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A NON-LINEAR EGOROV THEOREM AND POINCARÉ-BIRKHOFF NORMAL FORMS FOR QUASI-LINEAR PDES ON THE CIRCLE

ROBERTO FEOLA AND FELICE IANDOLI

ABSTRACT. In this paper we consider an abstract class of quasi-linear para-differential equations on the circle. For each equation in the class we prove the existence of a change of coordinates which conjugates the equation to a diagonal and constant coefficient para-differential one. In the case of Hamiltonian equations we also put them in Poincaré-Birkhoff normal form. By means of the Bony para-linearization formula we prove that it is possible to apply this transformation to quasi-linear Hamiltonian perturbations of the Schrödinger and beam equations. In this way we obtain the first long time existence result without requiring any symmetry on the initial data in the case of quasi-linear PDEs with super-linear dispersion law. We also prove the local in time existence of solutions for quasi-linear (not necessarily Hamiltonian) perturbations of the Benjamin-Ono equation.

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1. INTRODUCTION

In the past years numerous progresses have been done concerning the study of quasi-linear, dispersive, evolution equations on the circle. The local in time solvability has been proven for several models, as well as the existence of large sets of periodic and quasi-periodic in time solutions. More recently it has been established the long time existence and stability of small amplitude solutions for the gravity-capillary water waves equation in [9] and for quasi-linear perturbations of the Schrödinger equation in [26].

Due to the compactness of the circle the dispersive character of the equation is absent in the following sense: the solutions of the linear equation do not decay when the time goes to infinity. On the other hand, thanks, among several other things, to such a compactness, in the last decade it has been developed a very fruitful and systematic way to study the linearization of such equations at certain approximate solutions. The linearized equation coming from a quasi-linear problem is much more complicated compared with the one coming from semi-linear equations because it has *variable coefficients*. In this direction the first breakthrough result is due to Plotnikov-Toland [32] and Ioos-Plotnikov-Toland [29], the new idea is to apply a suitable diffeomorphism of the circle and *pseudo-differential* changes of coordinates in order to invert the linearized operator at an approximate solution. This is done in the

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context of a Nash-Moser iterative scheme in order to prove the existence of periodic in time solutions. The strategy in [32, 29] has been improved by Baldi-Berti-Montalto [3, 4] in order to show the existence of *quasi-periodic* solutions for quasi-linear perturbations of the KdV equation. These ideas has been used also for several other equations, for instance one can look at [12, 27, 2, 28, 23]. A similar method has been successfully applied also the context of reducibility of linear operators. We quote for instance [24, 7, 22].

This procedure, which has allowed to obtain so many results in the study of linearized equation, has been, very recently, transported to a non-linear level by Berti-Delort in [9] obtaining the aforementioned long-time existence result. Inspired by this work, we have proved first the local in time well-posedness in [25] and then the long time stability in [26] for quasi-linear perturbations of the Schrödinger equation. In order to obtain long-time stability of the solutions on compact manifolds the only helpful approach, due to the lack of dispersion, is the *normal forms* one. For semi-linear equations we have, nowadays, a very good knowledge of the topic, see for instance [16, 17, 5, 6, 20]. This approach does not apply directly in the quasi-linear case because of the loss of derivatives introduced by the nonlinearity. In this direction it is fundamental to *reduce* the equation to a constant coefficient one before performing a *normal form* procedure. This problem has been extensively explained in the introductions of [9] and [26], to which we refer for more details.

The non-linear adaptation of such techniques is a difficult problem. The main tool used is the *para-differential* calculus. Roughly speaking one wants to try to mimic the changes of coordinates previously operated on the linearized equations, on the *para-linearized* equations à la Bony [13], (see also [31]). In [9, 26] the authors prove that the original equation may be *reduced* to a new one which has constant coefficients up to very regularizing terms. They do not find *changes of coordinates* of the phase space, but they provide some *modified energies* which, in any case, are sufficient to establish the wanted results. In the following we shall fix some notation and we shall provide an example in order to explain precisely all the concepts that have been previously mentioned and not been introduced yet.

The main purpose of this paper is to prove a theorem which provides the existence of a *change of coordinates* of the phase space which transforms a quite general *para-differential* equation on the circle to another one which has constant coefficients up to very regularizing terms (see Theorem 3.1). For a comprehensive introduction to the importance of Hamiltonian PDEs in mathematics and physics we refer to the books by Berti [8], Kuksin [30], Zhakarov [34]. As a consequence we are able to put in Poincaré-Birkhoff normal form a class of abstract *Hamiltonian* para-differential equations satisfying some non-resonance conditions. Consequently we shall obtain a *long time* existence theorem (in the same sense of [9, 26]) for the latter class of para-differential equations posed on $H^s(\mathbb{T})$. We will apply this last theorem to some explicit examples of quasi-linear dispersive PDEs (with super-linear dispersion law) on the circle, obtaining, to the best of our knowledge, the first long time existence result for an equation of this kind without assuming any symmetry on the initial data (in [9, 26] the parity of the initial condition is necessary). Let us mention that, recently, Berti-Feola-Pusateri in [11], and Berti-Feola-Franzoi in [10], prove some long time existence results for gravity and gravity-capillary water waves equation without symmetry assumptions on the initial data. In the quoted papers the authors exploit an *a posteriori* identification argument of normal form which allows them to compute the Birkhoff normal form at the fourth order. With our method we will be able to compute the Birkhoff normal form, preserving the Hamiltonian structure, at *any* order.

In Section 1.1 we introduce the main Theorem of this paper, in 1.2 we give some ideas on the proof and we make a comparison between the strategy of this paper and the one in [9, 26]. In Section 1.3 we shall define a class of *Hamiltonian* para-differential equation which we are able to put in Poincaré-Birkhoff normal form. We shall also give some examples of PDEs which satisfy the hypotheses we shall require. Finally in 1.4 of this introduction we explain how the paper is organized.

1.1. Introduction to the Egorov theorem. Let us introduce some notation and a model problem in order to give an informal statement of our main theorem. An accurate definition, with an extensive analysis,

of the para-differential operators is given in Section 2. We shall deal with symbols $\mathbb{T} \times \mathbb{R} \ni (x, \xi) \rightarrow a(x, \xi)$ with limited smoothness in x satisfying, for some $m \in \mathbb{R}$, the following estimate

$$|\partial_\xi^\beta a(x, \xi)| \leq C_\beta \langle \xi \rangle^{m-\beta}, \quad \forall \beta \in \mathbb{N},$$

where $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$. These functions will have limited smoothness in x because they will depend on x through the dynamical variable U of a para-differential equation which is in $H^s(\mathbb{T})$ for some s . From the symbol $a(x, \xi)$ one can define the *para-differential* operator $Op^B(a(x, \xi))[\cdot]$, acting on periodic functions of the form $u(x) = \sum_{j \in \mathbb{Z}} \hat{u}(j) \frac{e^{ijx}}{\sqrt{2\pi}}$, in the following way:

$$Op^B(a(x, \xi))[u] := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{ikx} \left(\sum_{j \in \mathbb{Z}} \chi\left(\frac{k-j}{\langle j \rangle}\right) \hat{a}(k-j, j) \hat{u}(j) \right),$$

where $\hat{a}(k, j)$ is the k^{th} -Fourier coefficient of the 2π -periodic in x function $a(x, \xi)$, and where $\chi(\eta)$ is a C_0^∞ function supported in a sufficiently small neighborhood of the origin. This is the standard para-differential quantization of a symbol, in the paper we shall use the Weyl one, see formula (2.26), in this introduction we preferred to use it in order to simplify the presentation.

We consider a para-differential equation of the form

$$\partial_t U = iE\mathcal{A}(U; x)U + \mathcal{R}(U)U, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (1.1)$$

where $U = (u, \bar{u})$, $R(U)$ is a 2×2 -matrix of smoothing operators and \mathcal{A} is a 2×2 -matrix of para-differential operators of order $m > 1$. We assume that the initial datum is of the form $U(0) = U_0 = (u_0, \bar{u}_0)$, where u_0 belongs to a classical Sobolev space $H^s(\mathbb{T}; \mathbb{C})$ for s big enough, the smoothing remainder maps H^s in $H^{s+\rho}$ for a large number $\rho > 0$. We shall make in Section 3 some precise hypotheses on the matrix $\mathcal{A}(U; x)$, we do not recall them here in the complete generality. For simplicity one can think that $\mathcal{A}(U; x)$ is an elliptic, self-adjoint matrix of para-differential operators. We assume moreover that the matrices of operators $\mathcal{A}(U; x)[\cdot]$ and $\mathcal{R}(U)(\cdot)$ are *reality preserving*, i.e. they leave invariant the *real subspace* of $(C^\infty(\mathbb{T}; \mathbb{C}))^2$ made of those couples having the form (u, \bar{u}) .

We shall prove that there exists a non-linear map Ψ , which is *bounded* and *invertible* in a small neighborhood of the origin of H^s such that $Z := \Psi(U)$ solves the following problem

$$\partial_t Z = iE\mathcal{D}(Z)Z + \tilde{\mathcal{R}}(Z)Z, \quad (1.2)$$

where $\mathcal{D}(Z)$ is a reality preserving and *diagonal matrix*, whose entries are real and *constant in x* non-linear functions in Z ; $\tilde{\mathcal{R}}(Z)$ is a reality preserving matrix of smoothing operators mapping H^s in $H^{s+\tilde{\rho}}$ with $\tilde{\rho} > 0$ possibly smaller than ρ . This is essentially a rough statement of our main result which is Theorem 3.1. The hypotheses of Theorem 3.1 are slightly milder, it is not strictly necessary to require the self-adjointness of the matrix $\mathcal{A}(U; x)$, see (3.3) and (3.5).

To clarify things one can think about the following concrete example. Consider the quasi-linear perturbation of the Schrödinger equation

$$\partial_t u = i(1 + |u|^2)u_{xx}. \quad (1.3)$$

By applying the celebrated para-linearization formula of Bony one finds that such an equation is equivalent to the following system

$$\partial_t U = iEOp^B(A(U; x, \xi))U + R(U),$$

where the matrix $A(U; x, \xi)$ has the following form

$$A(U; x, \xi) = \begin{pmatrix} a(U; x, \xi) & b(U; x) \\ \bar{b}(U; x) & a(U; x, \xi) \end{pmatrix}, \quad \begin{cases} a(U; x, \xi) = (1 + |u|^2)(i\xi)^2 + \bar{u}u_{xx} \\ b(U; x, \xi) = uu_{xx}, \end{cases}$$

and the matrix R is a matrix of smoothing operators. In this case our theorem applies and provides a change of variable $Z = \Psi(U)$ such that the system in the new coordinates reads as

$$\partial_t Z = iEOp^B\left(A^+(Z)(i\xi^2)\right)Z + \ell.o.t., \quad A^+(Z) := \begin{pmatrix} a^+(Z) & 0 \\ 0 & a^+(Z) \end{pmatrix}, \quad (1.4)$$

and the function on the diagonal has the form

$$a^+(Z) := \int_{\mathbb{T}} f(Z) dx \in \mathbb{R}$$

for some regular and *real* function f . Essentially we have reduced the equation to a *Kirchhoff-type* one in the following sense. In the new coordinates system the equation is still a *quasi-linear* one, but the coefficients of the para-differential operators are constants. In this way the system enjoys a property which is peculiar of *semi-linear* equations: the linearized equation at any solution has constant coefficients at highest order. We are confident that it would be convenient to start a KAM procedure from the regularized system (1.4), instead of the original system (1.1), because it would be easier to study the invertibility of the linearized operator at periodic/quasi-periodic solutions. This is an advantage of having produced changes of coordinates of the phase space instead of modified energies.

A deeper advantage of our method is the following. In the case that (1.1) is an Hamiltonian equation (in the sense specified in section 1.3) it is possible to find a symplectic version of the aforementioned change of coordinates. In such a way the new system (1.4) is still Hamiltonian. Moreover, since it is reduced to constant coefficients up to smoothing reminders, it is possible to perform a Poincaré-Birkhoff normal argument by means, again, of a symplectic change of coordinates of the phase space. In this way standard arguments on *non-resonant* systems provide control on the growth of Sobolev norms for long time (for a detailed introduction on this topic see 1.3). In this way we obtain, to the best of our knowledge, the first results (in the case of super-linear dispersion law, for the linear see [15, 14]) of long time existence for quasi-linear equations without assuming any symmetry on the initial condition. This is a consequence of the fact that we are able to fully exploit the *Hamiltonian structure*. We shall consider quasi-linear Hamiltonian perturbation of the Schrödinger and beam equation and prove a long time existence in the spirit of [9, 26] but without the parity assumption therein. Besides these applications to long time solutions of Hamiltonian equations, we think that the Egorov theorem 3.1 is important *per se*, in order to stress this fact we prove a local well posedness theorem for quasi-linear, not necessarily Hamiltonian, perturbations of the Benjamin-Ono equation. This is deduced from the fact that a system like (1.4) admits energy inequality on $H^s(\mathbb{T})$. In a forthcoming paper we want to apply this theorem also to the gravity-capillary water waves system in order to obtain a long time result without any symmetry on the initial datum, the proof of this fact is much more involved compared with the examples we provide here.

1.2. Ideas of the proof of the Egorov theorem. The previously mentioned non-linear change of coordinates $Z = \Psi(U)$ will be obtained as the composition of numerous changes of coordinates. Each change of coordinates, let us call it ψ , will be generated as the *flow* of a para-differential operator as follows

$$\begin{cases} \partial_\tau \psi^\tau = G^\tau(\psi^\tau)\psi^\tau \\ \psi^0 = \text{Id}, \end{cases} \quad (1.5)$$

where G^τ is some non-linear vector field possibly depending explicitly on τ . We shall define $\psi := \psi^\tau|_{\tau=1}$. Note that the equation above is a non-linear para-differential equation, whose well-posedness has to be analyzed, this is done in Section 2.5. Then defining $Z := \psi(U)$ one has that the new unknown Z solves a system of the form

$$\partial_t Z = P^1(Z),$$

where $P^\tau(Z)$, for $\tau \in [0, 1]$, is the solution of the non-linear Heisenberg equation

$$\begin{cases} \partial_\tau P^\tau(Z) = [G^\tau(Z), P^\tau(Z)] \\ P^0(Z) = iE\mathcal{A}(Z; x)Z + \mathcal{R}(Z)Z, \end{cases} \quad (1.6)$$

where we have denoted by $[\cdot, \cdot]$ the non-linear commutator. More precisely, given two vector fields $X(u)$ and $Y(u)$, we define the non-linear commutator between the two as

$$[X, Y](u) := dX(u)[Y(u)] - dY(u)[X(u)]. \quad (1.7)$$

We have chosen to generate changes of coordinates in this way because there is a systematic method to prove that the equation for the new variable Z is still a para-differential one. For more details one should look at Section 5.

In Section 5 we perform all the changes of coordinates. First of all one has to diagonalize the matrix of symbols $\mathcal{A}(U)$, this is done by applying several changes of coordinates which diagonalize $\mathcal{A}(U)$ at each order, this is the content of Section 5.2.

Once achieved the diagonalization of $\mathcal{A}(U)$ we shall perform further changes of variables in order to put to constant coefficient the symbol at each order appearing on the diagonal of the new system, this is the content of Section 5.1. Among all the changes of coordinates the most difficult is the one which is needed to remove the x -dependence from the symbol at the highest order. We explain here the strategy we adopted for this step, this may be considered as a guideline also for all the other changes of coordinates which are performed in Section 5. To clarify things we explain the idea, having in mind equation (1.3), on the following toy model

$$\partial_t u = iOp^B((1 + |u|^2)(i\xi)^2)u. \quad (1.8)$$

We define $z = \psi^\tau(u)_{\tau=1}$, where ψ^τ solves (1.5) with $G^\tau(u) := iOp^B(b(\tau, u; x)\xi)u$ and $b(\tau, u; x)$ is a path of symbols with $\tau \in [0, 1]$. Then the equation in the new variable is $\partial_\tau z = P^1(z)$, where P^τ solves the Heisenberg equation

$$\begin{cases} \partial_\tau P^\tau(z) = [iOp^B(b(\tau, z; x)\xi)z, P^\tau(z)] \\ P^0(z) = iOp^B(1 + |z|^2)(i\xi)^2z. \end{cases}$$

At this point one can make the ansatz that P^τ is of the form $Op^B(a^+(\tau; z, x, \xi))z + \ell.o.t.$ for a symbol a^+ of order 2. By developing the commutator, by means of symbolic calculus (see Prop. 2.25), one obtains that the equation for the highest order symbol $a^+(\tau; z, x, \xi)$ is the following

$$\begin{cases} \partial_\tau a^+(\tau, z; x, \xi) = \{b(\tau, z; x)\xi, a^+(\tau, z; x, \xi)\} - d_z a^+(\tau, z; x, \xi)[Op^B(ib(\tau, z; x, \xi))[z]] \\ a^+(0; z, x, \xi) = (1 + |z|^2)(i\xi)^2, \end{cases} \quad (1.9)$$

where $\{f, g\} := \partial_\xi f \partial_x g - \partial_x f \partial_\xi g$ denotes the Poisson bracket between function. Such an equation is a non-linear transport one. In order to solve it we proceed as follows. Define the function

$$g(\tau) = a^+(\tau, z(\tau); x(\tau), \xi(\tau))$$

and note that this function is constant along the flow of the following non-linear system

$$\begin{cases} \partial_\tau x(\tau) = -b(\tau, z(\tau); x(\tau)) \\ \partial_\tau \xi(\tau) = b_x(\tau, z(\tau); x(\tau))\xi(\tau) \\ \partial_\tau z(\tau) = Op^B(ib(\tau, z(\tau); x(\tau))\xi(\tau))[z(\tau)]. \end{cases} \quad (1.10)$$

Note that the system above is a system of non-linear coupled equations, its well-posedness is not trivial and it is discussed in Section 4.1. Denote by

$$\tilde{Y}_b^\tau(z, x, \xi) := \left(\tilde{Y}_b^{(z)}, \tilde{Y}_b^{(x)}, \tilde{Y}_b^{(\xi)} \right)(\tau, z, x, \xi)$$

the inverse of the flow of system (1.10), then the solution of equation (1.9) is

$$a^+(z, x, \xi) = \left(1 + \left| \tilde{\Upsilon}_b^{(z)} \left(\tilde{\Upsilon}_b^{(x)} \right) \right|^2\right) \left(i \tilde{\Upsilon}_b^{(\xi)} \right)^2. \quad (1.11)$$

We want to find a function $b(\tau, z; x)$ and a constant (w.r.t. x) m_b such that one solves the equation $a^+(z, x, \xi) = m_b (i\xi)^2$. This is another non-linear and implicit equation which is solved in Theorem 4.2. Choosing such a b we have removed the x -dependence on the highest order term.

We are now in position to make a short comparison between this way of generating changes of coordinates and the ones used in [9, 26]. In those papers the authors look for *modified energies*, more precisely, starting from a solution $U(t, x)$ of the equation (1.1), they want to define a new unknown $W = \Phi(U)U$ in the following way. One requires that $\|W\|_{L^\infty H^s} \sim \|U\|_{L^\infty H^s}$ and that the time-dependent map $\Phi(U)[\cdot]$ is linearly invertible in a small neighborhood of the origin of the space of continuous function in time with values in H^s for any *fixed* function $U(t, x)$. The map $\Phi(U)[\cdot]$ is chosen in such a way that the equation for the new unknown W has constant coefficient in x . By using the existing results on the local well-posedness of the equations, they may recover information on the original variable U . Note that these modified energies make sense only in the case that one already owns a local Cauchy theory for the equation. An advantage of performing a complete change of coordinates is that we do not need any *a priori* local well-posedness result. As we shall see, these changes of coordinates can actually be used in order to prove local well-posedness results for several quasi-linear equations. Let us suppose that U is a solution of (1.1), then equation solved by the new unknown (*à la* Berti-Delort) $W = \Phi(U)U$ takes the following form

$$\begin{aligned} \partial_t W &= \Phi(U)U_t + [\partial_t \Phi(U)]U + \ell.o.t. \\ &= \Phi(U) iE\mathcal{A}(U; x)(\Phi(U))^{-1}W + (\partial_t \Phi(U))[(\Phi(U))^{-1}W] + \ell.o.t.. \end{aligned} \quad (1.12)$$

If, for instance, the operator $\Phi(U)[\cdot]$ is a para-differential one, then a time derivative falls on the symbol, this fact has to be taken into account in the definition of symbols in [9, 26]. In analogy we shall take into account that a differential with respect to the variable U falls on the symbols, as one can see from equations (1.6) and (1.7). Further comments on these differences are given in Section 2. In the case of the modified energies, since one looks only for a *linear invertibility* of the map $\Phi(U)[\cdot]$, one can realize it as the flow of a linear operator, for instance as follows

$$\begin{cases} \partial_\tau \Phi(\tau) = iOp^B(F(U; x, \xi))\Phi(\tau) \\ \Phi(\tau) = \text{Id}, \end{cases} \quad (1.13)$$

whose well-posedness is easier to discuss compared to the non-linear equation (1.5). (Note that here one has to reason at *fixed* U , instead by following our strategy one would have found the same equation with $U \rightsquigarrow \Phi(\tau)$). It turns out that, being the equations super-linear, the highest order term in the right hand side of (1.12) is the first one $Q := \Phi(U) iE\mathcal{A}(U; x)(\Phi(U))^{-1}$. Moreover one has that $Q = Q^\tau|_{\tau=1}$, where Q^τ solves the *linear* Heisenberg equation

$$\begin{cases} \partial_\tau Q^\tau(U)[\cdot] = \left[iOp^B(F(U; x, \xi))[\cdot], Q^\tau(U)[\cdot] \right]_- \\ Q^0(U)[\cdot] = iE\mathcal{A}(U; x)[\cdot], \end{cases} \quad (1.14)$$

where we denoted by $[\cdot, \cdot]_-$ the commutator between two linear operators. Consider again the toy model in (1.8), let us see how the procedure changes if one looks only for a modified energy in order to put the system to constant coefficients. One should consider the flow in (1.13) with $F(U; x, \xi) := b(u; x)\xi$ with $b \in \mathbb{R}$. Then the highest order operator in the r.h.s. of (1.12) is of the form $Q^\tau(u)w = Op^B(a^+(u; x)(i\xi)^2)w + \ell.o.t..$ By developing the linear commutators in (1.14) one finds that the equation solved by the new symbol $a^+(u; x)$ is

$$\begin{cases} \partial_\tau a^+(\tau, u; x, \xi) = \{b(\tau, u; x)\xi, a^+(\tau, u; x, \xi)\} \\ a^+(0; u, x, \xi) = (1 + |u|^2)(i\xi)^2. \end{cases} \quad (1.15)$$

For any *fixed* u one notes that the function

$$g(\tau, u; x, \xi) = a^+(\tau, u; x(\tau), \xi(\tau))$$

is constant along the solutions of the system

$$\begin{cases} \partial_\tau x(\tau) = -b(\tau, u; x(\tau)) \\ \partial_\tau \xi(\tau) = b_x(\tau, u; x(\tau))\xi(\tau), \end{cases} \quad (1.16)$$

which is different from (1.10) because here the function u is fixed and it does not depend on τ . At this point one denotes by $(\tilde{G}_b^{(x)}, \tilde{G}_b^{(\xi)})$ the inverse flow of (1.16) and finds the solutions of (1.15) as

$$a^+(\tau, u; x, \xi) = \left(1 + \left|u\left(\tilde{G}_b^{(x)}(\tau)\right)\right|^2\right) \left(i\tilde{G}_b^{(\xi)}(\tau)\right)^2. \quad (1.17)$$

One could show that at the first order of homogeneity the symbol in (1.17) and the one in (1.11) coincide by taking the Taylor expansion at $\tau = 0$. In other words the symbol *à la Berti-Delort* in (1.17) is an approximation of the symbol we find through changes of coordinates in (1.11) up to terms of higher homogeneity.

