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Rankin-Cohen brackets on tube-type domains

Jean-Louis Clerc

February 26, 2020

Abstract

A new formula is obtained for the holomorphic bi-differential operators on tube-type domains which are associated to the decomposition of the tensor product of two scalar holomorphic representations, thus generalizing the classical *Rankin-Cohen brackets*. The formula involves a family of polynomials of several variables which may be considered as a (weak) generalization the classical *Jacobi polynomials*.

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Key words : tube-type domains, Euclidean Jordan algebra, holomorphic discrete series, tensor product, weighted Bergman spaces, Rankin-Cohen brackets, Jacobi polynomials

Introduction

The tensor product of two holomorphic discrete series representations has been long studied (see e.g. [7]). For scalar ones, L. Peng and G. Zhang (see [6]) obtained fairly complete results for the decomposition into irreducible components. When the representations are realized in the weighted Bergman space model, the projectors on the irreducible components are given by bilinear holomorphic differential operators. In the case of the holomorphic discrete series representations of the groupe $SL(2, \mathbb{R})$ (or rather of its universal covering), these differential operators coincide with the classical *Rankin-Cohen brackets*. Recently, T. Kobayashi and M. Pevzner (see [5]) studied the same operators from a different point of view and pointed a link to classical *Jacobi polynomials*. The use of the L^2 -model for the representations, obtained from the weighted Bergman space model by an inverse Laplace transform plays an important rôle in their approach. Their paper was a source of inspiration for this work.

In the present paper, part of their results are generalized to a tube-type domain and its holomorphic automorphisms group. The group

can also be viewed as the conformal group of a Euclidean Jordan algebra. The connection between a family of covariant bi-differential operators on a Jordan algebra and a family of polynomials of several variables was observed by the present author in [2]. The holomorphic counterpart of these results yields on one hand a new formula for the holomorphic bi-differential operators on a tube-type domain which generalize the classical Rankin-Cohen brackets, and on the other hand a family of orthogonal polynomials which appears as a (weak) generalization of the Jacobi polynomials.

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1 Geometric and analytic background

1.1 Euclidean Jordan algebra

Let V be a simple Euclidean Jordan algebra. A general reference for notation and main results is [3]. Let n be the dimension of V , r its rank and d its characteristic number, which satisfy

$$n = r + \frac{r(r-1)}{2}d .$$

Let tr and \det be the trace and the determinant of V . The inner product on V is given by

$$(x, y) = \text{tr}(xy) .$$

The neutral element is denoted by e and satisfies $(e, x) = \text{tr } x$ for all $x \in V$.

The *structure group* $Str(V)$ is defined as the subgroup of $GL(V)$ of elements $\ell \in GL(V)$ for which there exists a scalar $\chi(\ell) \in \mathbb{R}^\times$ such that for all $x \in V$

$$\det(\ell x) = \chi(\ell) \det x . \quad (1)$$

Then $\chi : Str(V) \rightarrow \mathbb{R}^\times$ is a character of $Str(V)$.

Let Ω be the open cone of squares, and let $G(\Omega)$ be the subgroup of $GL(V)$ preserving the cone Ω . Then $G(\Omega) \subset Str(V)$ and in fact $Str(V)$ is the direct product of $G(\Omega)$ by $\{\pm \text{Id}\}$. Moreover, for $\ell \in G(\Omega)$, $\chi(\ell) > 0$ and

$$\text{Det}(\ell) = \chi(\ell)^{n/r} . \quad (2)$$

Let P be the *quadratic representation*. The following identity holds for x and $y \in V$:

$$\det(P(x)y) = (\det x)^2 \det y . \quad (3)$$

Recall the following equivalent propositions for an element $x \in V$:

- i) x is invertible
- ii) $\det x \neq 0$
- iii) $P(x)$ is invertible .

When any of these propositions is satisfied, then the inverse of x is given by

$$x^{-1} = P(x)^{-1}(x) .$$

The open set of invertible elements is denoted by V^\times . For $x \in V^\times$, $P(x)$ belongs to $Str(V)$, and if moreover $x \in \Omega$, $P(x)$ belongs to $G(\Omega)$.

The *Peirce decomposition* plays the rôle of the spectral theorem for the symmetric matrices. In particular it allows to define for any $z \in \Omega$ a square root $z^{1/2}$ which is the unique element in Ω such that $(z^{1/2})^2 = z$.

The following example may help the reader not familiar with Jordan algebras. Let r be an integer, $r \geq 1$, and consider the space $V = \text{Sym}(r, \mathbb{R})$ of $r \times r$ symmetric matrices with real entries, equipped with the Jordan product $x.y$ given by

$$x.y = \frac{1}{2}(xy + yx) .$$

The rank is r , the characteristic number is $d = 1$ and the dimension is $n = \frac{r(r+1)}{2}$. The trace and determinant coincide with the usual notions, the cone Ω is the cone of positive-definite symmetric matrices. The quadratic representation is given by

$$P(x) = xyx .$$

The group $G(\Omega)$ is isomorphic to $GL(r, \mathbb{R})/\{\pm \text{Id}\}$ acting by

$$(g, x) \longmapsto gxg^t .$$

Going back to the general situation, a parametrization of $\Omega \times \Omega$, akin to polar coordinates, will be needed later. Denote by $\llbracket -e, +e \rrbracket$ the "interval" between $-e$ and $+e$, i.e.

$$\llbracket -e, +e \rrbracket = (-e + \Omega) \cap (e - \Omega) = \{x \in V, e \pm x \in \Omega\} .$$

Proposition 1.1. *The map ι*

$$\Omega \times \llbracket -e, +e \rrbracket \ni (z, v) \longmapsto \iota(z, v) = \left(\frac{1}{2}(z - P(z^{\frac{1}{2}})v), \frac{1}{2}(z + P(z^{\frac{1}{2}})v) \right) \quad (4)$$

is a diffeomorphism from $\Omega \times \llbracket -e, +e \rrbracket$ onto $\Omega \times \Omega$. Moreover the Jacobian of ι is given by

$$\text{jac}(\iota)(z, v) = 2^{-n}(\det z)^{\frac{n}{r}} . \quad (5)$$

Proof. Let $z \in \Omega$ and $v \in \llbracket -e, +e \rrbracket$. Then, as $P(z^{1/2})$ belongs to $G(\Omega)$

$$z \pm P(z^{\frac{1}{2}})v = P(z^{\frac{1}{2}})(e \pm v) \in \Omega ,$$

and hence the image of ι is contained in Ω .

