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MAGNETIC WKB CONSTRUCTIONS ON SURFACES

Y. GUEDES BONTHONNEAU, T. NGUYEN DUC, N. RAYMOND, AND S. VŨ NGỌC

ABSTRACT. This article is devoted to the description of the eigenvalues and eigenfunctions of the magnetic Laplacian in the semiclassical limit via the complex WKB method. Under the assumption that the magnetic field has a unique and non-degenerate minimum, we construct the local complex WKB approximations for eigenfunctions on a general surface. Furthermore, in the case of the Euclidean plane, with a radially symmetric magnetic field, the eigenfunctions are approximated in an exponentially weighted space.

1. INTRODUCTION

1.1. Magnetic Laplacian on a Riemannian manifold. Let (M, g) be a two-dimensional connected oriented Riemannian manifold (possibly with boundary) equipped with a metric g . Let \mathbf{A} be a smooth real-valued 1-form defined on M . Since M is two-dimensional, there exists a smooth real-valued function \mathbf{B} such that

$$d\mathbf{A} = \mathbf{B} d\mathcal{V}_g,$$

in which $d\mathcal{V}_g$ is the Riemannian volume form on M . We call \mathbf{A} the magnetic potential and \mathbf{B} the magnetic field. Let $\hat{g} : TM \rightarrow T^*M$ be the canonical isomorphism induced by g , *i.e.* for each $p \in M$ and $V \in T_pM$,

$$(1.1) \quad \hat{g}_p(V)(W) = g_p(V, W) \quad \text{for all } W \in T_pM.$$

Let g^* be the dual metric on the set of 1-forms on M :

$$g_p^*(w_1, w_2) = g_p(\hat{g}_p^{-1}(w_1), \hat{g}_p^{-1}(w_2)) \quad \text{for all } w_1, w_2 \in T_p^*M.$$

The magnetic Laplacian can be defined as follows. When M is a compact manifold, possibly with boundary, consider the sesquilinear form defined for complex-valued functions $u, v \in H_0^1(M)$ by

$$Q_{h,\mathbf{A}}(u, v) = \int_M g^*((-ihd - \mathbf{A})u, (-ihd - \mathbf{A})v) d\mathcal{V}_g.$$

From the Lax-Milgram Theorem, $Q_{h,\mathbf{A}}$ is associated with a self-adjoint operator $\mathcal{L}_{h,\mathbf{A}}$ whose domain is given by

$$\begin{aligned} \text{Dom}(\mathcal{L}_{h,\mathbf{A}}) &= \left\{ u \in H_0^1(M) : \text{there exists } f \in L^2(M) \text{ such that} \right. \\ &\quad \left. Q_{h,\mathbf{A}}(u, v) = \langle f, v \rangle_{L^2(M)} \quad \text{for all } v \in H_0^1(M) \right\} \\ &= H_0^1(M) \cap H^2(M), \end{aligned}$$

and

$$\langle \mathcal{L}_{h,\mathbf{A}}u, v \rangle_{L^2(M)} = Q_{h,\mathbf{A}}(u, v) \quad \forall u \in \text{Dom}(\mathcal{L}_{h,\mathbf{A}}), \forall v \in H_0^1(M).$$

Choosing local coordinates (x^1, x^2) on M , the operator $\mathcal{L}_{h,\mathbf{A}}$ can be written explicitly as

$$(1.2) \quad \mathcal{L}_{h,\mathbf{A}} = \frac{1}{\sqrt{|G|}} \sum_{k,\ell=1}^2 (hD_k - A_k) \left[\sqrt{|G|} G^{k\ell} (hD_\ell - A_\ell) \right],$$

where $D_j := -i \frac{\partial}{\partial x^j}$, G is the matrix associated with g , and $G^{k\ell}$ are the matrix elements of G^{-1} .

Remark 1.1. When M is compact, we have $H_0^1(M) = H^1(M)$ and

$$\text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = H^2(M).$$

The readers may consult [13, 20, 9] for an introduction to the magnetic Laplacian on a Riemannian manifold. We also refer the reader to the book [18] for the study of the magnetic Laplacian under various aspects.

In this paper, we will also consider the non-compact case $M = \mathbb{R}^2$ with the Euclidean metric. In this case, the operator is characterized by

$$\begin{cases} \text{Dom}(\mathcal{L}_{h,\mathbf{A}}) = \{u \in H_{h,\mathbf{A}}^1(\mathbb{R}^2) : (-ih\nabla - \mathbf{A})^2 u \in L^2(\mathbb{R}^2)\}, \\ \mathcal{L}_{h,\mathbf{A}} u := (-ih\nabla - \mathbf{A})^2 u. \end{cases}$$

Since $\mathbf{A} \in \mathcal{C}^1(\mathbb{R}^2)$, the operator $(-ih\nabla - \mathbf{A})^2$ whose domain is $\mathcal{C}_0^\infty(\mathbb{R}^2)$ is essentially self-adjoint (see [5]).

1.2. Context and motivation. One of the initial motivations to study the spectral theory of the magnetic Schrödinger operator was the mathematical study of superconductivity, see [5]. The ground-energy is indeed related to the third critical field in the Ginzburg-Landau theory. From this initial motivation, the spectral theory of this operator acquired a life of its own, see the series of papers by Helffer-Morame, Helffer-Kordyukov, and Raymond-Vũ Ngọc [13, 14, 15, 16, 8, 17, 9, 10, 11, 19, 12, 7]. Among this vast literature, the case of the magnetic field having a unique and non-degenerate minimum was investigated very much. Namely, in [9, Theorem 1.2] (on a compact manifold) or in [12, Theorem 1.7] (on \mathbb{R}^2), Helffer and Kordyukov provided the following asymptotic expansions

$$(1.3) \quad \forall \ell \in \mathbb{N}, \quad \lambda_\ell(\mathcal{L}_{h,\mathbf{A}}) = \mathbf{B}(p_0)h + \left(2\ell \frac{\sqrt{\det H}}{\mathbf{B}(p_0)} + \frac{(\text{Tr } H^{\frac{1}{2}})^2}{2\mathbf{B}(p_0)} \right) h^2 + o(h^2),$$

where p_0 is the minimum point of \mathbf{B} and $H = \frac{1}{2}\text{Hess } \mathbf{B}(p_0)$.

We can also mention that, with the help of symplectic geometry and pseudo-differential techniques, Raymond and Vũ Ngọc recovered the eigenvalues expansions through a Birkhoff normal form and related them to the magnetic classical dynamics, see [19]. In [3], Bonthonneau and Raymond used a WKB analysis to recover (1.3) under analyticity assumptions on the magnetic field, when the metric is flat. Such expansions were not known before, except in a multi-scale context, see [1]. The reader might also want to consider the well-known results about the WKB analysis in the electric case, see [4, Chapter 3].

The aim of this paper is to extend the analysis of [3] to the case non-flat metrics, and to remove the analyticity assumptions. Moreover, in the case \mathbb{R}^2 and with radial magnetic fields, we will explicitly describe the magnetic eigenfunctions, and establish some optimal exponential decay away from the minimum of the magnetic field. Such an optimal decay is still a widely open problem in the general case, see however [6, 2].

2. STATEMENTS

2.1. General case. Let us state our main assumption on the magnetic field \mathbf{B} .

Assumption 2.1. *Let $p_0 \in M$, we assume that*

(i) *The magnetic field $B \in \mathcal{C}^\infty(M, \mathbb{R})$ has a positive minimum at p_0 , i.e.*

$$\mathbf{B}(p_0) = \min_{p \in M} \mathbf{B}(p) > 0.$$

(ii) *The Hessian of \mathbf{B} at p_0 is positive non-degenerate, i.e.*

$$(\text{d}^2 \mathbf{B})_{p_0}(V, V) > 0, \quad \text{for all } V \in T_{p_0} M \setminus \{0\}.$$

Remark 2.1. Note that, if M is a compact manifold without boundary, our first assumption cannot be satisfied since

$$\int_M \mathbf{B} d\mathcal{V}_g = \int_M d\mathbf{A} = 0.$$

Nevertheless, this is not a big issue since our constructions are local.

In order to state our main theorem, it is convenient to use the isothermal coordinates, see [22, Page 438].

Definition 2.1 (Isothermal coordinates). Let (M, g) be a Riemannian manifold of two dimensions, a local chart

$$(\Omega, \phi : \Omega \rightarrow \phi(\Omega) \subset \mathbb{R}^2)$$

is called an isothermal chart if there exists a function $\eta \in \mathcal{C}^\infty(\phi(\Omega))$ such that

$$(2.1) \quad \phi^*(e^{2\eta}g_0) = g,$$

where g_0 is the Euclidean metric on \mathbb{R}^2 .

Theorem 2.2. *Let $p^* \in M$ and assume that the magnetic field \mathbf{B} has a local positive minimum at p^* and its Hessian at p^* is positive non-degenerate. Then, there exists an isothermal local chart $(\Omega, \phi : \Omega \rightarrow U \subset \mathbb{R}^2)$ centered at p^* in which the magnetic field has the form*

$$(\mathbf{B} \circ \phi^{-1})(q) = b_0 + \alpha q_1^2 + \gamma q_2^2 + \mathcal{O}(\|q\|^3),$$

where $b_0 > 0$, $0 < \alpha \leq \gamma$ and for all $\ell \in \mathbb{N}$, there exist

(i) a smooth complex-valued function P defined on Ω satisfying

$$(2.2) \quad \operatorname{Re}(P \circ \phi^{-1})(q) = \frac{e^{2\eta(0)}b_0}{2} \left(\frac{\sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\gamma}} q_1^2 + \frac{\sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}} q_2^2 \right) + \mathcal{O}(\|q\|^3),$$

on U , where η is given in Definition 2.1,

(ii) a sequence of smooth complex-valued functions $(U_{\ell,j})_{j \in \mathbb{N}}$ defined on Ω ,

(iii) a sequence of real numbers $(\mu_{\ell,j})_{j \in \mathbb{N}}$ with

$$\mu_{\ell,0} = b_0, \quad \mu_{\ell,1} = \left(2\ell \frac{\sqrt{\det H}}{b_0} + \frac{(\operatorname{Tr} H^{\frac{1}{2}})^2}{2b_0} \right),$$

(iv) a sequence of smooth functions $(F_{\ell,j})_{j \in \mathbb{N}}$ defined on Ω and flat at p^* , such that, for all $J \in \mathbb{N}$,

$$e^{P/h} \left(\mathcal{L}_{h,\mathbf{A}} - h \sum_{j=0}^J \mu_{\ell,j} h^j \right) \left(e^{-P/h} \sum_{j=0}^J U_{\ell,j} h^j \right) = \sum_{j=0}^{J+1} h^j F_{\ell,j} + \mathcal{O}(h^{J+2}),$$

locally uniformly on Ω .

Corollary 2.3. *Let $p^* \in M$ and assume that the magnetic field satisfies the conditions of Theorem 2.2. For any $\ell \in \mathbb{N}$, there exist*

(i) a non-negative function $\widehat{P} \in \mathcal{C}_0^\infty(M)$,

(ii) a sequence of functions $(\widehat{U}_{\ell,j})_{j \in \mathbb{N}} \subset \mathcal{C}_0^\infty(M)$,

and for any $(\varepsilon, J) \in (0, 1) \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,

$$(2.3) \quad \|e^{\varepsilon \widehat{P}/h} (\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,\ell}^J) \Upsilon_{h,\ell}^J\|_{L^2(M)} \leq Ch^{J+2},$$

where

$$\lambda_{h,\ell}^J = h \sum_{j=0}^J \mu_{\ell,j} h^j \quad \text{and} \quad \Upsilon_{h,\ell}^J = \sum_{j=0}^J \widehat{U}_{\ell,j} h^j.$$

In particular,

$$(2.4) \quad \|(\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,\ell}^J) \Upsilon_{h,\ell}^J\|_{L^2(M)} \leq Ch^{J+2}.$$

Remark 2.2. Corollary 2.3 can be used to prove that there is no odd powers of $h^{\frac{1}{2}}$ in the expansion given by [9, Theorem 1.2]. Furthermore, Corollary 2.3 is a generalization of [9, Theorem 2.1] in the case $k = 0$.

Let us now turn to the description of the “true” eigenfunctions. For each $\ell \in \mathbb{N}$, let $\Upsilon_{h,\ell}$ be a normalized eigenfunction associated with $\lambda_\ell(\mathcal{L}_{h,\mathbf{A}})$. We introduce the projection into the eigenspace of $\lambda_\ell(\mathcal{L}_{h,\mathbf{A}})$:

$$\begin{aligned} \Pi_\ell : L^2(M) &\rightarrow \text{Dom}(\mathcal{L}_{h,\mathbf{A}}) \\ u &\mapsto \Pi_\ell u = \langle u, \Upsilon_{h,\ell} \rangle_{L^2(M)} \Upsilon_{h,\ell}. \end{aligned}$$

Using the asymptotic simplicity of the eigenvalues and the spectral theorem, we get the following corollary.

Corollary 2.4. *Assume that the magnetic field satisfies Assumption 2.1. For all $(J, \ell) \in \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$(2.5) \quad \|\Upsilon_{h,\ell}^J - \Pi_\ell \Upsilon_{h,\ell}^J\|_{L^2(M)} \leq Ch^{J+1}.$$

2.2. Radial magnetic fields on \mathbb{R}^2 . Let us now describe our results when $M = \mathbb{R}^2$ and when the magnetic field is radial.

Assumption 2.2. *We assume that the magnetic field \mathbf{B} has the form*

$$\mathbf{B}(q_1, q_2) = \beta \left(\frac{q_1^2 + q_2^2}{2} \right),$$

where $\beta : \mathbb{R} \rightarrow \mathbb{R}^+$ is a smooth function such that

$$(2.6) \quad \beta(r) > \beta(0), \quad \text{for all } r > 0.$$

$$(2.7) \quad \beta'(0) > 0.$$

In this case, the WKB analysis becomes quite explicit.

Theorem 2.5. *For all $m \in \mathbb{N}$, there exist*

(i) *a smooth positive function φ defined on $[0, +\infty)$,*

$$(2.8) \quad \varphi(\rho) := \frac{1}{2} \int_0^\rho \int_0^1 \beta(\xi\tau) \, d\xi \, d\tau,$$

(ii) *a sequence of smooth real-valued functions $(a_{m,j})_{j \in \mathbb{N}}$ defined on $[0, \infty)$, with $a_{m,0} > 0$,*

(iii) *a sequence of real numbers $(\mu_{m,j})_{j \in \mathbb{N}}$ with*

$$\mu_{m,0} = \beta(0), \quad \mu_{m,1} = \left(2m \frac{\sqrt{\det H}}{b_0} + \frac{(\text{Tr } H^{\frac{1}{2}})^2}{2b_0} \right), \quad H = \frac{1}{2} \text{Hess } \mathbf{B}(0).$$

We let

$$P(q) = \varphi \left(\frac{\|q\|^2}{2} \right), \quad U_{m,j}(q) = a_{m,j} \left(\frac{\|q\|^2}{2} \right), \quad \theta(q) = \arg(q_1 + iq_2).$$

Then, for all $J \in \mathbb{N}$,

$$\begin{aligned} e^{P/h} \left(\frac{\|q\|^2}{2} \right)^{\frac{-m}{2}} \left(\mathcal{L}_{h,\mathbf{A}} - h \sum_{j=0}^J \mu_{m,j} h^j \right) \left(e^{im\theta(q)} \left(\frac{\|q\|^2}{2} \right)^{\frac{m}{2}} e^{-P/h} \sum_{j=0}^J U_{m,j} h^j \right) \\ = \mathcal{O}(h^{J+2}), \end{aligned}$$

locally uniformly in \mathbb{R}^2 .