1.3. Poincaré-Birkhoff normal forms and applications to quasi-linear PDEs. The main consequence of the Egorov regularization theorem 3.1 we are interested in putting in Poincaré-Birkhoff normal forms a class of Hamiltonian equations in the following sense. We consider system of the form

$$\partial_t U = X_H(U), \quad X_H(U) = \begin{pmatrix} i\partial_{\bar{u}}H \\ -i\partial_u H \end{pmatrix} = iJ\nabla H, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (1.18)$$

where H is an Hamiltonian function of the form

$$H(U) = \int_{\mathbb{T}} \Omega u \cdot \bar{u} dx + \int_{\mathbb{T}} F(U) dx,$$

where Ω is a linear *pseudo-differential* operator satisfying some *non-resonance* conditions at order N (see Definition 3.4) and F is a non-linear, possibly unbounded, operator on H^s . The symplectic structure is the one induced by the non-degenerate symplectic form

$$\lambda(U, V) := \int_{\mathbb{T}} U \cdot iJV dx = \int_{\mathbb{T}} i(u\bar{v} - \bar{u}v) dx \quad (1.19)$$

on the space made by the couples $U = [\frac{u}{\bar{u}}]$, $V = [\frac{v}{\bar{v}}]$. The Poisson brackets between two Hamiltonian H, G are defined as

$$\{G, H\} := \lambda(X_G, X_H) \stackrel{(1.19)}{=} \int iJ\nabla G \cdot \nabla H dx = i \int \partial_u H \partial_{\bar{u}} G - \partial_{\bar{u}} H \partial_u G dx. \quad (1.20)$$

We have that

$$[X_G, X_H] = -X_{\{G, H\}}, \quad (1.21)$$

where $[\cdot, \cdot]$ is defined in (1.7).

We introduce informally our result. Assume that the system (1.18) satisfy the hypotheses of Theorem 3.1, then there exists a change of coordinates $\psi(U) = W$ such that the equation in the new variables reads

$$\partial_t W = iE\Omega W + X_{G_{res}}(W) + X_{\geq N}(W), \quad (1.22)$$

where $X_{G_{res}}(W)$ is the Hamiltonian vector-field of a *resonant* Hamiltonian $G_{res}(W)$, i.e.

$$\left\{G_{res}(W), \int_{\mathbb{T}} \Omega W \cdot \bar{W} dx\right\} = 0,$$

where the Poisson parenthesis are defined in (1.20) and $X_{\geq N}(W)$ is an Hamiltonian vector-field satisfying

$$\operatorname{Re} \left(\langle D \rangle^s X_{\geq N}(W), \langle D \rangle^s W \right)_{L^2} \leq \|W\|_{H^s}^{N+2},$$

where $\langle D \rangle$ is the Fourier multiplier defined by linearity in (2.4). We say that the system (1.22) is in Poincaré-Birkhoff normal form up to order N . Thanks to the *non-resonance* conditions in Definition 3.4, the resonant Hamiltonian is the sum of monomials of the form

$$w_{n_0} \cdots w_{n_p} \bar{w}_{n_{p+1}} \cdots \bar{w}_{n_{j+1}}, \quad \{|n_0|, \dots, |n_p|\} = \{|n_{p+1}|, \dots, |n_{j+1}|\}, \quad p = j/2.$$

Thanks to this structure one can prove that the *super-actions* $|w_j|^2 + |w_{-j}|^2$ are prime integrals of the Hamiltonian. This guarantees the long time stability for the whole system in the spirit of [19]. For a more detailed statement we refer to Theorem 3.7. In fact the classical dynamical consequence of having a system in such a form is a *long time* existence and stability theorem. More precisely, if we consider a solution whose initial datum has size ε , then it exists for a time of size, at least, ε^{-N} and moreover the solution remains of size ε for this long time. This is the consequence of Corollary 3.8 and a standard bootstrap argument.

The first step in the proof of this theorem is to find a symplectic correction to the change of coordinates given in Theorem 3.1. It turns out that this is possible up to correcting the generator of the transformation by a smoothing operator, this is the content of Theorem 3.3 whose proof is done in Section 6. At this point we obtain an Hamiltonian system having the following structure

$$\partial_t Z = iE(\Omega Z + \mathcal{D}(Z)Z) + \tilde{\mathcal{R}}(Z)Z, \quad (1.23)$$

where $\mathcal{D}(Z)Z$ and $\tilde{\mathcal{R}}(Z)Z$ are vector-fields satisfying the properties listed below equation (1.2). We shall perform two different Poincaré-Birkhoff normal forms, one on the matrix of para-differential (with constant coefficients) operators $\mathcal{D}(Z)Z$ and a second one on the matrix of the smoothing remainders $\tilde{\mathcal{R}}(Z)Z$. This is the content of Section 7.

We shall apply the abstract results we have introduced above to some quasi-linear PDEs on the circle. We conclude this introduction by stating the theorems we obtain on quasi-linear perturbations of the Schrödinger, beam and Benjamin-Ono equations.

The quasi-linear Schrödinger equation. Consider the following equation

$$i\partial_t u - \partial_{xx} u + P_{\tilde{m}} * u + f(u, u_x, u_{xx}) = 0, \quad u = u(x, t), \quad x \in \mathbb{T}, \quad t \in \mathbb{R} \quad (1.24)$$

where the $f(z_0, z_1, z_2)$ is a polynomial in the variables $(z_0, z_1, z_2) \in \mathbb{C}^3$ with a zero of order at least 2 at the origin. The potential $P_{\tilde{m}}(x) = (\sqrt{2\pi})^{-1} \sum_{j \in \mathbb{Z}} \hat{p}(j) e^{ijx}$ is a real function with real Fourier coefficients and the term $P_{\tilde{m}} * u$ denotes the convolution between the potential $P_{\tilde{m}}(x)$ and $u(x) = (\sqrt{2\pi})^{-1} \sum_{j \in \mathbb{Z}} \hat{u}(j) e^{ijx}$

$$P_{\tilde{m}} * u(x) = \int_{\mathbb{T}} P_{\tilde{m}}(x-y) u(y) dy = \sum_{j \in \mathbb{Z}} \hat{p}(j) \hat{u}(j) e^{ijx}.$$

Concerning the convolution potential $P_{\tilde{m}}(x)$ we define its j -th Fourier coefficient as follows. Fix $M > 0$ and set

$$\hat{p}(j) := \hat{p}_{\tilde{m}}(j) = \sum_{k=1}^M \frac{m_k}{\langle j \rangle^{2k+1}}, \quad (1.25)$$

where $\tilde{m} = (m_1, \dots, m_M)$ is a vector in $\mathcal{O} := [-1/2, 1/2]^M$ and $\langle j \rangle = \sqrt{1 + |j|^2}$. We assume that the nonlinearity in (1.24) has the form

$$f(u, u_x, u_{xx}) = (\partial_{\bar{z}_0} F)(u, u_x) - \frac{d}{dx} [(\partial_{\bar{z}_1} F)(u, u_x)], \quad (1.26)$$

where $\partial_{z_i} := (\partial_{\text{Re}(z_i)} - i\partial_{\text{Im}(z_i)})/\sqrt{2}$ and $\partial_{\bar{z}_i} := (\partial_{\text{Re}(z_i)} + i\partial_{\text{Im}(z_i)})/\sqrt{2}$ for $i = 0, 1$, and $F(z_0, z_1)$ is a *real valued* polynomial in $(z_0, z_1) \in \mathbb{C}^2$ vanishing at order 3 near the origin. Thanks to (1.26) the equation (1.24) is Hamiltonian with respect to the symplectic form (1.19), namely ita can be written in the complex Hamiltonian form

$$\partial_t u = i\nabla_{\bar{u}} \mathcal{H}(u) = i\Omega u + if(u, u_x, u_{xx}) \quad (1.27)$$

with Hamiltonian function

$$\mathcal{H}(u) := \int_{\mathbb{T}} \Omega u \cdot \bar{u} + F(u, u_x) dx, \quad (1.28)$$

where the operator $\Omega := -\partial_{xx} + P_{\vec{m}}*$ is the Fourier multiplier (recall (1.25))

$$\Omega e^{ijx} = \omega_j e^{ijx}, \quad \omega_j = \omega_j(\vec{m}) := j^2 + \hat{p}(j), \quad j \in \mathbb{Z}. \quad (1.29)$$

We have the following.

Theorem 1.1. (Long time existence for quasi-linear Schrödinger equations). *There is a zero Lebesgue measure set $\mathcal{N} \subset \mathcal{O}$ such that for any integer $0 \leq N \leq M$ and any $\vec{m} \in \mathcal{O} \setminus \mathcal{N}$ there exists $s_0 \in \mathbb{R}$ such that for any $s \geq s_0$ there are constants $r_0 \in (0, 1)$, $c_N > 0$ and $C_N > 0$ such that the following holds true. For any $0 < r \leq r_0$ and any function u_0 in the ball of radius r of $H^s(\mathbb{T}; \mathbb{C})$, the equation (1.24) with initial datum u_0 has a unique solution*

$$u(t, x) \in C^0([-T_r, T_r]; H^s(\mathbb{T}; \mathbb{C})) \cap C^1([-T_r, T_r]; H^{s-2}(\mathbb{T}; \mathbb{C})), \quad T_r \geq c_N r^{-N}. \quad (1.30)$$

Moreover one has that

$$\sup_{t \in (-T_r, T_r)} \|u(t, \cdot)\|_{H^s} \leq C_N r. \quad (1.31)$$

Notice that a similar Theorem is given in [26] in the case of a *parity-preserving* and *reversible* perturbation f . In that paper it is fundamental to assume that the initial condition is an even function of $x \in \mathbb{T}$. Here we do not need such an assumption since we fully exploit the Hamiltonian structure of the equation. Notice also that, by using Theorem 3.1, one could recover the local existence result given in [25] by following the strategy we suggest for the Benjamin-Ono equation in the following.

The quasi-linear beam equation. We consider the following *quasi-linear* beam equation

$$\psi_{tt} + \partial_{xx}^2 \psi + m\psi + p(\psi) = 0, \quad \psi = \psi(t, x), \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \quad (1.32)$$

where the *mass parameter* $m \in [1, 2]$ and the non linearity has the form

$$p(\psi) = g(\psi, \psi_x, \psi_{xx}, \psi_{xxx}, \psi_{xxxx}), \quad (1.33)$$

with g a polynomial function with a zero of order at least 2 at the origin. We shall also assume that

$$\begin{aligned} g(\psi, \psi_x, \psi_{xx}, \psi_{xxx}, \psi_{xxxx}) &= (\partial_\psi G)(\psi, \psi_x, \psi_{xx}) \\ &\quad - \frac{d}{dx} [(\partial_{\psi_x} G)(\psi, \psi_x, \psi_{xx})] + \frac{d^2}{dx^2} [(\partial_{\psi_{xx}} G)(\psi, \psi_x, \psi_{xx})], \end{aligned} \quad (1.34)$$

for some polynomial $G(\psi, \psi_x, \psi_{xx})$. For some information about the model we refer the reader to [18], [21], [33]. We have the following result.

Theorem 1.2. (Long time existence for quasi-linear beam equation). *There is a zero Lebesgue measure set $\mathcal{N} \subset [1, 2]$ such that for any $m \in [1, 2] \setminus \mathcal{N}$ any integer $1 \leq N$ there exists $s_0 \in \mathbb{R}$ such that for any $s \geq s_0$ there are constants $r_0 \in (0, 1)$, $c_N > 0$ and $C_N > 0$ such that the following holds true. For any $0 < r \leq r_0$ and any function ψ_0 in the ball of radius r of $H^s(\mathbb{T}; \mathbb{R})$, the equation (1.32) with initial datum ψ_0 has a unique solution and*

$$\psi(t, x) \in C^0([-T_r, T_r]; H^s(\mathbb{T}; \mathbb{R})) \cap C^1([-T_r, T_r]; H^{s-2}(\mathbb{T}; \mathbb{R})), \quad T_r \geq c_N r^{-N}. \quad (1.35)$$

Moreover one has that

$$\sup_{t \in (-T_r, T_r)} \|\psi(t, \cdot)\|_{H^s} \leq C_N r. \quad (1.36)$$

The quasi-linear Benjamin-Ono equation. We consider the following model

$$u_t + \mathcal{H}u_{xx} + uu_x + \mathcal{N}(u) = 0, \quad u = u(t, x), \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \quad (1.37)$$

where the unknown $u(t, x)$ is real valued, \mathcal{H} is the periodic Hilbert transform, namely the Fourier multiplier

$$\mathcal{H} e^{ijx} = -i \operatorname{sign}(j) e^{ijx}, \quad j \in \mathbb{Z}. \quad (1.38)$$

The non linearity $\mathcal{N}(u)$ has the form

$$\mathcal{N}(u) = g(u, \mathcal{H}u, u_x, \mathcal{H}u_x, \mathcal{H}u_{xx}) \quad (1.39)$$

where $g(z_0, z_1, z_2, z_3, z_4)$ is a real valued polynomial in the variables $(z_0, z_1, z_2, z_3, z_4) \in \mathbb{R}^5$ with a zero of order at least 3 in the origin. We assume that

$$(\partial_{z_3} g)(u, \mathcal{H}u, u_x, \mathcal{H}u_x, \mathcal{H}u_{xx}) = \frac{d}{dx} \left[(\partial_{z_4} g)(u, \mathcal{H}u, u_x, \mathcal{H}u_x, \mathcal{H}u_{xx}) \right]. \quad (1.40)$$

Examples of admissible non linearities are the following:

- (i) $g = u^2 \mathcal{H}u_{xx} + 2uu_x \mathcal{H}u_x$
- (ii) $g = g(u, \mathcal{H}u, u_x)$.

We remark that the non linearity $\mathcal{N}(u)$ is not necessarily Hamiltonian, i.e. equation (1.37) does not have necessarily the form

$$u_t = J \nabla H(u) + J \nabla K(u), \quad J = -\partial_x, \quad H(u) = \int_{\mathbb{T}} \left(\frac{u \mathcal{H}u_x}{2} + \frac{u^3}{6} \right) dx, \quad \nabla K(u) = \text{bounded operator}, \quad (1.41)$$

with ∇ the L^2 -gradient. Here $H(u)$ is the hamiltonian of the ‘‘unperturbed’’ Benjamin-Ono equation, namely equation (1.37) with $\mathcal{N} \equiv 0$. For further details regarding the admissible perturbations $\mathcal{N}(u)$ we refer the reader to [1]. In this paper we assume the (1.40) as an example, but of course other choices are possible. We have the following result.

Theorem 1.3 (Local well-posedness for quasi linear Benjamin-Ono-type equations). *Consider equation (1.37) with (1.39), (1.40). Then there exists $s_0 > 0$ such that for any $s \geq s_0$ there exists $r_0 > 0$ such that, for any $0 \leq r \leq r_0$, and for any u_0 in the ball of radius r of $H^s(\mathbb{T}; \mathbb{R})$ the following holds. The equation (1.37) with initial datum u_0 has a unique classical solution $u(t, x)$ such that*

$$u(t, x) \in C^0([0, T]; H^s(\mathbb{T}; \mathbb{R})) \cap C^1([0, T]; H^{s-2}(\mathbb{T}; \mathbb{R})), \quad T \gtrsim r^{-1}. \quad (1.42)$$

Moreover there is a constant $C > 0$

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^s} \leq C \|u_0\|_{H^s}. \quad (1.43)$$

1.4. Plan of the paper. The paper is organized as follows. In Section 2 we develop a para-differential calculus for symbols which depend non-linearly on a function U in a certain Sobolev space. In Section 3 we state our main Theorem 3.1 and its principal application to Poincaré-Birkhoff normal form (Theorem 3.7). Moreover we apply this theorem to some quasi-linear equations, obtaining a result of *long time* existence and stability. In Section 4 we prove the well-posedness of several non-linear equation which we need to solve in order to diagonalize and put to constant coefficient the original para-differential equation. In Section 5 we produce the changes of coordinates whose composition gives the proof of Theorem 3.1. In Section 6 we provide symplectic corrections to the previously found changes of coordinates, finally in Section 7 we prove Theorem 3.7.

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2. PARA-DIFFERENTIAL CALCULUS

In this section we develop a para-differential calculus following the ideas (and notation) in [9]. We shall introduce some classes of symbols and operators which slightly differ from the ones in [9]. As in [9] we shall define classes of multilinear and non-homogeneous symbols (and their relative para-differential quantization) and smoothing operators. The main difference between our classes and those in [9] is the following. The non-homogeneous symbols in [9] depend on some extra function U which depends on space-time variables, in the application the function U is indeed the solution of certain evolution PDEs. In our case, see item (ii) in Def. 2.1, the extra function U depends only on the space variable, however, in the case of space-time dependence of the function U , we recover the definition given in [9] as shown in Remark 2.3. Roughly speaking the conditions on the time derivative in [9] are replaced by some conditions on the differentials of the symbols with respect to U in (2.11) which makes our classes a generalization of the ones in [9].

In the following we fix some notation that will be kept until the end of the paper. For $s \in \mathbb{R}$ we denote by $H^s(\mathbb{T}; \mathbb{C})$ (respectively $H^s(\mathbb{T}; \mathbb{C}^2)$), with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, the Sobolev space of 2π -periodic functions with values in \mathbb{C} (respectively \mathbb{C}^2). Sometimes we shall simply write H^s when this does not create confusion. Moreover if $r > 0$ we define the ball of radius r

$$B_r(H^s) := \{U \in H^s(\mathbb{T}; \mathbb{C}) : \|u\|_{H^s} < r\}.$$

We expand a 2π -periodic function $u(x)$ in Fourier series as

$$u(x) = \sum_{n \in \mathbb{Z}} \hat{u}(n) \frac{e^{inx}}{\sqrt{2\pi}}, \quad \hat{u}(n) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} u(x) e^{-inx} dx. \quad (2.1)$$

We also use the notation

$$u_n^+ := u_n := \hat{u}(n) \quad \text{and} \quad u_n^- := \overline{u_n} := \overline{\hat{u}(n)}. \quad (2.2)$$

For $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ we denote by Π_n the orthogonal projector from $L^2(\mathbb{T}; \mathbb{C})$ to the subspace spanned by $\{e^{inx}, e^{-inx}\}$, i.e.

$$(\Pi_n u)(x) := \hat{u}(n) \frac{e^{inx}}{\sqrt{2\pi}} + \hat{u}(-n) \frac{e^{-inx}}{\sqrt{2\pi}},$$

while in the case $n = 0$ we define the mean $\Pi_0 u = \frac{1}{2\pi} \hat{u}(0) = \frac{1}{2\pi} \int_{\mathbb{T}} u(x) dx$. We denote by Π_n also the corresponding projector in $L^2(\mathbb{T}, \mathbb{C}^2)$. We shall identify the Sobolev norm $\|\cdot\|_{H^s(\mathbb{T}, \mathbb{C})} =: \|\cdot\|_{H^s}$ with the norms

$$\|u\|_{H^s}^2 := \|u\|_s^2 := \sum_{j \in \mathbb{Z}} \langle j \rangle^{2s} |u_j|^2, \quad \langle j \rangle := \sqrt{1 + |j|^2}, \quad j \in \mathbb{N}. \quad (2.3)$$

We introduce the operator $\langle D \rangle$ defined by linearity as

$$\langle D \rangle e^{ij \cdot x} = \langle j \rangle e^{ij \cdot x}. \quad (2.4)$$

With this notation we have that the norm in (2.3) reads

$$\|u\|_{H^s}^2 := \langle \langle D \rangle^s u, \langle D \rangle^s u \rangle_{L^2} \quad (2.5)$$

where $(\cdot, \cdot)_{L^2}$ denotes the standard complex L^2 -scalar product

$$(u, v)_{L^2} := \int_{\mathbb{T}} u \cdot \bar{v} dx, \quad \forall u, v \in L^2(\mathbb{T}^d, \mathbb{C}). \quad (2.6)$$

If $\mathcal{U} = (U_1, \dots, U_p)$ is a p -tuple of functions, $\vec{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$, we set

$$\Pi_{\vec{n}} \mathcal{U} := (\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p). \quad (2.7)$$

For a family $(n_1, \dots, n_{p+1}) \in \mathbb{N}^{p+1}$ we denote by $\max_2(\langle n_1 \rangle, \dots, \langle n_{p+1} \rangle)$, the second largest among the numbers $\langle n_1 \rangle, \dots, \langle n_{p+1} \rangle$.

Notation. $A \lesssim_s B$ means $A \leq C(s)B$ where $C(s) > 0$ is a constant depending on $s \in \mathbb{R}$.

2.1. Classes of operators.

2.1.1. *Classes of symbols.* We give the definition of a class of symbols we shall use along the paper.

Definition 2.1. (Classes of symbols). Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}$, $r > 0$.