Conversely, assume $x, y \in \Omega$. Necessarily, $z = x + y$ and as $z \in \Omega$, $z^{\frac{1}{2}}$ is well-defined. Next, $y - x = P(z^{1/2})v$, and as $P(z^{1/2})$ is invertible, again necessarily $v = P(z^{-\frac{1}{2}})(y - x)$. Now, as $e = P(z^{-\frac{1}{2}})(z)$

$$e \pm v = P(z^{-\frac{1}{2}})(x + y \pm (y - x)) \in P(z^{-\frac{1}{2}})(\Omega) = \Omega .$$

Hence $(z, v) \in \Omega \times \llbracket -e, +e \rrbracket$. Moreover,

$$\frac{1}{2}(z \pm P(z^{\frac{1}{2}})v) = \frac{1}{2}(x + y \pm (P(z^{\frac{1}{2}})P(z^{-\frac{1}{2}}))(y - x)) = \frac{1}{2}(x + y \pm (y - x)),$$

or equivalently $\iota(z, v) = (x, y)$, showing that ι is both injective and surjective.

The map $z \longmapsto z^{\frac{1}{2}}$ is known to be a diffeomorphism of Ω , so that ι is a diffeomorphism. To compute the Jacobian of ι , set for a while $s = x + y$ and $d = -x + y$, so that

$$s = z, \quad d = P(z^{\frac{1}{2}})v,$$

and the differential of the map $(z, v) \longmapsto (s, d)$ is equal to

$$\begin{pmatrix} I & 0 \\ \star & P(z^{\frac{1}{2}}) \end{pmatrix} .$$

Now by (1), (2) and (3), $\text{Det } P(z^{\frac{1}{2}}) = (\det(z^{\frac{1}{2}}))^{\frac{2n}{r}} = (\det z)^{\frac{n}{r}}$ and the result follows \square

There is a corresponding integration formula for the change of variables.

Proposition 1.2. *Let f be an integrable function on $\Omega \times \Omega$. Then*

$$\iint_{\Omega \times \Omega} f(x, y) dx dy = 2^{-n} \int_{\Omega} \int_{\llbracket -e, +e \rrbracket} f(\iota(z, v)) (\det z)^{\frac{n}{r}} dz dv . \quad (6)$$

1.2 The conformal group and the degenerate principal series

Let j be the mapping defined on V^\times by

$$j(x) = -x^{-1} .$$

The differential of j at $x \in V^\times$ is given by

$$Dj(x) = P(x^{-1}) = P(x)^{-1} .$$

In [1], in a larger context and for technical reasons, was introduced the subgroup L of $Str(V)$ defined by

$$L = \{\ell \in Str(V), \chi(\ell) > 0\} .$$

Remark. $G(\Omega)$ is contained in L and, thanks to (3), for any $x \in V^\times$, $P(x)$ also belongs to L .

Let N be the group of translations of V , which is isomorphic to V as an Abelian Lie group, and let $P = L \ltimes N$.

Following [1], define the *proper conformal group* $Co(V)^+$ of the Jordan algebra V as the group of rational transforms of V generated by L , N and j . The group $Co(V)^+$ is a simple Lie group. The subgroup P consists of all elements of $Co(V)^+$ which are affine transformations of V and is a maximal parabolic subgroup of $Co(V)^+$.

Let $g \in Co(V)^+$, and assume that g is defined at $x \in V$ (hence also in a neighborhood of x). Then the differential $Dg(x)$ of g at x belongs to L . Define

$$c_g(x) = \chi(Dg(x))^{-1} .$$

The Jacobian $j(g, x)$ of g at x is given by

$$j(g, x) = c_g(x)^{-n/r} .$$

Proposition 1.3. *Let $g \in Co(V)^+$.*

i) The function c_g a priori defined on some open dense subset of V can be extended as a polynomial function on V .

ii) The function c_g is nonnegative and

$$c_g(x) \neq 0 \iff g \text{ is defined at } x .$$

iii) There exists a polynomial function $a_g(x)$, unique up to a sign \pm , such that

$$c_g(x) = a_g(x)^2 .$$

For a proof of iii), see [1].

A twofold covering of $Co(V)^+$ was constructed in [1] using the cocycle $a(g, x) := a_g(x)$.

Let $\tilde{G}^2 = \{(g, a_g), g \in Co(V)^+\}$ with the group law

$$(g_1, a_{g_1})(g_2, a_{g_2}) = (g_1 g_2, (a_{g_1} \circ g_2) a_{g_2}) .$$

The group \tilde{G}^2 is a twofold covering of $Co(V)^+$.

There is a family of representations $(\rho_{\lambda, \epsilon})$ induced from the parabolic subgroup \tilde{P}^2 of \tilde{G}^2 covering P , indexed by $(\lambda, \epsilon) \in \mathbb{C} \times \{\pm\}$, sometimes called the *scalar degenerate principal series*. In the *noncompact picture*, the representation is given by

$$(\rho_{\lambda, \epsilon}((g, a_g))f)(x) = a(g^{-1}, x)^{-\lambda, \epsilon} f(g^{-1}(x)) , \quad (7)$$

where $x \in V$ and $f \in C^\infty(V)$.¹

1.3 The automorphisms group of the tube-type domain and the series of holomorphic representations

Form the tube domain $T_\Omega = V \oplus i\Omega \subset \mathbb{V} = V \otimes \mathbb{C}$. Let $G(T_\Omega)$ be the group of holomorphic automorphisms of T_Ω . Some subgroups of T_Ω are easy to describe. Firstly, any $g \in G(\Omega)$ can be extended to a complex linear automorphism of T_Ω and hence $G(\Omega)$ can be regarded a a subgroup of $G(T_\Omega)$. Next, for $u \in V$, the translation

$$\tau_u : z \mapsto z + u$$

is a holomorphic automorphism of T_Ω , and the group of all translations $\tau_u, u \in V$ is an Abelian subgroup N^+ of $G(T_\Omega)$. Finally, any element of T_Ω is invertible in \mathbb{V} and the map

$$\iota : z \mapsto -z^{-1}$$

is an holomorphic automorphism of T_Ω .

Proposition 1.4. *The subgroups $G(\Omega)$, N^+ together with the element j generate the group $G(T_\Omega)$.*

For a proof see [3] Theorem X.5.6.

¹with the usual convention for $t \in \mathbb{R}^\times$, $t^{\lambda, +} = |t|^\lambda$, $t^{\lambda, -} = \text{sgn}(t)|t|^\lambda$.