Let $K > 0$. We consider a smooth cut-off function such that

$$(2.9) \quad \chi(\rho) = \begin{cases} 1 & \text{on } [0, K] \\ 0 & \text{on } [K+1, +\infty) \end{cases}.$$

Corollary 2.6. *For all $(\varepsilon, m, J) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$, there exist a constant $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$(2.10) \quad \|e^{\varepsilon P/h} (\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,m}^J) \Upsilon_{h,m}^J\|_{L^2(\mathbb{R}^2)} \leq Ch^{J+2},$$

where

$$\lambda_{h,m}^J := h \sum_{j=0}^J \mu_{m,j} h^j,$$

$$\Upsilon_{h,m}^J = \chi \left(\frac{\|\cdot\|^2}{2} \right) e^{im\theta(q)} \left(\frac{\|q\|^2}{2} \right)^{\frac{m}{2}} e^{-P/h} \sum_{j=0}^J U_{m,j} h^j.$$

In particular,

$$(2.11) \quad \|(\mathcal{L}_{h,\mathbf{A}} - \lambda_{h,m}^J) \Upsilon_{h,m}^J\|_{L^2(\mathbb{R}^2)} \leq Ch^{J+2}.$$

Let $\Upsilon_{h,m}$ be an eigenfunction associated with $\lambda_m(\mathcal{L}_{h,\mathbf{A}})$. We introduce the projection into the eigenspace of $\lambda_m(\mathcal{L}_{h,\mathbf{A}})$

$$\Pi_m : L^2(\mathbb{R}^2) \rightarrow \text{Dom}(\mathcal{L}_{h,\mathbf{A}})$$

$$u \mapsto \Pi_m u = \langle u, \Upsilon_{h,m} \rangle_{L^2(\mathbb{R}^2)} \Upsilon_{h,m}.$$

As in Corollary 2.4, we get an approximation of the eigenfunctions.

Corollary 2.7. *For all $(J, m) \in \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$(2.12) \quad \|\Upsilon_{h,m}^J - \Pi_m \Upsilon_{h,m}^J\|_{L^2(\mathbb{R}^2)} \leq Ch^{J+1}.$$

By using an Agmon estimates, we can prove that the eigenfunctions of the magnetic Laplacian decay exponentially.

Theorem 2.8. *Let $U_{h,m}$ be an eigenfunction associated with $\lambda_m(\mathcal{L}_{h,\mathbf{A}})$. Then, for all $\varepsilon \in (0, 1)$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$\|e^{\varepsilon P/h} U_{h,m}\|_{L^2(\mathbb{R}^2)} \leq C \|U_{h,m}\|_{L^2(\mathbb{R}^2)}.$$

Actually, we can even prove a stronger approximation of the eigenfunctions.

Theorem 2.9. *For all $(\varepsilon, J, m) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$(2.13) \quad \|e^{\varepsilon P/h} (\Upsilon_{h,m}^J - \Pi_m \Upsilon_{h,m}^J)\|_{L^2(\mathbb{R}^2)} \leq Ch^{J+1}.$$

Without the radial symmetry, it is expected that such a result holds only when the magnetic field satisfies some analyticity assumption.

2.3. Organization of the article. The article is organized as follows. Section 3 is devoted to the proof of Theorem 2.2. In Section 4, we prove Theorems 2.5 and 2.9.

3. PROOF OF THEOREM 2.2 AND OF ITS CONSEQUENCES

3.1. The magnetic Laplacian in isothermal coordinates. Let $p^* \in M$ be the point in Theorem 2.2. There exists an isothermal chart $(\Omega, \phi : \Omega \rightarrow \phi(\Omega))$ centered at p^* . We set $U := \phi(\Omega) \subset \mathbb{R}^2$ and $\tilde{g} := e^{2\eta} g_0$ (the metric on U). We let $\mathcal{M} = \phi_* \mathbf{A}$.

Lemma 3.1. *Consider the operator acting on $L^2(U, e^{2\eta} dq)$ defined by*

$$\mathcal{L}_{h,\mathcal{M}} = e^{-2\eta} [(-ih\partial_{q_1} - \mathcal{M}_1)^2 + (-ih\partial_{q_2} - \mathcal{M}_2)^2].$$

We have

$$(3.1) \quad \mathcal{L}_{h,\mathbf{A}} = \phi^* \mathcal{L}_{h,\mathcal{M}}.$$

Moreover,

$$\left(\frac{\partial \mathcal{M}_2}{\partial q_1} - \frac{\partial \mathcal{M}_1}{\partial q_2} \right) dq_1 \wedge dq_2 = d(\phi_* \mathbf{A}) = \phi_* d\mathbf{A} = \phi_* (\mathbf{B} d\mathcal{V}_g) = e^{2\eta} (\mathbf{B} \circ \phi^{-1}) dq_1 \wedge dq_2.$$

Proof. For all $u, v \in \mathcal{C}_0^\infty(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} g^*((-ihd - \mathbf{A})u, (-ihd - \mathbf{A})v) d\mathcal{V}_g &= \int_U \phi_* (g^*((-ihd - \mathbf{A})u, (-ihd - \mathbf{A})v) d\mathcal{V}_g) \\ &= \int_U \tilde{g}^*((-ihd - \varphi^* \mathbf{A})\tilde{u}, (-ihd - \varphi^* \mathbf{A})\tilde{v}) |\tilde{G}|^{\frac{1}{2}} dq, \end{aligned}$$

where $\tilde{u} := \phi_* u$, $\tilde{v} := \phi_* v$ and $\tilde{G} = \begin{pmatrix} e^{-2\eta} & 0 \\ 0 & e^{-2\eta} \end{pmatrix}$ is the matrix of \tilde{g} .

By considering \mathcal{M} as a vector field $(\mathcal{M}_1, \mathcal{M}_2)^T$, we have

$$\begin{aligned} &\int_{\Omega} g^*((-ihd - \mathbf{A})u, (-ihd - \mathbf{A})v) d\mathcal{V}_g \\ &= \int_U \left\langle \tilde{G}^{-1}(-ih\nabla_q - \mathcal{M})\tilde{u}, (-ih\nabla_q - \mathcal{M})\tilde{v} \right\rangle_{\mathbb{C}^2} |\tilde{G}|^{\frac{1}{2}} dq \\ &= \int_U [(-ih\partial_{q_1} - \mathcal{M}_1)^2 + (-ih\partial_{q_2} - \mathcal{M}_2)^2] \tilde{u}\tilde{v} dq, \end{aligned}$$

where the last equality is obtained by an integration by parts. \square

3.2. Spectral analysis with the WKB method.

3.2.1. *A choice of the magnetic potential.* Let us denote $\mathcal{B}(q) = \mathbf{B}(\phi^{-1}(q))$. Through a linear change of variable in \mathbb{R}^2 , without loss of generality, we can write

$$\mathcal{B}(q_1, q_2) = b_0 + \alpha q_1^2 + \gamma q_2^2 + \mathcal{O}(\|q\|^3) \quad \text{with } b_0 > 0 \text{ and } 0 < \alpha \leq \gamma.$$

The following lemma will be useful to define a special vector potential.

Lemma 3.2. *There exists a smooth solution of*

$$(3.2) \quad \Delta \Psi = e^{2\eta} \mathcal{B},$$

in a neighborhood of U such that

$$\Psi(q_1, q_2) = \frac{e^{2\eta(0)} \mathcal{B}(0)}{4} (q_1^2 + q_2^2) + \mathcal{O}(\|q\|^3).$$

Proof. It is well-known that the Poisson equation (3.2) always has smooth solutions modulo harmonic functions. Consider such a solution u . We have

$$u(q_1, q_2) = u(0) + \frac{\partial u(0)}{\partial q_1} q_1 + \frac{\partial u(0)}{\partial q_2} q_2 + \frac{1}{2} \frac{\partial^2 u(0)}{\partial q_1^2} q_1^2 + \frac{\partial^2 u(0)}{\partial q_1 q_2} q_1 q_2 + \frac{1}{2} \frac{\partial^2 u(0)}{\partial q_2^2} q_2^2 + \mathcal{O}(\|x\|^3),$$

and consider the harmonic function

$$\varphi(q) = - \left[u(0) + \frac{\partial u(0)}{\partial q_1} q_1 + \frac{\partial u(0)}{\partial q_2} q_2 + \frac{\partial^2 u(0)}{\partial q_1 q_2} q_1 q_2 \right] + \frac{1}{4} \left(\frac{\partial^2 u(0)}{\partial q_2^2} - \frac{\partial^2 u(0)}{\partial q_1^2} \right) (q_1^2 - q_2^2).$$

Then, $\Psi = u + \varphi$ satisfies

$$\Psi(q) = a(q_1^2 + q_2^2) + \mathcal{O}(\|q\|^3), \quad a = \frac{\partial^2 \Psi(0)}{\partial q_1^2} = \frac{\partial^2 \Psi(0)}{\partial q_2^2}.$$

Therefore, from (3.2) at 0, we see that $a = \frac{e^{2\eta(0)} \mathcal{B}(0)}{4}$. \square

Let Ψ be the function given by Lemma 3.2, we let $A := (-\partial_{q_2}\Psi, \partial_{q_1}\Psi)$. Then A satisfies

$$\frac{\partial \mathcal{M}_2}{\partial q_1} - \frac{\partial \mathcal{M}_1}{\partial q_2} = \frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}.$$

The Poincaré lemma implies the existence of a function $\theta \in C^\infty(U)$ such that

$$A = \mathcal{M} + \nabla\theta.$$

By gauge invariance, we get

$$(3.3) \quad \mathcal{L}_{h,A} = e^{i\theta/h} \mathcal{L}_{h,\mathcal{M}} e^{-i\theta/h}.$$

Note that, with the choice of the magnetic potential $A = (-\partial_{q_2}\Psi, \partial_{q_1}\Psi)$, we have $\nabla \cdot A = 0$.

3.2.2. *WKB analysis.* The eigenvalue equation reads

$$(3.4) \quad \mathcal{L}_{h,A} u(q, h) = \lambda(h) u(q, h).$$

We look for a solution $u(q, h)$ in the form

$$u(q, h) = e^{-S(q)/h} a(q, h),$$

where a and S are complex-valued functions. We have

$$(3.5) \quad (\mathcal{L}_{h,A}^S - \lambda(h)) a(q, h) = 0, \quad \mathcal{L}_{h,A}^S = e^{S/h} \mathcal{L}_{h,A} e^{-S/h}.$$

We have

$$\begin{aligned} \mathcal{L}_{h,A}^S &= e^{-2\eta} e^{S/h} (-ih\nabla - A)^2 e^{-S/h} \\ &= e^{-2\eta} [(-A_1 + i\partial_{q_1}S)^2 + (-A_2 + i\partial_{q_2}S)^2 + ih\nabla \cdot A + h\Delta S + 2h(\nabla S + iA) \cdot \nabla - h^2\Delta]. \end{aligned}$$

Since $\nabla \cdot A = 0$, gathering the terms of same order in h , we can write $\mathcal{L}_{h,A}^S$ as

$$(3.6) \quad \mathcal{L}_{h,A}^S = E_0^S + hE_1^S - h^2\Delta,$$

where

$$(3.7) \quad \begin{aligned} E_0^S &= e^{-2\eta} [(-A_1 + i\partial_{q_1}S)^2 + (-A_2 + i\partial_{q_2}S)^2], \\ E_1^S &= e^{-2\eta} (\Delta S + 2(\nabla S + iA) \cdot \nabla). \end{aligned}$$

We look for $\lambda(h)$ and $a(q, h)$ in the form

$$(3.8) \quad \lambda(h) = h \sum_{j=0}^{\infty} \mu_j h^j, \quad a(q, h) = \sum_{j=0}^{\infty} a_j(q) h^j,$$

where $(a_j)_{j \geq 0}$ are smooth complex-valued functions and $(\mu_j)_{j \in \mathbb{N}} \subset \mathbb{R}$.

Let us substitute (3.8) into (3.5), we get the sequence of equations

$$\begin{aligned} h^0 : & \quad E_0^S a_0 = 0 \\ h^1 : & \quad E_0^S a_1 + (E_1^S - \mu_0) a_0 = 0, \\ h^2 : & \quad E_0^S a_2 + (E_1^S - \mu_0) a_1 = (\mu_1 + e^{-2\eta} \Delta) a_0, \\ & \quad \dots\dots\dots \\ h^n : & \quad E_0^S a_n + (E_1^S - \mu_0) a_{n-1} = (\mu_1 + e^{-2\eta} \Delta) a_{n-2} + \sum_{j=2}^{n-1} \mu_j a_{n-1-j}. \end{aligned}$$

In [3], under an analyticity assumption, a similar system of equations was solved exactly in a neighbourhood of 0. Here, without an analyticity assumption, we will only be able to solve this exactly in a space of formal series at 0.

3.3. WKB construction.

Notation 3.1. Let \hat{f} be the Taylor formal series of $f \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{C})$ at zero, *i.e.*,

$$\hat{f}(q_1, q_2) = \sum_{m,n \geq 0} \frac{1}{m!n!} \frac{\partial^{m+n} f(0)}{\partial q_1^m \partial q_2^n} q_1^m q_2^n.$$

Let \tilde{f} be the formal series after changing variable $(q_1, q_2) = \left(\frac{z+w}{2}, \frac{z-w}{2i}\right)$ with $(z, w) \in \mathbb{C}^2$. $\mathbb{C}[[z]]$ denotes the ring of formal series in the variable z with coefficients in \mathbb{C} and $\mathbb{C}[[z, w]]$ is the ring of formal series in the variable (z, w) with coefficients in \mathbb{C} .