(i) **p -homogeneous symbols.** We denote by $\tilde{\Gamma}_p^m$ the space of symmetric p -linear maps from $(H^\infty(\mathbb{T}; \mathbb{C}^2))^p$ to the space of C^∞ functions of $(x, \xi) \in \mathbb{T} \times \mathbb{R}$, $\mathcal{U} \rightarrow ((x, \xi) \rightarrow a(\mathcal{U}; x, \xi))$, satisfying the following. There is $\mu > 0$ and, for any $\alpha, \beta \in \mathbb{N}$, there is $C > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(\Pi_{\tilde{n}} \mathcal{U}; x, \xi)| \leq C |\tilde{n}|^{\mu+\alpha} \langle \xi \rangle^{m-\beta} \prod_{j=1}^p \|\Pi_{n_j} U_j\|_{L^2} \quad (2.8)$$

for any $\mathcal{U} = (U_1, \dots, U_p)$ in $(H^\infty(\mathbb{T}; \mathbb{C}^2))^p$, and $\tilde{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p$. Moreover we assume that, if for some $(n_0, \dots, n_p) \in \mathbb{N} \times (\mathbb{N}^*)^p$,

$$\Pi_{n_0} a(\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p; \cdot) \neq 0, \quad (2.9)$$

then there exists a choice of signs $\sigma_0, \dots, \sigma_p \in \{-1, 1\}$ such that $\sum_{j=0}^p \sigma_j n_j = 0$. For $p = 0$ we denote by $\tilde{\Gamma}_0^m$ the space of constant coefficients symbols $\xi \mapsto a(\xi)$ which satisfy (2.8) with $\alpha = 0$ and the right hand side replaced by $C \langle \xi \rangle^{m-\beta}$. In addition we require the translation invariance property

$$a(\tau_\theta \mathcal{U}; x, \xi) = a(\mathcal{U}; x + \theta, \xi), \quad \forall \theta \in \mathbb{R}. \quad (2.10)$$

(ii) **Non-homogeneous symbols.** Let $p \geq 1$, $d \geq 0$. We denote by $\Gamma_p^{m,d}[r] := \Gamma_p^m[r]$ the space of functions $(U; x, \xi) \mapsto a(U; x, \xi)$, defined for $U \in B_r(H^{s_0})$, for some large enough s_0 , with complex values such that for any $s \geq s_0$, there are $C > 0$, $0 < r_0 < r$ and for any $U \in B_{r_0}(H^{s_0}) \cap H^s$ and any $\alpha, \beta \in \mathbb{N}$, with $\alpha + d \cdot k \leq s - s_0$, the following holds

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta (d_u^k a(U; x, \xi)[h_1, \dots, h_k])| &\leq C \langle \xi \rangle^{m-\beta} \left[\max\{0, p-k\} \|U\|_{H^{s_0}}^{\max\{0, p-k-1\}} \|U\|_{H^{s_0+\alpha}} \prod_{j=1}^k \|h_j\|_{H^{s_0}} + \right. \\ &\left. + \|U\|_{H^{s_0}}^{\max\{0, p-k\}} \sum_{i=1}^k \prod_{j=1, j \neq i}^k \|h_j\|_{H^{s_0}}^v \|h_i\|_{H^{s_0+\alpha}} \right] \end{aligned} \quad (2.11)$$

for any $h_j \in H^s$, where $v = 1$ if $k \geq 2$ and $v = 0$ otherwise.

(iii) **Symbols.** We denote by $\Sigma \Gamma_p^m[r, N]$ the space of functions $(U, x, \xi) \rightarrow a(U; x, \xi)$ such that there are homogeneous symbols $a_q \in \tilde{\Gamma}_q^m$, $q = p, \dots, N-1$, and a non-homogeneous symbol $a_N \in \Gamma_N^m[r]$ such that

$$a(U; t, x, \xi) = \sum_{q=p}^{N-1} a_q(U, \dots, U; x, \xi) + a_N(U; x, \xi). \quad (2.12)$$

We denote by $\Sigma \Gamma_p^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ the space 2×2 matrices with entries in $\Sigma \Gamma_p^m[r, N]$.

Remark 2.2. We omit the dependence on the number d in the definition of $\Gamma_p^m[r, N]$ because this is a number that will be fixed once for all in the procedure that we shall implement in Sections 5 and 7 of our paper. In the classical definition of symbols one does not have the loss $d \cdot k$ appearing in the definition above. We have to include this loss in our definition because of the following reason. In Sections 5 and 7 we shall perform some changes of variable, in one of them we shall redefine the variable of the ambient space $x \in \mathbb{T}$ in function of a solution of a para-differential equation whose principal symbol, in these notation, has order d . For this reason the smoothness in x and in U of the symbols is linked by the relation $\alpha + d \cdot k \leq s - s_0$. In such a way our classes will be closed for the changes of coordinates that we shall define in Sections 5 and 7.

Remark 2.3. Let us make a comment about the **non-homogeneous symbols** defined in the item (ii) of Def. 2.1. We claim that if we plug a function $U(t, x)$ (depending on the space-time couple (x, t)) in a symbol $a(U; x, \xi)$ we recover the definition given in the previous papers [26] and [9], more precisely the following holds true. Let $K - K' > 0$, suppose for simplicity that $K - K' < p$, and consider a function $U \in C_*^{K-K'}(I, H^s)$

defined in Section 2 in [26]. Let a be a symbol satisfying (2.11) with $s_0 := \sigma_0 - 2(K - K') \gg 1$. For any $0 \leq k \leq K - K'$ we have

$$\begin{aligned} |\partial_t^k \partial_x^\alpha \partial_\xi^\beta a(U; t, x, \xi)| &\leq C \sum_{\gamma=1}^k \sum_{k_1+\dots+k_\gamma=k} \langle \xi \rangle^{m-\beta} \\ &\quad \left[(p-\gamma) \|U\|_{\sigma_0-2(K-K')}^{p-\gamma-1} \|U\|_{\sigma_0-2(K-K')+\alpha} \prod_{j=1}^{\gamma} \|\partial_t^{k_j} U\|_{\sigma_0-2(K-K')} \right. \\ &\quad \left. + \|U\|_{\sigma_0}^{p-\gamma} \sum_{i=1}^{\gamma} \prod_{j=1, j \neq i}^{\gamma} \|\partial_t^{k_j} U\|_{\sigma_0-2(K-K')} \|\partial_t^{k_i} U\|_{\sigma_0-2(K-K')+\alpha} \right], \end{aligned}$$

therefore, by recalling that $0 \leq k \leq K - K'$ and $k_j \leq k$, one obtains the thesis. In order to recover the definition of non-homogeneous smoothing operator given in [9] the computation is the same.

Remark 2.4. Let $s_0 > \mu + 1/2$ and $s \geq s_0$. Consider a function U in $B_r(H^{s_0}) \cap H^s$, then for any $0 \leq \alpha \leq s - s_0$ and $\beta \in \mathbb{N}$ the function $\tilde{a}(U; \xi) := a(U, \dots, U; \xi)$ defines a non-homogeneous symbol as in item (i) of Definition 2.1. To see this let us develop in Fourier series

$$a(U, \dots, U; x, \xi) = \sum_{n_1, \dots, n_p \in \mathbb{N}} a(\Pi_{n_1} U, \dots, \Pi_{n_p} U; \xi),$$

therefore by using condition (2.8) and supposing, for simplicity, that $n_1 \geq \dots \geq n_p$ we deduce that

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta a(U, \dots, U; x, \xi)| &\leq \\ &C \sum_{n_1, \dots, n_p \in \mathbb{N}} \langle \xi \rangle^{m-\beta} \langle n_1 \rangle^{\mu+\alpha} \prod_{j=1}^p \|\Pi_{n_j} U\|_{L^2} = \\ &C \sum_{n_1, \dots, n_p \in \mathbb{N}} \langle \xi \rangle^{m-\beta} \langle n_1 \rangle^{\mu+\alpha} \prod_{j=1}^p \langle n_j \rangle^{-\sigma_0} \prod_{j=1}^p \langle n_j \rangle^{\sigma_0} \|\Pi_{n_j} U\|_{L^2}. \end{aligned}$$

From the latter inequality it is easy to obtain the condition (2.11) by using the Cauchy-Schwartz inequality and the fact that $\mu - s_0 < -1/2$ in the case that no differentials, with respect to U , on the symbols are taken. The estimate for the differentials of the symbol $\tilde{a}(U; \xi)$, i.e. the case $k > 0$ in the notation of item (i) of Definition 2.1, is trivial thanks to multi-linearity of the symbol a with respect its arguments.

We also introduce the following class of “functions”, i.e. those symbols which are independent of the variable ξ .

Definition 2.5 (Functions). Fix $N \in \mathbb{N}$, $p \in \mathbb{N}$ with $p \leq N$, with $r > 0$. We denote by $\widetilde{\mathcal{F}}_p$ (resp. $\mathcal{F}_p[r]$, resp. $\Sigma \mathcal{F}_p[r, N]$) the subspace of $\widetilde{\Gamma}_p^0$ (resp. $\Gamma_p^0[r]$, resp. $\Sigma \Gamma_p^0[r, N]$) made of those symbols which are independent of ξ . We shall write $\widetilde{\mathcal{F}}_p^{\mathbb{R}}$ (resp. $\mathcal{F}_p^{\mathbb{R}}[r]$, resp. $\Sigma \mathcal{F}_p^{\mathbb{R}}[r, N]$) to denote those functions in the class $\widetilde{\mathcal{F}}_p$ (resp. $\mathcal{F}_p[r]$, resp. $\Sigma \mathcal{F}_p[r, N]$) which are real valued.

Remark 2.6. The class of symbols $\Gamma_p^m[r]$, defined in item (i) of Definition 2.1, restricted to $B_r(H^s)$, is a Fréchet space equipped with the family of semi-norms

$$|a|_{\alpha, \beta, k}^{\Gamma, s} := \inf\{C > 0 : (2.11) \text{ holds true}\} \quad (2.13)$$

with $\alpha + d \cdot k \leq s - s_0$. The semi-norms on the space of functions $\mathcal{F}_p[r]$ are analogously defined and we shall denote them by $|a|_{\alpha, k}^{\mathcal{F}, s}$. Moreover we shall use the following notation

$$|a|_s^{\mathcal{F}} := \sum_{\alpha + d \cdot k \leq s - s_0} |a|_{\alpha, k}^{\mathcal{F}, s}. \quad (2.14)$$

The quantity defined in (2.14) is a norm on the space $\mathcal{F}_p[r]$. The couple $(\mathcal{F}_p[r], |a|_s^{\mathcal{F}})$ defines a Banach space. We shall also write

$$|a|_{s,0}^{\mathcal{F}} := \sum_{\alpha \leq s-s_0} |a|_{\alpha,0}^{\mathcal{F},s}. \quad (2.15)$$

Remark 2.7. Let $a \in \mathcal{F}_1[r]$ restricted to $B_r(H^s)$. Then, recalling the definition (2.13)-(2.14), we have for any $\alpha \leq s - s_0$

$$\|\partial_x^\alpha a(U; x)\|_{L_x^\infty} \lesssim_s |a|_{\alpha,0}^{\mathcal{F},s} r \lesssim_s |a|_s^{\mathcal{F}} r.$$

More in general, for $a \in \Gamma_1^m[r]$, we have

$$\langle \xi \rangle^{-m} \|\partial_x^\alpha a(U; x)\|_{L_x^\infty} \lesssim_s |a|_{\alpha,0}^{\Gamma^m,s} r.$$

Remark 2.8. Let $a \in \Gamma_p^m[r]$, $b \in \Gamma_p^{m'}[r]$. One can check, by using the Leibniz rule, that

$$|ab|_{\alpha,\beta,k}^{\Gamma^{m+m'},s} \lesssim_s \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta \\ k_1+k_2=k}} |a|_{\alpha_1,\beta_1,k_1}^{\Gamma^m,s} |b|_{\alpha_2,\beta_2,k_2}^{\Gamma^{m'},s} \lesssim_s |a|_{\alpha,\beta,k}^{\Gamma^m,s} |b|_{\alpha,\beta,k}^{\Gamma^{m'},s}.$$

Consider also a function $f(x)$ which is analytic in some neighborhood of the origin of \mathbb{C} and let $c \in \mathcal{F}_1[r]$. Then, using the formula of Faa di Bruno (i.e. the formula for the derivatives of the the composition of functions), one has that $d := f(c)$ is a function in $\mathcal{F}_1[r]$. In particular

$$|d|_s^{\mathcal{F}} \lesssim_s C,$$

where $C > 0$ is a constant depending only on s and the semi-norm $|c|_s^{\mathcal{F}}$. If $f(x) = 1/(1+x) - 1$ and $c \in \Gamma_1^m[r]$, then $d = f(c) \in \Gamma_1^m[r]$ and

$$|d|_{\alpha,\beta,k}^{\Gamma^m,s} \lesssim_s C,$$

for some constant $C > 0$ depending only on s and the seminorm $|c|_{\alpha,\beta,k}^{\Gamma^m,s}$.

Remark 2.9. (Expansion of Homogeneous symbols). Let $a_p \in \tilde{\Gamma}_p^m$ (see Def. 2.1). Notice that, by the autonomous condition (2.9) and the x -translation invariant property (2.10), we can write the symbol a_p , expanding u as in (2.1), (2.2), as

$$a_p(U; x, \xi) = \sum_{j \in \mathbb{Z}} e^{ijx} \sum_{\substack{\sigma_i \in \{\pm\}, i=1,\dots,p \\ \sum_{i=1}^p \sigma_i j_i = j}} (a_p)_{j_1,\dots,j_p}^{\sigma_1 \dots \sigma_p}(\xi) u_{j_1}^{\sigma_1} \dots u_{j_p}^{\sigma_p}, \quad (2.16)$$

for some coefficients $(a_p)_{j_1,\dots,j_p}^{\sigma_1 \dots \sigma_p}(\xi) \in \tilde{\Gamma}_0^m$.

2.1.2. *Spaces of Smoothing operators.* We now introduce some classes of smoothing operators.

Definition 2.10 (Classes of smoothing operators). Let $\rho \in \mathbb{R}$, with $\rho \geq 0$, $p, N \in \mathbb{N}$, $r > 0$.

(i) **p -homogeneous smoothing operator.** We denote by $\tilde{\mathcal{R}}_p^{-\rho}$ the space of $(p+1)$ -linear maps from the space $(C^\infty(\mathbb{T}; \mathbb{C}^2))^p \times C^\infty(\mathbb{T}; \mathbb{C})$ to the space $C^\infty(\mathbb{T}; \mathbb{C})$ symmetric in (U_1, \dots, U_p) , of the form $(U_1, \dots, U_{p+1}) \rightarrow R(U_1, \dots, U_p)U_{p+1}$, that satisfy the following. There is $\mu \geq 0$, $C > 0$ such that

$$\|\Pi_{n_0} R(\Pi_{\tilde{n}} \mathcal{U}) \Pi_{n_{p+1}} U_{p+1}\|_{L^2} \leq C \frac{\max_2(\langle n_1 \rangle, \dots, \langle n_{p+1} \rangle)^{\rho+\mu}}{\max(\langle n_1 \rangle, \dots, \langle n_{p+1} \rangle)^\rho} \prod_{j=1}^{p+1} \|\Pi_{n_j} U_j\|_{L^2}, \quad (2.17)$$

for any $\mathcal{U} = (U_1, \dots, U_p) \in (C^\infty(\mathbb{T}; \mathbb{C}^2))^p$, any $U_{p+1} \in C^\infty(\mathbb{T}; \mathbb{C})$, any $\tilde{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$, any $n_0, n_{p+1} \in \mathbb{N}$. Moreover, if

$$\Pi_{n_0} R(\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p) \Pi_{n_{p+1}} U_{p+1} \neq 0, \quad (2.18)$$

then there is a choice of signs $\sigma_0, \dots, \sigma_{p+1} \in \{-1, 1\}$ such that $\sum_{j=0}^{p+1} \sigma_j n_j = 0$. In addition we require the translation invariance property

$$R(\tau_\theta \mathcal{U})[\tau_\theta U_{p+1}] = \tau_\theta (R(\mathcal{U})U_{p+1}), \quad \forall \theta \in \mathbb{R}. \quad (2.19)$$

(ii) **Non-homogeneous smoothing operators.** We define the class of remainders $\mathcal{R}_N^{-\rho}[r]$ as the space of maps $(V, u) \mapsto R(V)u$ defined on $B_r(H^{s_0}) \times H^{s_0}(\mathbb{T}, \mathbb{C})$ which are linear in the variable u and such that the following holds true. For any $s \geq s_0$ there exist a constant $C > 0$ and $r(s) \in]0, r[$ such that for any $V \in B_r(H^{s_0}) \times H^s(\mathbb{T}, \mathbb{C})$, any $u \in H^s(\mathbb{T}, \mathbb{C})$, any $0 \leq k \cdot d \leq s - s_0$ the following estimate holds true

$$\begin{aligned} \|((d_V^k R(V))u)[h_1, \dots, h_k]\|_{H^{s+\rho-dk}} &\leq C \left[\max\{0, p-k\} \|V\|_{s_0}^{\max\{0, p-k-1\}} \|V\|_s \|u\|_{s_0} \prod_{j=1}^k \|h_j\|_{s_0} + \right. \\ &+ \|V\|_{s_0}^{\max\{0, p-k\}} \|u\|_{s_0} \sum_{i=1}^k \prod_{j=1, j \neq i}^k \|h_j\|_{H^{s_0}}^v \|h_i\|_{H^s} \\ &\left. + \|V\|_{s_0}^{\max\{0, p-k\}} \|u\|_s \prod_{j=1}^k \|h_j\|_{s_0} \right] \end{aligned} \quad (2.20)$$

for any h_j and $U \in H^s$, where $v = 1$ if $k \geq 2$ and $v = 0$ otherwise. Here d is the same number appearing in the definition of the non-homogeneous symbols, see also the Remark 2.2.

(iii) **Smoothing operators.** We denote by $\Sigma \mathcal{R}_p^{-\rho}[r, N]$ the space of maps $(V, t, u) \rightarrow R(V, t)u$ that may be written as

$$R(V; t)u = \sum_{q=p}^{N-1} R_q(V, \dots, V)u + R_N(V; t)u,$$

for some $R_q \in \tilde{\mathcal{R}}_q^{-\rho}$, $q = p, \dots, N-1$ and R_N belongs to $\mathcal{R}_N^{-\rho}[r]$. We denote by $\Sigma \mathcal{R}_p^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ the space 2×2 matrices with entries in $\Sigma \mathcal{R}_p^{-\rho}[r, N]$.

Remark 2.11. Let $R_1(U)$ be a smoothing operator in $\Sigma \mathcal{R}_{p_1}^{-\rho_1}[r, N]$ and $R_2(U)$ in $\Sigma \mathcal{R}_{p_2}^{-\rho_2}[r, N]$, then the operator $R_1(U) \circ R_2(U)[\cdot]$ belongs to $\Sigma \mathcal{R}_{p_1+p_2}^{-\rho}[r, N]$, where $\rho = \min(\rho_1, \rho_2)$.

Remark 2.12. We remark that if R is in $\tilde{\mathcal{R}}_p^{-\rho}$, $p \geq N$, then $(V, U) \rightarrow R(V, \dots, V)U$ is in $\mathcal{R}_N^{-\rho}[r]$. This inclusion follows by the multi-linearity of R in each argument, and by estimate (2.17). The proof of this fact is very similar to the one given in the remark after Definition 2.2.3 in [9].

Remark 2.13. In analogy with Remark 2.3 we make a comparison between the **non-homogenous** smoothing remainders defined in [9] or [26] and the ones defined in item (ii) in Def. 2.10. Let $U(t, x)$ be a function in $C_*^{K-K'}(I, H^s)$ (defined in Section 2 in [26]) and $(V, u) \mapsto R(V)u$ be a smoothing operator satisfying (2.20) with $s_0 := \bar{s}_0 - 2(K - K') \gg 1$. Let us suppose for simplicity that $p - k - 1 \geq 0$ and $k \geq 2$, in the other cases the proof may be easily adapted. For any $2 \leq k \leq K - K'$ we have

$$\begin{aligned} \|\partial_t^k R(V)u\|_{s-2k} &\leq \left\| \sum_{k'+k''=k} \sum_{\gamma=1}^{k'} \sum_{k_1+\dots+k_\gamma=k''} d_V^{k'} R(V) \left[\partial_t^{k_1} V, \dots, \partial_t^{k_\gamma} V, \partial_t^{k''} u \right] \right\|_{s-2k} \\ &\leq C \sum \left\{ \|V\|_{s_0}^{p-\gamma-1} \|V\|_{s-2k} \|\partial_t^{k''} u\|_{s_0} \prod_{j=1}^{\gamma} \|\partial_t^{k_j} V\|_{s_0} + \right. \\ &+ \|V\|_{s_0}^{p-\gamma} \|\partial_t^{k''} u\|_{s_0} \sum_{i=1}^{\gamma} \prod_{j=1, j \neq i}^{\gamma} \|\partial_t^{k_j} V\|_{s-2k} \|\partial_t^{k_j} V\|_{s_0} + \\ &\left. + \|V\|_{s_0}^{p-\gamma} \prod_{j=1}^{\gamma} \|\partial_t^{k_j} V\|_{s_0} \|\partial_t^{k''} u\|_{s-2k} \right\}, \end{aligned}$$

where the sum in the r.h.s. is taken over the set of indices such that $k' + k'' = k$, $k_1 + \dots + k_\gamma = k''$ and $\gamma = 1, \dots, k'$. The r.h.s. of the above inequality may be bounded from above by the r.h.s. of (2.7) in [26]. Therefore we recover the definition of non-homogeneous smoothing operators given in [26]. To recover the definition in [9] the computation is the same.

Remark 2.14. (Expansion of Homogeneous remainders). Let $R_p \in \widetilde{\mathcal{R}}_p^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$. Then by (2.18), (2.19) we deduce the following. First of all write

$$R_p(U) = \begin{pmatrix} (\mathbb{R}_p(U))_+^+ & (\mathbb{R}_p(U))_+^- \\ (\mathbb{R}_p(U))_-^+ & (\mathbb{R}_p(U))_-^- \end{pmatrix}, \quad (\mathbb{R}_i(U))_\sigma^{\sigma'} \in \widetilde{\mathcal{R}}_p^{-\rho}, \quad (2.21)$$

for $\sigma, \sigma' \in \{\pm\}$ and $i = 1, 2$. Then, expanding u as in (2.1), (2.2), we have and

$$(\mathbb{R}_p(U))_\sigma^{\sigma'} z^{\sigma'} = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} (\mathbb{R}_p(U))_{\sigma, j}^{\sigma', k} z_k^{\sigma'} \right) e^{i\sigma j x} \quad (2.22)$$

with entries

$$(\mathbb{R}_p(U))_{\sigma, j}^{\sigma', k} := \frac{1}{(2\pi)^p} \sum_{\substack{\sigma_i \in \{\pm\}, j_i \in \mathbb{Z} \\ \sum_{i=1}^p \sigma_i j_i = \sigma j - \sigma' k}} ((\mathbb{r}_p)^{\sigma_1 \dots \sigma_p})_{j_1, \dots, j_p}^{\sigma', k} u_{j_1}^{\sigma_1} \dots u_{j_p}^{\sigma_p}, \quad j, k \in \mathbb{Z} \setminus \{0\}, \quad (2.23)$$

and suitable scalar coefficients $((\mathbb{r}_p)^{\sigma_1 \dots \sigma_p})_{j_1, \dots, j_p}^{\sigma', k} \in \mathbb{C}$.