Let $G = G(T_\Omega)^0$ be the connected component of $G(T_\Omega)$ containing the neutral element. Notice that the inversion ι belongs to G , as ι belongs to the connected subgroup of $G(T_\Omega)$ given by

$$\{g_{\alpha,\beta,\gamma,\delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, \quad \alpha\delta - \beta\gamma = 1\} / \{\pm \text{id}\}$$

where

$$g_{\alpha,\beta,\gamma,\delta} : z \mapsto (\alpha z + \beta e)(\gamma z + \delta e)^{-1} .$$

See [3] p. 208.

For $g \in G$ and $z \in T_\Omega$ let

$$j(g, z) = \text{Det}_{\mathbb{C}}(Dg(z)) .$$

For fixed g , the function $z \mapsto j(g, z)$ does not vanish on T_Ω . As T_Ω is simply connected, it is possible to define a logarithm, and two determinations differ by a multiple of $2i\pi$.

Now the *universal covering* \tilde{G} of G can be described as follows :

$$\tilde{G} = \{(g, \psi_g), g \in G, \psi_g : T_\Omega \longrightarrow \mathbb{C}, e^{\Psi_g(z)} = j(g, z)\}$$

with group law given by

$$(g_1, \psi_{g_1})(g_2, \psi_{g_2}) = (g_1 g_2, \psi_{g_1}(g_2(z)) + \psi_{g_2}(z)) .$$

We omit the description of the topology and the Lie group structure of \tilde{G} . Notice that the fiber of the covering is isomorphic to \mathbb{Z} .

Denote by $\mathcal{O}(T_\Omega)$ the space of holomorphic functions on T_Ω equipped with its Montel topology. Let $\nu \in \mathbb{C}$. Let $(g, \psi_g) \in \tilde{G}$ and let $F \in \mathcal{O}(T_\Omega)$. The formula

$$\pi_\nu((g, \psi_g))F(z) = e^{\frac{\nu}{2\pi} \nu \psi_{g^{-1}}(z)} F(g^{-1}(z)) \quad (8)$$

defines a (smooth) representation π_ν of G on $\mathcal{O}(T_\Omega)$. The factor $\frac{\nu}{2\pi}$ is introduced to correspond to the classical normalization and will be commented later.

1.4 The derived representations

The groups $G(\Omega)$ and L have the same neutral component. Hence the group $Co(V)^+$ and $G(T_\Omega)$ have the same neutral components, so that the groups \tilde{G}^2 and \tilde{G} are locally isomorphic, in particular have the same Lie algebra simply denoted as \mathfrak{g} .

The purpose of the next proposition is to compare the derived representations $d\pi_\nu$ and $d\rho_{\nu,\epsilon}$ of the Lie algebra $\mathfrak{g} = \text{Lie}(G(T_\Omega)) = \text{Lie}(Co(V)^+) = \text{Lie}(\tilde{G}) = \text{Lie}(\tilde{G}^2)$. It is well-known that in both cases,

these derived representations are given by first order differential operators with polynomial coefficients. For π_ν , the derived representation is given by holomorphic differential operators on \mathbb{V} . For $\rho_{\nu,\epsilon}$ we may extend the differential operators on V of the derived representation as holomorphic differential operators on \mathbb{V} .

Proposition 1.5. *The derived representations $d\pi_\nu$ and $d\rho_{\nu,\epsilon}$ coincide.*

Proof. In order to compare holomorphic differential operators, it is enough to test them on holomorphic polynomials. So let F be a holomorphic polynomial on \mathbb{V} . Let x_0 be a point in V . For g in the common neutral component of $Co(V)^+$ and of $G(T_\Omega)$, close enough to the neutral element, g is defined in a neighborhood \mathcal{V}_0 of x_0 in \mathbb{V} , and $j(g^{-1}, z) \in \mathbb{C} \setminus (-\infty, 0]$ for $z \in \mathcal{V}_0$. Denote by Ln the principal determination of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$. Then $\text{Ln}(j(g^{-1}, z))$ is well defined on \mathcal{V}_0 and there exists a global determination ψ_g of $\log j(g^{-1}, z)$ on T_Ω which coincides on $\mathcal{V}_0 \cap T_\Omega$ with $\text{Ln}(j(g^{-1}, z))$. Then

$$\pi_\nu((g, \psi_g))F(z) = e^{\frac{\nu}{2n} \text{Ln} j(g^{-1}, z)} F(g^{-1}(z)). \quad (9)$$

On the other hand, $j(g^{-1}, x) > 0$ for $x \in \mathcal{V}_0 \cap V$ and

$$c_{g^{-1}}(x) = j(g^{-1}, x)^{-\frac{\nu}{n}} = e^{-\frac{\nu}{n} \text{Ln}(j(g^{-1}, x))}.$$

Hence $e^{-\frac{\nu}{2n} \text{Ln}(j(g^{-1}, x))}$ is a local determination of $c_g(x)^{1/2}$. By Proposition 1.3 *iii*), this local determination has a global extension $a_g(x)$ to V . Hence for $x \in \mathcal{V}_0 \cap V$,

$$\rho_{\nu,\epsilon}((g, a_g))F(x) = e^{\frac{\nu}{2n} \text{Ln} j(g^{-1}, x)} F(g^{-1}(x)). \quad (10)$$

Extend by continuity (9) to $z = x$ where $x \in \mathcal{V}_0 \cap V$ and compare with (10), to get

$$\pi_\nu((g, \psi_g))F(x) = \rho_{\nu,\epsilon}((g, a_g))F(x)$$

for $x \in \mathcal{V}_0 \cap V$. By differentiation, for any $X \in \mathfrak{g}$

$$d\pi_\nu(X)F(x) = d\rho_{\nu,\epsilon}F(x)$$

for $x \in \mathcal{V}_0 \cap V$. As x_0 was arbitrary in V , $d\pi_\nu(X)F$ and $d\rho_{\nu,\epsilon}(X)F$ coincide on V . As F was an arbitrary holomorphic polynomial, this shows that $d\pi_\nu(X)$ and $d\rho_{\nu,\epsilon}(X)$ are equal for any $X \in \mathfrak{g}$. The statement follows. \square

1.5 Weighted Bergman spaces and the holomorphic discrete series

Let dx be the Lebesgue measure on V associated to the Euclidean structure on V . The *Gamma function* of the positive cone Ω is given by

$$\begin{aligned}\Gamma_{\Omega}(\nu) &= \int_{\Omega} e^{-\text{tr}(x)} \det(x)^{\nu - \frac{n}{r}} dx \\ &= (2\pi)^{\frac{n-r}{2}} \Gamma(\nu) \Gamma\left(\nu - \frac{d}{2}\right) \dots \Gamma\left(\nu - (r-1)\frac{d}{2}\right) .\end{aligned}$$

See [3] chapter VII.