3.3.1. *The eikonal equation.* Let us find \hat{S} in $\mathbb{C}[[q_1, q_2]]$ such that

$$\left(-\hat{A}_1 + i\partial_{q_1}\hat{S}\right)^2 + \left(-\hat{A}_2 + i\partial_{q_2}\hat{S}\right)^2 = 0,$$

and thus such that

$$\left(-\hat{A}_1 + i\partial_{q_1}\hat{S} + i(-\hat{A}_2 + i\partial_{q_2}\hat{S})\right) \left(-\hat{A}_1 + i\partial_{q_1}\hat{S} - i(-\hat{A}_2 + i\partial_{q_2}\hat{S})\right) = 0.$$

Let us consider an \hat{S} such that

$$-\hat{A}_1 + i\partial_{q_1}\hat{S} + i(-\hat{A}_2 + i\partial_{q_2}\hat{S}) = 0.$$

It satisfies

$$2\partial_{\bar{z}}\hat{S} = -i\hat{A}_1 + \hat{A}_2, \quad \text{with} \quad \partial_{\bar{z}} := \frac{1}{2}(\partial_{q_1} + i\partial_{q_2}).$$

Notice that we also have $\partial_{\bar{z}}\hat{\Psi} = -i\hat{A}_1 + \hat{A}_2$. It implies that

$$(3.9) \quad \partial_{\bar{z}}\hat{S} = \partial_{\bar{z}}\hat{\Psi}.$$

After changing the variable $q_1 = \frac{z+w}{2}$ and $q_2 = \frac{z-w}{2i}$ in the formal series \hat{S} and $\hat{\Psi}$, we have $\partial_w = \partial_{\bar{z}}$. Thus, \hat{S} satisfies (3.9) if and only if \tilde{S} satisfies

$$\partial_w\tilde{S}(z, w) = \partial_w\tilde{\Psi}(z, w),$$

and thus \tilde{S} has the form

$$(3.10) \quad \tilde{S}(z, w) = \tilde{\Psi}(z, w) + f(z),$$

where $f(z) = \sum_{m \geq 0} f_m z^m$ be a formal series in $\mathbb{C}[[z]]$ to be determined later. Next, we write the

transport equations. With the choice of \tilde{S} in (3.10), we have

$$(3.11) \quad \Delta\tilde{S} = \Delta\tilde{\Psi} = \widetilde{\mathcal{E}^{-1}\tilde{\mathcal{B}}} \quad \mathcal{E} = e^{-2\eta}.$$

The term $(\nabla S + iA) \cdot \nabla$ in E_1^S can be written as

$$(\partial_{q_1}\hat{S} + i\hat{A}_1)\partial_{q_1} + (\partial_{q_2}\hat{S} + i\hat{A}_2)\partial_{q_2} = 2 \left(2\partial_z\tilde{\Psi} + f'(z)\right) \partial_w.$$

The operator E_1^S becomes

$$E_1^{\tilde{S}} = 4\tilde{\mathcal{E}} \left(2\partial_z\tilde{\Psi} + f'(z)\right) \partial_w + \tilde{\mathcal{B}}.$$

Finally, we obtain the system of the transport equations:

$$h^1 : \quad \left[4\tilde{\mathcal{E}} \left(2\partial_z\tilde{\Psi} + f'(z)\right) \partial_w + \tilde{\mathcal{B}} - \mu_0\right] A^{(0)} = 0,$$

$$h^2 : \quad \left[4\tilde{\mathcal{E}} \left(2\partial_z\tilde{\Psi} + f'(z)\right) \partial_w + \tilde{\mathcal{B}} - \mu_0\right] A^{(1)} = \left(\mu_1 + 4\tilde{\mathcal{E}}\partial_z\partial_w\right) A^{(0)},$$

.....

$$h^n : \quad \left[4\tilde{\mathcal{E}} \left(2\partial_z\tilde{\Psi} + f'(z)\right) \partial_w + \tilde{\mathcal{B}} - \mu_0\right] A^{(n-1)} = \left(\mu_1 + 4\tilde{\mathcal{E}}\partial_z\partial_w\right) A^{(n-1)} + \sum_{j=2}^{n-1} \mu_j A^{(n-1-j)}.$$

3.3.2. *Some lemmas.* In this subsection, we prove some useful lemmas for solving the transport equations.

Lemma 3.3. *There exists a formal series $w(z) = \sum_{k \geq 1} w_k z^k$ in $\mathbb{C}[[z]]$ satisfying*

$$(3.12) \quad \tilde{\mathcal{B}}(z, w(z)) = b_0,$$

and such that $w_1 = \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}}$.

Proof. We write $\tilde{\mathcal{B}}(z, w) = \sum_{m, n \geq 0} \tilde{b}_{mn} z^m w^n$ and

$$\sum_{m, n \geq 0} \tilde{b}_{mn} z^m \left(\sum_{k \geq 1} w_k z^k \right)^n = b_0.$$

Note that

$$\tilde{\mathcal{B}}(z, w) = b_0 + \frac{1}{4}(\alpha - \gamma)z^2 + \frac{1}{2}(\alpha + \gamma)zw + \frac{1}{4}(\alpha - \gamma)w^2 + \dots$$

Collecting the various terms, we have

- (a) Term z^0 : $\tilde{b}_{00} = b_0$.
- (b) Term z^1 : $\tilde{b}_{10} + \tilde{b}_{01}w_1 = 0$.
- (c) Term z^2 : $\tilde{b}_{02}w_1^2 + \tilde{b}_{11}w_1 + \tilde{b}_{20} = 0$. There are two solutions for w_1 :

$$\frac{\sqrt{\gamma} + \sqrt{\alpha}}{\sqrt{\gamma} - \sqrt{\alpha}}, \quad \text{and} \quad \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}}.$$

We choose $w_1 = \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}}$.

- (d) Term z^3 : notice that the equation obtained by collecting the coefficients of the term z^3 does not contain w_k for $k \geq 3$ since $\tilde{b}_{01} = 0$. This equation is linear in w_2 whose prefactor is

$$\tilde{b}_{11} + 2\tilde{b}_{02}w_1 = \frac{1}{2}(\alpha + \gamma) + \frac{1}{2}(\alpha - \gamma) \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}} = \sqrt{\alpha\gamma} > 0.$$

By induction, let $p \in \mathbb{N} \setminus \{0\}$, we assume that $(w_k)_{1 \leq k \leq p-1}$ are determined and we need to look for w_p . We collect all coefficients of z^{p+1} , and since $b_{01} = 0$, we get an equation containing only a finite number of terms $(w_k)_{1 \leq k \leq p}$ and $(\tilde{b}_{mn})_{0 \leq m, n \leq p+1}$. The equation is linear in w_p whose prefactor is

$$\tilde{b}_{11} + 2\tilde{b}_{02}w_1 = \sqrt{\alpha\gamma} \neq 0.$$

So, w_p is determined. □

Lemma 3.4. *Let $V(s, t)$ and $F(s, t)$ be formal series in $\mathbb{C}[[s, t]]$. We write $V(s, t)$ and $F(s, t)$ in the form*

$$V(s, t) := \sum_{m \geq 0} v_m(s) t^m, \quad F(s, t) := \sum_{m \geq 0} f_m(s) t^m,$$

where $(v_n(s))_{n \in \mathbb{N}}$ and $(f_n(s))_{n \in \mathbb{N}}$ are the sequences in $\mathbb{C}[[s]]$. We assume that $v_0(s) = 0$, $v_1(s) = v_1$, $f_0(s) = f_0$ with $v_1 \in \mathbb{R} \setminus \{0\}$ and $f_0 \in \mathbb{R}$. Let $\ell \in \mathbb{N}$, then

(i) *the homogeneous equation*

$$(3.13) \quad (V(s, t)\partial_t + F(s, t))u(s, t) = 0,$$

has solutions in the set

$$W(\ell) = \left\{ \sum_{m \geq 0} w_m(s) t^m \in \mathbb{C}[[s, t]] : w_k(s) = 0 \text{ for } k \in \{0, \dots, (\ell - 1)\} \text{ and } w_\ell(s) \neq 0 \right\}$$

if and only if $f_0 + \ell v_1 = 0$.

(ii) Under this condition, there exists a family $(c_k(s))_{k=0\dots\ell} \subset \mathbb{C}[[s]]$ such that the inhomogeneous equation

$$(3.14) \quad (V(s, t)\partial_z + F(s, t))u(s, t) = G(s, t) = \sum_{m \geq 0} g_m(s)t^m,$$

has a formal solution in the form

$$(3.15) \quad u(s, t) = \sum_{m \geq 0} u_m(s)t^m,$$

if and only if

$$(3.16) \quad c_\ell(s)g_0(s) + c_{\ell-1}(s)g_1(s) + \dots + c_0(s)g_\ell(s) = 0.$$

Here, the coefficients $(c_k(s))_{k=0\dots\ell} \subset \mathbb{C}[[s]]$ are determined by $(v_j(s))_{1 \leq j \leq (\ell+1)}$ and $(f_j(s))_{1 \leq j \leq \ell}$, and $c_0(s) = 1$. Furthermore, assume that (3.16) is satisfied, if $u_\ell(s)$ is given, the formal series solution u will be determined uniquely by the relation

$$u_m(s) = \frac{g_m(s) - \sum_{j=0}^{m-1} (jv_{m-j+1}(s) + f_{m-j}(s))u_j(s)}{(m-\ell)v_1},$$

for all $m \in \mathbb{N} \setminus \{\ell\}$.

Proof. Let us start with the homogeneous equation. We look for a solution $u(s, t)$ in the form

$$u(s, t) = \sum_{m \geq 0} u_m(s)t^m,$$

of the equation

$$\left(\sum_{m \geq 1} v_m(s)t^m \right) \left(\sum_{m \geq 1} mu_m(s)t^{m-1} \right) + \left(\sum_{m \geq 0} f_m(s)t^m \right) \left(\sum_{m \geq 0} u_m(s)t^m \right) = 0.$$

For arbitrary $k \in \mathbb{N}$, we get the equations corresponding to t^k

$$(kv_1 + f_0)u_k(s) + \sum_{j=0}^{k-1} (jv_{k-j+1}(s) + f_{k-j}(s))u_j(s) = 0.$$

Let us write the first three equations

$$\begin{aligned} t^0 : & \quad f_0u_0(s) = 0, \\ t^1 : & \quad [v_1 + f_0]u_1(s) + f_1(s)u_0(s) = 0, \\ t^2 : & \quad [2v_1 + f_0]u_2(s) + f_2(s)u_0(s) + (v_2(s) + f_1(s))u_1(s) = 0. \end{aligned}$$

For $\ell \in \mathbb{N}$, consider a non-zero solution u in $W(\ell)$. Then $u_k(s) = 0$ for $0 \leq k \leq \ell - 1$ and $u_\ell(s) \neq 0$. We have $\ell v_1 + f_0 = 0$. Now, if $\ell v_1 + f_0 = 0$, then for all $k \neq \ell$, we get

$$u_k(s) = -\frac{\sum_{j=0}^{k-1} (jv_{k-j+1}(s) + f_{k-j}(s))u_j(s)}{(k-\ell)v_1}.$$

We consider two cases:

a) $\ell = 0$. It leads to $f_0 = 0$. Since the first equation is $f_0u_0(s) = 0$, we can choose any $u_0(s) \neq 0$, and we have

$$u_k(s) = -\frac{\sum_{j=0}^{k-1} (jv_{k-j+1}(s) + f_{k-j}(s))u_j(s)}{kv_1},$$

for all $k \geq 1$.

b) $\ell \neq 0$. Then f_0 has to be non-zero. From the first equation, it leads to $u_0(s) = 0$. From the recursion formula, it implies that $u_k(s) = 0$ for all $k < \ell$. For $k = \ell$, we get the equation

$$(k - \ell)v_1(s)u_\ell(s) = 0,$$

we can choose any $u_\ell(s) \neq 0$.

In any case, we always get non-trivial solutions u in the set $W(\ell)$.

Now we consider the inhomogeneous case:

$$\left(\sum_{m \geq 1} v_m(s)t^m \right) \left(\sum_{m \geq 1} mu_m(s)t^{m-1} \right) + \left(\sum_{m \geq 0} f_m(s)t^m \right) \left(\sum_{m \geq 0} u_m(s)t^m \right) = \sum_{m \geq 0} g_m(s)t^m.$$

For all $k \in \mathbb{N}$, the equations corresponding to t^k are

$$(kv_1 + f_0)u_k(s) + \sum_{j=0}^{k-1} (jv_{k-j+1}(s) + f_{k-j}(s))u_j(s) = g_k(s).$$

Assume that there exists $\ell \in \mathbb{N}$ such that $\ell v_1 + f_0 = 0$, these equations become

$$(3.17) \quad (k - \ell)v_1 u_k(s) + \sum_{j=0}^{k-1} (jv_{k-j+1}(s) + f_{k-j}(s))u_j(s) = g_k(s).$$

The equation corresponding to t^ℓ is

$$(\ell - \ell)u_\ell(s) + \sum_{j=0}^{\ell-1} (jv_{\ell-j+1}(s) + f_{\ell-j}(s))u_j(s) = g_\ell(s).$$

The inhomogeneous equation has solutions in the form (3.15) if and only if

$$g_\ell(s) - \sum_{j=0}^{\ell-1} (jv_{\ell-j+1}(s) + f_{\ell-j}(s))u_j(s) = 0.$$

This relation is in the form (3.16) after computing $(u_k(s))$ as a function of $g_k(s)$, $f_k(s)$ and $v_k(s)$ for $k = 0, \dots, \ell - 1$. For example, we can compute $c_1(s)$ by collecting all coefficients connecting with $g_{\ell-1}(s)$. Notice that $g_{\ell-1}(s)$ only appears in the formula of $u_{\ell-1}$ and its coefficient in $u_{\ell-1}$ is $\frac{-1}{v_1}$, then we can compute

$$c_1(s) = -[(\ell - 1)v_2(s) + f_1(s)] \frac{-1}{v_1} = \frac{(\ell - 1)v_2(s) + f_1(s)}{v_1}.$$

The statement at the end of the lemma is obtained from (3.17). \square

3.3.3. *The first transport equation.* We consider the first transport equation

$$(3.18) \quad v(z, w)\partial_w A^{(0)}(z, w) + \left(\tilde{\mathcal{B}}(z, w) - \mu_0 \right) A^{(0)}(z, w) = 0,$$

where $v(z, w) = 4\tilde{\mathcal{E}}(z, w) \left(f'(z) + 2\partial_z \tilde{\Psi}(z, w) \right)$.