2.1.3. Spaces of Maps. Below we deal with classes of operators without keeping track of the number of lost derivatives in a precise way (see Definition 3.9 in [9]). The class $\widetilde{\mathcal{M}}_p^m$ denotes multilinear maps that lose m derivatives and are p -homogeneous in U , while the class \mathcal{M}_p^m contains non-homogeneous maps which lose m derivatives, vanish at degree at least p in U .

Definition 2.15. (Classes of maps). Let $p, N \in \mathbb{N}$, with $p \leq N$, $N \geq 1$ and $m \geq 0$.

(i) **p -homogeneous maps.** We denote by $\widetilde{\mathcal{M}}_p^m$ the space of $(p+1)$ -linear maps M from $(H^\infty(\mathbb{T}; \mathbb{C}^2))^p \times H^\infty(\mathbb{T}; \mathbb{C})$ to the space $H^\infty(\mathbb{T}; \mathbb{C})$ which are symmetric in (U_1, \dots, U_p) , of the form

$$(U_1, \dots, U_{p+1}) \rightarrow M(U_1, \dots, U_p) U_{p+1}$$

and that satisfy the following. There is $C > 0$ such that

$$\|\Pi_{n_0} M(\Pi_{\vec{n}} \mathcal{U}) \Pi_{n_{p+1}} U_{p+1}\|_{L^2} \leq C(n_0 + n_1 + \dots + n_{p+1})^m \prod_{j=1}^{p+1} \|\Pi_{n_j} U_j\|_{L^2}$$

for any $\mathcal{U} = (U_1, \dots, U_p) \in (H^\infty(\mathbb{T}; \mathbb{C}^2))^p$, any $U_{p+1} \in H^\infty(\mathbb{T}; \mathbb{C})$, any $\vec{n} = (n_1, \dots, n_p) \in (\mathbb{N}^*)^p$, any $n_0, n_{p+1} \in \mathbb{N}^*$. Moreover the properties (2.18)-(2.19) hold.

(ii) **Non-homogeneous maps.** We denote by $\mathcal{M}_N^m[r]$ the space of maps $(V, u) \mapsto M(V)U$ defined on $B_r(H^{s_0}) \times H^{s_0}(\mathbb{T}, \mathbb{C})$ which are linear in the variable U and such that the following holds true. For any $s \geq s_0$ there exist a constant $C > 0$ and $r(s) \in]0, r[$ such that for any $V \in B_r(H^s) \cap H^s(\mathbb{T}, \mathbb{C}^2)$, any $U \in H^s(\mathbb{T}, \mathbb{C})$, any $0 \leq dk \leq s - s_0$, we have $\|d_V^k(M(V)U)[h_1, \dots, h_k]\|_{H^{s-m-dk}}$ is bounded by the right hand side of (2.20).

(iii) **Maps.** We denote by $\Sigma \mathcal{M}_p^m[r, N]$ the space of maps $(V, t, U) \rightarrow M(V, t)U$ that may be written as

$$M(V; t)U = \sum_{q=p}^{N-1} M_q(V, \dots, V)U + M_N(V; t)U$$

for some M_q in $\widetilde{\mathcal{M}}_q^m$, $q = p, \dots, N-1$ and M_N in $\mathcal{M}_N^m[r]$. Finally we set $\widetilde{\mathcal{M}}_p := \cup_{m \geq 0} \widetilde{\mathcal{M}}_p^m$, $\mathcal{M}_p[r] := \cup_{m \geq 0} \mathcal{M}_p^m[r]$ and $\Sigma \mathcal{M}_p[r, N] := \cup_{m \geq 0} \Sigma \mathcal{M}_p^m[r]$.

We denote by $\Sigma \mathcal{M}_p^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices whose entries are maps in the class $\Sigma \mathcal{M}_p^m[r, N]$. We also set $\Sigma \mathcal{M}_p[r, N] \otimes \mathcal{M}_2(\mathbb{C}) = \cup_{m \in \mathbb{R}} \Sigma \mathcal{M}_p^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$.

- If M is in $\widetilde{\mathcal{M}}_p^m$, $p \geq N$, then $(V, U) \rightarrow M(V, \dots, V)U$ is in $\mathcal{M}_{N,0}^m[r]$.
- If $a \in \Sigma \Gamma_{K, K', p}^m[r, N]$ for $p \geq 1$, then $Op^{\text{BW}}(a(V; t, \cdot))U$ is in $\Sigma \mathcal{M}_p^{m'}[r, N]$ for some $m' \geq m$.
- Any $R \in \Sigma \mathcal{R}_p^{-\rho}[r, N]$ defines an element of $\Sigma \mathcal{M}_p^m[r, N]$ for some $m \geq 0$.
- If $M \in \Sigma \mathcal{M}_p[r, N]$, $\tilde{M} \in \Sigma \mathcal{M}_1[r, N-p]$, then $(V, t, U) \rightarrow M(V + \tilde{M}(V; t)V; t)U$ is in $\Sigma \mathcal{M}_p[r, N]$.

- If $M \in \Sigma \mathcal{M}_p^m[r, N]$ and $\tilde{M} \in \Sigma \mathcal{M}_q^{m'}[r, N]$, then $M(U; t) \circ \tilde{M}(U; t)$ is in $\Sigma \mathcal{M}_{p+q}^{m+m'}[r, N]$.

2.2. Quantization of symbols. Given a smooth symbol $(x, \xi) \rightarrow a(x, \xi)$, we define, for any $\sigma \in [0, 1]$, the quantization of the symbol a as the operator acting on functions u as

$$\text{Op}_\sigma(a(x, \xi))u = \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} e^{i(x-y)\xi} a(\sigma x + (1-\sigma)y, \xi) u(y) dy d\xi. \quad (2.24)$$

This definition is meaningful in particular if $u \in C^\infty(\mathbb{T})$ (identifying u to a 2π -periodic function). By decomposing u in Fourier series as $u = \sum_{j \in \mathbb{Z}} \hat{u}(j) (1/\sqrt{2\pi}) e^{ijx}$, we may calculate the oscillatory integral in (2.24) obtaining

$$\text{Op}_\sigma(a)u := \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \hat{a}(k-j, (1-\sigma)k + \sigma j) \hat{u}(j) \right) \frac{e^{ikx}}{\sqrt{2\pi}}, \quad \forall \sigma \in [0, 1], \quad (2.25)$$

where $\hat{a}(k, \xi)$ is the k^{th} -Fourier coefficient of the 2π -periodic function $x \mapsto a(x, \xi)$. For convenience in the paper we shall use two particular quantizations:

Standard quantization. We define the standard quantization by specifying formula (2.25) for $\sigma = 1$:

$$\text{Op}(a)u := \text{Op}_1(a)u = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \hat{a}(k-j, j) \hat{u}(j) \right) \frac{e^{ikx}}{\sqrt{2\pi}};$$

Weyl quantization. We define the Weyl quantization by specifying formula (2.25) for $\sigma = \frac{1}{2}$:

$$\text{Op}^W(a)u := \text{Op}_{\frac{1}{2}}(a)u = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \hat{a}(k-j, \frac{k+j}{2}) \hat{u}(j) \right) \frac{e^{ikx}}{\sqrt{2\pi}}. \quad (2.26)$$

Moreover the above formulas allow to transform the symbols between different quantizations, in particular we have

$$\text{Op}(a) = \text{Op}^W(b), \quad \text{where } \hat{b}(j, \xi) = \hat{a}(j, \xi - \frac{j}{2}).$$

We want to define a *para-differential* quantization. First we give the following definition.

Definition 2.16. (Admissible cut-off functions). Fix $p \in \mathbb{N}$ with $p \geq 1$. We say that $\chi_p \in C^\infty(\mathbb{R}^p \times \mathbb{R}; \mathbb{R})$ and $\chi \in C^\infty(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ are admissible cut-off functions if they are even with respect to each of their arguments and there exists $\delta > 0$ such that

$$\begin{aligned} \text{supp } \chi_p &\subset \{(\xi', \xi) \in \mathbb{R}^p \times \mathbb{R}; |\xi'| \leq \delta \langle \xi \rangle\}, & \chi_p(\xi', \xi) &\equiv 1 \text{ for } |\xi'| \leq \frac{\delta}{2} \langle \xi \rangle, \\ \text{supp } \chi &\subset \{(\xi', \xi) \in \mathbb{R} \times \mathbb{R}; |\xi'| \leq \delta \langle \xi \rangle\}, & \chi(\xi', \xi) &\equiv 1 \text{ for } |\xi'| \leq \frac{\delta}{2} \langle \xi \rangle. \end{aligned}$$

We assume moreover that for any derivation indices α and β

$$\begin{aligned} |\partial_\xi^\alpha \partial_{\xi'}^\beta \chi_p(\xi', \xi)| &\leq C_{\alpha, \beta} \langle \xi \rangle^{-\alpha - |\beta|}, \quad \forall \alpha \in \mathbb{N}, \beta \in \mathbb{N}^p, \\ |\partial_\xi^\alpha \partial_{\xi'}^\beta \chi(\xi', \xi)| &\leq C_{\alpha, \beta} \langle \xi \rangle^{-\alpha - \beta}, \quad \forall \alpha, \beta \in \mathbb{N}. \end{aligned}$$

An example of function satisfying the condition above, and that will be extensively used in the rest of the paper, is $\chi(\xi', \xi) := \tilde{\chi}(\xi'/\langle \xi \rangle)$, where $\tilde{\chi}$ is a function in $C_0^\infty(\mathbb{R}; \mathbb{R})$ having a small enough support and equal to one in a neighborhood of zero. For any $a \in C^\infty(\mathbb{T})$ we shall use the following notation

$$(\chi(D)a)(x) = \sum_{j \in \mathbb{Z}} \chi(j) \Pi_j a.$$

Definition 2.17. (The Bony quantization). Let χ be an admissible cut-off function according to Definition 2.16. If a is a symbol in $\tilde{\Gamma}_p^m$ and b is in $\Gamma_p^m[r]$, we set, using notation (2.7),

$$a_\chi(\mathcal{U}; x, \xi) = \sum_{\tilde{n} \in \mathbb{N}^p} \chi_p(\tilde{n}, \xi) a(\Pi_{\tilde{n}} \mathcal{U}; x, \xi), \quad b_\chi(U; x, \xi) = \frac{1}{2\pi} \int_{\mathbb{T}} \chi(\eta, \xi) \hat{b}(U; \eta, \xi) e^{i\eta x} d\eta.$$

We define the Bony quantization as

$$Op^B(a(\mathcal{U}; \cdot)) = Op(a_\chi(\mathcal{U}; \cdot)), \quad Op^B(b(U; \cdot)) = Op(b_\chi(U; t, \cdot)).$$

and the Bony-Weyl quantization as

$$Op^{BW}(a(\mathcal{U}; \cdot)) = Op^W(a_\chi(\mathcal{U}; \cdot)), \quad Op^{BW}(b(U; t, \cdot)) = Op^W(b_\chi(U; \cdot)). \quad (2.27)$$

Finally, if a is a symbol in the class $\Sigma\Gamma_p^m[r, N]$, that we decompose as in (2.12), we define its Bony quantization as

$$Op^B(a(U; \cdot)) = \sum_{q=p}^{N-1} Op^B(a_q(U, \dots, U; \cdot)) + Op^B(a_N(U; \cdot)),$$

and its Bony-Weyl quantization as

$$Op^{BW}(a(U; \cdot)) = \sum_{q=p}^{N-1} Op^{BW}(a_q(U, \dots, U; \cdot)) + Op^{BW}(a_N(U; \cdot)).$$

Remark 2.18. Let $a \in \Sigma\Gamma_p^m[r, N]$. We note that

$$\overline{Op^B(a(U; x, \xi)[v])} = Op^B(\overline{a^\vee(U; x, \xi)})[\bar{v}], \quad \overline{Op^{BW}(a(U; x, \xi)[v])} = Op^{BW}(\overline{a^\vee(U; x, \xi)})[\bar{v}],$$

where

$$a^\vee(U; x, \xi) := a(U; x, -\xi).$$

Moreover if we define the operator $A(U)[\cdot] := Op^{BW}(a(U; x, \xi))[\cdot]$ we have that $A^*(U)$, its adjoint operator w.r.t. the $L^2(\mathbb{T}; \mathbb{C})$ scalar product, can be written as

$$A^*(U)[v] = Op^{BW}(\overline{a(U; x, \xi)})[v]. \quad (2.28)$$

Remark 2.19. By formula (2.28) one has $Op^{BW}(a(U; x, \xi))[\cdot]$ is self-adjoint, w.r.t. the $L^2(\mathbb{T}; \mathbb{C})$ scalar product, if and only if the symbol $a(U; x, \xi)$ is real valued for any $x \in \mathbb{T}$, $\xi \in \mathbb{R}$.

The next proposition states boundedness properties on Sobolev spaces of the para-differential operators (see Proposition 3.8 in [9]).

Proposition 2.20. (Action of para-differential operator 1). Let $r > 0$, $m \in \mathbb{R}$, $p \in \mathbb{N}$. Then: (i) There is $s_0 > 0$ such that for any symbol $a \in \tilde{\Gamma}_p^m$, for any $s \geq s_0$ there is a constant $C > 0$, depending only on s and on (2.8) with $\alpha = \beta = 0$, such that for any $\mathcal{U} = (U_1, \dots, U_p)$

$$\|Op^{BW}(a(\mathcal{U}; \cdot))U_{p+1}\|_{H^{s-m}} \leq C \prod_{j=1}^p \|U_j\|_{H^{s_0}} \|U_{p+1}\|_{H^s}, \quad (2.29)$$

for $p \geq 1$, while for $p = 0$ the (2.29) holds by replacing the right hand side with $C\|U_{p+1}\|_{H^s}$.

(ii) There is $s_0 > 0$ such that for any symbol $a \in \Gamma_p^m[r]$ and any $s \geq s_0$ there is a constant $C > 0$, depending only on s, r and (2.11) with $0 \leq \alpha \leq 2$, $\beta = 0$, such that, for any $t \in I$, any $0 \leq dk + 2 \leq s - s_0$,

$$\|Op^{BW}((\partial_u^k a)(U; \cdot)[h_1, \dots, h_k])\|_{\mathcal{L}(H^s, H^{s-m})} \leq C\|U\|_{H^{s_0}}^{\max\{0, p-k\}} \prod_{i=1}^k \|h_i\|_{s_0}. \quad (2.30)$$

Proof. The proof of item (i) may be found at pag. 47 of the book [9], the item (i) of this theorem is indeed the same of item (i) of Prop. 3.8 in [9]. Note that the proof of item (ii) in the case that $k = 0$ is the same given in item (ii) of Prop 3.8 in [9], in the case $k > 0$ is very similar, however we give the proof for completeness. Let χ an admissible cut-off function in the sense of Def. 2.16. We set

$$b_k(x, \xi) := (d_u^k a(U; \cdot)[h_1, \dots, h_k])_\chi = \sum_{n \in \mathbb{Z}} \chi(n, \xi) \Pi_n d_u^k a(U; x, \xi)[h_1, \dots, h_k]$$

for any fixed $\xi \in \mathbb{R}$. Let V be a regular function on the torus, we have the following

$$Op^{\text{BW}}(\partial_u^k a(U; x, \xi)[h_1, \dots, h_k])v = \frac{1}{\sqrt{2\pi}} \sum_{n' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} \widehat{b}_k(k' - n', \frac{k' + n'}{2}) \widehat{v}(n') \frac{e^{ik'x}}{\sqrt{2\pi}},$$

where by $\widehat{b}_k(\ell, n)$ we have denoted the ℓ -Fourier coefficient with respect to x of the function $b_k(x, \xi)$ restricted at $\xi = n$. We need to estimate the general Fourier coefficient $\widehat{b}_k(\ell, \xi)$. By definition we have

$$\widehat{b}_k(\ell, \xi) = \int_{\mathbb{T}} b_k(x, \xi) e^{-i\ell x} dx,$$

therefore, by integrating twice in x and by using (2.11) with $\alpha = 2$ (relabelling $s_0 \rightsquigarrow s_0 + 2$), we obtain

$$|\widehat{b}_k(\ell, \xi)| \leq C \langle \xi \rangle^m \frac{1}{\langle \ell \rangle^2} \|U\|_{s_0}^{p-k} \prod_{j=1}^k \|h_j\|_{s_0}.$$

Having this inequality one can conclude the proof as done in the case of Prop. 3.8 in [9]. \square

Remark 2.21. Notice that the following holds.

- Let $m \leq 0$ and $p \geq 1$ and let a be a symbol in $\Sigma\Gamma_p^m[r, N]$, then the map $(U, V) \mapsto Op^{\text{BW}}(a(U; x, \xi))V$ is in $\Sigma\mathcal{R}_p^m[r, N]$;
- let a be a symbol in $\Sigma\Gamma_p^m[r, N]$, then the map $(U, V) \mapsto Op^{\text{BW}}(a(U; x, \xi))V$ is in $\Sigma\mathcal{M}_p^{m'}[r, N]$ for any $m' \geq m$;
- any smoothing operator in $\Sigma\mathcal{R}_p^{-\rho}[r, N]$ defines an element of $\Sigma\mathcal{M}_p^{m'}[r, N]$ for some $m' > 0$.

We now state a classical version of the action-theorem on Sobolev spaces for para-differential operators, whose proof can be found in the book by Metivier [31] (see Theorem 5.1.15 and formula (5.1.25) therein).

Proposition 2.22. (Action of para-differential operators 2). Consider a symbol $a \in \Gamma_p^m[r]$, then it defines a bounded operator from H^s to H^{s-m} with the following estimate

$$\|Op^{\text{BW}}(a(U; x, \xi))\|_{\mathcal{L}(H^s, H^{s-m})} \leq \sup_{\xi \in \mathbb{R}} \langle \xi \rangle^{-m} \|a(U; x, \xi)\|_{L_x^\infty}.$$

Remark 2.23. Note that in Prop. 2.22 we have better estimates in terms of regularity of the function $a(U; \cdot)$ in the r.h.s. compared to Prop. 2.20. In Prop. 2.20 we have more information on the smallness of the symbol in terms of its dependence of the function U but we have to pay the price of losing some derivatives in the r.h.s.. In Section 2.5 we shall need the optimal estimate in term of regularity given by Prop. 2.22 in several proofs.

2.3. Symbolic calculus and Compositions theorems. We introduce the following differential operator

$$\sigma(D_x, D_\xi, D_y, D_\eta) := D_\xi D_y - D_x D_\eta,$$

where $D_x := \frac{1}{i} \partial_x$ and D_ξ, D_y, D_η are similarly defined. Given two symbols a and b , in the following we define a new symbol $a\#_\rho b$ which turns out to be the symbol of the composition of the para-differential operators generated by a and b modulo smoothing operators of order $-\rho$.

Definition 2.24. (Asymptotic expansion of composition symbol). Let ρ, p, q be in \mathbb{N} , $m, m' \in \mathbb{R}$, $r > 0$. Consider $a \in \Sigma\Gamma_p^m[r, N]$ and $b \in \Sigma\Gamma_q^{m'}[r, N]$. For U in $B_r(H^\sigma)$ we define, for $\rho < \sigma - s_0$, the symbol

$$(a\#_\rho b)(U; x, \xi) := \sum_{k=0}^{\rho} \frac{1}{k!} \left(\frac{i}{2} \sigma(D_x, D_\xi, D_y, D_\eta) \right)^k \left[a(U; x, \xi) b(U; y, \eta) \right]_{|_{x=y, \xi=\eta}} \quad (2.31)$$

modulo symbols in $\Sigma\Gamma_{p+q}^{m+m'-\rho}[r, N]$.

- The symbol $a\#_\rho b$ belongs to $\Sigma\Gamma_{p+q}^{m+m'}[r, N]$.
- We have the expansion $a\#_\rho b = ab + \frac{1}{2i} \{a, b\} + \dots$, up to a symbol in $\Sigma\Gamma_{p+q}^{m+m'-2}[r, N]$, where $\{a, b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b$ denotes the Poisson bracket.

Proposition 2.25. (Composition of Bony-Weyl operators). *Let ρ, p, q be in \mathbb{N} , $m, m' \in \mathbb{R}$, $r > 0$. Consider $a \in \Sigma\Gamma_p^m[r, N]$ and $b \in \Sigma\Gamma_q^{m'}[r, N]$. Then*

$$R(U) := Op^{\text{BW}}(a(U; x, \xi)) \circ Op^{\text{BW}}(b(U; x, \xi)) - Op^{\text{BW}}((a\#b)(U; x, \xi)) \quad (2.32)$$

is a non-homogeneous smoothing remainder in $\Sigma\mathcal{R}_{p+q}^{-\rho+m+m'}[r, N]$.