For a real parameter ν , define $\mathcal{H}_{\nu}(T_{\Omega})$ as the space of holomorphic functions $f : T_{\Omega} \rightarrow \mathbb{C}$ such that

$$\|F\|_{\nu}^2 = \int_{T_{\Omega}} |F(x + iy)|^2 \det(y)^{\nu - \frac{2n}{r}} dx dy < +\infty .$$

For $\nu \leq 1 + d(r-1)$, the space \mathcal{H}_{ν} is reduced to $\{0\}$, so assume that $\nu > 1 + d(r-1)$. Then \mathcal{H}_{ν} is a Hilbert space $\neq \{0\}$. For these results, see [3], chapter XIII.

For $(g, \psi_g) \in \tilde{G}$ and $F \in \mathcal{H}_{\nu}$ the formula

$$\rho_{\nu}(g)F(z) = e^{\frac{2\nu}{2n}\psi_g^{-1}(z)} F(g^{-1}(z)) \quad (11)$$

defines now a *unitary* representation ρ_{ν} on \mathcal{H}_{ν} . These representations belong to the *holomorphic discrete series*.

There is another useful realization of these representations, using the *Laplace transform*. For $f \in C_c^{\infty}(\Omega)$, let for $z \in T_{\Omega}$

$$\mathcal{L}f(z) = \int_{\Omega} f(\xi) e^{i(z, \xi)} d\xi .$$

The integral converges for $z \in T_{\Omega}$ and the function $\mathcal{L}f$ is holomorphic on T_{Ω} .

Let $L^2(\Omega)_{\nu} = L^2(\Omega, \det(\xi)^{-\nu + \frac{n}{r}})$ be the Hilbert space of functions on Ω which are square-integrable w.r.t. the measure $\det(\xi)^{-\nu + \frac{n}{r}} d\xi$.

Proposition 1.6. *For $\nu \in \mathbb{R}, \nu > 1 + d(r-1)$, the Laplace transform can be extended by continuity to yield (up to a scalar) an isometry from $L^2(\Omega)_{\nu}$ onto $\mathcal{H}_{\nu}(T_{\Omega})$. Furthermore*

$$\|\mathcal{L}f\|_{\nu}^2 = 2^{\frac{n}{r} - \nu} \Gamma_{\Omega}\left(\nu - \frac{n}{r}\right) \|f\|_{\nu}^2 .$$

For this result, see again [3] ch. XIII. Notice however some changes in notation.

2 Rankin-Cohen brackets in a tube-type domain

2.1 Definition of the Rankin-Cohen brackets

We now proceed to the construction of a family of holomorphic bi-differential operators with constant coefficients on $\mathbb{V} \times \mathbb{V}$, called the *generalized Rankin-Cohen brackets*.

Proposition 2.1. *Let $s, t \in \mathbb{C}$ and $k \in \mathbb{N}$. The expression*

$$c_{s,t}^{(k)}(x, y) := (\det x)^{-s} (\det y)^{-t} \det \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)^k ((\det x)^{s+k} (\det y)^{t+k}) \quad (12)$$

a priori defined on $\Omega \times \Omega$ can be extended as a polynomial function on $V \times V$.

This proposition, formulated in [2] is essentially a by-product of an important result of [1].

Lemma 2.1. *The polynomial $c_{s,t}^{(k)}$ satisfies the identity*

$$\forall \ell \in G(\Omega), \quad c_{s,t}^{(k)}(\ell x, \ell y) = \chi(\ell)^k c_{s,t}^{(k)}(x, y) . \quad (13)$$

Proof. Let $D = \det \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)$. For $\ell \in G(\Omega)$, let D_ℓ be the differential operator given by

$$D_\ell f = D(f \circ \ell^{-1}) \circ \ell$$

The symbol of D is $d(\xi, \zeta) = i^r \det(\xi - \zeta)$, and the symbol of D_ℓ is equal to $d_\ell = d \circ (\ell^t)^{-1}$. From (1) follows for $h \in G(\Omega)$

$$d(h\xi, h\zeta) = \chi(h) d(\xi, \zeta) .$$

Apply to $h = (\ell^t)^{-1}$ to obtain

$$d_\ell = \chi(\ell)^{-1} d \quad \text{and hence} \quad D_\ell = \chi(\ell)^{-1} D ,$$

which implies

$$D_\ell^k = \chi(\ell)^{-k} D^k . \quad (14)$$

Let for a while $e_{s,t}(x, y) = \det(x)^s \det(y)^t$. By definition,

$$D^k e_{s+k, t+k} = c_{s,t} e_{s,t} .$$

On the other hand,

$$\begin{aligned} D_\ell^k e_{s+k, t+k} &= D^k (e_{s+k, t+k} \circ \ell^{-1}) \circ \ell = (\chi(\ell)^{-s-t-2k} (D^k e_{s+k, t+k})) \circ \ell \\ &= \chi(\ell)^{-s-t-2k} (c_{s,t} e_{s,t}) \circ \ell = \chi(\ell)^{-2k} (c_{s,t} \circ \ell) e_{s,t} . \end{aligned}$$

Combining these two calculations with (14) yields (13). Notice that this result implies that $c_{s,t}^{(k)}$ is homogeneous of degree rk . \square

Let $\text{res} : \mathcal{O}(T_\Omega \times T_\Omega) \longrightarrow \mathcal{O}(T_\Omega)$ be the *restriction map* to the diagonal given by

$$f \in \mathcal{O}(T_\Omega \times T_\Omega), \quad \text{res}(f)(z) = f(z, z) .$$

Definition 2.1. For $\lambda, \mu \in \mathbb{C}$ and $k \in \mathbb{N}$ define the k -th generalized Rankin-Cohen bracket $B_{\lambda, \mu}^{(k)}$ by

$$B_{\lambda, \mu}^{(k)} = \text{res} \circ c_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial w} \right) . \quad (15)$$

Theorem 2.1. Let $\lambda, \mu \in \mathbb{C}$. The Rankin-Cohen brackets are covariant bi-differential operators with respect to $(\pi_\lambda \otimes \pi_\mu, \pi_{\lambda+\mu+2k})$, i.e. for any $g \in \tilde{G}$

$$B_{\lambda, \mu}^{(k)} \circ (\pi_\lambda(g) \otimes \pi_\mu(g)) = \pi_{\lambda+\mu+2k}(g) \circ B_{\lambda, \mu}^{(k)} . \quad (16)$$