Let $w(z)$ be the formal series given by Lemma 3.3. Using the change of variables $(z, w) = (z, y + w(z))$, which is allowed in $\mathbb{C}[[z, w]]$ since $w_0 = 0$, we get the equation

$$(3.19) \quad V(z, y)\partial_y A^{(0)}(z, y + w(z)) + F(z, y)A^{(0)}(z, y + w(z)) = 0,$$

where

$$V(z, y) = v(z, y + w(z)) = 4\tilde{\mathcal{E}}(z, y + w(z)) \left(f'(z) + 2\partial_z \tilde{\Psi}(z, y + w(z)) \right),$$

and

$$F(z, y) = \tilde{\mathcal{B}}(z, y + w(z)) - \mu_0.$$

3.3.4. *Choosing formal series f and determining \tilde{S} .* We recall that \tilde{S} is given in (3.10). The formal series $\tilde{\Psi}(z, w)$ is given by Lemma 3.2 and the formal series $f(z) \in \mathbb{C}[[z]]$ has to be determined. We choose $f(z)$ such that we can apply Lemma 3.4, *i.e.*,

$$(3.20) \quad f'(z) + 2\partial_z \tilde{\Psi}(z, w(z)) = 0.$$

According to Lemma 3.2,

$$\tilde{\Psi}(z, w) = \frac{e^{2\eta(0)}b_0}{4}zw + \sum_{m+n \geq 3} \psi_{mn}z^m w^n,$$

so that

$$\partial_z \tilde{\Psi}(z, w(z)) = \frac{e^{2\eta(0)}b_0}{4}w(z) + \sum_{\substack{m+n \geq 3 \\ m \geq 1}} m\psi_{mn}z^{m-1}(w(z))^n.$$

Let $f(z) = \sum_{k \geq 0} \hat{f}_k z^k$. Since there is no restriction for $f(0)$, we can choose $\hat{f}_0 = 0$. Moreover, a straightforward computation using Lemma 3.3 gives

$$\hat{f}_1 = 0, \quad \hat{f}_2 = \frac{e^{2\eta(0)}b_0}{4} \frac{\sqrt{\alpha} - \sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}}.$$

Now, $\tilde{S}(z, w)$ is completely determined and

$$(3.21) \quad \tilde{S}(z, w) = \frac{e^{2\eta(0)}b_0}{4}zw + \frac{e^{2\eta(0)}b_0}{4} \frac{\sqrt{\alpha} - \sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}} z^2 + \sum_{m+n \geq 3} [\tilde{S}]_{mn} z^m w^n.$$

3.3.5. *Solving the first transport equation.* Let us come back to the transport equation (3.18). We write

$$V(z, y) = \sum_{m \geq 0} v_m(z)y^m \quad \text{and} \quad F(z, y) = \sum_{m \geq 0} f_m(z)y^m.$$

We now check the assumptions of Lemma 3.4. From (3.20), we have

$$v_0(z) = V(z, 0) = 4\tilde{\mathcal{E}}(z, w(z)) \left(f'(z) + 2\partial_z \tilde{\Psi}(z, w(z)) \right) = 0.$$

From Lemma 3.2, we obtain

$$\begin{aligned} v_1(z) &= \partial_y V(z, 0) \\ &= 4\partial_w \tilde{\mathcal{E}}(z, y + w(z)) \left(f'(z) + 2\partial_z \tilde{\Psi}(z, y + w(z)) \right) \Big|_{y=0} \\ &\quad + 8\tilde{\mathcal{E}}(z, y + w(z)) \partial_w \partial_z \tilde{\Psi}(z, y + w(z)) \Big|_{y=0} \\ &= 2\tilde{\mathcal{B}}(z, w(z)) = 2b_0. \end{aligned}$$

Finally, from (3.3), we get

$$f_0(z) = F(z, 0) = \tilde{\mathcal{B}}(z, w(z)) - \mu_0 = b_0 - \mu_0.$$

Thanks to Lemma 3.4 for the homogeneous case, we see that (3.19) has solutions in the form $\sum_{m \geq 0} A_m^{(0)}(z)y^m$ with $A_0^{(0)}(z) \neq 0$ if and only if $f_0(z) = 0$, *i.e.* $\mu_0 = b_0$. In this case, the solution of the first transport equation (3.18) is in the form $A^{(0)}(z, w) = \sum_{m \geq 0} A_m^{(0)}(z)(w - w(z))^m$, where $A_m^{(0)}(z)$ can be computed for $m \geq 1$ thanks to

$$A_m^{(0)}(z) = -\frac{\sum_{j=0}^{m-1} (jv_{m-j+1}(z) + f_{m-j}(z))A_j^{(0)}(z)}{2mb_0}.$$

The series $A_0^{(0)}(z)$ will be determined later.

3.3.6. *The second transport equation.* We consider the second transport equation

$$(3.22) \quad \left(v(z, w) \partial_w + \tilde{B}(z, w) - \mu_0 \right) A^{(1)} = (\mu_1 + 4\tilde{\mathcal{E}}(z, w) \partial_z \partial_w) A^{(0)}.$$

By changing variables $(z, w) = (z, y + w(z))$, we obtain

$$(3.23) \quad V(z, y) \partial_y A^{(1)}(z, y + w(z)) + F(z, y) A^{(1)}(z, y + w(z)) = G^{(1)}(z, y),$$

where $G^{(1)}(z, y) = \left(\mu_1 + 4\tilde{\mathcal{E}}(z, y + w(z)) \partial_z \partial_w \right) A^{(0)}(z, y + w(z))$.

We write $G^{(1)}(z, y) = \sum_{m \geq 0} g_m^{(1)}(z) y^m$. Since $v_0(z) = 0$, $v_1(z) = 2b_0 \neq 0$ and $f_0(z) = 0$, and thanks to Lemma 3.4 (with $\ell = 0$), the equation (3.23) has solutions if and only if $g_0^{(1)}(z) = 0$, i.e.,

$$G^{(1)}(z, 0) = (\mu_1 + 4\tilde{\mathcal{E}}(z, w(z)) \partial_z \partial_w) A^{(0)}(z, w(z)) = 0,$$

or, equivalently,

$$\mu_1 A_0^{(0)}(z) + 4\tilde{\mathcal{E}}(z, w(z)) \left(\partial_z A_1^{(0)}(z) - 2A_2^{(0)}(z) w'(z) \right) = 0.$$

Since

$$A_1^{(0)}(z) = -\frac{f_1(z)}{2b_0} A_0^{(0)}(z),$$

and

$$\begin{aligned} A_2^{(0)}(z) &= -\frac{(v_2(z) + f_1(z)) A_1^{(0)}(z) + f_2(z) A_0^{(0)}(z)}{4b_0} \\ &= \frac{1}{8b_0^2} f_1(z) (v_2(z) + f_1(z)) A_0^{(0)}(z) - \frac{1}{4b_0} f_2(z) A_0^{(0)}(z), \end{aligned}$$

the equation related to μ_1 can be rewritten as

$$(3.24) \quad V(z) \partial_z A_0^{(0)}(z) + F(z) A_0^{(0)}(z) = 0, \quad V(z) := \frac{2}{b_0} \tilde{\mathcal{E}}(z, w(z)) f_1(z),$$

with

$$F(z) := \tilde{\mathcal{E}}(z, w(z)) \left(\frac{2}{b_0} f_1'(z) + \frac{1}{b_0^2} f_1(z) (f_1(z) + v_2(z)) w'(z) + \frac{2}{b_0} f_2(z) w'(z) \right) - \mu_1.$$

Notation 3.2. Below, with a given formal series $X(z) = \sum_{k \geq 0} x_k z^k \in C[[z]]$, we use the notation $[X(z)]_k$ to extract the coefficient of z^k , so that $[X(z)]_k = x_k$.

We have

$$\begin{aligned} f_1(z) &= \partial_2 \tilde{\mathcal{B}}(z, w(z)) = \sum_{\substack{m \geq 0 \\ n \geq 1}} n \tilde{b}_{mn} z^m (w(z))^{n-1}, \\ f_2(z) &= \frac{1}{2} \partial_2^2 \tilde{\mathcal{B}}(z, w(z)) = \sum_{\substack{m \geq 0 \\ n \geq 2}} \frac{n(n-1)}{2} \tilde{b}_{mn} z^m (w(z))^{n-2}. \end{aligned}$$

It is easy to check that

$$[V(z)]_0 = \frac{2}{b_0} [\tilde{\mathcal{E}}(z, w(z))]_0 [f_1(z)]_0 = 0 \quad (\text{since } \tilde{b}_{01} = 0),$$

and

$$\begin{aligned}
[V(z)]_1 &= \frac{2}{b_0} [\tilde{\mathcal{E}}(z, w(z))]_0 [f_1(z)]_1 + \frac{2}{b_0} [\tilde{\mathcal{E}}(z, w(z))]_1 [f_1(z)]_0 \\
&= \frac{2e^{-2\eta(0)}}{b_0} \left(2\tilde{b}_{02}w_1 + \tilde{b}_{11} \right) \\
&= \frac{2e^{-2\eta(0)}}{b_0} \left(\frac{1}{2}(\alpha - \gamma) \frac{\sqrt{\gamma} - \sqrt{\alpha}}{\sqrt{\gamma} + \sqrt{\alpha}} + \frac{1}{2}(\alpha + \gamma) \right) \\
&= \frac{2e^{-2\eta(0)}\sqrt{\alpha\gamma}}{b_0}.
\end{aligned}$$

Furthermore, we can compute

$$[F(z)]_0 = \frac{2e^{-2\eta(0)}}{b_0} \left(\tilde{b}_{11} + \tilde{b}_{02}w_1 \right) - \mu_1 = \frac{e^{-2\eta(0)}(\sqrt{\gamma} + \sqrt{\alpha})^2}{2b_0} - \mu_1.$$

Applying Lemma 3.4, we see that (3.24) has solutions if and only if there exists $\ell \in \mathbb{N}$ such that

$$\mu_1 = e^{-2\eta(0)} \left(2\ell \frac{\sqrt{\alpha\gamma}}{b_0} + \frac{(\sqrt{\gamma} + \sqrt{\alpha})^2}{2b_0} \right).$$

Then, $A_0^{(0)}$ can be determined as a formal series whose ℓ first terms vanish and $[A_0^{(0)}(z)]_\ell = 1$.

Once the right-hand side of (3.22) is determined, we can find a particular solution in the form

$$\sum_{m \geq 0} \alpha_m^{(1)}(z)(w - w(z))^m,$$

where the $\alpha_m^{(1)}(z)$ are determined by a recursion formula starting with $\alpha_0^{(1)}(z) = 0$. The solution of (3.22) takes the form

$$(3.25) \quad A^{(1)}(z, w) = \sum_{m \geq 0} \alpha_m^{(1)}(z)(w - w(z))^m + \sum_{m \geq 0} A_m^{(1)}(z)(w - w(z))^m,$$

where $\sum_{m \geq 0} A_m^{(1)}(z)(w - w(z))^m$ is the solution of the first transport equation (3.18), and in which only $A_0^{(1)}(z)$ remains to be determined.

3.3.7. Induction. Let $p \in \mathbb{N} \setminus \{0\}$. We assume that the sequences $(\mu_j)_{0 \leq j \leq p}$ and $(A^{(j)})_{0 \leq j \leq p-1}$ are determined from the first $(p+1)$ transport equations:

$$\begin{aligned}
&\left(v(z, w)\partial_w + \tilde{B}(z, w) - \mu_0 \right) A^{(0)} = 0 \\
&\left(v(z, w)\partial_w + \tilde{B}(z, w) - \mu_0 \right) A^{(1)} = (\mu_1 + 4\tilde{\mathcal{E}}(z, w)\partial_z\partial_w)A^{(0)} \\
&\dots\dots\dots \\
&\left(v(z, w)\partial_w + \tilde{B}(z, w) - \mu_0 \right) A^{(p)} = (\mu_1 + 4\tilde{\mathcal{E}}(z, w)\partial_z\partial_w)A^{(p-1)} + \sum_{j=2}^p \mu_j A^{(p-j)}.
\end{aligned}$$

Let us also assume that the $A^{(j)}$'s, for $j \in \{1, \dots, p\}$, are in the form

$$A^{(j)}(z, w) = \sum_{m \geq 0} \alpha_m^{(j)}(z)(w - w(z))^m + \sum_{m \geq 0} A_m^{(j)}(z)(w - w(z))^m,$$

where

- (i) $\sum_{m \geq 0} \alpha_m^{(j)}(z)(w - w(z))^m$ is a particular solution of the j -th transport equation and satisfies $\alpha_0^{(j)}(z) = 0$ in $\mathbb{C}[[z]]$ for $j \in \{1, \dots, p\}$.
- (ii) $\sum_{m \geq 0} A_m^{(j)}(z)(w - w(z))^m$, which is a solution of the first transport equation (3.18), is also determined and satisfies $[A_0^{(j)}(z)]_\ell = 0$ in \mathbb{C} for $j \in \{1, \dots, p-1\}$.

At the rank p , only $A_0^{(p)}(z)$ needs to be determined.

Let us now consider the equation satisfied by $A^{(p+1)}$:

$$(3.26) \quad \left(v \partial_w + \tilde{B} - b_0\right) A^{(p+1)} = (\mu_1 + 4\tilde{\mathcal{E}}\partial_z\partial_w)A^{(p)} + \mu_{p+1}A^{(0)} + \sum_{j=2}^p \mu_j A^{(p+1-j)}.$$

As before, the fact that this equation has solutions determines the value of μ_{p+1} and the series $A_0^{(p)}(z)$. We leave these details to the reader.

3.3.8. *Conclusion.* From the above analysis, we get the following theorem.

Theorem 3.1 (WKB construction for $\mathcal{L}_{h,\mathcal{M}}$). *For all $\ell \in \mathbb{N}$, there exist*

(i) *a smooth complex-valued function T on U satisfying*

$$(3.27) \quad \operatorname{Re}(T)(q) = \frac{e^{2\eta(0)}b_0}{2} \left(\frac{\sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\gamma}} q_1^2 + \frac{\sqrt{\gamma}}{\sqrt{\alpha} + \sqrt{\gamma}} q_2^2 \right) + \mathcal{O}(\|q\|^3),$$

(ii) *a sequence of smooth complex-valued function $(a_{\ell,j})_{j \in \mathbb{N}}$ on U ,*

(iii) *a sequence of real numbers $(\mu_{\ell,j})_{j \in \mathbb{N}}$ with*

$$\mu_{\ell,0} = b_0, \quad \mu_{\ell,1} = e^{-2\eta(0)} \left(2\ell \frac{\sqrt{\alpha\gamma}}{b_0} + \frac{(\sqrt{\gamma} + \sqrt{\alpha})^2}{2b_0} \right),$$

(iv) *a sequence of a flat function $(f_j)_{j \in \mathbb{N}}$ on U ,*

such that, for all $J \in \mathbb{N}$,

$$e^{T/h} \left(\mathcal{L}_{h,\mathcal{M}} - h \sum_{j=0}^J \mu_{\ell,j} h^j \right) \left(e^{-T/h} \sum_{j=0}^J a_{\ell,j} h^j \right) = \sum_{j=0}^{J+1} h^j f_j + \mathcal{O}(h^{J+2}),$$

local uniformly in U .