Proof. We give the proof of the theorem in the case that the symbols a and b are non-homogenous in the classes $\Gamma_p^m[r]$ and $\Gamma_q^{m'}[r]$ respectively. In the case of composition between operators generated by homogeneous symbols we refer to the proof of Prop. 3.12 in [9] since our classes coincide with the ones therein. We set

$$\begin{aligned} \tilde{a}_\chi(x, \xi) &= d_U^{k_1} a(U; x, \xi) [h_{n_1}, \dots, h_{n_{k_1}}] \\ \tilde{b}_\chi(x, \xi) &= d_U^{k_2} b(U; x, \xi) [h_{n_{k_1}}, \dots, h_{n_{k_1+k_2}}]. \end{aligned}$$

We have that the $(k_1 + k_2)$ -differential with respect to U of the expression in (2.32) applied to the vector $[h_{n_1}, \dots, h_{n_{k_1+k_2}}]$ may be written as $Op^W(r_1(U; x, \xi) + r_2(U; x, \xi))$ where

$$\begin{aligned} r_1(U; \cdot) &= \tilde{a}_\chi \# \tilde{b}_\chi - (\tilde{a}_\chi \# \tilde{b}_\chi)_\rho, \\ r_2(U; \cdot) &= (\tilde{a}_\chi \# \tilde{b}_\chi)_\rho - (\tilde{a} \# \tilde{b})_{\rho, \chi}, \end{aligned}$$

where $a\#b$ is the symbol of the composition given in Lemma 3.14 in [9]. We estimate the term coming from $r_1(U; \cdot)$, the other one is similar. At this point one has to use Lemma 3.13 in [9] by using the fact the the symbols a and b satisfy the estimate (2.11). By following the proof of the *action*-theorem 2.20 one can estimate the $H^{s-m-m'+\rho-2}$ of $Op^W(r_1(U; \cdot))v$ for some regular function v obtaining the estimate of smoothing remainder (2.20) (up to renaming $\rho - 2 \rightsquigarrow \rho$). The translation invariance property (2.19) follows as in [11]. \square

Remark 2.26. *As proved in the remark after the proof of Proposition 3.12 in [9], the remainder obtained by the composition of para-differential operators in Proposition 2.25 has actually better estimates than (2.20), i.e. it is bounded from H^s to $H^{s+\rho-(m+m')}$ for any s , with operator norm bounded by $\|U\|_{s_0}^{p+q}$.*

Proposition 2.27. (Compositions). *Let $m, m', m'' \in \mathbb{R}$, $N, p_1, p_2, p_3, \rho \in \mathbb{N}$, $p_1 + p_2 < N$, $p_2 + p_3 < N$, $\rho \geq 0$ and $r > 0$. Let $a \in \Sigma\Gamma_{p_1}^m[r, N]$, $R \in \Sigma\mathcal{R}_{p_2}^{-\rho}[r, N]$ and $M \in \Sigma\mathcal{M}_{p_3}^{m''}[r, N]$. Then*

- (i) $R(U) \circ Op^{\text{BW}}(a(U; x, \xi))$, $Op^{\text{BW}}(a(U; x, \xi)) \circ R(U; t)$ are in $\Sigma\mathcal{R}_{p_1+p_2}^{-\rho+m}[r, N]$.
- (ii) $R(U) \circ M(U)$ and $M(U; t) \circ R(U)$ are smoothing operators in $\Sigma\mathcal{R}_{p_2+p_3}^{-\rho+m''}[r, N]$.
- (iii) If $R_2 \in \widetilde{\mathcal{R}}_{p_2}^{-\rho}$ then $R_2(U, M(U)U)$ belongs to $\Sigma\mathcal{R}_{p_2+p_3}^{-\rho+m''}[r, N]$.
- (iv) Let c be in $\widetilde{\Gamma}_p^m$, $p \in \mathbb{N}$. Then

$$U \rightarrow c_M(U; x, \xi) := c(U, \dots, U, M(U)U; x, \xi)$$

is in $\Sigma\Gamma_{p+p_3}^m[r, N]$. If the symbol c is independent of ξ (i.e. c is in $\widetilde{\mathcal{F}}_p$), so is the symbol c_M (thus it is a function in $\Sigma\mathcal{F}_{p+p_3}[r, N]$). Moreover if c is a symbol in $\Gamma_N^m[r]$ then the symbol c_M is in $\Gamma_N^m[r]$.

- (v) $Op^{\text{BW}}(c(U, \dots, U, W; x, \xi))|_{W=M(U)U} = Op^{\text{BW}}(b(U; x, \xi)) + R(U)$ where

$$b(U; x, \xi) := c(U, \dots, U, M(U)U; x, \xi)$$

and $R(U)$ is in $\Sigma\mathcal{R}_{K, K', p+p_1}^{-\rho}[r, N]$.

Proof. We prove (i) in the case of the operator $H(U) := R(U) \circ Op^{\text{BW}}(a(U; x, \xi))$. Let $s_0 - m \gg 1$, we have to estimate the quantity $\|d_U^k H(U)(h_1, \dots, h_k)\|_{s+\rho-dk-m}$. We give the proof in the case that the symbol a is in $\Gamma_{p_1}^m[r, N]$ and R is in $\mathcal{R}_{p_2}^{-\rho}[r, N]$. By using estimate (2.20) for the operator $R(U)$ we may bound from

above, modulo the constant appearing in (2.20), the previous quantity by

$$\begin{aligned}
& \sum_{k_1+k_2=k} \left\{ \mathbf{1}_{\{p-k \geq 0\}} \|U\|_{s_0-m}^{\max\{0, p-k-1\}} \|U\|_{s-m} \|Op^{\text{BW}}(d_U^{k_2} a(U; x, \xi)[h_1, \dots, h_{k_2}])V\|_{s_0-m} \prod_{j=k_2+1}^k \|h_{k_j}\|_{s_0-m} \right. \\
& + \|U\|_{s_0-m}^{\max\{0, p-k\}} \|Op^{\text{BW}}(d_U^{k_2} a(U; x, \xi)[h_1, \dots, h_{k_2}])V\|_{s_0-m} \sum_{i=k_2+1}^k \prod_{j=k_2+1, j \neq i}^k \|h_j\|_{s_0-m}^v \|h_i\|_{s-m} \\
& \left. + \|U\|_{s_0-m}^{\max\{0, p-k\}} \|Op^{\text{BW}}(d_U^{k_2} a(U; x, \xi)[h_1, \dots, h_{k_2}])V\|_{s-m} \prod_{j=k_2+1}^k \|h_j\|_{s_0-m}, \right. \quad (2.33)
\end{aligned}$$

where by $\mathbf{1}_A$ we denoted the characteristic function of the set A . We prove the result in for the addendum in (2.33), the others may be similarly bounded. We just have to use Theorem 2.20 obtaining

$$\begin{aligned}
& \mathbf{1}_{\{p-k \geq 0\}} \|U\|_{s_0-m}^{\max\{0, p-k-q\}} \|U\|_{s-m} \prod_{j=k_2+1}^k \|h_j\|_{s_0-m} \\
& \times \|U\|_{s_0-m}^{\max\{0, p-k-1\}} \|V\|_{s_0} \prod_{j=1}^{k_2} \|h_j\|_{s_0-m},
\end{aligned}$$

which is the first line of (2.20) up to renaming $s_0 \rightsquigarrow s_0 - m$. See Proposition 3.16, 3.17, 3.18 in [9]. The translation invariance properties for the composed operators and symbols in items (i)-(v) follow as in the proof of Proposition 2.25 in [11]. \square

2.4. Real to real and self-adjoint matrices of operators. We discuss some algebraic properties of matrices of operators.

Real-to-real operators. Given a linear operator $R(U)[\cdot]$ acting on \mathbb{C}^2 (it may be a smoothing operator in $\Sigma \mathcal{R}_1^{-\rho}[r, N]$ or $Op^{\text{BW}}(a(U; x, \xi))$ with $a \in \Sigma \Gamma_1^m[r, N]$) we associate the linear operator defined by the relation

$$\overline{R(U)[V]} := \overline{R(U)[\overline{V}]}, \quad \forall V \in \mathbb{C}^2. \quad (2.34)$$

We say that a matrix of operators acting in \mathbb{C}^2 is *real-to-real*, if it has the form

$$R(U) = \begin{pmatrix} R_1(U) & R_2(U) \\ \overline{R_2(U)} & \overline{R_1(U)} \end{pmatrix}. \quad (2.35)$$

Notice that

- if $R(U)$ is a real-to-real matrix of operators then, given $V = \begin{bmatrix} v \\ \overline{v} \end{bmatrix}$, the vector $Z := R(U)[V]$ has the form $Z = \begin{bmatrix} z \\ \overline{z} \end{bmatrix}$, namely the second component is the complex conjugated of the first one.
- If a matrix of symbols $A(U; x, \xi)$, in some class $\Sigma \Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$, has the form

$$A(U; x, \xi) = \begin{pmatrix} a(U; x, \xi) & b(U; x, \xi) \\ \overline{b(U; x, -\xi)} & \overline{a(U; x, -\xi)} \end{pmatrix}, \quad (2.36)$$

then the matrix of operators $Op^{\text{BW}}(A(U; x, \xi))$ is real-to-real.

Definition 2.28. (Classical symbol). A symbol a of order d is called *classical* if $a(x, \xi) \sim \sum a_j(x, \xi)$ and a_j is positive homogeneous with respect to ξ of order $d - j$.

Self-adjoint para-differential operators. We now study self-adjoint matrices para-differential operators. We shall restrict to the case that such matrices are reality preserving, i.e. matrices of the form (2.35). Consider an operator \mathfrak{F} of the form

$$\mathfrak{F} := \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}, \quad (2.37)$$

for A, B linear operators and denote by \mathfrak{F}^* its adjoint with respect to the scalar product

$$(U, V)_{\mathbf{H}^0} := \int_{\mathbb{T}} U \cdot \bar{V} dx, \quad U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}, V = \begin{bmatrix} v \\ \bar{v} \end{bmatrix} \in H^s(\mathbb{T}, \mathbb{C}^2),$$

i.e.

$$(\mathfrak{F}U, V)_{\mathbf{H}^0} = (U, \mathfrak{F}^*V)_{\mathbf{H}^0}.$$

One can check that

$$\mathfrak{F}^* := \begin{pmatrix} A^*(U; t) & \bar{B}^*(U; t) \\ B^*(U; t) & \bar{A}^*(U; t) \end{pmatrix},$$

where A^* and B^* are respectively the adjoints of the operators A and B with respect to the complex scalar product on $L^2(\mathbb{T}; \mathbb{C})$ in (2.6).

Definition 2.29. (Self-adjointness). *Let \mathfrak{F} be a reality preserving linear operator of the form (2.37). We say that \mathfrak{F} is self-adjoint if $A, A^*, B, B^* : H^s \rightarrow H^{s'}$, for some $s, s' \in \mathbb{R}$ and*

$$A^* = A, \quad \bar{B} = B^*. \quad (2.38)$$

We consider para-differential operators of the form:

$$\begin{aligned} Op^{\text{BW}}(A(U; x, \xi)) &:= Op^{\text{BW}} \begin{pmatrix} a(U; x, \xi) & b(U; x, \xi) \\ b(U; x, -\xi) & a(U; x, -\xi) \end{pmatrix} \\ &:= \begin{pmatrix} Op^{\text{BW}}(a(U; x, \xi)) & Op^{\text{BW}}(b(U; x, \xi)) \\ Op^{\text{BW}}(b(U; x, -\xi)) & Op^{\text{BW}}(a(U; x, -\xi)) \end{pmatrix}, \end{aligned} \quad (2.39)$$

where a and b are symbols in $\Gamma_p^m[r]$ and U is a function belonging to $B_r^K(H^{s_0})$ for some s_0 large enough. Note that the matrix of operators in (2.39) has the form (2.37). Moreover it is self-adjoint if and only if

$$a(U; x, \xi) = \overline{a(U; x, \xi)}, \quad b(U; x, -\xi) = \overline{b(U; x, \xi)}, \quad (2.40)$$

indeed conditions (2.38) on these operators read

$$\begin{aligned} (Op^{\text{BW}}(a(U; x, \xi)))^* &= Op^{\text{BW}}(\overline{a(U; x, \xi)}), \\ \overline{Op^{\text{BW}}(b(U; x, \xi))} &= Op^{\text{BW}}(\overline{b(U; x, -\xi)}). \end{aligned} \quad (2.41)$$

2.5. Flows of para-differential operators. The main result of this section is Theorem 2.31. In this Theorem we analyze the well-posedness of some non linear flows generated by para-differential operators. These flows will be used in the next Sections as auxiliary flows in order to generate non linear changes of coordinates. More precisely we study the system (2.51) in the cases of generators as in (2.52), (2.53), (2.54) and (2.55). We shall define a sequence of linear problems approximating the non linear one. In section 2.5.1 we study the linear problem associated to problem (2.51). In section 2.5.2 we prove Theorem 2.31.

2.5.1. *Linear flows.* Let $0 < r \ll 1$, $u \in B_r(H^s)$ we define the symbols

$$\begin{aligned} \tilde{A}_1(\tau, u; x, \xi) &:= A(\tau, u; x)\xi, \quad \text{with } A \in \mathcal{F}_1^{\mathbb{R}}[2r], \\ A_j(\tau, u; \xi) &\in \Sigma\Gamma_p^j[r, N], \quad j \geq 0, \quad \text{and } \begin{cases} 1 \leq j, \text{ real and independent of } x \\ j < 1, A_j - \bar{A}_j \in \Sigma\Gamma_p^0[r, N], \end{cases} \end{aligned}$$

and we consider the problem

$$\begin{cases} \partial_\tau z = Op^{\text{BW}}(i\mathfrak{A}(\tau, u; x, \xi))[z], \\ z(0) = u, \end{cases} \quad (2.42)$$

where \mathfrak{A} equals either \tilde{A}_1 or A_j . We have the following.

Lemma 2.30. *Let d be the number appearing in the definition of symbols 2.1 item (ii). Consider the problem (2.42) with $\mathfrak{A} = A_j$, then for any $0 \leq j \leq d$ the following holds true. For any $s \geq s_0$ there is $r_0, C > 0$ such that for $0 \leq r \leq r_0$, one has:*

$$\begin{aligned} \|z(\tau)\|_{H^s} &\leq \|u\|_{H^s} (1 + C\|u\|_{H^{s_0}}) \quad \forall 0 \leq \tau \leq 1, \\ \|\partial_\tau^k z(\tau)\|_{H^{s-jk}} &\leq C\|u\|_{H^s} \|u\|_{H^{s_0}}^k, \quad 1 \leq dk \leq s - s_0. \end{aligned} \quad (2.43)$$

Moreover

$$\|(d_u z)(u)[h]\|_{H^{s-j}} \leq \|h\|_{H^s} (1 + C\|u\|_{H^s}), \quad \forall 0 \leq \tau \leq 1 \quad \forall h \in H^s, \quad (2.44)$$

and, for any $2 \leq dk \leq s - s_0$, we have that

$$\|(d_u^k z)(u)[h_1, \dots, h_k]\|_{H^{s-jk}} \leq C\|h_1\|_{H^s} \cdots \|h_k\|_{H^s}, \quad \forall 0 \leq \tau \leq 1 \quad \forall h_i \in H^s \quad i = 1, \dots, k. \quad (2.45)$$

for some constant $C = C(s, k) > 0$ independent of \mathfrak{A} . The same estimates hold true with $j = 1$ in the case that $\mathfrak{A} = \tilde{A}_1$.

Proof. We give the proof in the case of \tilde{A}_1 , for A_j with $j > 0$ the proof is similar and for $j = 0$ it is standard theory of Banach spaces ODEs. By reasoning as in Lemma 3.22 in [9] one can prove that the flow $\Phi^\tau = \Phi^\tau(u)$

$$\partial_\tau \Phi^\tau(u) = Op^{BW}(iA(\tau, u; x)\xi)[\Phi^\tau(u)], \quad \Phi^0(u) = \text{Id} \quad (2.46)$$

is well posed on H^s and satisfies

$$\|\Phi^\tau(u)h\|_{H^s} \leq C\|h\|_{H^s} (1 + \|u\|_{H^{s_0}}) \quad \forall 0 \leq \tau \leq 1. \quad (2.47)$$

In particular setting $z(\tau) = \Phi^\tau(u)u$ the (2.43) holds.

To prove (2.44) we argue as follows. By differentiating in u the problem (2.42) we have

$$\begin{cases} \partial_\tau (d_u z)(u)[h] = Op^{BW}(iA(\tau, u; x)\xi)[(d_u z)(u)[h]] + f(\tau; u), \\ (d_u z(0))(u)[h] = h, \end{cases} \quad (2.48)$$

where

$$f(\tau; u) := Op^{BW}(i(d_u A)(\tau, u; x)[h]\xi)[z]. \quad (2.49)$$

By estimate (2.11) and the Lemma of action of para-differential operators on Sobolev spaces we have

$$\|f(\tau, u)\|_{H^{s-1}} \lesssim_s \|h\|_{s_0} \|z\|_{H^s} \stackrel{(2.43)}{\lesssim_s} \|u\|_{H^s} \|h\|_{H^{s_0}}, \quad \forall 0 \leq \tau \leq 1. \quad (2.50)$$

By Duhamel formula we have (recall (2.46))

$$(d_u z)(u)[h] = \Phi^\tau h + \Phi^\tau \int_0^\tau (\Phi^\sigma)^{-1} f(\sigma; u) d\sigma$$

which, together with (2.47), (2.50), implies (2.44). Iterating this reasoning one gets the (2.45). \square

2.5.2. Non-linear flows. Consider the Cauchy problem

$$\begin{cases} \partial_\tau z = Op^{BW}(if(\tau, z; x, \xi))[z], \\ z(0) = u, \quad u \in B_r(H^s), \end{cases} \quad (2.51)$$

for some $r > 0$ small and $s > 0$ large, where f is a symbol assuming one of the following forms:

$$f(\tau, u; x, \xi) := B(\tau, u; x)\xi, \quad B(\tau, u; x) \in \Sigma_{\Gamma_1^{\mathbb{R}}}^{\mathbb{R}}[r, N], \quad (2.52)$$

$$f(\tau, u; x, \xi) \in \Sigma_{\Gamma_1^m}^m[r, N], \quad 0 < m < 1, \quad \overline{f(\tau, u; x, \xi) - f(\tau, u; x, \xi)} \in \Sigma_{\Gamma_1^0}^0[r, N], \quad (2.53)$$

$$f(\tau, u; x, \xi) \in \Sigma_{\Gamma_1^m}^m[r, N], \quad m \leq 0, \quad (2.54)$$

$$f(\tau, u; \xi) \in \Sigma_{\Gamma_1^m}^m[r, N], \quad m \geq 0, \quad f(\tau, u; x) \text{ real and independent of } x, \quad (2.55)$$

with $\tau \in [0, 1]$. We also assume that the symbol $f(\tau, u; x, \xi)$ satisfies the estimates (2.8)-(2.11) uniformly in $\tau \in [0, 1]$. The key result of this section is the following.

Theorem 2.31. (The non-linear transport flow). *For any $s \geq s_0$ there is $r_0, C > 0$ such that for $0 \leq r \leq r_0$, the following holds. There exists a unique solutions of (2.51) (with generator in (2.52), (2.53), (2.54), (2.55)) $z(\tau) := \Phi^\tau(u)$, defined for $\tau \in [0, 1]$ such that*

$$z(\tau) \in \bigcap_{k=0}^K C^k([0, 1]; H^{s-jk}), \quad 0 \leq jK \leq s - s_0. \quad (2.56)$$

In particular we have

$$\sup_{\tau \in [0, 1]} \|z(\tau)\|_{H^s} \leq \|u\|_{H^s} (1 + C\|u\|_{H^{s_0}}), \quad (2.57)$$

$$\sup_{\tau \in [0, 1]} \|\partial_\tau^k z(\tau)\|_{H^{s-jk}} \leq C\|u\|_{H^s} \|u\|_{H^{s_0}}^k, \quad 1 \leq k \leq s - s_0, \quad (2.58)$$

$$\sup_{\tau \in [0, 1]} \|(d_u^k \Phi^\tau)(u)[h_1, \dots, h_k]\|_{H^{s-jk}} \leq C\|h_1\|_{H^s} \cdots \|h_k\|_{H^s}, \quad \forall h_i \in H^s \quad i = 1, \dots, k, \quad (2.59)$$

with $j = 1$ in the case of (2.52), $j = m$ for the cases (2.53), (2.54), (2.55). Finally we have that $\Phi^\tau(u) = u + M(\tau; u)[u]$ with $M(\tau; u) \in \Sigma \mathcal{M}_1[r, N]$ with estimates uniform in $\tau \in [0, 1]$.

The proof of Theorem 2.31 relies on an iterative scheme based on the ideas used in [25]. We give the proof of the result of the case $f(\tau, Z; x, \xi)$ in (2.52), the others are similar. Let us introduce the following sequence of linear problems. Let $u^{(0)} \in H^s$ such that $\|u^{(0)}\|_{H^s} \leq r$ for some $r > 0$. For $n = 0$ we set

$$\mathcal{A}_0 := \begin{cases} \partial_\tau u_0 = 0, \\ u_0(0) = u^{(0)}. \end{cases} \quad (2.60)$$

The solution of this problem exists and it is unique, defined for any $\tau \in \mathbb{R}$ by standard linear theory. For $n \geq 1$, assuming u_n satisfies (2.56) for some $s_0, K > 0$ and $s \geq s_0$, we define the Cauchy problem

$$\mathcal{A}_n := \begin{cases} \partial_\tau u_n - Op^{\text{BW}}(iB(\tau, u_{n-1}; x)\xi)u_n = 0, \\ u_n(0) = u^{(0)}, \end{cases} \quad (2.61)$$

where the symbol $B(\tau, z; x)$ is defined in (2.52). One has to show that each problem \mathcal{A}_n admits a unique solution U_n defined for $\tau \in [0, 1]$. We use Lemma 2.30 in order to prove the following.

Lemma 2.32. *If r is sufficiently small, then there exists $s_0 > 0$ such that for all $s \geq s_0$ the following holds. There exists a constant θ , depending on r and s , such that for any $n \geq 0$ one has:*

(S1)_n *for $0 \leq m \leq n$ there exists a function u_m in $u_m \in \bigcap_{k=0}^K C^k([0, 1]; H^{s-k})$, $0 \leq K \leq s - s_0$ such that*

$$\begin{aligned} \|u_m(\tau)\|_{H^s} &\leq \|u_0\|_{H^s} (1 + C\|u_0\|_{H^{s_0}}) \quad \forall 0 \leq \tau \leq 1, \\ \|\partial_\tau^k u_m(\tau)\|_{H^{s-k}} &\leq C\|u_0\|_{H^s} \|u_0\|_{H^{s_0}}^k, \quad 1 \leq dk \leq s - s_0, \end{aligned} \quad (2.62)$$

$$\|(d_u^k u_m)(u)[h_1, \dots, h_k]\|_{H^{s-k}} \leq C\|h_1\|_{H^s} \cdots \|h_k\|_{H^s}, \quad \forall 0 \leq \tau \leq 1 \quad \forall h_i \in H^s \quad i = 1, \dots, k.$$

for some $C > 0$ independent of m, n , which is the unique solution of the problem \mathcal{A}_m ;

(S2)_n *for $0 \leq m \leq n$ one has*

$$\sum_{k=0}^K \|\partial_\tau (u_m - u_{m-1})\|_{H^{s'-k}} \leq 2^{-m} r, \quad s_0 \leq s' \leq s - 1, \quad \forall 0 \leq \tau \leq 1, \quad (2.63)$$

where $U_{-1} := 0$.