Proof. For arbitrary $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$ the authors constructed in [1] a family of bi-differential operators from $C^\infty(V \times V)$ into $C^\infty(V)$ which are covariant for \tilde{G}^2 with respect to $((\rho_{\lambda, \epsilon} \otimes \rho_{\mu, \eta}), \rho_{\lambda+\mu+2k, \epsilon\eta})$. In [2] the present author proved that these operators are equal (up to scalars) to

$$\text{res} \circ c_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) ,$$

where "res" here stands for the restriction from $C^\infty(V \times V)$ to $C^\infty(V)$. The covariance relation obtained for the group \tilde{G}^2 can be translated to the derived representations. Then use Proposition 1.5, i.e. the identification of the derived representations $d(\rho_{\lambda, \epsilon} \otimes \rho_{\mu, \eta})$ (resp. $d\rho_{\lambda+\mu+2k, \epsilon\eta}$) with $d(\pi_\lambda \otimes \pi_\mu)$ (resp. $d\pi_{\lambda+\mu+2k}$) to obtain a covariance relation for the derived representations, namely for any $X \in \mathfrak{g}$

$$B_{\lambda, \mu}^{(k)} \circ d(\pi_\lambda \otimes \pi_\mu)(X) = d\pi_{\lambda+\mu+2k}(X) \circ B_{\lambda, \mu}^{(k)} .$$

This relation can now be integrated along the one-parameter subgroups of \tilde{G} . Use the connectedness of \tilde{G} to obtain the covariance relation (16) for any element $g \in \tilde{G}$. \square

2.2 Expression in the L^2 -model

Assume now that $\lambda, \mu > 1 + d(r-1)$. The (completed) tensor product $\mathcal{H}_\lambda \hat{\otimes} \mathcal{H}_\mu$ is identified with the space $\mathcal{H}_{\lambda, \mu}$ of holomorphic functions F on $T_\Omega \times T_\Omega$ such that

$$\iint_{T_\Omega \times T_\Omega} |F(x + iy, u + it)|^2 \det(y)^{\lambda - \frac{2n}{r}} \det(t)^{\mu - \frac{2n}{r}} dx dy du dt < \infty .$$

The corresponding (completed) tensor product $L_\lambda^2(\Omega) \widehat{\otimes} L_\mu^2(\Omega)$ is identified with

$$L_{\lambda,\mu}^2(\Omega \times \Omega) = L^2(\Omega \times \Omega, \det(\xi)^{-\lambda+\frac{n}{r}} \det(\zeta)^{-\mu+\frac{n}{r}} d\xi d\zeta) .$$

For $f \in C_c^\infty(\Omega \times \Omega)$ define its Laplace transform $\mathcal{L}_2 f$ to be

$$\mathcal{L}_2 f(z, w) = \iint_{\Omega \times \Omega} f(\xi, \zeta) e^{i((z,\xi)+(w,\zeta))} d\xi d\zeta .$$

Then \mathcal{L}_2 extends as an isometry (up to a scalar) between $L_{\lambda,\mu}^2(\Omega \times \Omega)$ and $\mathcal{H}_{\lambda,\mu}$.

The Rankin-Cohen operators have a counterpart when working with the L^2 -model instead of the weighted Bergman spaces. Define

$$\widehat{B}_{\lambda,\mu}^{(k)} = \mathcal{L}^{-1} \circ B_{\lambda,\mu}^{(k)} \circ \mathcal{L}_2 = \mathcal{L}^{-1} \circ \text{res} \circ c_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial w} \right) \circ \mathcal{L}_2 .$$

The composition of these operators is *a priori* formal, but will be shown to make sense on functions in $C_c^\infty(\Omega \times \Omega)$. Recall the elementary formula for the Laplace transform, valid for any holomorphic polynomial p on \mathbb{V}

$$p \left(\frac{\partial}{\partial z} \right) (\mathcal{L}f)(z) = \int_{\Omega} p(i\xi) f(\xi) e^{i(z,\xi)} d\xi$$

The following consequence is then immediate.

Lemma 2.2.

$$c_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial w} \right) \circ \mathcal{L}_2 = i^{rk} \mathcal{L}_2 \circ c_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}} . \quad (17)$$

Notice that the multiplication by the polynomial $c_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}$ is a continuous operator on $C_c^\infty(\Omega \times \Omega)$.

The next step is to calculate $\text{res} \circ \mathcal{L}_2$ (cf [5]). For $f \in C_c^\infty(\Omega \times \Omega)$, define for $\xi \in \Omega$

$$\mathcal{J}f(\xi) = 2^{-n} (\det \xi)^{\frac{n}{r}} \int_{-]e, e[} f(\iota(\xi, v)) dv . \quad (18)$$

By elementary arguments, the integral converges, the resulting function $\mathcal{J}f$ belongs to $C_c^\infty(\Omega)$ and the operator $\mathcal{J} : C_c^\infty(\Omega \times \Omega) \rightarrow C_c^\infty(\Omega)$ is continuous.

Lemma 2.3.

$$\text{res} \circ \mathcal{L}_2 = \mathcal{L} \circ \mathcal{J} . \quad (19)$$

Proof. Let $f \in C_c^\infty(\Omega \times \Omega)$. Then, for $z \in T_\Omega$

$$\begin{aligned} \text{res}(\mathcal{L}_2 f)(z) &= \int_\Omega \int_\Omega f(\xi, \zeta) e^{i(z, \xi + \zeta)} d\xi d\zeta \\ &= \int_\Omega \left(2^{-n} (\det \eta)^{\frac{n}{r}} \int_{\llbracket -e, +e \rrbracket} f(\boldsymbol{\iota}(\eta, v)) dv \right) e^{i(z, \eta)} d\xi, \end{aligned}$$

by using the change of variables $(\xi, \zeta) = \boldsymbol{\iota}(\eta, v)$ and the integration formula (6). This finishes the proof. \square

From these two lemmas follows the main result.