Proof. We have constructed the formal series $\tilde{S}(z, w)$, $(A^{(j)}(z, w))_{j \in \mathbb{N}}$ in $\mathbb{C}[[z, w]]$ and $\mu_{\ell,j}$, all depending on ℓ . By coming back to the original variables (q_1, q_2) , we obtain formal series in $\mathbb{C}[[q_1, q_2]]$. Applying Borel's Lemma, we get a smooth complex-valued functions S and $(a_j)_{j \in \mathbb{N}}$ so that for each $J \in \mathbb{N}$, there exist flat functions f_0, f_1, \dots, f_{J+1} and a smooth function F on \mathbb{R}^2 (F is a polynomial in h whose coefficients are smooth functions depending on a_j and $\mu_{\ell,j}$) such that

$$\left(\mathcal{L}_{h,A}^S - h \sum_{j=0}^J \mu_{\ell,j} h^j \right) \left(\sum_{j=0}^J a_{\ell,j} h^j \right) = \sum_{j=0}^{J+1} h^j f_j + h^{J+2} F.$$

Note that $\mathcal{L}_{h,A}^S = e^{S/h} \mathcal{L}_{h,A} e^{-S/h}$ so that

$$(3.28) \quad e^{S/h} \left(\mathcal{L}_{h,A} - h \sum_{j=0}^J \mu_{\ell,j} h^j \right) \left(e^{-S/h} \sum_{j=0}^J a_{\ell,j} h^j \right) = \sum_{j=0}^{J+1} h^j f_j + h^{J+2} F.$$

We recall that we used the WKB method for the operator $\mathcal{L}_{h,A}$, with the special magnetic potential A , see Section 3.2.1 and especially (3.3). Letting $T = S + i\theta$, we get

$$(3.29) \quad e^{T/h} \left(\mathcal{L}_{h,\mathcal{M}} - h \sum_{j=0}^J \mu_{\ell,j} h^j \right) \left(e^{-T/h} \sum_{j=0}^J a_{\ell,j} h^j \right) = \sum_{j=0}^{J+1} h^j f_j + h^{J+2} F.$$

Since $\operatorname{Re}(T) = \operatorname{Re}(S)$, the Taylor expansion of $\operatorname{Re}(T)$ directly comes from (3.21). \square

Proof of Theorem 2.2. Let us recall (3.1). Consider a fixed cut-off function χ_1 equal to 1 in a neighbourhood of 0 and with support in U . Then, we set

- i) $P = \phi^*(\chi_1 T)$,
- ii) $U_{\ell,j} = \phi^*(\chi_1 a_{\ell,j})$ for all $(\ell, j) \in \mathbb{N}^2$,

Using Appendix B, we deduce Theorem 2.2. □

In order to prove Corollary 2.3, we just need to shrink the supports of functions in Theorem 2.2 by inserting appropriate cutoff functions and use the fact that there exist $R > 0$ and $\delta > 0$ such that

$$(3.30) \quad \operatorname{Re}(T)(q) \geq \delta \|q\|^2 \text{ for all } q \in D(0; R).$$

This coercivity of the phase implies that the Ansatz has an exponential decay.

4. PROOF OF THEOREM 2.5 AND OF ITS CONSEQUENCES

4.1. Radial magnetic Laplacian and Fourier decomposition. Consider the polar coordinates

$$(4.1) \quad \psi : \begin{cases} \mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^2 \setminus \{0\} \\ (r, \theta) \mapsto (r \cos \theta, r \sin \theta) = (q_1, q_2) \end{cases}.$$

The following proposition is easy to get.

Proposition 4.1. *Letting $\mathbf{A} = A_1 dq_1 + A_2 dq_2$, we have*

$$\mathbf{A} = A_r dr + A_\theta d\theta,$$

with $\tilde{\mathbf{A}} = (A_r, A_\theta)^\top := (d\psi)^\top (A_1, A_2)^\top$, where $d\psi$ denotes the Jacobian matrix of ψ . The magnetic Laplacian $\mathcal{L}_{h, \mathbf{A}}$ is unitary equivalent to the operator

$$(4.2) \quad \mathcal{K}_{h, \tilde{\mathbf{A}}} = r^{-2} (r(-ih\partial_r - A_r))^2 + r^{-2} (-ih\partial_\theta - A_\theta)^2,$$

whose domain is

$$\operatorname{Dom}(\mathcal{K}_{h, \tilde{\mathbf{A}}}) = \{w \in L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta) : \mathcal{K}_{h, \tilde{\mathbf{A}}} w \in L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta)\}.$$

Thanks to the gauge of invariance, we can choose a magnetic potential compatible with the radial symmetry (the one given by the Poincaré lemma):

$$(4.3) \quad A_1(q) = -q_2 \alpha(q), \quad A_2(q) = q_1 \alpha(q),$$

where

$$\alpha(q) := \int_0^1 t \mathbf{B}(tq) dt.$$

With the choice, we get

$$A_r(r, \theta) = 0, \quad A_\theta(r, \theta) = G(r) := \int_0^r \tau \beta\left(\frac{\tau^2}{2}\right) d\tau,$$

and the magnetic Laplacian becomes

$$\mathcal{K}_h = -h^2 r^{-2} (r\partial_r)^2 + r^{-2} (-ih\partial_\theta - G(r))^2.$$

Via the Fourier decomposition, the magnetic Laplacian \mathcal{K}_h can be written as the direct sum of radial electric Schrödinger operators

$$(4.4) \quad \mathcal{K}_h = \bigoplus_{m \in \mathbb{Z}} \mathfrak{L}_{h, m}, \quad \mathfrak{L}_{h, m} := -h^2 r^{-2} (r\partial_r)^2 + r^{-2} (hm - G(r))^2.$$

Therefore, we can focus on the spectral analysis of $\mathfrak{L}_{h, m}$, especially on its ground-energy.

4.2. Compact resolvent.

Proposition 4.2. *For any $m \in \mathbb{Z}$ and $h > 0$, the operator $\mathfrak{L}_{h,m}$ has compact resolvent.*

Proof. We fix $m \in \mathbb{Z}$ and $h > 0$, consider the sesquilinear

$$\mathcal{Q}_{h,m}(u, v) = 2h^2 \int_0^{+\infty} \partial_r u \overline{\partial_r v} r dr + \int_0^{+\infty} \frac{(hm - G(r))^2}{r^2} u \bar{v} r dr + \int_0^{+\infty} u \bar{v} r dr,$$

defined on the domain

$$\text{Dom}(\mathcal{Q}_{h,m}) = \left\{ u \in L^2(\mathbb{R}^+, r dr) : \partial_r u \in L^2(\mathbb{R}^+, r dr), \sqrt{V_{h,m}} u \in L^2(\mathbb{R}^+, r dr) \right\},$$

with

$$V_{h,m}(r) := \frac{(hm - G(r))^2}{r^2}.$$

$\text{Dom}(\mathcal{Q}_{h,m})$ is a Hilbert space equipped with the inner product $\mathcal{Q}_{h,m}$. The sesquilinear form $\mathcal{Q}_{h,m}$ induces a positive self-adjoint operator $\mathcal{S}_{h,m}$. From (2.6), we have

$$(4.5) \quad \lim_{r \rightarrow +\infty} \frac{(hm - G(r))^2}{r^2} = +\infty.$$

The conclusion easily follows by means of the Riesz-Fréchet-Kolmogorov criterion for compactness in L^p spaces. \square

4.3. Spectrum of rescaled radial electric Schrödinger operators. By the change of variable $\rho = \frac{r^2}{2}$, we have

$$G(r) = \int_0^r \tau \beta \left(\frac{\tau^2}{2} \right) d\tau = \int_0^\rho \beta(s) ds.$$

We let

$$a(\rho) := \int_0^\rho \beta(s) ds.$$

Using the unitary transformation

$$(4.6) \quad \begin{aligned} T_1 : L^2(\mathbb{R}^+, d\rho) &\rightarrow L^2(\mathbb{R}^+, r dr), \\ v(\rho) &\mapsto (T_1 v)(r) = v(r^2/2), \end{aligned}$$

we get the new operator, acting on $L^2(\mathbb{R}^+, d\rho)$,

$$(4.7) \quad \mathcal{N}_{h,m} := T_1^{-1} \mathfrak{L}_{h,m} T_1 = -2h^2 \partial_\rho \rho \partial_\rho + \frac{(hm - a(\rho))^2}{2\rho}.$$

4.3.1. Rescaling. The rescaling $\rho = ht$ is convenient to analyse the expansion of the eigenvalues and some decay properties of the eigenfunctions. Consider

$$(4.8) \quad \begin{aligned} T_2 : L^2(\mathbb{R}^+, dt) &\rightarrow L^2(\mathbb{R}^+, d\rho), \\ v(t) &\mapsto (T_2 v)(\rho) = h^{-1/2} v(h^{-1} \rho). \end{aligned}$$

and

$$(4.9) \quad \mathcal{M}_{h,m} := T_2^{-1} \mathcal{N}_{h,m} T_2 = -2h \partial_t t \partial_t + \frac{(hm - a(ht))^2}{2ht},$$

We have

$$a(ht) = \beta(0)ht + \frac{\beta'(0)}{2} h^2 t^2 + \mathcal{O}(h^3 t^3).$$

and thus

$$\mathcal{M}_{h,m} = h \mathfrak{L}_m^{[0]} + h^2 \mathfrak{L}_m^{[1]} + \frac{R(ht)}{t},$$

where

$$\begin{aligned}\mathfrak{L}_m^{[0]} &:= -2\partial_t t \partial_t + \frac{\beta(0)^2 t}{2} + \frac{m^2}{2t} - m\beta(0) \\ \mathfrak{L}_m^{[1]} &:= -\frac{m\beta'(0)}{2}t + \frac{\beta(0)\beta'(0)}{2}t^2,\end{aligned}$$

and R is a remainder such that there exist a constant $C > 0$ and $\delta > 0$ such that

$$(4.10) \quad |R(s)| \leq Cs^3 \quad \text{for all } s \in [0, \delta].$$

We consider $\mathfrak{L}_m^{[0]}$ defined through the sesquilinear form

$$\mathcal{Q}_m^{[0]}(u, v) = \int_0^{+\infty} 2t\partial_t u \overline{\partial_t v} dt + \int_0^{+\infty} \left(\frac{\beta(0)^2 t}{2} + \frac{m^2}{2t} - m\beta(0) \right) u \overline{v} dt.$$

The operator $\mathfrak{L}_m^{[0]}$ has clearly compact resolvent. Furthermore, the spectrum of $\mathfrak{L}_m^{[0]}$ is well known and related to the Laguerre operator. Indeed, by letting $t = \frac{s}{\beta(0)}$, the operator $\mathfrak{L}_m^{[0]}$ becomes

$$(4.11) \quad \beta(0) \left(-2\partial_s s \partial_s + \frac{s}{2} + \frac{m^2}{2s} - m \right).$$

We consider the following operator \mathcal{T}_m on the Hilbert space $L^2(\mathbb{R}^+, s^{-|m|} e^s ds)$

$$\mathcal{T}_m = s^{-\frac{|m|}{2}} e^{\frac{s}{2}} \left(-2\partial_s s \partial_s + \frac{s}{2} + \frac{m^2}{2s} - m \right) s^{\frac{|m|}{2}} e^{-\frac{s}{2}}.$$

It is unitarily equivalent to $\mathfrak{L}_m^{[0]}$. A computation gives

$$\mathcal{T}_m = -2s\partial_s^2 + (2s - 2 - 2|m|)\partial_s + |m| - m + 1.$$

The spectrum of operator \mathcal{T}_m is described in Appendix B and we get

$$\text{Sp}(\mathfrak{L}_m^{[0]}) = \{(2k + 1 + |m| - m)\beta(0) : k \in \mathbb{N}\}.$$

Remark 4.1. When $m \geq 0$, the first and the second eigenpairs of $\mathfrak{L}_m^{[0]}$ are respectively

$$\left(\beta(0), t^{\frac{m}{2}} e^{-\frac{\beta(0)t}{2}} \right) \quad \text{and} \quad \left(3\beta(0), [\beta(0)t - m - 1] t^{\frac{m}{2}} e^{-\frac{\beta(0)t}{2}} \right).$$

By using the $t^{\frac{m}{2}} e^{-\frac{\beta(0)t}{2}}$ as test function and then the Spectral Theorem, we get the following (this result will be recovered later by using a WKB analysis).

Proposition 4.3. *For all $m \in \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$\text{dist} \left(\mu_{m,0}h + \mu_{m,1}h^2, \text{Sp}(\mathcal{M}_{h,m}) \right) \leq Ch^3,$$

where $\mu_{m,0} = \beta(0)$ and $\mu_{m,1} = \frac{(m+1)\beta'(0)}{\beta(0)}$.

4.3.2. Semiclassical Agmon estimate. The above proposition states that, for each $m \in \mathbb{N}$, one can find an eigenvalue of $\mathcal{M}_{h,m}$ near $\beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)}h^2$. We will use Agmon estimates to prove that this eigenvalue is in fact $\lambda_0(\mathcal{M}_{h,m})$. In this section, let us denote

$$\mathcal{V}_{h,m}(t) = \frac{(hm - a(ht))^2}{2ht},$$

and consider the quadratic form associated with $\mathcal{M}_{h,m}$:

$$Q_{h,m}(u) = 2h \int_0^\infty t |\partial_t u|^2 dt + \int_0^\infty \mathcal{V}_{h,m}(t) |u(t)|^2 dt.$$

Since we only deal with the Hilbert space $L^2(\mathbb{R}^+, dt)$ in this section, we denote $\|\cdot\|_{L^2}$ (resp. $\langle \cdot, \cdot \rangle_{L^2}$) instead of $\|\cdot\|_{L^2(\mathbb{R}^+, dt)}$ (resp. $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^+, dt)}$).

Let us recall that $a(\rho) = \int_0^\rho \beta(s) ds$. From (2.6), we get

$$a(ht) \geq \beta(0)ht.$$

For t large enough, we have

$$(4.12) \quad \mathcal{V}_{h,m}(t) = \frac{(hm - a(ht))^2}{2ht} \geq h \frac{(\beta(0)t - m)^2}{2t}.$$

Proposition 4.4. *Let $m \in \mathbb{N}$ and let $\Phi \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$, then for all $u \in \text{Dom}(\mathcal{M}_{h,m})$, we have*

$$(4.13) \quad Q_{h,m}(e^\Phi u) = \text{Re} \langle \mathcal{M}_{h,m} u, e^{2\Phi} u \rangle_{L^2} + 2h \int_0^{+\infty} t(\Phi'(t))^2 e^{2\Phi} |u|^2 dt.$$

Proof. We have

$$(4.14) \quad 2h \langle \sqrt{t} \partial_t u, \sqrt{t} \partial_t (e^{2\Phi} u) \rangle_{L^2} + \int_0^{+\infty} \mathcal{V}_{h,m}(t) e^{2\Phi} |u|^2 dt = \langle \mathcal{M}_{h,m} u, e^{2\Phi} u \rangle_{L^2}.$$

We set $P := \sqrt{t} \partial_t$. Notice that the commutator

$$[P, e^\Phi] = \sqrt{t} \Phi'(t) e^\Phi$$

is a multiplication operator. We have

$$\begin{aligned} \langle Pu, Pe^{2\Phi} u \rangle_{L^2} &= \langle Pu, [P, e^\Phi] e^\Phi u \rangle_{L^2} + \langle Pu, e^\Phi P e^\Phi u \rangle_{L^2} \\ &= \langle e^\Phi Pu, [P, e^\Phi] u \rangle_{L^2} + \langle e^\Phi Pu, P e^\Phi u \rangle_{L^2} \\ &= \langle e^\Phi Pu, [P, e^\Phi] u \rangle_{L^2} + \langle P e^\Phi u, P e^\Phi u \rangle_{L^2} + \langle [e^\Phi, P] u, P e^\Phi u \rangle_{L^2} \\ &= \|P e^\Phi u\|_{L^2}^2 - \|[P, e^\Phi] u\|_{L^2}^2 + \langle e^\Phi Pu, [P, e^\Phi] u \rangle_{L^2} - \langle [P, e^\Phi] u, e^\Phi Pu \rangle_{L^2}. \end{aligned}$$

Taking the real part of (4.14), we get

$$2h \|P e^\Phi u\|_{L^2}^2 - 2h \|[P, e^\Phi] u\|_{L^2}^2 + \int_0^{+\infty} \mathcal{V}_{h,m}(t) e^{2\Phi} |u|^2 dt = \text{Re} \langle \mathcal{M}_{h,m} u, e^{2\Phi} u \rangle_{L^2}.$$

□

Proposition 4.5. *Under the assumption (2.6), for all $m \in \mathbb{N}$, and for all $\varepsilon \in (0, \frac{\beta(0)}{2})$, there exists $M > 0$ such that*

$$\|e^{\varepsilon t} \Psi\|_{L^2}^2 \leq M \|\Psi\|_{L^2}^2, \quad \text{and} \quad Q_{h,m}(e^{\varepsilon t} \Psi) \leq M h \|\Psi\|_{L^2}^2,$$

for all eigenfunctions Ψ of the operator $\mathcal{M}_{h,m}$, with eigenvalue of order h .