Proof. We argue by induction. The $(S1)_0$ and $(S2)_0$ are trivial (see the problem (2.60)). Suppose that $(S1)_{n-1}, (S2)_{n-1}$ hold with a constant $C = C(s) \gg 1$. We show that $(S1)_n, (S2)_n$ hold with the same constant C . By estimates (2.62) on u_{n-1} we deduce that $\|u_{n-1}\|_{H^s} \leq 2r$ (if $r > 0$ is small enough) and the symbol

$B(\tau, u_{n-1}, x)\xi$ satisfies the hypotheses of Lemma 2.30. Then the (2.62) on u_n follows by estimates (2.43)-(2.45). Let us check $(S2)_n$. Setting $v_n = v_n - v_{n-1}$ we have that

$$\begin{cases} \partial_\tau v_n - Op^{\text{BW}}(iB(\tau, u_{n-1}; x)\xi)v_n + f_n = 0, \\ v_n(0) = 0, \end{cases} \quad (2.64)$$

where

$$f_n := iOp^{\text{BW}}\left(B(\tau, u_{n-1}; x)\xi - B(\tau, u_{n-2}; x)\xi\right)U_{n-1}. \quad (2.65)$$

Notice that

$$B(\tau, u_{n-1}; x) - B(\tau, u_{n-2}; x) = (d_u B)(\tau, u_{n-1} + \sigma v_{n-1})[v_{n-1}]$$

for some $\sigma \in [0, 1]$. Moreover, by (2.11), we have

$$\|(d_u B)(\tau, u_{n-1} + \sigma v_{n-1})[v_{n-1}]\|_{L_x^\infty} \lesssim_s C \|v_{n-1}\|_{H^{s_0}}.$$

Therefore, by Proposition 2.22, we have

$$\|f_n\|_{H^{s'}} \leq \|iOp^{\text{BW}}\left(B(u_{n-1}; x)\xi - B(u_{n-2}; x)\xi\right)U_{n-1}\|_{H^{s'}} \lesssim_s C \|v_{n-1}\|_{H^{s_0}} \|u_{n-1}\|_{H^{s'+1}}. \quad (2.66)$$

Let $\psi_{u_{n-1}}(\tau)$ be the flow of system (2.64) with $f_n = 0$, which is given by Lemma 2.30. The Duhamel formulation of (2.64) is

$$v_n(\tau) = \psi_{u_{n-1}}(\tau) \int_0^\tau (\psi_{u_{n-1}}(\sigma))^{-1} f_n(\sigma) d\sigma. \quad (2.67)$$

Then using the inductive hypothesis (2.62), inequality (2.43) we get

$$\|v_n\|_{H^{s'}} \leq C_s r \|v_{n-1}\|_{H^{s'}}, \quad \forall t \in [0, 1], \quad (2.68)$$

where $C_s > 0$ is a constant depending s . If $C_s r \leq 1/2$ then we have $\|v_n\|_{H^{s'}} \leq 2^{-n} r$ for any $t \in [0, 1]$ which is the $(S2)_n$. \square

Proof of Theorem 2.31. By Lemma 2.32 we know that the sequence u_n defined by the problem (2.61) converges strongly to a function z in $C^0([0, 1], H^{s'})$ for any $s' \leq s - 1$ and, up to subsequences,

$$u_n(\tau) \rightharpoonup U(\tau), \quad \text{in } H^s, \quad \partial_\tau u_n(\tau) \rightharpoonup \partial_\tau u(\tau), \quad \text{in } H^{s-1}, \quad (2.69)$$

for any $\tau \in [0, 1]$, moreover the function u is in $L^\infty([0, 1], H^s) \cap \text{Lip}([0, 1], H^{s-1})$. We claim that, z solves the (2.51), it belongs to $C^0([0, 1]; H^s) \cap C^1([0, 1]; H^{s-1})$ and it is unique. This can be proved by classical arguments, for instance following the proof of Theorem 1.1 in section 6 in [25]. The (2.58), (2.59) can be deduced by differentiating the equation (2.51) (or using that z is weak limit of the sequence u_n satisfying the estimates (2.62)). The theorem in the cases (2.53), (2.54), (2.55) may be proved exactly in the same way modifying Lemma 2.32 according to Lemma 2.30 in the case that $\mathfrak{A} = A_j = f$ in (2.53), (2.54), (2.55), with $j \rightsquigarrow m$. \square

3. MAIN RESULTS AND APPLICATIONS TO PDES

In this section we state the main results of this paper. Consider

$$m = \frac{k}{2}, \quad k \in \mathbb{N}, \quad (3.1)$$

and let $f_m \in \Gamma_0^m$ be a real valued, even in $\xi \in \mathbb{R}$ classical symbol, i.e. it admits an expansion in decreasing homogeneous symbols. Let us define the operator Ω as

$$\Omega e^{ijx} = \omega_j e^{ijx}, \quad \omega_j := f_m(j), \quad \forall j \in \mathbb{Z}. \quad (3.2)$$

In other words $\Omega := Op^{\text{BW}}(f_m(\xi))$. We consider symbols of the following form

$$\begin{aligned} a(U; x, \xi) &= (1 + a_m(U; x))f_m(\xi) + a_{m'}(U; x, \xi), \quad m > 1, \quad m' = m - \frac{1}{2} \text{ or } m' = m - 1, \\ b(U; x, \xi) &= b_m(U; x)f_m(\xi) + b_{m'}(U; x, \xi), \\ a_m &\in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N], \quad b_m \in \Sigma\mathcal{F}_1[r, N], \quad a_{m'}, b_{m'} \in \Sigma\Gamma_1^{m'}[r, N], \\ a_{m'}(U; x, \xi) - \overline{a_{m'}(U; x, \xi)} &\in \Sigma\Gamma_1^0[r, N], \end{aligned} \tag{3.3}$$

and the system

$$\begin{cases} \dot{U} = X(U) := iEOp^{\text{BW}}(A(U; x, \xi))[U] + R(U)[U] \\ U(0) = U_0 \in H^s \times H^s \end{cases} \tag{3.4}$$

with $R \in \Sigma\mathcal{R}_1^{-p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and

$$A(U; x, \xi) := \begin{pmatrix} a(U; x, \xi) & b(U; x, \xi) \\ b(U; x, -\xi) & a(U; x, -\xi) \end{pmatrix}. \tag{3.5}$$

3.1. Regularization of para-differential vector fields. The main result of the paper is the following.

Theorem 3.1. (Non-linear Egorov). *There exist $s_0 > 0$, $r_0 > 0$ such that, for $s \geq s_0$, $r \leq r_0$ the following holds true. There exist an invertible map*

$$\Psi : B_r(H^s(\mathbb{T}; \mathbb{C}^2)) \cap \mathcal{U} \rightarrow H^s(\mathbb{T}; \mathbb{C}^2) \cap \mathcal{U},$$

such that, setting

$$Z := \Psi(U), \quad \mathcal{Y}(Z) := d\Psi(\Psi^{-1}(Z))[X(\Psi^{-1}(Z))], \tag{3.6}$$

we have that

$$\begin{cases} \dot{Z} = \mathcal{Y}(Z) := iE\mathcal{L}(Z)[Z] + \mathcal{Q}(Z)[Z] \\ Z(0) = \Psi(U_0) \end{cases} \tag{3.7}$$

where $\mathcal{Q} \in \Sigma\mathcal{R}_1^{-p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and (see (3.2))

$$\begin{aligned} \mathcal{L}(Z) &:= Op^{\text{BW}} \begin{pmatrix} f_m(\xi) & 0 \\ 0 & f_m(\xi) \end{pmatrix} + Op^{\text{BW}}(\mathfrak{M}(Z; \xi)), \quad \mathfrak{M}(Z; \xi) := \begin{pmatrix} \mathfrak{m}(Z; \xi) & 0 \\ 0 & \overline{\mathfrak{m}(Z; -\xi)} \end{pmatrix}, \\ \mathfrak{m}(Z; \xi) &:= \mathfrak{m}_m(Z)f_m(\xi) + \mathfrak{m}_{m'}(Z; \xi), \quad \mathfrak{m}_m(Z) \in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N], \quad m' = m - \frac{1}{2} \\ \mathfrak{m}_{m'}(Z; \xi) &\in \Sigma\Gamma_1^{m'}[r, N], \quad \mathfrak{m}_{m'}(Z; \xi) - \overline{\mathfrak{m}_{m'}(Z; \xi)} \in \Sigma\Gamma_1^0[r, N]. \end{aligned} \tag{3.8}$$

Moreover, for any $s \geq s_0$, the maps $\Psi^{\pm 1}$ satisfy

$$\|\Psi^{\pm 1}(U)\|_{H^s} \leq \|U\|_{H^s} (1 + C\|U\|_{H^{s_0}}), \tag{3.9}$$

for some constant $C > 0$ depending on s .

Proof. We shall apply iteratively Theorems 5.3, 5.4, 5.1 and 5.2. □

Some comments on the theorem above are in order

- The Theorem above shows that a *system* as (3.4) can be reduced to a *diagonal* system with constant coefficients plus a smoothing remainder. This will be achieved into two steps: (i) a *block-diagonalization* of the system (which is the content of Theorems 5.3, 5.4); (ii) a *reduction to constant coefficients* of the diagonal terms (which is the content of Theorems 5.1, 5.2).

- The parity assumption of the Fourier multiplier $f_m(\xi)$ is used only in the *block-diagonalization* procedure in sections 5.2. Therefore, Theorem 3.1 applies also to *scalar* equations of the form

$$\dot{u} = iOp^{BW}(a(u; x, \xi))u + Q(u)[u], \quad (3.10)$$

where $a(u; x, \xi)$ is a scalar symbol as in (3.3) and $Q \in \Sigma\mathcal{R}_1^{-\rho}[r, N]$. In this case, we assume that $f_m(-\xi) = f_m(\xi)$, i.e. it is *odd* in $\xi \in \mathbb{R}$, then we have the following result which is a consequence of Theorem 3.1.

Corollary 3.2. *There exist $s_0 > 0$, $r_0 > 0$ such that, for $s \geq s_0$, $r \leq r_0$ the following holds true. There exist an invertible map $\Psi : B_r(H^s(\mathbb{T}; \mathbb{R})) \rightarrow B_r(H^s(\mathbb{T}; \mathbb{R}))$ satisfying estimates like (3.9) and*

$$\dot{z} = iOp^{BW}(f_m(\xi) + m(z; \xi))z + \tilde{Q}(z)[z], \quad (3.11)$$

where $z = \Psi(u)$, $m(z; \xi)$ is as in (3.8) and $\tilde{Q} \in \Sigma\mathcal{R}_1^{-\rho}[r, N]$.

In the case that the vector field $X(U)$ in (3.4) has an Hamiltonian structure (see (1.18)) we also have a version of Theorem 3.1 which preserves the symplectic structure of the vector field.

Theorem 3.3. (Symplectic structure). *Assume that the vector field $X(U)$ in (3.4) is Hamiltonian, i.e.*

$$X(U) := X_H(U) := iJ\nabla H(U),$$

for some Hamiltonian $H(U) : B_R(H^s(\mathbb{T}; \mathbb{C})) \rightarrow \mathbb{R}$. Then the result of Theorem 3.1 holds with a symplectic map Ψ and the vector field \mathcal{Y} in (3.6) is Hamiltonian with respect to the symplectic form λ in (1.19). Moreover the operator $\mathcal{L}(Z)$ in (3.8) is self-adjoint.

3.2. Poincaré-Birkhoff normal forms. In this section we state an abstract Birkhoff normal form result for vector a field $\mathcal{Y}(Z)$ as in (3.7) given by Theorem 3.1 assuming that the starting vector field X is Hamiltonian. Recalling (3.2), thanks to Theorems 3.1 and 3.3 the system (3.7) is rewritten as

$$\dot{Z} = \mathcal{Y}(Z) = iE\Omega Z + iEOp^{BW}(\mathfrak{M}(Z; \xi))Z + \mathcal{Q}(Z)[Z], \quad (3.12)$$

where $\mathfrak{M}(Z; \xi) \in \Sigma\Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is given in (3.8), \mathcal{Q} is in $\Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathcal{Y}(Z)$ in (3.12) is an Hamiltonian vector field. Furthermore the matrix $\mathfrak{M}(Z; \xi)$ is self-adjoint, i.e. it satisfies (2.40). Recalling the Definitions 2.1, 2.10 and the remarks under the Definition 2.15 we have that

$$\mathcal{M}(Z) := iEOp^{BW}(\mathfrak{M}(Z; \xi)) + \mathcal{Q}(Z) \in \Sigma\mathcal{M}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C}),$$

hence $\dot{Z} = iE\Omega Z + \mathcal{M}(Z)[Z]$. We also assume that the frequencies (3.2) are such that

$$\omega_0 \neq 0. \quad (3.13)$$

In order to state the main result of the section we need some further definitions.

Definition 3.4. (Non-resonance conditions). *We say that the linear frequencies ω_j in (3.2) are not resonant, at order $N \geq 1$, if the following holds. There are $N_0 > 0$ and $c > 0$ such that, for any $1 \leq p \leq N$, one has*

$$|\sigma_1\omega_{j_1} + \dots + \sigma_p\omega_{j_p}| \geq c \max\{\langle j_1 \rangle, \dots, \langle j_p \rangle\}^{-N_0}, \quad \forall \sigma_i = \pm, j_i \in \mathbb{Z}, i = 1, \dots, p, \quad (3.14)$$

unless p is even and, up to permutations, one has

$$\sigma_i = \sigma_{\frac{p}{2}+i}, \quad |j_i| = |j_{\frac{p}{2}+i}|, \quad i = 1, \dots, p. \quad (3.15)$$

Let $\vec{\sigma} := (\sigma_1, \dots, \sigma_p) \in \{\pm\}^p$, $\vec{j} = (j_1, \dots, j_p) \in \mathbb{Z}^p$, we define the resonant set \mathcal{S}_p as

$$\mathcal{S}_p := \{(\vec{\sigma}, \vec{j}) \in \{\pm\}^p \times \mathbb{Z}^d : (3.15) \text{ holds}\} \quad (3.16)$$

for p even and $\mathcal{S}_p = \emptyset$ for p odd.

The aim of this section is to conjugate, if Ω is non-resonant, the system in (3.12) to another paradiifferential system of the same form whose symbols and smoothing remainders are *resonant*, up to terms of degree of homogeneity N , according to the following definition.

Definition 3.5. Let $a_p \in \tilde{\Gamma}_p^m$ independent of $x \in \mathbb{T}$, $R_p \in \tilde{\mathcal{R}}_p^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ and recall the homogeneity expansions in Remarks 2.9, 2.14.

(i) Given the symbol a_p (recall the expansion (2.16)) we define the symbol $[[a_p]]$ as

$$[[a_p]](U; \xi) = \sum_{\substack{\sum_{i=1}^p \sigma_i j_i = 0 \\ (\vec{\sigma}, \vec{j}) \in \mathcal{S}_p}} (a_p)^{\sigma_1 \dots \sigma_p}_{j_1, \dots, j_p}(\xi) u_{j_1}^{\sigma_1} \dots u_{j_p}^{\sigma_p}. \quad (3.17)$$

We say that a_p is resonant if $a_p \equiv [[a_p]]$. Let $a \in \Sigma \Gamma_1^m[r, N]$ (independent of x) of the form

$$a(U; \xi) = \sum_{p=1}^{N-1} a_p(U; \xi) + a_N(U; \xi), \quad a_p \in \tilde{\Gamma}_p^m, \quad a_N \in \Sigma \Gamma_N^m[r],$$

we define the symbol $[[a]](U; \xi)$ as

$$[[a]](U; \xi) := \sum_{p=1}^{N-1} [[a_p]](U; \xi) + a_N(U; \xi),$$

where $[[a_p]](U; \xi)$ is in (3.17). For a diagonal matrix of symbols $A \in \Sigma \Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ of the form

$$A(U; \xi) = \begin{pmatrix} a(U; \xi) & \mathbf{0} \\ \mathbf{0} & \overline{a(U; -\xi)} \end{pmatrix},$$

we define

$$[[A]](U; \xi) := \begin{pmatrix} [[a]](U; \xi) & \mathbf{0} \\ \mathbf{0} & [[\overline{a}]](U; -\xi) \end{pmatrix}. \quad (3.18)$$

(ii) Given an operator R_p (recall the expansion (2.21), (2.22), (2.23)) we define the operator $[[R_p]]$ as the operator with the form (2.21), (2.22), with coefficients

$$([[R_p]](U))_{\sigma, j}^{\sigma', k} := \frac{1}{(2\pi)^p} \sum_{\substack{\sum_{i=1}^p \sigma_i j_i = \sigma j - \sigma' k \\ (\vec{\mu}, \vec{j}) \in \mathcal{S}_{p+2}}} ((r_p)^{\sigma_1 \dots \sigma_p}_{j_1, \dots, j_p})_{\sigma, j}^{\sigma', k} u_{j_1}^{\sigma_1} \dots u_{j_p}^{\sigma_p}, \quad j, k \in \mathbb{Z} \setminus \{0\}, \quad (3.19)$$

where

$$\vec{\mu} := (\vec{\sigma}, \sigma, \sigma') = (\sigma_1, \dots, \sigma_p, \sigma, \sigma'), \quad \vec{j} := (\vec{j}, j, k) = (j_1, \dots, j_p, j, k).$$

We say that R_p is resonant if $R_p \equiv [[R_p]]$. Let $R \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ of the form

$$R(U) = \sum_{p=1}^{N-1} R_p(U) + R_N(U), \quad R_p \in \tilde{\mathcal{R}}_p^{-\rho}, \quad R_N \in \Sigma \mathcal{R}_N^{-\rho}[r],$$

we define the operator $[[R]](U)$ as

$$[[R]](U) := \sum_{p=1}^{N-1} [[R_p]](U) + R_N(U),$$

where $[[R_p]](U)$ are matrices of operators with entries given by the r.h.s. of (3.19).

Remark 3.6. Consider a multilinear and constant coefficients in x symbol a_p in $\tilde{\Gamma}_p^m$, and consider the case of an Hamiltonian vector-field of the form

$$iE[\text{Op}^{\text{BW}}([[A_p]](U, \dots, U; \xi))U + R(U, \dots, U)U] \quad (3.20)$$

for a smoothing reminder R in $\tilde{\mathcal{R}}_p^{-\rho}$, where the matrix $[[A_p]](U, \dots, U; \xi)$ is defined as in (3.18). Then its Hamiltonian function has the form

$$\int_{\mathbb{T}} \text{Op}^{\text{BW}}([[A_p(U, \dots, U; \xi)])U \cdot \overline{U} dx + \int_{\mathbb{T}} M(U)U \cdot \overline{U} dx, \quad (3.21)$$

where $M(U) = M_p(U, \dots, U)$ for a multilinear map M_p in $\widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$. We know by Proposition 6.4 that there exists \widetilde{R} smoothing remainder in $\widetilde{\mathcal{R}}_p$ such that the Hamiltonian vector field of (3.21) equals (recall (1.18))

$$iE(O p^{\text{BW}}([[A_p]](U, \dots, U))U + \widetilde{R}(U, \dots, U)U) + iJ\nabla_{\overline{U}} \left(\int_{\mathbb{T}} M(U)U \cdot \overline{U} dx \right).$$

By (3.20) we must have $R = iE\widetilde{R}(U, \dots, U)U + iJ\nabla_{\overline{U}}(\int_{\mathbb{T}} M(U)U \cdot \overline{U} dx)$. Since we are considering a matrix whose entries are resonant symbols (they are $[[a_p]]$, see Definition 3.5), we have $\widetilde{R} = [[\widetilde{R}]]$ and therefore $R^\perp := R - [[R]] = [[iJ\nabla_{\overline{U}}(\int_{\mathbb{T}} M(U)U \cdot \overline{U} dx)]]$ is an Hamiltonian vector field.

More precisely we prove the following result.

Theorem 3.7. (Poincaré-Birkhoff normal form). *Assume that ω_j in (3.2) are non-resonant at order N according to Definition 3.4 and that (3.13) holds. There exist $s_0 > 0$, $r_0 > 0$ (possibly larger and smaller resp. with respect to the ones in Theorem 3.1) such that, for $s \geq s_0$, $r \leq r_0$ the following holds true. There exist an invertible and symplectic map*

$$\mathfrak{B} : B_r(H^s(\mathbb{T}; \mathbb{C}^2)) \cap \mathcal{U} \rightarrow H^s(\mathbb{T}; \mathbb{C}^2) \cap \mathcal{U},$$

such that, setting (recall (3.6))

$$W := \mathfrak{B}(Z), \quad \mathcal{Y}_N(Z) := d\mathfrak{B}(\mathfrak{B}^{-1}(Z))[\mathcal{Y}(\mathfrak{B}^{-1}(Z))],$$

we have that

$$\begin{cases} \dot{W} = \mathcal{Y}_N(W) := iE\Omega W + iEO p^{\text{BW}}([[M^{(N)}]](W; \xi))[W] + [[\mathcal{Q}_N]](W)[W] \\ W(0) = \mathfrak{B}(Z_0) \end{cases} \quad (3.22)$$

where $\mathcal{Q}_N \in \Sigma \mathcal{R}_2^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$, $M^{(N)} \in \Sigma \Gamma_2^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is independent of $x \in \mathbb{T}$ and it has the form

$$\begin{aligned} M^{(N)}(Z; \xi) &:= \begin{pmatrix} m^{(N)}(W; \xi) & 0 \\ 0 & m^{(N)}(W; -\xi) \end{pmatrix}, \\ m^{(N)}(W; \xi) &:= m_m^{(1)}(W) f_m(\xi) + m_{m'}^{(N)}(W; \xi), \quad m_m^{(1)}(W) \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N], \\ m_{m'}^{(N)}(W; \xi) &\in \Sigma \Gamma_1^{m'}[r, N], \end{aligned} \quad (3.23)$$

with $m^{(N)}(W; \xi)$ real valued. Moreover the vector field \mathcal{Y}_N is Hamiltonian. Finally, for any $s \geq s_0$, the maps $\mathfrak{B}^{\pm 1}$ satisfy

$$\|\mathfrak{B}^{\pm 1}(Z)\|_{H^s} \leq \|Z\|_{H^s} (1 + C\|Z\|_{H^{s_0}}), \quad (3.24)$$

for some constant $C > 0$ depending on s .

Proof. It follows by applying the result of section 7.2 and then, iteratively, the results of Lemmata 7.5, 7.7. \square

The Theorem above is the key step to obtain a long time existence and stability result for a system of the form (3.4). In particular the following result is consequence of Theorem 3.7.

Corollary 3.8. (A priori energy estimate). *Let $W = [\frac{w}{\overline{w}}] \in C^0([0, T]; H^s(\mathbb{T}; \mathbb{C}^2)) \cap C^1([0, T]; H^{s-m}(\mathbb{T}; \mathbb{C}^2))$ for $T > 0$ be a solution of the system (3.22) with initial condition $W_0 = [\frac{w_0}{\overline{w_0}}] \in B_r(H^s(\mathbb{T}; \mathbb{C}^2))$, $0 < r \ll 1$. Then there exists $C = C(s) > 0$ such that*

$$\|w(t)\|_{H^s}^2 \leq \|w_0\|_{H^s}^2 + C \int_0^t \|w(\sigma)\|_{H^{s+2}}^{N+2} d\sigma, \quad \forall t \in [0, T]. \quad (3.25)$$

The proof of the corollary is given in Section 7.

