1635-1654.
- [11] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, 1979.
- [12] Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., (1998).
- [13] M. C. Lin, M. J. Mizrahi (2015), *On the complexity of the minimum domination problem restricted by forbidden induced subgraphs of small size*, Discr. Appl. Math. 197, 53-58.
- [14] M. Yannakakis, F. Gavril (1980) *Edge dominating sets in graphs*, SIAM J. Appl. Math. 38 (3), 364-372.