Proof. Let us consider a sequence of functions $(\chi_k)_{k \geq 1}$ defined as follows

$$(4.15) \quad \chi_k(t) = \begin{cases} t & \text{for } 0 \leq t \leq k, \\ 2k - t & \text{for } k \leq t \leq 2k, \\ 0 & \text{for } t \geq 2k. \end{cases}$$

Notice that $\chi_k \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$ and $|\chi_k'(t)| \leq 1$ for all $t \in \mathbb{R}^+$.

Let us consider the equation

$$(4.16) \quad \mathcal{M}_{h,m} \Psi = \lambda \Psi \quad \text{with} \quad \lambda \leq Ch.$$

Using Proposition 4.4 with $\Phi = \varepsilon \chi_k(t)$, we get

$$\begin{aligned} Q_{h,m}(e^{\varepsilon \chi_k(t)} \Psi) &\leq \lambda \|e^{\varepsilon \chi_k(t)} \Psi\|_{L^2}^2 + 2h \varepsilon^2 \int_0^{+\infty} t |\chi_k'(t)|^2 e^{2\varepsilon \chi_k(t)} |\Psi(t)|^2 dt \\ &\leq Ch \|e^{\varepsilon \chi_k(t)} \Psi\|_{L^2}^2 + 2h \varepsilon^2 \int_0^{+\infty} t e^{2\varepsilon \chi_k(t)} |\Psi(t)|^2 dt. \end{aligned}$$

It implies that

$$\int_0^{+\infty} \mathcal{V}_{h,m}(t) e^{2\varepsilon \chi_k(t)} |\Psi|^2 dt \leq Ch \|e^{\varepsilon \chi_k(t)} \Psi\|_{L^2}^2 + 2h \varepsilon^2 \int_0^{+\infty} t e^{2\varepsilon \chi_k(t)} |\Psi(t)|^2 dt,$$

so that

$$\int_0^{+\infty} (\mathcal{V}_{h,m}(t) - Ch - 2h\varepsilon^2 t) e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \leq 0.$$

From (4.12), there exist $R > 0$ and $C_1(R, \varepsilon) > 0$ such that for all $t \geq R$, we have

$$\begin{aligned} \mathcal{V}_{h,m}(t) - Ch - 2\varepsilon^2 th &\geq h \left[\frac{(b_0 t - m)^2}{2t} - 2\varepsilon^2 t - C \right] \\ &= h \left[\left(\frac{b_0^2}{2} - 2\varepsilon^2 \right) t + \frac{m^2}{2t} - b_0 m - C \right] \\ (4.17) \qquad \qquad \qquad &\geq C_1(R, \varepsilon) h. \end{aligned}$$

We deduce the existence of $C_2(R, \varepsilon) > 0$ such that, for all $k \geq 1$,

$$\begin{aligned} C_1(R, \varepsilon) h \int_R^{+\infty} e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt &\leq \int_R^{+\infty} (\mathcal{V}_{h,m}(t) - Ch - 2h\varepsilon^2 t) e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \\ &\leq \int_0^R (2h\varepsilon^2 t + Ch - \mathcal{V}_{h,m}(t)) e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \\ &\leq C_2(R, \varepsilon) h \|\Psi\|_{L^2}^2. \end{aligned}$$

Then, there exists a constant $C(R, \varepsilon)$ such that, for all $k \geq 1$,

$$\begin{aligned} \int_0^{+\infty} e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt &= \int_0^R e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt + \int_R^{+\infty} e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \\ &\leq \int_0^R e^{2\varepsilon R} |\Psi|^2 dt + \int_R^{+\infty} e^{2\varepsilon\chi_k(t)} |\Psi|^2 dt \\ &\leq C(R, \varepsilon) \|\Psi\|_{L^2}^2. \end{aligned}$$

Letting $k \rightarrow +\infty$ and using Fatou's lemma give

$$\int_0^{+\infty} e^{2\varepsilon t} |\Psi|^2 dt \leq C(R, \varepsilon) \|\Psi\|_{L^2}^2.$$

Using again (4.17), we notice that

$$\mathcal{V}_{h,m}(t) - Ch - 2\varepsilon^2 th \geq C_1(R, \varepsilon) ht.$$

Following the same steps as above, we get

$$\int_0^{+\infty} t e^{2\varepsilon\chi_k(t)} |\Psi(t)|^2 dt \leq C \|\Psi\|_{L^2}^2.$$

Then, it leads to

$$Q_{h,m}(e^{\varepsilon\chi_k(t)} \Psi) \leq Ch \|\Psi\|_{L^2}^2,$$

and we get the result. □

Proposition 4.6. *For all $m \in \mathbb{N}$,*

$$(4.18) \qquad \lambda_0(\mathcal{M}_{h,m}) = \beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)} h^2 + o(h^2).$$

Proof. Let us fix $m \in \mathbb{N}$. We can choose the first eigenpairs $(\lambda_i(\mathcal{M}_{h,m}), \Psi_{i,h})_{i=1,2}$ such that $\Psi_{0,h}$ and $\Psi_{1,h}$ are orthogonal. We consider the two-dimensional space

$$(4.19) \qquad E(h) = \text{span}(\Psi_{0,h}, \Psi_{1,h}).$$

Let $\Psi \in E(h)$, we have

$$\begin{aligned}
Q_{h,m}(\Psi) &= 2h \int_0^{+\infty} t |\partial_t(\Psi)|^2 dt + \int_0^{+\infty} \mathcal{V}_{h,m}(t) |\Psi|^2 dt \\
&= h \int_0^{+\infty} 2t |\partial_t(\Psi)|^2 + \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) |\Psi|^2 dt \\
&\quad + \int_0^{+\infty} \left[\mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) \right] |\Psi|^2 dt \\
&\geq h \mathcal{Q}_m^{[0]}(\Psi) - \int_0^{+\infty} \left| \mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) \right| |\Psi|^2 dt,
\end{aligned}$$

where $\mathcal{Q}_m^{[0]}$ is the quadratic form associated with $\mathfrak{L}_m^{[0]}$. We have

$$\mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) = \frac{\mathcal{O}((ht)^2)}{t}.$$

There exist $C_1 > 0$ and $\delta > 0$ such that, for all (h, t) satisfying $ht \leq \delta$,

$$\left| \mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) \right| \leq C_1 t h^2.$$

We have

$$\begin{aligned}
\int_0^{\delta/h} \left| \mathcal{V}_{h,m}(t) - h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) \right| |\Psi|^2 dt &\leq C_1 h^2 \int_0^{\delta/h} t |\Psi|^2 dt \\
&\leq C_2 h^2 \int_0^{+\infty} e^{-2\epsilon t} |\Psi|^2 dt \\
&\leq C h^2 \|\Psi\|_{L^2}^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\delta/h}^{+\infty} |\mathcal{V}_{h,m}(t)| |\Psi|^2 dt &= \int_{\delta/h}^{+\infty} e^{-2\epsilon t} |\mathcal{V}_{h,m}(t)| e^{2\epsilon t} |\Psi|^2 dt \\
&\leq \max_{t \geq \delta/h} e^{-2\epsilon t} \int_0^{+\infty} |\mathcal{V}_{h,m}(t)| e^{2\epsilon t} |\Psi|^2 dt \\
&\leq C h^2 \|\Psi\|_{L^2}^2.
\end{aligned}$$

Similarly, we also have

$$\int_{\delta/h}^{+\infty} h \left(\frac{\beta(0)^2}{2} t + \frac{m^2}{2t} - m\beta(0) \right) dt \leq C h^2 \|\Psi\|_{L^2}^2.$$

Thus, we have the estimate

$$Q_{h,m}(\Psi) \geq h \mathcal{Q}_m^{[0]}(\Psi) - C h^2 \|\Psi\|_{L^2}^2.$$

Since $\Psi \in E(h)$, there exists $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ such that $\Psi = \alpha \Psi_{0,h} + \beta \Psi_{1,h}$ and

$$Q_{h,m}(\Psi) = \lambda_0(\mathcal{M}_{h,m}) |\alpha_1|^2 \|\Psi_{0,h}\|_{L^2}^2 + \lambda_1(\mathcal{M}_{h,m}) |\alpha_2|^2 \|\Psi_{1,h}\|_{L^2}^2 \leq \lambda_1(\mathcal{M}_{h,m}) \|\Psi\|_{L^2}^2.$$

By the min-max principle, we deduce that

$$(4.20) \quad \lambda_1(\mathcal{M}_{h,m}) \geq 3\beta(0)h - C h^2.$$

From Proposition 4.3, we see that there exists an eigenvalue of $\mathcal{M}_{h,m}$ which is near $\beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)} h^2$ modulo $o(h^2)$. This eigenvalue can not be $\lambda_k(\mathcal{M}_{h,m})$ with $k \geq 1$, because if it were, the estimate

$$3\beta(0)h - C h^2 \leq \lambda_k(\mathcal{M}_{h,m}) = \beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)} h^2 + o(h^2),$$

would give a contradiction. The conclusion follows. \square

4.4. Eigenfunctions of the magnetic Laplacian and of its fibration. We recall the expression of the magnetic Laplacian operator \mathcal{K}_h

$$(4.21) \quad \mathcal{K}_h = -h^2 r^{-2} (r \partial_r)^2 + r^{-2} (-ih \partial_\theta - G(r))^2.$$

For each $m \in \mathbb{N}$, let $\lambda_0(\mathcal{N}_{h,m})$ be the first eigenvalue of the operator $\mathcal{N}_{h,m}$ and $\Psi_{h,m}$ be the associated eigenfunction. Since $\mathcal{M}_{h,m}$ and $\mathcal{N}_{h,m}$ are unitary equivalent, from Proposition 4.6, we obtain

$$(4.22) \quad \lambda_0(\mathcal{N}_{h,m}) = \lambda_0(\mathcal{M}_{h,m}) = \beta(0)h + \frac{(m+1)\beta'(0)}{\beta(0)}h^2 + o(h^2).$$

On the other hand, $\mathfrak{L}_{h,m}$ and $\mathcal{N}_{h,m}$ also are equivalent by

$$\mathfrak{L}_{h,m} = T_1 \mathcal{N}_{h,m} T_1^{-1},$$

where T_1 is defined in (4.6). Therefore, $\lambda_0(\mathcal{N}_{h,m})$ is also the first eigenvalue of $\mathfrak{L}_{h,m}$ and $T_1(\Psi_{h,m})$ is the associated eigenfunction. It results that

$$[\mathcal{K}_h - \lambda_0(\mathcal{N}_{h,m})] e^{im\theta} T_1(\Psi_{h,m}) = [\mathfrak{L}_{h,m} - \lambda_0(\mathcal{M}_{h,m})] T_1(\Psi_{h,m}) e^{im\theta} = 0.$$

Thus $\lambda_0(\mathcal{N}_{h,m})$ belongs to the spectrum of \mathcal{K}_h . Now, the result by Helffer and Kordyukov tells us that, for all $k \in \mathbb{N}$, the k -th eigenvalue of \mathcal{K}_h , denoted by $\lambda_k(\mathcal{K}_h)$ satisfies

$$(4.23) \quad \lambda_k(\mathcal{K}_h) = \mathbf{B}(0,0)h + \left(2k \frac{\sqrt{\det H}}{\mathbf{B}(0,0)} + \frac{(\text{Tr} H^{1/2})^2}{2\mathbf{B}(0,0)} \right) h^2 + o(h^2),$$

where $H = \frac{1}{2} \text{Hess}_{(0,0)} \mathbf{B}$, so that

$$(4.24) \quad \lambda_k(\mathcal{K}_h) = \beta(0)h + \frac{(k+1)\beta'(0)}{\beta(0)}h^2 + o(h^2).$$

Since $\lambda_0(\mathcal{N}_{h,m})$ is the eigenvalue of \mathcal{K}_h , thus there exists $k \in \mathbb{N}$ such that

$$\lambda_0(\mathcal{M}_{h,m}) = \lambda_k(\mathcal{K}_h).$$

Considering (4.22) and taking h small enough, we immediately obtain $k = m$ and

$$\lambda_m(\mathcal{K}_h) = \lambda_0(\mathcal{M}_{h,m}).$$

Since $\lambda_m(\mathcal{K}_h)$ is a simple eigenvalue, we have the following statement

Proposition 4.7. *When h is small enough, the m -th eigenvalue of the magnetic Laplacian \mathcal{K}_h is exactly the first eigenvalue of the operator $\mathcal{N}_{h,m}$:*

$$\lambda_m(\mathcal{K}_h) = \lambda_0(\mathcal{N}_{h,m}).$$

The m -th eigenfunction of \mathcal{K}_h is in the form

$$(4.25) \quad c e^{im\theta} \Psi_{h,m} \left(\frac{r^2}{2} \right),$$

where $\Psi_{h,m}$ is a ground-state of the operator $\mathcal{N}_{h,m}$ and $c \in \mathbb{C} \setminus \{0\}$.