3.3. Applications to some PDEs. In this section, by using Theorems 3.1, 3.3 and 3.7, we shall prove Theorems 1.1, 1.2, 1.3.

3.3.1. *The quasi-linear Schrödinger equation.* We give the proof of Theorem 1.1.

Proof of Theorem 1.1. In order to prove Theorem 1.1 we shall apply Theorems 3.1, 3.3, 3.7 and Corollary 3.8. Let us show that equation (1.24) satisfies the assumptions of these abstract results.

First of all we recall that the nonlinearity $f(u, u_x, u_{xx})$ is a polynomial of maximum degree $\bar{q} \geq 2$. Therefore, by Lemmata 3.2, 3.3 in [26] and Lemma 4.1 in [25] we have

$$\begin{aligned} f(u, u_x, u_{xx}) &= Op^{\text{BW}}(g_2(U; x))\partial_{xx}u + Op^{\text{BW}}(h_2(U; x))\partial_{xx}\bar{u} \\ &\quad + Op^{\text{BW}}(g_1(U; x))\partial_x u + Op^{\text{BW}}(h_1(U; x))\partial_x \bar{u} \\ &\quad + Op^{\text{BW}}(g_0(U; x))u + Op^{\text{BW}}(h_0(U; x))\bar{u} + R(U)[U], \end{aligned} \quad (3.26)$$

where $R(U) = \sum_{j=1}^{\bar{q}} R_j(U)$ and R_j are 1×2 matrices of operators in $\widetilde{\mathcal{R}}_j^{-\rho}$, and

$$g_i(U; x) = \sum_{j=1}^{\bar{q}} g_i^{(j)}(U; x), \quad h_i(U; x) = \sum_{j=1}^{\bar{q}} h_i^{(j)}(U; x), \quad g_i^{(j)}, h_i^{(j)} \in \widetilde{\mathcal{F}}_j, \quad i = 0, 1, 2.$$

Moreover

$$g_i(U; x) := (\partial_{\partial_x^i u} f)(u, u_x, u_{xx}), \quad h_i(U; x) := (\partial_{\partial_x^i \bar{u}} f)(u, u_x, u_{xx}).$$

More precisely, using the (1.26), we have

$$\begin{aligned} g_2 &:= -\partial_{u_x \bar{u}_x} F, & h_2 &:= -\partial_{\bar{u}_x u_x} F, \\ g_1 &:= -\frac{d}{dx}(\partial_{u_x \bar{u}_x} F) + \partial_{u_x \bar{u}} F - \partial_{u \bar{u}_x} F, & h_1 &:= -\frac{d}{dx}(\partial_{\bar{u}_x u_x} F) + \partial_{\bar{u}_x u} F - \partial_{\bar{u} u_x} F, \\ g_0 &:= \partial_{u \bar{u}} F - \frac{d}{dx}(\partial_{u \bar{u}_x} F), & h_0 &:= \partial_{\bar{u} u} F - \frac{d}{dx}(\partial_{\bar{u} u_x} F). \end{aligned} \quad (3.27)$$

Recall now that $\partial_x^p := Op^{\text{BW}}((i\xi)^p)$, $p = 0, 1, 2$. Then, using the composition Proposition 2.25 (see also (2.31)) and the formulæ (1.26), (3.27), we obtain

$$\begin{aligned} f(u, u_x, u_{xx}) &= Op^{\text{BW}}\left(\widetilde{a}_2(U; x)(i\xi)^2 + \widetilde{a}_1(U; x)(i\xi) + \widetilde{a}_0(U; x)\right)u \\ &\quad + Op^{\text{BW}}\left(\widetilde{b}_2(U; x)(i\xi)^2 + \widetilde{b}_1(U; x)(i\xi) + \widetilde{b}_0(U; x)\right)\bar{u} + R(U)U, \end{aligned} \quad (3.28)$$

where $R(U)$ is a 1×2 matrix of operators in $\Sigma\mathcal{R}_1^{-\rho}[r, N]$ (for any $N \geq 1$), and

$$\begin{aligned} \widetilde{a}_2 &:= -\partial_{u_x \bar{u}_x} F, & \widetilde{b}_2 &:= -\partial_{\bar{u}_x u_x} F, \\ \widetilde{a}_1 &:= \partial_{u_x \bar{u}} F - \partial_{u \bar{u}_x} F, & \widetilde{a}_0 &:= \partial_{u \bar{u}} F - \frac{1}{2}\partial_x(\partial_{u_x \bar{u}} F + \partial_{u \bar{u}_x} F), \\ \widetilde{b}_1 &:= h_1 - \frac{1}{2}h_2, & \widetilde{b}_0 &:= h_0 + \frac{1}{2}\partial_{xx}h_2 - \frac{1}{2}\partial_x h_1. \end{aligned} \quad (3.29)$$

Notice that $\widetilde{a}_i, \widetilde{b}_i \in \Sigma\mathcal{F}_1[r, N]$, $i = 0, 1, 2$. Moreover, by (1.25), we can also note that

$$p(\xi) := \widehat{p}(\xi), \quad \xi \in \mathbb{R} \quad p \in \widetilde{\Gamma}_0^0. \quad (3.30)$$

Then we write

$$\Omega := Op^{\text{BW}}(f_2(\xi)), \quad f_2(\xi) := \xi^2 + p(\xi) \in \widetilde{\Gamma}_0^2. \quad (3.31)$$

The symbol $f_2(\xi)$ is real valued, even in x and *classical*, namely satisfies the properties of f_m in Theorem 3.1. By (3.31), (3.28), we have that equation (1.27) reads

$$\begin{aligned} \dot{u} &= iOp^{\text{BW}}\left((1 - \widetilde{a}_2(U; x))f_2(\xi) + \widetilde{a}_1(U; x)i\xi + \widetilde{a}_0(U; \xi) + \widetilde{a}_2(U; x)p(\xi)\right)u \\ &\quad + Op^{\text{BW}}\left(-\widetilde{b}_2(U; x)f_2(\xi) + \widetilde{b}_1(U; x)i\xi + \widetilde{b}_0(U; \xi) + \widetilde{b}_2(U; x)p(\xi)\right)\bar{u} + R(U)U \end{aligned} \quad (3.32)$$

where R is a 1×2 matrix of operators in $\Sigma\mathcal{R}_1^{-\rho}[r, N]$. Then, setting

$$\begin{aligned} a_2 &:= -\tilde{a}_2, & b_2 &:= -\tilde{b}_2, & a_0 &:= \tilde{a}_0 + \tilde{a}_2 p(\xi), & b_0 &:= \tilde{b}_0 + \tilde{b}_2 p(\xi), \\ a &= a(U; x, \xi) := (1 + a_2(U; x))f_2(\xi) + a_1(U; x, \xi), & a_1(U; x, \xi) &:= \tilde{a}_1(U; x)(i\xi) + a_0(U; x, \xi) \\ b(U; x, \xi) &= b_2(U; x)f_2(\xi) + b_1(U; x, \xi), & b_1(U; x, \xi) &:= \tilde{b}_1(U; x)(i\xi) + b_0(U; x, \xi), \end{aligned} \quad (3.33)$$

we have that (3.32) is equivalent to the system on the variables $U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$

$$\dot{U} = iEOP^{\text{BW}}(A(U; x, \xi))U + R(U)U, \quad A(U; x, \xi) = \begin{pmatrix} a(U; x, \xi) & b(U; x, \xi) \\ b(U; x, -\xi) & a(U; x, -\xi) \end{pmatrix} \quad (3.34)$$

where $R(U)$ is some smoothing remainder in $\Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ which is *real-to-real* (see (2.35)). By (3.33), (3.29) and using that $F(u, u_x)$ is real valued, we deduce that

$$a(U; x, \xi) = \overline{a(U; x, \xi)}. \quad (3.35)$$

Therefore system (3.34) has the same form of (3.4), (3.3) and Theorem 3.1 applies. Moreover the equation (1.27) (and hence (3.34)) is Hamiltonian with respect to the symplectic form (1.19). Hence Theorem 3.3 guarantees that the map Ψ given by Theorem 3.1 is symplectic. Then, setting $Z = \Psi(U)$, we have that system (3.34) conjugates to a system of the form (3.6), (3.7) with $\mathcal{Y}(Z)$ an Hamiltonian vector field. The local well-posedness on the system (3.7) can be deduced as in Theorem 2.31. Furthermore the linear frequencies of oscillations in (1.29) are *non-resonant* according to Definition 3.4 for any choice of parameters $\tilde{m} = (m_1, \dots, m_M)$ (see (1.25)) in $[-1/2, 1/2]^M \setminus \mathcal{N}$ where \mathcal{N} has zero Lebesgue measure. This is a consequence of Proposition 5.5 in [26]. Theorem 3.7 applies and provide a symplectic map \mathfrak{B} such that the system for the variables $W := \mathfrak{B}(Z)$ has the form (3.22). Namely we conjugate, in a symplectic way the system (3.34) to its resonant *Poincaré-Birkhoff normal form* up to order N (see (3.22)). We used the map $\mathfrak{B} \circ \Psi$. In particular, by estimates (3.9), (3.24), we deduce that

$$\|W\|_{H^s} \sim \|U\|_{H^s}, \quad W = \mathfrak{B} \circ \Psi(U), \quad (3.36)$$

if $r > 0$ is small enough. Let W be the solution of problem (3.22) and assume it is defined on a time interval $[0, T)$, $T > 0$. By the estimate (3.25) in Corollary 3.8 and using a standard bootstrap argument (see for instance the proof of Theorem 5.1 in [26]) one can prove that actually

$$\|W(t)\|_{H^s} \lesssim_s \|W(0)\|_{H^s}, \quad t \in [0, T), \quad T \gtrsim r^{-N}.$$

The latter estimate combined with (3.36) proves the estimate (1.31) over a time scale as in (1.30). This concludes the proof. \square

3.3.2. Quasi-linear perturbations of the beam equation. Introducing the variable $v = \psi = \partial_t \psi$ we can rewrite equation (1.32) as

$$\begin{cases} \dot{\psi} = -v, \\ \dot{v} = \Omega^2 \psi + p(\psi), \end{cases} \quad \Omega := (\partial_{xx}^2 + m)^{\frac{1}{2}}. \quad (3.37)$$

Notice that the operator Ω is the Fourier multiplier defined as

$$\Omega e^{ijx} = \omega_j e^{ijx}, \quad \omega_j = \omega_j(m) := \sqrt{|j|^4 + m}, \quad j \in \mathbb{Z}. \quad (3.38)$$

We define the complex variable

$$u := \frac{1}{\sqrt{2}} (\Omega^{\frac{1}{2}} \psi + i\Omega^{-\frac{1}{2}} v). \quad (3.39)$$

Therefore the (3.37) reads

$$\dot{u} = i\Omega u + \frac{i}{\sqrt{2}} \Omega^{-\frac{1}{2}} p\left(\Omega^{-\frac{1}{2}} \left(\frac{u + \bar{u}}{\sqrt{2}}\right)\right). \quad (3.40)$$

Notice that (3.40) has the form $\dot{u} = i\partial_{\bar{u}}H(u, \bar{u})$, i.e. is the Hamiltonian equation (w.r.t. the symplectic form (1.19)) of the Hamiltonian

$$H(u, \bar{u}) = \int_{\mathbb{T}} \Omega u \cdot \bar{u} dx + \int_{\mathbb{T}} P\left(\Omega^{-\frac{1}{2}}\left(\frac{u + \bar{u}}{\sqrt{2}}\right)\right) dx \quad (3.41)$$

where

$$P(\psi) := G(\psi, \psi_x, \psi_{xx}). \quad (3.42)$$

We now prove the following.

Lemma 3.9. (Paralinearization of the beam equation). *The equation (3.40) can be written in the form $\dot{U} = X(U)$, $U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$ where $X(U)$ is an Hamiltonian vector field of the form (3.4), (3.3) where $f_m \rightsquigarrow f_2(\xi)$ defined as*

$$f_2(\xi) := \sqrt{\xi^4 + m} \in \tilde{\Gamma}_0^2. \quad (3.43)$$

Proof. We reason as done in section 3.3.1. First of all consider the function $G(\psi, \psi_x, \psi_{xx})$ appearing in (1.34). Since G is a real valued polynomial in the variables $(\psi, \psi_x, \psi_{xx})$ it is easy to check that

$$c_{jk}(\psi; x) := \left(\partial_{\partial_x^k \psi \partial_x^j \bar{\psi}} G\right)(\psi, \psi_x, \psi_{xx}) \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N], \quad N \geq 1, \quad k, j = 0, 1, 2. \quad (3.44)$$

Then, using the Bony paralinearization formula (see also Lemmata 3.2, 3.3 in [26] and Lemma 4.1 in [25]), we can deduce that (see (1.34))

$$\begin{aligned} g(\psi, \psi_x, \psi_{xx}, \psi_{xxx}, \psi_{xxxx}) &= C(\psi)\psi + Q(\psi)\psi, \\ C(\psi) &:= \sum_{k,j=0}^2 C_{kj}(\psi), \quad C_{kj}(\psi) := (-1)^k \partial_x^k Op^{\text{BW}}(c_{jk}(\psi; x)) \partial_x^j \end{aligned} \quad (3.45)$$

for some $Q(\psi) \in \Sigma \mathcal{R}_1^{-\rho}[r, N]$. We note that the operator $C(\psi)$ is self-adjoint. Indeed $C_{00} = Op^{\text{BW}}(c_{00}(\psi; x))$ is self-adjoint since the symbol is real valued (recall (2.41)). Reasoning similarly we have that the operators

$$\begin{aligned} C_{01}(\psi) + C_{10}(\psi) &= Op^{\text{BW}}(c_{01}(\psi; x)) \partial_x - \partial_x Op^{\text{BW}}(c_{10}(\psi; x)), \\ C_{02}(\psi) + C_{11}(\psi) + C_{20}(\psi) &= Op^{\text{BW}}(c_{02}(\psi; x)) \partial_{xx} - \partial_x Op^{\text{BW}}(c_{11}(\psi; x)) \partial_x + \partial_{xx} Op^{\text{BW}}(c_{20}(\psi; x)) \\ C_{12}(\psi) + C_{21}(\psi) &= -\partial_x Op^{\text{BW}}(c_{12}(\psi; x)) \partial_{xx} + \partial_{xx} Op^{\text{BW}}(c_{21}(\psi; x)) \partial_x \\ C_{22}(\psi) &= \partial_{xx} Op^{\text{BW}}(c_{22}(\psi; x)) \partial_{xx}, \end{aligned}$$

are self-adjoint since $c_{jk} = c_{kj}$. Using the Definition 2.1, 2.10 and the (3.39), one can check that

$$\tilde{c}_{jk}(U; x) := c_{jk}\left(\Omega^{-\frac{1}{2}} \frac{u + \bar{u}}{\sqrt{2}}; x\right) \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N], \quad \tilde{Q}(U) := Q\left(\Omega^{-\frac{1}{2}} \frac{u + \bar{u}}{\sqrt{2}}\right) \in \Sigma \mathcal{R}_1^{-\rho}[r, N]. \quad (3.46)$$

We define

$$\begin{aligned} B(U) &:= \sum_{j,k=0}^2 B_{kj}(U), \quad B_{kj}(U) := \frac{(-1)^k}{2} \Omega^{-\frac{1}{2}} \partial_x^k Op^{\text{BW}}(\tilde{c}_{kj}(U; x)) \partial_x^j \Omega^{-\frac{1}{2}}, \\ \widehat{Q}(U) &:= \frac{1}{2} \Omega^{-\frac{1}{2}} \tilde{Q}(U) \Omega^{-\frac{1}{2}} \end{aligned} \quad (3.47)$$

With this notation, recalling (1.33), (3.45), we have that equation (3.40) reads

$$\dot{u} = i\Omega u + iB(U)u + iB(U)\bar{u} + i\widehat{Q}(U)[u] + i\widehat{Q}(U)[\bar{u}]. \quad (3.48)$$

In order to show that equation (3.48) can be written in the form (3.4) we provide a more explicit description of the operator $B(U)$ at the highest order. Let us write $\Omega := Op^{\text{BW}}(f_2(\xi))$, $\partial_x^p = Op^{\text{BW}}((i\xi)^p)$ where $f_2(\xi)$ is in (3.43). Notice also that

$$(\xi^4 + m)^{-1} (i\xi)^4 = 1 - m(\xi^4 + m)^{-1}.$$

Then, using (3.47) and the expansion (2.31), we have

$$B(U) := Op^{BW} \left(a_2(U; x) f_2(\xi) + a_1(U; x, \xi) \right), \quad a_2(U; x) = -\frac{1}{2} \tilde{c}_{22}(U; x) \quad (3.49)$$

up to smoothing remainders in $\Sigma \mathcal{R}_1^{-\rho}[r, N]$ and where $a_1(U; x, \xi)$ is a symbol in $\Sigma \Gamma_1^1[r, N]$. Moreover, since $B(U)$ is self-adjoint and $\tilde{c}_{22}(U; x)$ is real-valued (see (3.46)), we must have that $a_1(U; x, \xi)$ is real-valued. Using (3.49) we rewrite (3.48) as

$$\dot{U} = iEOp^{BW} \begin{pmatrix} (1 + a_2) f_2(\xi) & a_2 f_2(\xi) \\ a_2 f_2(\xi) & (1 + a_2) f_2(\xi) \end{pmatrix} U + iEOp^{BW} \begin{pmatrix} a_1(U; x, \xi) & a_1(U; x, \xi) \\ a_1(U; x, -\xi) & a_1(U; x, -\xi) \end{pmatrix} U + R(U)U$$

which has the form (3.4). \square

We now state a result regarding the *non-resonance* of the linear frequencies of oscillations.

Lemma 3.10. *There exists a zero Lebesgue measure set $\mathcal{N} \subset [1, 2]$ such that, for any $m \in [1, 2] \setminus \mathcal{N}$, the frequencies $\omega_j = \omega_j(m)$ in (3.38) are non-resonant according to Definition 3.4.*

Proof. It follows by Proposition 3.1 in [18] reasoning as in the proof of Proposition 5.5 in [26] \square

Following almost word by word the proof of the long time existence Theorem 1.1 using Lemmata 3.9, 3.10 one can deduce the proof of Theorem 1.2.

3.3.3. *Benjamin-Ono type equations.* We prove the following.

Lemma 3.11. (Parilinearization of the Benjamin-Ono equation) *Let $0 < r \ll 1$ and $\rho > 0$. Then there exist a remainder $R \in \mathcal{R}_1^{-\rho}[r]$ and a symbol $a(u; x, \xi)$ in $\Gamma_1^2[r]$ of the form*

$$\begin{aligned} a(u; x, \xi) &= a_2(u; x) |\xi| \xi + a_1(u; x, \xi), \quad a_2(u; x) := (\partial_{z_4} g)(u, \mathcal{H}u, u_x, \mathcal{H}u_x, \mathcal{H}u_{xx}) \in \mathcal{F}_1^{\mathbb{R}}[r], \\ a_1(u; x, \xi) &\in \Gamma_1^1[r], \quad a_1(u; x, \xi) - \overline{a_1(u; x, \xi)} \in \Gamma_1^0[r], \end{aligned} \quad (3.50)$$

such that the following holds. The equation (1.37) can be written as

$$u_t = -iOp^{BW} \left((1 + a_2(u; x)) |\xi| \xi + a_1(u; x, \xi) \right) u + R(u)[u]. \quad (3.51)$$

Proof. Since $g(z_0, z_1, z_2, z_3, z_4)$ is a polynomial in the variables $(z_0, z_1, z_2, z_3, z_4)$, it is easy to check (recall Def. 2.1) that

$$b_j(u; x) := (\partial_{z_j} g)(u, \mathcal{H}u, u_x, \mathcal{H}u_x, \mathcal{H}u_{xx}) \in \mathcal{F}_1^{\mathbb{R}}[r]. \quad (3.52)$$

Then, using the Bony parilinearization formula (see also Lemmata 3.2, 3.3 in [26] and Lemma 4.1 in [25]), we can deduce that (see (1.39))

$$\begin{aligned} \mathcal{N}(u) &= B(u)u + Q(u)u, \quad Q \in \mathcal{R}_1^{-\rho}[r], \\ B(u) &= Op^{BW}(b_4(u; x)) \mathcal{H} \partial_{xx} + Op^{BW}(b_3(u; x)) \mathcal{H} \partial_x + Op^{BW}(b_2(u; x)) \partial_x \\ &\quad + Op^{BW}(b_1(u; x)) \mathcal{H} + Op^{BW}(b_0(u; x)). \end{aligned}$$

Recalling (1.38) we write $\mathcal{H} = Op^{BW}(-i \operatorname{sign}(\xi))$. Using the expansion (2.31) we get, up to smoothing remainders in $\mathcal{R}_1^{-\rho}[r]$

$$B(u) = Op^{BW} \left(b_4 i |\xi| \xi + \frac{1}{2i} \{b_4, i |\xi| \xi\} + b_3 |\xi| + b_2 i \xi + \tilde{b}(u; x, \xi) \right)$$

for some symbol $\tilde{b} \in \Gamma_1^0[r]$. We also have

$$\frac{1}{2i} \{b_4(u; x), i |\xi| \xi\} + b_3(u; x) |\xi| = -\partial_x \left(b_4(u; x) \right) |\xi| + b_3(u; x) |\xi| \stackrel{(3.52), (1.40)}{=} 0.$$

We define $a_2(u; x) := b_4(u; x)$ and

$$a_1(u; x, \xi) := (u + b_2(u; x)) \xi + u_x + \tilde{b}(u; x, \xi). \quad (3.53)$$

Therefore the equation (1.37) assumes the form

$$u_t = -\mathcal{H}\partial_{xx}u - iOp^{\text{BW}}(a_2(u; x)|\xi|\xi + a_1(u; x, \xi)) + R(u)u,$$

for some $R \in \mathcal{R}_1^{-\rho}[r]$. Since $-\mathcal{H}\partial_{xx} = -i\xi|\xi|$ we get the (3.51). The symbol a_1 in (3.53) satisfies the property in (3.50) by explicit computation. \square

We are in position to prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 3.11 we have that equation (1.37) has the form (3.10) (see (3.51)) for some symbol $a(u; x, \xi)$ satisfying the properties in (3.3). Therefore the Corollary 3.2 of Theorem 3.1 applies. Then there is a map Ψ such that the equation for the variable $z = \Psi(u)$ has the form (3.11). The existence result for such an equation over a time interval $[0, T]$, can be deduce by Theorem 2.31 in the case of generator as in (2.55). Moreover, reasoning as in the proof of Corollary 3.8 one can prove an a priori energy estimate for the equation (3.11) of the form (3.25) with $N = 1$. Then, reasoning as in the proof of Theorem 1.1, one can prove that estimate (1.43) holds over a time interval $[0, T]$ with $T \gtrsim r^{-1}$. \square

4. SOME NON LINEAR PARA-DIFFERENTIAL EQUATIONS

This section is the core of our paper. We study four non linear problems arising in implementing an iterative procedure which diagonalizes and conjugates to constant coefficients a system like (3.4). These problems involve symbols transported along the flows of some para-differential equations and can be considered as a non linear counterpart of problems arising the Egorov Theory for *pseudo-differential* operators.