Theorem 2.2. *The operator $\widehat{B}_{\lambda, \mu}^{(k)}$ maps $C_c^\infty(\Omega \times \Omega)$ into $C_c^\infty(\Omega)$ and satisfies*

$$\widehat{B}_{\lambda, \mu}^{(k)} = i^{rk} \mathcal{J} \circ c_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}}^{(k)}. \quad (20)$$

The expression for the operator $\widehat{B}_{\lambda, \mu}^{(k)}$ can be given a slightly different form. Introduce the polynomial $C_{\lambda, \mu}^{(k)}$ on V defined by

$$C_{\lambda, \mu}^{(k)}(x) = c_{\lambda, \mu}^{(k)} \left(\frac{e-x}{2}, \frac{e+x}{2} \right).$$

Lemma 2.4. *For $(\eta, v) \in \Omega \times \llbracket -e, +e \rrbracket$*

$$c_{\lambda, \mu}^{(k)}(\boldsymbol{\iota}(\eta, v)) = (\det \eta)^k C_{\lambda, \mu}^{(k)}(v). \quad (21)$$

Proof. Use the covariance property (13) of the polynomials $c_{\lambda, \mu}^{(k)}$ under the action of $G(\Omega)$ to obtain

$$\begin{aligned} c_{\lambda, \mu}^{(k)}(\boldsymbol{\iota}(\eta, v)) &= c_{\lambda, \mu}^{(k)} \left(P(\eta^{1/2}) \frac{e-v}{2}, P(\eta^{1/2}) \frac{e+v}{2} \right) \\ &= \chi \left(P(\eta^{1/2}) \right)^k c_{\lambda, \mu}^{(k)} \left(\frac{e-v}{2}, \frac{e+v}{2} \right) = (\det \eta)^k C_{\lambda, \mu}^{(k)}(v). \end{aligned}$$

\square

Hence we may rewrite Theorem 2.2 as follows.

Theorem 2.3.

$$\widehat{B}_{\lambda, \mu}^{(k)} f(\xi) = 2^{-n} i^{rk} (\det \xi)^{k + \frac{n}{r}} \int_{\llbracket -e, e \rrbracket} C_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}}(v) f(\boldsymbol{\iota}(\xi, v)) dv. \quad (22)$$

3 Continuity and unitarity of the Rankin-Cohen brackets

When λ, μ are real numbers and $\lambda, \mu > 1 + d(r - 1)$, it is possible to study continuity and unitarity properties of the Rankin-Cohen brackets using their simple expression in the L^2 -model.

3.1 The formal adjoint of $\widehat{B}_{\lambda, \mu}^{(k)}$

Let $h \in C_c^\infty(\Omega)$. The formula

$$\begin{aligned} \left(\Phi_{\lambda, \mu}^{(k)} h \right) (\xi, \zeta) = & i^{rk} (\det \xi)^{\lambda - \frac{n}{r}} (\det \zeta)^{\mu - \frac{n}{r}} \det(\xi + \zeta)^{-\lambda - \mu - 2k + \frac{n}{r}} \\ & c_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}}(\xi, \zeta) h(\xi + \zeta) \end{aligned} \quad (23)$$

defines an operator from $C_c^\infty(\Omega)$ into $C^\infty(\Omega \times \Omega)$.

The Hilbert product on $L^2_\nu(\Omega)$ (resp. on $L^2_{\lambda, \nu}(\Omega \times \Omega)$) induces a duality between $C_c^\infty(\Omega)$ and $C^\infty(\Omega)$ (resp. $C_c^\infty(\Omega \times \Omega)$ and $C^\infty(\Omega \times \Omega)$), denoted by $(h_1, h_2)_{L^2_\nu}$ (resp. $(f_1, f_2)_{L^2_{\lambda, \nu}}$).

Proposition 3.1. *The operator $\Phi_{\lambda, \mu}^{(k)}$ is the formal adjoint of $\widehat{B}_{\lambda, \mu}^{(k)}$ with respect to the dualities induced by the Hilbert products on $L^2_{\lambda + \mu + 2k}(\Omega)$ and $L^2_{\lambda, \mu}(\Omega \times \Omega)$.*

Proof. Let $h \in C_c^\infty(\Omega)$ and $f \in C_c^\infty(\Omega \times \Omega)$. Then

$$\begin{aligned} \left(h, \widehat{B}_{\lambda, \mu}^{(k)} f \right)_{L^2_{\lambda + \mu + 2k}} = \\ 2^{-n} i^{rk} \int_{\Omega} h(\xi) (\det \xi)^{k + \frac{n}{r}} \det(\xi)^{-\lambda - \mu - 2k + \frac{n}{r}} \int_{\mathbb{J}_{-e, +e}} C_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}}^{(k)}(v) \overline{f(\boldsymbol{\nu}(\xi, v))} dv d\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left(\Phi_{\lambda, \mu}^{(k)} h, f \right)_{L^2_{\lambda, \mu}} = \\ i^{rk} \iint_{\Omega \times \Omega} h(\xi + \zeta) c_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}}(\xi, \zeta) \det(\xi + \zeta)^{-\lambda - \mu - 2k + \frac{n}{r}} \overline{f(\xi, \zeta)} d\xi d\zeta \\ = 2^{-n} i^{rk} \int_{\Omega} \int_{\mathbb{J}_{-e, +e}} h(\eta) (\det \eta)^{\frac{n}{r}} (\det \eta)^{-\lambda - \mu - 2k + \frac{n}{r}} (\det \eta)^k C_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}}(v) \overline{f(\boldsymbol{\nu}(\eta, v))} dv d\eta \end{aligned}$$

and the result follows. \square

3.2 Unitarity and continuity properties

Theorem 3.1. *The operator $\Phi_{\lambda,\mu}^{(k)}$ is a partial isometry (up to a scalar) from $L_{\lambda+\mu+2k}^2(\Omega)$ into $L_{\lambda,\mu}^2(\Omega \times \Omega)$.*

Proof.

$$\begin{aligned} & \|\Phi_{\lambda,\mu}^{(k)} h\|^2 = \\ & \int_{\Omega} \int_{\Omega} |\Phi_{\lambda,\mu}^{(k)} h(\xi, \zeta)|^2 (\det \xi)^{-\lambda+\frac{n}{r}} (\det \zeta)^{-\mu+\frac{n}{r}} d\xi d\zeta \\ & = \int_{\Omega} \int_{\Omega} |h(\xi+\zeta)|^2 |c_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(k)}(\xi, \zeta)|^2 (\det \xi)^{\lambda-\frac{n}{r}} (\det \zeta)^{\mu-\frac{n}{r}} \det(\xi+\zeta)^{-2\lambda-2\mu-4k+\frac{2n}{r}} d\xi d\zeta. \end{aligned}$$