4.5. Radial magnetic WKB construction. In this section, we focus on constructing a WKB Ansatz. Thanks to Proposition 4.7, we just need to do the WKB analysis for the operator $\mathcal{N}_{h,m}$. Since $\mathcal{N}_{h,m}$ is a real electric Schrödinger operator in dimension 1, one can easily find a WKB approximation of $\Psi_{h,m}$. We recall that

$$(4.26) \quad \mathcal{N}_{h,m} = -2h^2 \partial_\rho \rho \partial_\rho + \frac{(hm - a(\rho))^2}{2\rho}, \quad a(\rho) = \int_0^\rho \beta(\tau) d\tau.$$

We consider the conjugated operator with real-valued smooth function φ :

$$\widehat{\mathcal{N}}_{h,m} = e^{\frac{\varphi(\rho)}{h}} \rho^{-\frac{m}{2}} \mathcal{N}_{h,m} \rho^{\frac{m}{2}} e^{-\frac{\varphi(\rho)}{h}},$$

and get

$$\begin{aligned} \widehat{\mathcal{N}}_{h,m} = & \left[\frac{(a(\rho))^2}{2\rho} - 2\rho(\varphi'(\rho))^2 \right] + h \left[4\varphi'(\rho)\rho\partial_\rho + 2\varphi'(\rho) + 2\rho\varphi''(\rho) + 2m\varphi'(\rho) - m\frac{a(\rho)}{\rho} \right] \\ & + h^2 \left[-2\rho\partial_\rho^2 - (2m+2)\partial_\rho \right]. \end{aligned}$$

4.5.1. *The eikonal equation.* The eikonal equation reads

$$(4.27) \quad (\varphi'(\rho))^2 = \frac{(a(\rho))^2}{4\rho^2}.$$

We choose a positive solution

$$\varphi(\rho) = \int_0^\rho \frac{a(\tau)}{2\tau} d\tau,$$

which is a smooth function on $[0, +\infty)$ because it can be rewritten in the form

$$\varphi(\rho) = \int_0^\rho \frac{1}{2\tau} \int_0^\tau \beta(\xi) d\xi d\tau = \frac{1}{2} \int_0^\rho \int_0^1 \beta(\xi\tau) d\xi d\tau.$$

Then the operator $\widehat{\mathcal{N}}_{h,m}$ becomes

$$\widehat{\mathcal{N}}_{h,m} = h\mathcal{N}^1 + h^2\mathcal{N}_m^2,$$

where

$$\mathcal{N}^1 = 4\varphi'(\rho)\rho\partial_\rho + 2\varphi'(\rho) + 2\rho\varphi''(\rho) = 2a(\rho)\partial_\rho + \beta(\rho),$$

and

$$\mathcal{N}_m^2 = -2\rho\partial_\rho^2 - (2m+2)\partial_\rho.$$

We now look for a WKB Ansatz and a quasi-eigenvalue

$$\begin{aligned} a(\rho, h) &\sim a_0(\rho) + ha_1(\rho) + h^2a_2(\rho) + \dots, \\ \lambda(h) &\sim h(\mu_0 + h\mu_1 + h^2\mu_2 + \dots). \end{aligned}$$

We substitute these formal series into the equation

$$\left(\widehat{\mathcal{N}}_{h,m} - \lambda(h) \right) a(\rho, h) = 0,$$

and get

$$\begin{aligned} h : & (\mathcal{N}^1 - \mu_0) a_0 = 0 \\ h^2 : & (\mathcal{N}^1 - \mu_0) a_1 = (\mu_1 - \mathcal{N}_m^2) a_0 \\ \dots & \end{aligned}$$

4.5.2. *The first transport equation.* Collecting all terms of order h^1 , we have the first transport equation

$$(4.28) \quad (2a(\rho)\partial_\rho + \beta(\rho) - \mu_0)a_0 = 0.$$

This equation has smooth solutions which do not vanish at 0 if and only if

$$\mu_0 = \beta(0).$$

Indeed, since $a(0) = 0$, the function

$$F(\rho) := \frac{\beta(0) - \beta(\rho)}{2a(\rho)},$$

is actually smooth on $[0, +\infty)$. Then,

$$a_0(\rho) = a_0(0) \exp\left(\int_0^\rho F(s) ds\right),$$

with $a_0(0) \neq 0$. We choose $a(0) = 1$.

4.5.3. *The second transport equation.* Let us gather all terms of order h^2 to get the second transport equation

$$(4.29) \quad (2a(\rho)\partial_\rho + \beta(\rho) - \mu_0)a_1 = (\mu_1 + (2m + 2)\partial_\rho + 2\rho\partial_\rho^2)a_0.$$

Thus, (4.29) has a smooth solution if and only if

$$(\mu_1 + (2m + 2)\partial_\rho)a_0(0) = 0,$$

or

$$\mu_1 = -\frac{(2m + 2)\partial_\rho a_0(0)}{a_0(0)}.$$

From (4.28), we get

$$\frac{\partial_\rho a_0(0)}{a_0(0)} = \lim_{\rho \rightarrow 0} \frac{\beta(0) - \beta(\rho)}{2a(\rho)} = \frac{-\beta'(0)}{2\beta(0)}.$$

Thus,

$$(4.30) \quad \mu_1 = (m + 1) \frac{\beta'(0)}{\beta(0)}.$$

With this choice, (4.29) can be rewritten as

$$(4.31) \quad \partial_\rho a_1 - F(\rho)a_1 = g_1(\rho) := \frac{(\mu_1 + (2m + 2)\partial_\rho + 2\rho\partial_\rho^2)a_0}{2a(\rho)}.$$

This equation has solutions in the form

$$\begin{aligned} a_1(\rho) &= \exp\left(\int_0^\rho F(s) ds\right) \int_0^\rho \exp\left(-\int_0^\tau F(s) ds\right) g_1(\tau) d\tau \\ &\quad + a_1(0) \exp\left(\int_0^\rho F(s) ds\right). \end{aligned}$$

We impose $a_1(0) = 0$ so that

$$a_1(\rho) = \exp\left(\int_0^\rho F(s) ds\right) \int_0^\rho \exp\left(-\int_0^\tau F(s) ds\right) g_1(\tau) d\tau.$$

4.5.4. *Induction.* Let $n \in \mathbb{N}$ and $n \geq 2$. We assume that $(\mu_j)_{0 \leq j \leq n}$ and $(a_j)_{0 \leq j \leq n}$ are determined and $(a_j)_{1 \leq j \leq n}$ are smooth function on $[0, +\infty)$ and vanish at $\rho = 0$. Let us show that we can determine μ_{n+1} and a_{n+1} by the $(n+1)$ -th transport equation

$$(4.32) \quad (2a(\rho)\partial_\rho + \beta(\rho) - \mu_0) a_{n+1} = ((2m+2)\partial_\rho + 2\rho\partial_\rho^2) a_n + \sum_{j=1}^{n+1} \mu_j a_{n+1-j}.$$

The equation has a smooth solution at 0 if and only if

$$(2m+2)\partial_\rho a_n(0) + \sum_{j=1}^n \mu_j a_{n+1-j}(0) + \mu_{n+1} a_0(0) = 0.$$

Since $a_0(0) = 1$, μ_{n+1} is completely determined by

$$\mu_{n+1} = -(2m+2)\partial_\rho a_n(0).$$

With this value of μ_{n+1} , we can rewrite the equation (4.32) as

$$(4.33) \quad \partial_\rho a_{n+1} - F(\rho) a_{n+1} = g_n(\rho),$$

where g_n is the smooth extension of the function

$$G_n(\rho) = \frac{((2m+2)\partial_\rho + 2\rho\partial_\rho^2) a_n + \sum_{j=1}^{n+1} \mu_j a_{n+1-j}}{2a(\rho)},$$

on $[0, +\infty)$.

There is only one solution a_{n+1} such that $a_{n+1}(0) = 0$, that is

$$a_{n+1}(\rho) = \exp\left(\int_0^\rho F(s) ds\right) \int_0^\rho \exp\left(-\int_0^\tau F(s) ds\right) g_n(\tau) d\tau.$$

Proof of Theorem 2.5. We fix $m \in \mathbb{N}$. The WKB analysis provided us with functions and sequences as follows:

(i) The function $\varphi(\rho)$ is given by (4.27):

$$(4.34) \quad \varphi(\rho) = \frac{1}{2} \int_0^\rho \int_0^1 \beta(\xi\tau) d\xi d\tau.$$

(ii) The transport equations give us the existence of a sequence of smooth functions $(a_{m,j})_{j \in \mathbb{N}}$ defined on $[0, +\infty)$ and the sequence $(\mu_{m,j})_{j \in \mathbb{N}}$ which depends on m . Notice that $a_{m,0}$ is positive since

$$a_{m,0}(\rho) = \exp\left(\int_0^\rho F(s) ds\right).$$

For each $J \in \mathbb{N}$, from the WKB construction, there exists a smooth function $f_{m,J}(\rho)$ defined on $[0, +\infty)$ such that

$$e^{\frac{\varphi(\rho)}{h}} \rho^{\frac{-m}{2}} \left(\mathcal{N}_{h,m} - h \sum_{j=0}^J \mu_{m,j} h^j \right) \left(\rho^{\frac{m}{2}} e^{-\frac{\varphi(\rho)}{h}} \sum_{j=0}^J a_{m,j} h^j \right) = f_{m,J}(\rho) h^{J+2}.$$

After changing of variable $\rho = \frac{r^2}{2}$, we obtain

$$\begin{aligned} e^{\frac{\varphi\left(\frac{r^2}{2}\right)}{h}} \left(\frac{r^2}{2}\right)^{\frac{-m}{2}} \left(\mathfrak{L}_{h,m} - h \sum_{j=0}^J \mu_{m,j} h^j \right) \left(\left(\frac{r^2}{2}\right)^{\frac{m}{2}} e^{-\frac{\varphi\left(\frac{r^2}{2}\right)}{h}} \sum_{j=0}^J a_{m,j} \left(\frac{r^2}{2}\right) h^j \right) \\ = f_{m,J} \left(\frac{r^2}{2}\right) h^{J+2}. \end{aligned}$$

By multiplying $\left(\frac{r^2}{2}\right)^{\frac{m}{2}} e^{-\frac{\varphi\left(\frac{r^2}{2}\right)}{h}} \sum_{j=0}^J a_{m,j} \left(\frac{r^2}{2}\right) h^j$ with $e^{im\theta}$ and using the fact that

$$\mathcal{K}_h(e^{im\theta}u) = \mathfrak{L}_{h,m}(e^{im\theta}u),$$

we deduce Theorem 2.5 is easily deduced. \square

Corollary 4.8. *For all $(\varepsilon, m, J) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$, there exist a constant $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$(4.35) \quad \left\| e^{\varepsilon\varphi(\rho)/h} (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+2},$$

where

$$(4.36) \quad \lambda_{h,m}^J := h \sum_{j=0}^J \mu_{j,m} h^j \quad \text{and} \quad \Psi_{h,m}^J(\rho) := \chi e^{-\varphi(\rho)/h} \rho^{\frac{m}{2}} \left(\sum_{j=0}^J a_{j,m} h^j \right),$$

where χ is defined in (2.9).

In particular,

$$(4.37) \quad \left\| (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+2}.$$

We may now provide an approximation of the ground-state eigenfunction of the operator $\mathcal{N}_{h,m}$ by the WKB construction $\Psi_{h,m}^J$ defined in (4.36). Let $\Psi_{h,m}$ be an eigenfunction according to $\lambda_0(\mathcal{N}_{h,m})$, we introduce the orthogonal projection of $\Psi_{h,m}^J$ on the eigenspace of $\lambda_0(\mathcal{N}_{h,m})$

$$\Gamma_m \Psi_{h,m}^J = \langle \Psi_{h,m}^J, \Psi_{h,m} \rangle \Psi_{h,m}.$$

Corollary 4.9. *For all $(m, J) \in \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$(4.38) \quad \left\| \Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+1}.$$

4.6. A stronger WKB approximation. Let us recall the expression of the operator $\mathcal{N}_{h,m}$

$$\mathcal{N}_{h,m} = -2h^2 \partial_\rho \rho \partial_\rho + \tilde{V}_{h,m}(\rho),$$

where

$$\tilde{V}_{h,m}(\rho) := \frac{(hm - a(\rho))^2}{2\rho}.$$

Proposition 4.10. *Let $m \in \mathbb{N}$ and let $(\Phi_k)_{k \in \mathbb{N}} \subset W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$. Assume that there exist $M > 0$, $K_1 > 0$, $K_2 > 0$ and $R_0 > 0$ such that for all $h \in (0, 1)$, $k \in \mathbb{N}$*

$$(4.39) \quad \tilde{V}_{h,m}(\rho) - 2\rho |\Phi'_k(\rho)|^2 \geq Mh \quad \text{for all } \rho \in [R_0h, +\infty),$$

$$(4.40) \quad |\Phi'_k(\rho)| \leq K_1, \quad |\Phi_k(\rho)| \leq K_2h \quad \text{for all } \rho \in [0, R_0h].$$

Then, for all $c_0 \in (0, M)$, there exists a positive constant $C > 0$ such that, for all $h \in (0, 1)$, $k \in \mathbb{N}$, $z \in [0, c_0h]$, and $u \in \text{Dom}(\mathcal{N}_{h,m})$,

$$(4.41) \quad \|e^{\Phi_k/h} u\|_{L^2(\mathbb{R}^+)} \leq \frac{C}{h} \|e^{\Phi_k/h} (\mathcal{N}_{h,m} - z) u\|_{L^2(\mathbb{R}^+)} + C \|u\|_{L^2(\mathbb{R}^+)}.$$

Proof. We have

$$(4.42) \quad \langle \mathcal{N}_{h,m} u, e^{2\Phi_k/h} u \rangle_{L^2(\mathbb{R}^+)} = 2h^2 \langle \sqrt{\rho} \partial_\rho u, \sqrt{\rho} \partial_\rho (e^{2\Phi_k/h} u) \rangle_{L^2(\mathbb{R}^+)} + \int_0^\infty \tilde{V}_{h,m}(\rho) e^{2\Phi_k/h} |u|^2 d\rho.$$

Setting $P = \sqrt{\rho} \partial_\rho$, we get

$$\text{Re} \left(\langle Pu, P e^{2\Phi_k/h} u \rangle_{L^2(\mathbb{R}^+)} \right) = \|P e^{\Phi_k/h} u\|_{L^2(\mathbb{R}^+)}^2 - \|[P, e^{\Phi_k/h}] u\|_{L^2(\mathbb{R}^+)}^2.$$

Noticing that $[P, e^{\Phi_k/h}] = \frac{\sqrt{\rho}\Phi'_k}{h}e^{\Phi_k/h}$ and (4.42) gives

$$\begin{aligned} \operatorname{Re} \langle \mathcal{N}_{h,m}u, e^{2\Phi_k/h}u \rangle &= 2h^2 \int_0^{+\infty} \rho |\partial_\rho(e^{\Phi_k/h}u)|^2 d\rho \\ &\quad + \int_0^{+\infty} \left(\tilde{V}_{h,m} - 2\rho|\Phi'_k(\rho)|^2 \right) e^{2\Phi_k/h}|u|^2 d\rho. \end{aligned}$$