Now use the coordinate change $(\xi, \zeta) = \iota(\eta, v)$. Notice that

$$\det \xi = 2^{-r} \det \eta \det(e-v), \quad \det \zeta = 2^{-r} \det \eta \det(e+v),$$

$$c_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(k)}(\xi, \zeta) = (\det \eta)^k C_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(k)}(v)$$

so that the integral becomes

$$\begin{aligned} & 2^{-r\lambda-r\mu+n} \int_{\llbracket -e, +e \rrbracket} |C_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(k)}(v)|^2 \det(e-v)^{\lambda-\frac{n}{r}} \det(e+v)^{\mu-\frac{n}{r}} dv \\ & \quad \times \int_{\Omega} |h(\eta)|^2 (\det \eta)^{-\lambda-\mu-2k+\frac{n}{r}} d\eta \\ & = c(\lambda, \mu; k) \|h\|_{\lambda+\mu+2k}^2, \end{aligned}$$

where

$$c(\lambda, \mu; k) = 2^{-r\lambda-r\mu+n} \int_{\llbracket -e, +e \rrbracket} |C_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(k)}(v)|^2 \det(e-v)^{\lambda-\frac{n}{r}} \det(e+v)^{\mu-\frac{n}{r}} dv.$$

□

Corollary 3.1. *The operator $B_{\lambda,\mu}^{(k)}$ is continuous from $\mathcal{H}_{\lambda,\mu}$ into $\mathcal{H}_{\lambda+\mu+2k}$. Its adjoint $B_{\lambda,\mu}^{(k)*}$ is (up to a scalar) a partial isometry from $\mathcal{H}_{\lambda+\mu+2k}$ into $\mathcal{H}_{\lambda,\mu}$.*

Proof. Recall that $\Phi_{\lambda,\mu}^{(k)}$ is the dual of the operator $\widehat{B}_{\lambda,\mu}^{(k)}$. From Theorem 3.1 follows that $\widehat{B}_{\lambda,\mu}^{(k)}$ is continuous from $L_{\lambda,\mu}^2(\Omega \times \Omega)$ into $L_{\lambda+\mu+2k}^2(\Omega)$. Now via Laplace transform/inverse Laplace transform, $B_{\lambda,\mu}^{(k)}$ is continuous from $\mathcal{H}_{\lambda,\mu}$ into $\mathcal{H}_{\lambda+\mu+2k}$. Similarly, its adjoint $B_{\lambda,\mu}^{(k)*}$ corresponds to $\Phi_{\lambda,\mu}^{(k)}$ and hence is a partial isometry (up to scalars). □

Said differently, the last result shows that the orthogonal decomposition of the tensor product $\pi_{\lambda} \otimes \pi_{\mu}$ into irreducible representations contains a copy of $\pi_{\lambda+\mu+2k}$ for each $k \in \mathbb{N}$.

4 Applications

In this last section, we continue to assume that $\lambda, \mu > 1 + (r - 1)d$. The corresponding representations π_λ or π_μ belong to the holomorphic discrete series, a fact which was not used before. The existence of a *lowest weight vector* in the space of the representation is the only result really used thereafter. For more information, see [3], [4], [7].

4.1 Lowest weight vectors

Let

$$\begin{aligned}\psi_\lambda(\xi) &= e^{-\text{tr} \xi} (\det \xi)^{-\lambda}, & x \in \Omega, \\ \varphi_\lambda(z) &= \det(z + ie)^{-\lambda}, & z \in T_\Omega.\end{aligned}$$

Lemma 4.1.

- i) ψ_λ belongs to $L^2_\lambda(\Omega)$
- ii) φ_λ belongs to \mathcal{H}_λ
- iii) $\mathcal{L}(\psi_\lambda) = i^{r\lambda} \Gamma_\Omega(\lambda) \varphi_\lambda$.

These results are contained in [3] chapter XIII. They give the expression of a (unique up to a scalar) lowest weight vector in the two realizations of the holomorphic discrete series representation π_λ .

Proposition 4.1. *The lowest weight vector (unique up to a scalar) in the subspace of $\mathcal{H}_{\lambda, \mu}$ corresponding to the component isomorphic to $\mathcal{H}_{\lambda + \mu + 2k}$ is given by*

$$\det(z - w)^k \det(z + ie)^{-\lambda - k} \det(w + ie)^{-\mu - k}. \quad (24)$$

Proof. The lowest weight vector in the space $L^2_{\lambda + \mu + 2k}(\Omega)$ is $\psi_{\lambda + \mu + 2k}$. The corresponding vector in $L^2_{\lambda, \mu}$ (which is the lowest weight vector of the component in $L^2_{\lambda, \mu}$ isomorphic to $L^2_{\lambda + \mu + 2k}(\Omega \times \Omega)$) is

$$\Phi_{\lambda, \mu}^{(k)}(\psi_{\lambda + \mu + 2k})(\xi, \zeta) = (\det \xi)^{\lambda - \frac{n}{r}} (\det \zeta)^{\mu - \frac{n}{r}} e^{-\text{tr} \xi} e^{-\text{tr} \zeta} c_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}}^{(k)}(\xi, \zeta). \quad (25)$$

The Laplace transform of this function is

$$\int_\Omega \int_\Omega e^{i((z, \xi) + (w, \zeta))} \det \xi^{\lambda - \frac{n}{r}} \det \zeta^{\mu - \frac{n}{r}} e^{-\text{tr} \xi} e^{-\text{tr} \zeta} c_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}}^{(k)}(\xi, \zeta) d\xi d\zeta.$$

Recall that

$$\det \xi^{\lambda - \frac{n}{r}} \det \zeta^{\mu - \frac{n}{r}} c_{\lambda - \frac{n}{r}, \mu - \frac{n}{r}}^{(k)}(\xi, \zeta) = \det \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \zeta} \right)^k (\det \xi^{\lambda + k - \frac{n}{r}} \det \zeta^{\mu + k - \frac{n}{r}}),$$

and substitute this expression in the integral to obtain

$$\int_\Omega \int_\Omega e^{i((z, \xi) + (w, \zeta))} e^{-\text{tr}(\xi + \zeta)} \det \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \zeta} \right)^k (\det \xi^{\lambda + k - \frac{n}{r}} \det \zeta^{\mu + k - \frac{n}{r}}) d\xi d\zeta.$$

Next integrate by parts. Notice that for any smooth function f on Ω and any polynomial p on V

$$p\left(\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\zeta}\right)\right) f(\xi + \zeta) = 0,$$

so that

$$\begin{aligned} & \det\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\zeta}\right)^k \left(e^{i((z,\xi)+(w,\zeta))} e^{-\text{tr}(\xi+\zeta)} \right) \\ &= i^rk \det(z-w)^k e^{i((z,\xi)+(w,\zeta))} e^{-\text{tr}(\xi+\zeta)}. \end{aligned}$$

Moreover, the condition on λ and μ implies $\lambda - \frac{n}{r}, \mu - \frac{n}{r} > (r-1)\frac{d}{2}$, which in turn implies the vanishing of the border contributions when performing the integrations by parts. Thus the integral is equal to

$$\begin{aligned} & (-1)^{kr} i^{kr} \det(z-w)^k \int_{\Omega} \int_{\Omega} e^{-\text{tr}\xi} \det\xi^{\lambda+k-\frac{n}{r}} e^{-\text{tr}\zeta} \det\zeta^{\mu+k-\frac{n}{r}} d\xi d\zeta \\ &= \text{const.} \det(z-w)^k \det(z+ie)^{-\lambda-k} \det(w+ie)^{-\mu-k}. \end{aligned}$$

□

4.2 Orthogonality of the family $C_{\lambda,\mu}^{(k)}$

Proposition 4.2. *The polynomials $C_{\lambda,\mu}^{(k)}$ form an orthogonal family in $L^2(\llbracket -e, +e \rrbracket, \det(e-v)^\lambda \det(e+v)^\mu)$.*

Proof. For $k \neq l \in \mathbb{N}$, the subspaces $\text{Im}\left(B_{\lambda,\mu}^{(k)*}\right)$ and $\text{Im}\left(B_{\lambda,\mu}^{(l)*}\right)$ are two non equivalent (hence orthogonal) components in the decomposition of the tensor product $\mathcal{H}_{\lambda,\mu}$. In particular, their lowest weight vectors are orthogonal. Hence, by isometry, the corresponding lowest weight vectors in the L^2 -model of the representations are orthogonal, i.e.

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} \Phi_{\lambda,\mu}^{(k)}(\psi_{\lambda+\mu+2k})(\xi, \zeta) \Phi_{\lambda,\mu}^{(l)}(\psi_{\lambda+\mu+2l})(\xi, \zeta) (\det\xi)^{-\lambda+\frac{n}{r}} (\det\zeta)^{-\mu+\frac{n}{r}} d\xi d\zeta \\ &= 0. \end{aligned}$$

Using (25) and after simplification, the integral becomes

$$\int_{\Omega} \int_{\Omega} (\det\xi)^{\lambda-\frac{n}{r}} (\det\zeta)^{\mu-\frac{n}{r}} e^{-2\text{tr}(\xi+\zeta)} c_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(k)}(\xi, \zeta) c_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(l)}(\xi, \zeta) d\xi d\zeta.$$

Use the change of coordinates $(\xi, \zeta) = \boldsymbol{\nu}(\eta, v)$ to write the integral as

$$\int_{\Omega} \int_{\llbracket -e, +e \rrbracket} e^{-2\text{tr}\eta} (\det\eta)^{k+l+\frac{n}{r}} \det(e-v)^{\lambda-\frac{n}{r}} \det(e+v)^{\mu-\frac{n}{r}} \dots$$

$$\dots C_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(k)}(v) C_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(l)}(v) d\eta dv .$$

As

$$\int_{\Omega} e^{-2 \operatorname{tr} \eta} (\det \eta)^{k+l+\frac{n}{r}} d\eta > 0$$

follows

$$\int_{\mathbb{I}_{-e, +e}} C_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(k)}(v) C_{\lambda-\frac{n}{r}, \mu-\frac{n}{r}}^{(l)}(v) \det(e-v)^{\lambda-\frac{n}{r}} \det(e+v)^{\mu-\frac{n}{r}} dv = 0 .$$

Up to the change of $\lambda - \frac{n}{r}, \mu - \frac{n}{r}$ to λ, μ this proves the statement. \square

4.3 Final remark

The family of polynomials $C_{\lambda, \mu}^{(k)}$ reminds of the classical Jacobi polynomials, appearing in the simplest example $V = \mathbb{R}$ (cf [5]). They have in common at least three properties (see [2] for details) :

- a *Rodrigues formula*
- a recurrence relation relating $C_{\lambda, \mu}^{(k)}$ and $C_{\lambda+1, \mu+1}^{(k-1)}$
- for appropriate values of the parameters λ, μ , orthogonality properties with respect to a certain measure $d\nu_{\lambda, \mu}$ on an "interval".

For these appropriate values of the parameters, the Jacobi polynomials are not only orthogonal but form an *orthogonal basis* of the Hilbert space of square-integrable functions for the measure $d\nu_{\lambda, \mu} = (1-x)^{\lambda}(1+x)^{\mu} dx$. This is lacking in our more general case. The polynomials $C_{\lambda, \mu}^{(k)}$ are invariant by the automorphisms group $Aut(V)$ of the Jordan algebra V . From the spectral theorem, it is easy to see that such an invariant polynomial (or more generally invariant function) depends on r variables. So one can guess that, in order to be a basis of the Hilbert space of invariant functions square-integrable w.r.t. $d\nu_{\lambda, \mu} = \det(e-v)^{\lambda} \det(e+v)^{\mu} dv$, a family of polynomials should depend on a r -tuple of integers. The decomposition of the tensor product of two scalar holomorphic representations contains many non scalar components (see [6]), indexed by a r -tuple of integers, which are not considered in the present article. To these components correspond more lowest weight vectors, which presumably could enlarge the family of the orthogonal polynomials $C_{\lambda, \mu}^{(k)}$, in order to obtain an orthogonal *basis* of the Hilbert space of square-integrable functions with respect to the measure $d\nu_{\lambda, \mu}$.

Let me address another question. The classical Jacobi polynomials have expressions in terms of certain hypergeometric functions. A theory of generalized hypergeometric functions exists for Euclidean Jordan algebra, see [3] ch. XV. A complementary investigation would be to try to relate the polynomials $C_{\lambda, \mu}^{(k)}$ with these generalized hypergeometric functions.

References

- [1] S. Ben Saïd, J.-L. Clerc, Kh. Koufany, *Conformally covariant bi-differential operators on a simple Jordan algebra*, Int. Math. Res. Notes, doi:10.193/imrn/rny082, 65 pages (2018)
- [2] J.-L. Clerc, *Symmetry breaking differential operators, the source operator and Rodrigues formulæ*, accepted for publication in Pac. J. Math. (2020)
- [3] J. Faraut, A. Korányi, *Analysis on Symmetric Cones*, Oxford Science Publications, Clarendon Press (1994)
- [4] J. Faraut, A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88**, 6–89 (1990)
- [5] T. Kobayashi, M. Pevzner, *Inversion of Rankin-Cohen operators via holographic transform*, arXiv:1812.09733 (2018)
- [6] L. Peng, G. Zhang, *Tensor products of holomorphic representations and bilinear differential operators*, J. Funct. Anal. **210**, 171–192 (2004)
- [7] J. Repka, *Tensor products of holomorphic discrete series representations*, Canad.J. Math. **31** (1979), 836–844

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