Since $\tilde{V}_{h,m}(\rho) \geq 0$, we get

$$\begin{aligned} \int_{R_0h}^{+\infty} \left(\tilde{V}_{h,m} - 2\rho|\Phi'_k(\rho)|^2 \right) |e^{\Phi_k/h}u|^2 d\rho &\leq \|e^{\Phi_k/h}\mathcal{N}_{h,m}u\|_{L^2(\mathbb{R}^+)} \|e^{\Phi_k/h}u\|_{L^2(\mathbb{R}^+)} \\ &\quad + \int_0^{R_0h} 2\rho|\Phi'_k(\rho)|^2 e^{2\Phi_k/h}|u|^2 d\rho. \end{aligned}$$

Using (4.39), we deduce that

$$\begin{aligned} Mh \int_{R_0h}^{+\infty} |e^{\Phi_k/h}u|^2 d\rho &\leq \|e^{\Phi_k/h}\mathcal{N}_{h,m}u\|_{L^2(\mathbb{R}^+)} \|e^{\Phi_k/h}u\|_{L^2(\mathbb{R}^+)} \\ &\quad + \int_0^{R_0h} 2\rho|\Phi'_k(\rho)|^2 e^{2\Phi_k/h}|u|^2 d\rho. \end{aligned}$$

Thanks to (4.40), Φ_k/h and Φ'_k are uniformly bounded with respect to h and to k on $[0, R_0h]$. Therefore, there exists a constant $L > 0$ (independent of h and k) such that

$$Mh \int_0^{+\infty} |e^{\Phi_k/h}u|^2 d\rho \leq \|e^{\Phi_k/h}\mathcal{N}_{h,m}u\|_{L^2(\mathbb{R}^+)} \|e^{\Phi_k/h}u\|_{L^2(\mathbb{R}^+)} + Lh \int_0^{R_0h} |u|^2 d\rho.$$

For $z \in [0, c_0h]$, we get

$$(M - c_0)h \|e^{\Phi_k/h}u\|_{L^2(\mathbb{R}^+)}^2 \leq \|e^{\Phi_k/h}(\mathcal{N}_{h,m} - z)u\|_{L^2(\mathbb{R}^+)} \|e^{\Phi_k/h}u\|_{L^2(\mathbb{R}^+)} + Lh \|u\|_{L^2(\mathbb{R}^+)}^2.$$

Since $M > c_0$, this gives (4.41). \square

The first application of the above Agmon estimate is to prove the decay of the eigenfunctions.

Theorem 4.11. *For all $\varepsilon \in (0, 1)$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$ and all eigenfunctions Ψ with eigenvalue of order h of the operator $\mathcal{N}_{h,m}$,*

$$(4.43) \quad \|e^{\varepsilon\varphi/h}\Psi\|_{L^2(\mathbb{R}^+)} \leq C \|\Psi\|_{L^2(\mathbb{R}^+)},$$

where $\varphi(\rho) = \int_0^\rho \frac{a(\tau)}{2\tau} d\tau$ is given by (4.34).

Proof. Let $(\chi_k)_{k \in \mathbb{N}}$ be a sequence of functions as in the proof of Proposition (4.5). In order to apply Proposition 4.10, we consider

$$\Phi_k(\rho) = \varepsilon\chi_k(\varphi(\rho)).$$

For each $k \in \mathbb{N}$, we have $\Phi_k \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$. Furthermore, one has

$$|\Phi'_k(\rho)| \leq \varepsilon|\varphi'(\rho)| = \frac{\varepsilon a(\rho)}{2\rho} \quad \text{a.e. on } \mathbb{R}_+.$$

Let us consider an eigenvalue $\lambda = (\mathcal{O}(h))$ and an associated eigenfunction Ψ . Then, there exist $c_0 > 0$ and $h_0 > 0$ such that

$$|\lambda| \leq c_0h \quad \text{for all } h \in (0, h_0).$$

Let M and R_0 be numbers such that

$$\begin{cases} M > c_0, \\ R_0 \geq \frac{2\beta(0)m + 2M}{\beta(0)^2(1 - \varepsilon^2)}. \end{cases}$$

Using (2.6), we have $a(\rho) \geq b_0\rho$ for all $\rho \in \mathbb{R}_+$. From the definition of R_0 , we have the estimate, for all $h \in (0, h_0)$, $k \in \mathbb{N}$ and $\rho \geq R_0h$,

$$\begin{aligned} \tilde{V}_{h,m}(\rho) - 2\rho|\Phi'_k(\rho)|^2 &\geq \tilde{V}_{h,m}(\rho) - \varepsilon^2 \frac{a^2(\rho)}{2\rho} \\ &\geq \frac{(1 - \varepsilon^2) \left(\beta(0)\rho - \frac{hm}{1-\varepsilon^2} \right)^2 - \frac{h^2m^2}{1-\varepsilon^2}}{2\rho} \\ &\geq \left(\frac{(1 - \varepsilon^2)\beta(0)^2 R_0}{2} - \beta(0)m \right) h \\ &\geq Mh. \end{aligned}$$

On the other hand, there exist $K_1 > 0$ and $K_2 > 0$ such that, for all $h \in (0, h_0)$, $k \in \mathbb{N}$ and $\rho \in [0, R_0h)$,

$$|\Phi'_k(\rho)| \leq \frac{\varepsilon a(\rho)}{2\rho} = \frac{\varepsilon}{2} \int_0^1 \beta(\rho s) ds \leq K_1$$

and

$$|\Phi_k(\rho)| \leq \varepsilon \phi(\rho) = \varepsilon \int_0^\rho \int_0^1 \beta(\tau s) ds d\tau \leq K_2 h.$$

Now, we can apply Proposition 4.10 for $z = \lambda$, there exists a constant $C > 0$ such that, for all eigenfunction Ψ associated with λ ,

$$\int_0^{+\infty} e^{2\varepsilon\chi_k(\varphi/h)} |\Psi|^2 d\rho \leq C \int_0^{+\infty} |\Psi|^2 d\rho.$$

By letting $k \rightarrow \infty$ and using Fatou's lemma, we get

$$\int_0^{+\infty} e^{2\varepsilon\varphi/h} |\Psi|^2 d\rho \leq C \int_0^{+\infty} |\Psi|^2 d\rho.$$

□

Proof of Theorem 2.8. Let $T : L^2(\mathbb{R}^2, dq) \rightarrow L^2(\mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}, r dr d\theta)$ be the unitary operator associated with the polar coordinates. Then, $T(U_{h,m})$ is the eigenfunction associated with the eigenvalue $\lambda_m(\mathcal{K}_h) = \lambda_m(\mathcal{L}_{h,\mathbf{A}})$ of the operator \mathcal{K}_h . From Proposition 4.7,

$$T(U_{h,m}) = \frac{1}{\sqrt{2\pi}} e^{-im\theta} \Psi_{h,m} \left(\frac{r^2}{2} \right),$$

where $\Psi_{h,m}$ is a eigenfunction associated with the first eigenvalue $\lambda_0(\mathcal{N}_{h,m})$ of the operator $\mathcal{N}_{h,m}$. Theorem 2.8 easily follows. □

Theorem 4.12. *For all $(\varepsilon, m, J) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$(4.44) \quad \|e^{\varepsilon\varphi(\rho)/h} (\Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J)\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+1},$$

where $\varphi(\rho) = \int_0^\rho \frac{a(\tau)}{2\tau} d\tau$ is given by (4.34).

Proof. Let us fix $(\varepsilon, m, J) \in (0, 1) \times \mathbb{N} \times \mathbb{N}$. We recall that $\Gamma_m \Psi_{h,m}^J$ is the eigenfunction with the eigenvalue $\lambda_0(\mathcal{N}_{h,m})$ that has order h . We consider again the sequence $(\Phi_k)_{k \in \mathbb{N}}$. Applying Proposition 4.10 to $u = \Psi_{h,m}^J - \Gamma_m \Psi_{h,m}^J$ we find

$$(4.45) \quad \|e^{\Phi_k/h} u\| \leq \frac{C}{h} \|e^{\Phi_k/h} (\mathcal{N}_{h,m} - \lambda_0(\mathcal{N}_{h,m})) u\| + C \|u\|.$$

Thanks to (4.35), we have

$$\begin{aligned}
& \left\| e^{\Phi_k/h} (\mathcal{N}_{h,m} - \lambda_0(\mathcal{N}_{h,m})) u \right\|_{L^2(\mathbb{R}^+)} \\
& \leq \left\| e^{\Phi_k/h} (\mathcal{N}_{h,m} - \lambda_{h,m}^J) \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} + |\lambda_0(\mathcal{N}_{h,m}) - \lambda_{h,m}^J| \left\| e^{\Phi_k/h} \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \\
& \leq Ch^{J+2} + Ch^{J+1} |\lambda_0(\mathcal{N}_{h,m}) - \lambda_{h,m}^J| + C |\lambda_0(\mathcal{N}_{h,m}) - \lambda_{h,m}^J| \left\| \Psi_{h,m}^J \right\|_{L^2(\mathbb{R}^+)} \\
& \leq Ch^{J+2}.
\end{aligned}$$

Using Corollary 4.9 and (4.45), we get

$$\left\| e^{\varepsilon\chi_k(\varphi(\rho))/h} u \right\|_{L^2(\mathbb{R}^+)} \leq Ch^{J+1},$$

for all $k \geq 1$. Then, we take to limit $k \rightarrow +\infty$ and use Fatou's lemma. \square

APPENDIX A. SPECTRUM OF THE LAGUERRE OPERATOR

This appendix is devoted to the Laguerre operators

$$\mathcal{T}_m = -2s\partial_s^2 + (2s - 2 - 2|m|)\partial_s + |m| - m + 1,$$

for each $m \in \mathbb{Z}$. We denote by $L_n^{(m)}$ the generalized Laguerre polynomials: these are solutions of the differential equation

$$(A.1) \quad s\partial_s^2 y + (|m| + 1 - s)\partial_s y + ny = 0,$$

with $n \in \mathbb{Z}$, see [21]. Then, for each $m \in \mathbb{Z}$, we have

$$(A.2) \quad \mathcal{T}_m(L_n^{(m)}) = (2n + 1 + |m| - m)L_n^{(m)}.$$

In particular,

$$(A.3) \quad \{2n + |m| - m + 1 : n \in \mathbb{N}\} \subset \text{sp}(\mathcal{T}_m).$$

These polynomials are orthogonal with the inner product of space $L^2(\mathbb{R}^+, s^{|m|}e^{-s}ds)$ and satisfy

$$\int_0^{+\infty} L_k^{(m)}(s)L_n^{(m)}(s) s^{|m|}e^{-s}ds = \frac{\Gamma(n + |m| + 1)}{n!} \delta_{k,n},$$

where $\delta_{k,n}$ denotes Kronecker symbol.

Theorem A.1. *For each $m \in \mathbb{Z}$, the family $(L_n^{(m)})_{n \in \mathbb{N}}$ is total in $L^2(\mathbb{R}^+, s^{|m|}e^{-s}ds)$. Moreover, the spectrum of the operator \mathcal{T}_m is*

$$\text{Sp}(\mathcal{T}_m) = \{2k + 1 + |m| - m : k \in \mathbb{N}\}.$$

APPENDIX B. ABOUT THE DETERMINANT AND THE TRACE OF THE HESSIAN

From the definition of isothermal coordinates, we have

$$g_{p^*}(U, V) = e^{2\eta(0)} g_0(d\phi_{p^*}U, d\phi_{p^*}V), \quad \text{for all } U, V \in T_{p^*}M.$$

In the matrix expression, we have

$$(B.1) \quad G_{p^*} = e^{2\eta(0)} (D\phi)_{p^*}^T (D\phi)_{p^*},$$

where $(D\phi)_{p^*}$ is the matrix of $d\phi_{p^*}$. The relation between the Hessian of \mathbf{B} on manifold and the Hessian of $\mathcal{B} = B \circ \phi^{-1}$ is given by

$$(B.2) \quad d^2B_{p^*}(V_1, V_2) = \langle \text{Hess}\mathcal{B}(0)d\phi_{p^*}V_1, d\phi_{p^*}V_2 \rangle_{\mathbb{R}^2}.$$

In order to compute the trace and determinant of the Hessian at p^* , we consider the endomorphism \mathcal{H} of $T_{p^*}M$ defined by

$$(B.3) \quad (d^2B)_{p^*}(V_1, V_2) = g_{p^*}(\mathcal{H}V_1, V_2) \quad \forall V_1, V_2 \in T_{p^*}M.$$

Additionally, (B.2) and (B.3) imply that

$$(B.4) \quad \langle \text{Hess}\mathcal{B}(0)d\phi_{p^*}V_1, d\phi_{p^*}V_2 \rangle_{\mathbb{R}^2} = g_{p^*}(\mathcal{H}V_1, V_2) \quad \forall V_1, V_2 \in T_{p^*}M,$$

or

$$(D\phi)_{p^*}^T \text{Hess } \mathcal{B}(0) (D\phi)_{p^*} = \mathcal{H}G_p.$$

Using (B.1), we get

$$(D\phi)_{p^*}^T \text{Hess } \mathcal{B}(0) [(D\phi)_{p^*}^T]^{-1} = e^{2\eta(0)} \mathcal{H}.$$

Notice that

$$\text{Hess } \mathcal{B}(0) = \begin{pmatrix} 2\alpha & 0 \\ 0 & 2\gamma \end{pmatrix}.$$

Let $H = \frac{1}{2}\mathcal{H}$, then we can easily compute the determinant of H and the trace of $H^{\frac{1}{2}}$:

$$\det(H) = \frac{e^{-4\eta(0)}}{4} \det(\text{Hess } \mathcal{B}(0)) = e^{-4\eta(0)} \alpha \gamma,$$

and

$$\begin{aligned} \text{Tr } H^{1/2} &= \text{Tr} \left[\frac{e^{-\eta(0)}}{\sqrt{2}} (D\phi)_{p^*}^T (\text{Hess } \mathcal{B}(0))^{1/2} [(D\phi)_{p^*}^T]^{-1} \right] \\ &= \frac{e^{-\eta(0)}}{\sqrt{2}} \text{Tr} [(\text{Hess } \mathcal{B}(0))^{1/2}] \\ &= e^{-\eta(0)} (\sqrt{\alpha} + \sqrt{\gamma}). \end{aligned}$$

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(Y. Guedes Bonthonneau) UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE
Email address: `yannick.bonthonneau@univ-rennes1.fr`

(T. Nguyen Duc) UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE
Email address: `duc-tho.nguyen@univ-rennes1.fr`

(N. Raymond) LABORATOIRE ANGEVIN DE RECHERCHE EN MATHÉMATIQUES, LAREMA, UMR 6093,
UNIV ANGERS, SFR MATH-STIC, 2 BOULEVARD LAVOISIER 49045 ANGERS CEDEX 01, FRANCE
Email address: `nicolas.raymond@univ-angers.fr`

(S. Vũ Ngọc) UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE
Email address: `san.vu-ngoc@univ-rennes1.fr`