4.1. Diagonal terms at highest order. The equation that we study in this subsection is the one appearing in conjugating to constant coefficient the principal symbol on the diagonal of the system (3.4). The equation we need to solve, and that will be used in section 5.1.1, is the (4.8). This equation involves a non linear auxiliary flow which is the solution of the system (4.2). Such a system is a highly non-linear system of coupled equations. Given a function $b(\tau, u; x)$, we study the well-posedness of such flow in Section 4.1.1 (see Theorem 4.3). In section 4.1.2 (see Theorem 4.6) we provide a more explicit expression of such flow in the case that the function $b(\tau, u; x)$ is the one associated to a torus diffeomorphism $x \rightarrow x + \beta(u; x)$ (see equations (4.32), (4.33)). This is necessary in order to solve (4.8) which depends on the flow generated by β . A similar problem is faced also in the paper [9]. In such paper the authors do not look for an invertible change of coordinates of the phase space, as a consequence the equation they need to solve is "essentially linear". We explain in Remark 4.8 the link between our and their solution, another comparison between the two methods is made in Remark 4.7.

Finally, in section 4.1.3, we look for a solution of equation (4.8). Being (4.8) a non-linear equation we use an iterative scheme (see system (4.45)-(4.47), which are well-posed thanks to Section 4.1.1) which converges to a solution of (4.8). In the proof of the convergence of the aforementioned iterative scheme one can note the n^{th} approximate solution of the non-linear system (4.50), (4.51) are "close" (up h.o.t. in degree of homogeneity) to the "linear" solutions found in [9], see equations (5.1.8) and (5.1.9) therein. In analogy with the papers [9], [25], [26] this is the most delicate part of our analysis.

Before entering in the core of the section we fix the following notation.

Notation 4.1. *Throughout this section (and also the throughout the next ones) we fix a number $1 < m$ in $\frac{1}{2}\mathbb{N}$ and we work with symbols in $\Sigma\Gamma_p^{m'}[r, N]$ with $m' \leq m$. We fix $d = m$, where d is the number appearing in item (ii) of Definition 2.1, in other words, in the notation of item (ii) of Definition 2.1 we shall work with symbols in the classes $\Sigma\Gamma_p^{m', m}[r, N]$ with $m' \leq m$. The role of this number is discussed in Remark 2.2.*

Consider a symbol $a(u; x, \zeta)$ in the class $\Sigma\Gamma_1^m[r, N]$, $m \in \mathbb{R}$, $0 < r \ll 1$ and assume that it is *classical* according to Definition 2.28. Assume also that its principal part a_m has the following structure

$$a_m(z_0, x_0; \xi_0) := (1 + \tilde{a}_m(z_0, x_0))f_0(\xi_0), \quad \tilde{a}_m(z_0, x_0) \in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N], \quad (4.1)$$

where f_0 is a m -homogeneous $C^\infty(\mathbb{R}^+, \mathbb{R})$ function and z_0 is in H^s for s large enough. Let $b(\tau, w, y) \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N]$, $\tau \in [0, 1]$, and consider the system

$$\begin{cases} \partial_\tau x(\tau) = -b(\tau, z(\tau); x(\tau)) \\ \partial_\tau \xi(\tau) = b_x(\tau, z(\tau); x(\tau))\xi(\tau) \\ \partial_\tau z(\tau) = Op^{\text{BW}}(ib(\tau, z(\tau); x(\tau))\xi(\tau))[z(\tau)], \end{cases} \quad (4.2)$$

with initial condition $(z(0), x(0), \xi(0)) = (z_0, x_0, \xi_0)$. We have the following.

Theorem 4.2. *Assume (4.1). For $r > 0$ small enough there exists a symbol $b(\tau, w; y) \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N]$, $\tau \in [0, 1]$, such that, the following holds.*

(i) *The flow*

$$(z(\tau), x(\tau), \xi(\tau)) = \Phi_b(\tau, z_0, x_0, \xi_0) = (\Phi_b^{(z)}(\tau), \Phi_b^{(x)}(\tau), \Phi_b^{(\xi)}(\tau))(z_0, x_0, \xi_0) \quad (4.3)$$

of (4.2) is well-posed and

$$z(\tau) = \Phi_b^{(z)}(\tau, z_0) \in \cap_{k=0}^K C^k([0, 1]; H^{s-k}), \quad 0 \leq K \leq s, \quad (4.4)$$

$$x(\tau) = \Phi_b^{(x)}(\tau, z_0, x_0) = x_0 + \Psi_b^{(x)}(\tau, z_0, x_0), \quad \Psi_b^{(x)} \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N] \quad (4.5)$$

$$\xi(\tau) = \Phi_b^{(\xi)}(\tau, z_0, x_0, \xi_0) = \xi_0(1 + \Psi_b^{(\xi)}(\tau, z_0, x_0)), \quad \Psi_b^{(\xi)} \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N]. \quad (4.6)$$

It is invertible, we denote by $(\tilde{\Phi}_b^{(z)}(\tau, \tilde{z}_0), \tilde{\Phi}_b^{(x)}(\tau, \tilde{z}_0, \tilde{x}_0), \tilde{\Phi}_b^{(\xi)}(\tau, \tilde{z}_0, \tilde{x}_0, \tilde{\xi}_0))$ its inverse where

$$\tilde{z}_0 = \Phi_b^{(z)}(1, z_0), \quad \tilde{x}_0 = \Phi_b^{(x)}(1, z_0, x_0), \quad \tilde{\xi}_0 = \Phi_b^{(\xi)}(1, z_0, x_0, \xi_0).$$

We have that

$$\tilde{\Phi}_b^{(\xi)}(\tau, \tilde{z}_0, \tilde{x}_0, \tilde{\xi}_0) = (1 + \tilde{\Psi}_b^{(\xi)}(\tau, \tilde{z}_0, \tilde{x}_0))\tilde{\xi}_0. \quad (4.7)$$

(ii) *There is m_b in the class $\Sigma \mathcal{F}_0^{\mathbb{R}}[r, N]$ independent of $x \in \mathbb{T}$ such that we have*

$$F(b) := (1 + \tilde{a}_m(\tilde{\Phi}_b^{(z)}(1, \tilde{z}_0), \tilde{\Phi}_b^{(x)}(1, \tilde{z}_0, \tilde{x}_0)))(1 + \tilde{\Psi}_b^{(\xi)}(1, \tilde{z}_0, \tilde{x}_0))^d = m_b. \quad (4.8)$$

The proof of Theorem 4.2 involves many different arguments that we shall study in the following subsections.

4.1.1. *Well-posedness of the flow (4.2).* In this subsection we study the existence of the flow of (4.2) for any generator $b(\tau; w, y)$ in the class $\mathcal{F}_1^{\mathbb{R}}[r]$.

Theorem 4.3. (WP of (4.2)). *Consider the problem (4.2) with b in the class $\mathcal{F}_1^{\mathbb{R}}[r]$ for some small enough $r > 0$. Then there exists $0 < \tilde{r} \ll r$ such that if z_0 is in $B_{\tilde{r}}(H^s)$ the following holds true. There exists a solution of the problem (4.2) with initial condition $z(0) = z_0$, $x(0) = x_0$, $\xi(0) = \xi_0$ of the form (4.4)-(4.6) for $\tau \in [0, 1]$. In particular (recall (2.13), (2.14)) one has*

$$\sup_{\tau \in [0, 1]} |\Psi_b^{(x)}(\tau)|_{\alpha, k}^{\mathcal{F}, s-1} \leq C|b|_s^{\mathcal{F}}; \quad (4.9)$$

$$\sup_{\tau \in [0, 1]} |\Psi_b^{(\xi)}(\tau)|_{\alpha, k}^{\mathcal{F}, s-2} \leq C|b|_{s-1}^{\mathcal{F}}; \quad (4.10)$$

$$z(\tau) \in C^0(I, H^s) \cap C^1(I, H^{s-1}), \quad \sup_{\tau \in [0, 1]} \|z(\tau)\|_s \leq C\|z_0\|_s, \quad (4.11)$$

$$\|(d_u^k z)(z_0)[h_1, \dots, h_k]\|_{H^{s-mk}} \leq C\|h_1\|_{H^s} \cdots \|h_k\|_{H^s}, \quad \forall 0 \leq \tau \leq 1 \quad \forall h_i \in H^s \quad i = 1, \dots, k. \quad (4.12)$$

The proof of the theorem above is based on the following iterative scheme. Define for any $n \geq 1$ the following system of equations

$$\partial_\tau x_n = -b(\tau, z_{n-1}; x_n) \quad (4.13)$$

$$\partial_\tau \xi_n = (\partial_x b)(\tau, z_{n-1}; x_n) \xi_n \quad (4.14)$$

$$\partial_\tau z_n = Op^{BW}(ib(\tau, z_{n-1}; x_{n-1}) \xi_{n-1})[z_n], \quad (4.15)$$

$$(x_n(0), \xi_n(0), z_n(0)) = (x_0, \xi_0, z_0).$$

We shall prove that the sequence of solutions of the problem above converges to a solution of the system (4.2). To start, in the following lemma, we shall prove that, if z_0 is small enough in H^s for some $s \gg 1$, for any n in \mathbb{N}^* there are functions $\Psi_n^{(x)}, \Psi_n^{(\xi)}$ in $\mathcal{F}_1^{\mathbb{R}}[r]$ such that the solutions of (4.13) and (4.14) are of the form

$$x_n(\tau) = x_0 + \Psi_n^{(x)}(\tau, z_0, x_0), \quad \Psi_n^{(x)} \in \mathcal{F}_1^{\mathbb{R}}[r], \quad (4.16)$$

$$\xi_n(\tau) = \xi_0 \left(1 + \Psi_n^{(\xi)}(\tau, z_0, x_0) \right), \quad \Psi_n^{(\xi)} \in \mathcal{F}_1^{\mathbb{R}}[r]. \quad (4.17)$$

Moreover we shall prove that $\{\Psi_n^{(x)}\}_{n \in \mathbb{N}^*}, \{\Psi_n^{(\xi)}\}_{n \in \mathbb{N}^*}$ are converging sequences in the space $\mathcal{F}_1^{\mathbb{R}}[r]$. Furthermore we prove that for any n in \mathbb{N}^* the equation (4.15) admits a solution z_n such that $\{z_n\}_{n \in \mathbb{N}^*}$ is a Cauchy sequence in H^{s-1} and it is bounded in H^s .

Lemma 4.4. (Iterative Lemma). *Consider b a function in $\mathcal{F}_1^{\mathbb{R}}[r]$. There exist $\tilde{r} > 0$ and $s > 0$ respectively small and big enough, such that if $z_0 \in B_{\tilde{r}}(H^s)$ the following holds true. For any $n \in \mathbb{N}^*$ there exist a unique solution $(x_n(\tau), \xi_n(\tau), z_n(\tau))$ of the system made of equations (4.13), (4.14), (4.15) with initial condition $(x_n(0) = x_0, \xi_n(0) = \xi_0, z_n(0) = z_0)$ satisfying the following properties.*

(S1) $_n$: $x_n(\tau)$ and $\xi_n(\tau)$ have the form (4.16) and (4.17) respectively. Moreover, recalling the notation (2.14), there exists a constant $C > 0$ depending on \tilde{r} and s (independent on n) such that

$$\sup_{\tau \in [0,1]} |\Psi_n^{(x)}(\tau)|_{\alpha,k}^{\mathcal{F},s} \leq C |b|_s^{\mathcal{F}}; \quad (4.18)$$

$$\sup_{\tau \in [0,1]} |\Psi_n^{(\xi)}(\tau)|_{\alpha,k}^{\mathcal{F},s-1} \leq C |b|_s^{\mathcal{F}}; \quad (4.19)$$

$$z_n(\tau) \in C^0(I, H^s) \cap C^1(I, H^{s-1}), \quad \sup_{\tau \in I} \|z_n(\tau)\|_s \leq C \|z_0\|_s; \quad (4.20)$$

$$b(\tau, z_{n-1}(\tau), x_{n-1}(\tau)) \text{ satisfies (2.11) for any } \alpha \text{ and } k \text{ such that } \alpha + mk \leq s - s_0. \quad (4.21)$$

(S2) $_n$: We have the following estimates, recall (2.15),

$$|\Psi_n^{(x)} - \Psi_{n-1}^{(x)}|_{s-1,0}^{\mathcal{F}} \leq 2^{-n}; \quad (4.22)$$

$$|\Psi_n^{(\xi)} - \Psi_{n-1}^{(\xi)}|_{s-2,0}^{\mathcal{F}} \leq 2^{-n}; \quad (4.23)$$

$$\|z_n - z_{n-1}\|_{H^{s-1}} \leq 2^{-n} \tilde{r}. \quad (4.24)$$

Proof. We proceed by induction over n in \mathbb{N}^* . In the case that $n = 1$ we have

$$x_1(\tau) = x_0 - \int_0^\tau b(s, z_0; x_0) ds, \quad \xi_1(\tau) = \xi_0 \left(1 + \int_0^\tau (\partial_x b)(s, z_0; x_0) ds \right). \quad (4.25)$$

Concerning the solution of (4.15), in the case that $n = 1$, one has to reason as done in Lemma 3.22 in [9], obtaining a solution $z_1(\tau)$ belonging to $C^0([0, 1], H^s) \cap C^1([0, 1], H^{s-1})$ and such that

$$\sup_{\tau \in I} \|z_1(\tau)\|_{H^s} \leq C_{s,r} \|z_0\|_{H^s},$$

therefore if z_0 is small enough in H^s the statement $(S1)_1$ is proved by setting

$$\begin{aligned}\Psi_1^{(x)}(\tau, z_0; x_0) &= - \int_0^\tau b(s, z_0; x_0) ds, \\ \Psi_1^{(\xi)}(\tau, z_0; x_0) &= \int_0^\tau (\partial_x b)(s, z_0; x_0) ds.\end{aligned}$$

We recall that the (4.21) is trivial and follows immediately from the fact that function b belongs to $\mathcal{F}_1^{\mathbb{R}}[r]$. We now prove $(S2)_1$. The function $v(\tau) := z_1(\tau) - z_0$ solves the problem

$$\partial_\tau v(\tau) = Op^{\text{BW}}(ib(\tau; z_0, x_0)\xi_0)v(\tau) + Op^{\text{BW}}(ib(\tau; z_0, x_0)\xi_0)z_0,$$

therefore, by using the Duhamel formulation and Lemma 3.22 in [9] we obtain

$$\|v(t)\|_{s-1} \leq C\|z_0\|_{s_0}\|z_0\|_s,$$

therefore it is enough to choose $\|z_0\|_{s_0}$ small enough to satisfy the third condition in $(S2)_1$. We obtain the first line in $(S2)_1$ by using equation (4.25). The reasoning for the second line in $(S2)_1$ is similar.

We suppose that $(S1)_n, (S2)_n$ hold true and we prove $(S1)_{n+1}, (S2)_{n+1}$. We start by showing the (4.18) of $(S1)_{n+1}$, we have that

$$\begin{aligned}x_{n+1}(\tau) - x_0 &= - \int_0^\tau b(\sigma, z_n(\sigma), x_n(\sigma)) d\sigma \\ &= - \int_0^\tau b(\sigma, z_n(\sigma), x_0 + \Psi_n^{(x)}(\tau, z_0, x_0)) d\sigma := \Psi_{n+1}^{(x)}(\tau, z_0, x_0).\end{aligned}$$

We want to bound the semi-norm $|\Psi_{n+1}^{(x)}(\tau, z_0, x_0)|_{\alpha, k}^{\mathcal{F}}$ for any α and k satisfying $\alpha + mk \leq s - s_0$. We shall do the computation, for simplicity, in the case $\alpha = 0$. If $\alpha > 0$ the reasoning is similar but the computation is much more tedious. Let $K \leq k$ where k is as in (4.21), we have

$$D_z^K \Psi_{n+1}^{(x)}(\tau, z, x) = - \int_0^\tau D_z^K b(\sigma, z_n; x + \Psi_n(z, x)) [h_1, \dots, h_K] d\sigma.$$

We expand the term inside the integral in the equation above by using the formula for the derivatives of the composition of functions, obtaining

$$\begin{aligned}\sum_{k=1}^K \sum_{k_1+k_2=K} \sum_{v_1=1}^{k_1} \sum_{v_2=1}^{k_2} \sum_{p_1+\dots+p_{v_1}=k_1} \sum_{q_1+\dots+q_{v_2}=k_2} C_{q_1, \dots, q_{v_2}}^{p_1, \dots, p_{v_1}} \prod_{j=1}^{v_1} \left(D_{z_0}^{p_j} (x_0 + \Psi_n^{(x)}(x_0, z_0)) [h_{p_j, 1}, \dots, h_{p_j, p_j}] \right) \\ \partial_y^{k_1} D_{z_n}^{k_2} b(z_n, x_0 + \Psi_n^{(x)}(x_0, z_0)) \left[D_{z_0}^{q_1} z_n [h_{q_1, 1}, \dots, h_{q_1, q_1}], \dots, D_{z_0}^{q_{v_2}} z_n [h_{q_{v_2}, 1}, \dots, h_{q_{v_2}, q_{v_2}}] \right],\end{aligned}\quad (4.26)$$

where we denoted $y = x_0 + \Psi_n^{(x)}(x_0, z_0)$ and by $C_{q_1, \dots, q_{v_2}}^{p_1, \dots, p_{v_1}}$ some combinatorial coefficients. We estimate the absolute value of the general term in the sum above. The first factor may be bounded from above by

$$\prod_{j=1}^{v_1} \left| D_z^{p_j} (x_0 + \Psi_n^{(x)}(x_0, z_0)) [h_{p_j, 1}, \dots, h_{p_j, p_j}] \right| \leq C(1 + |\Psi_n^{(x)}|_{p_j, 0}^{\mathcal{F}, s}) \|z_0\|_{s_0}^{\max\{0, 1-p_j\}} \prod_{j=1}^{v_1} \|h_j\|_{s_0}, \quad (4.27)$$

here we have used just the definition of symbol (more precisely the definition of a function independent on ξ) and of semi-norm, i.e. (2.11) and (2.13). For the second factor we have

$$\begin{aligned}\left| \partial_y^{k_1} D_{z_n}^{k_2} b(z_n; x_0 + \Psi_n^{(x)}(x_0, z_0)) \left[D_{z_0}^{q_1} z_n [h_{q_1, 1}, \dots, h_{q_1, q_1}], \dots, D_{z_0}^{q_{v_2}} z_n [h_{q_{v_2}, 1}, \dots, h_{q_{v_2}, q_{v_2}}] \right] \right| \leq \\ |b|_{k_1, k_2}^{\mathcal{F}, s} \left\{ \max\{0, 1-k_2\} \|z_n\|_{s_0}^{\max\{0, p-k_2-1\}} \|z_n\|_{s_0+k_1} \prod_{j=1}^{k_2} \|D_{z_0}^{q_j} z_n [h_{q_j, 1}, \dots, h_{q_j, q_j}]\|_{s_0} \right. \\ \left. + \|z_n\|_{s_0}^{\max\{0, 1-k_2\}} \sum_{i=1}^{k_2} \prod_{j=1, j \neq i}^{k_2} \|D_{z_0}^{q_j} z_n [h_{q_j, 1}, \dots, h_{q_j, q_j}]\|_{s_0} \|D_{z_0}^{q_i} z_n [h_{q_i, 1}, \dots, h_{q_i, q_i}]\|_{s_0+k_1} \right\}.\end{aligned}\quad (4.28)$$

Thanks to the inductive hypothesis (4.21) we can apply the Lemma 2.30, therefore we can estimate from above the r.h.s. of (4.28) by

$$C|b|_{k_1, k_2}^{\mathcal{F}, s} \left\{ \max\{0, 1 - k_2\} \|z_0\|_{s_0}^{\max\{0, p - k_2 - 1\}} \|z_0\|_{s_0 + k_1} \prod_{j=1}^{k_2} \prod_{i=1}^j \|h_{j,i}\|_{s_0} \right. \\ \left. + \|z_0\|_{s_0}^{\max\{0, 1 - k_2\}} \prod_{j=1}^{k_2} \prod_{i=1}^j \|h_{j,i}\|_{s_0 + k_1} \right\}. \quad (4.29)$$

Putting together (4.27), (4.28) and (4.29) we obtain the bound (2.11) for the general term of the sum in (4.26) up to renaming, with abuse of notation, $s_0 \rightsquigarrow s_0 + k_1$ with the constant C in (2.11) replaced by

$$C \prod_{j=1}^{v_1} (1 + |\Psi_n^{(x)}|_{p_j, 0}^{\mathcal{F}}) |b|_{k_1, k_2}^{\mathcal{F}},$$

which, by using the inductive hypothesis is bounded by $C(1 + |b|_{k_1, 0}^{\mathcal{F}})^{v_1} |b|_{k_1, k_2}^{\mathcal{F}}$. Therefore one obtains (4.18) by using the smallness of r .

The proof of (4.19) of $(S1)_{n+1}$ is similar. One can also deduce the (4.21) of $(S1)_{n+1}$ by equation (4.26). Concerning (4.20) of $(S1)_{n+1}$ one has to reason as done in Section 3 of [9] by recalling that (4.21) and (4.20) of $(S1)_n$ hold true.

We now pass to the proof of $(S2)_{n+1}$, starting from (4.22). By using the fundamental calculus' theorem we obtain

$$x_{n+1} - x_n = \int_0^\tau b(\sigma, z_{n-1}; x_{n-1}) - b(\sigma, z_n; x_n) d\sigma \\ = \int_0^\tau \int_0^1 \partial_x b(\sigma, z_{n-1}; x_{n-1} + \gamma(x_n - x_{n-1}))(x_{n-1} - x_n) \\ + D_z b(\sigma, z_{n-1} + \gamma(z_n - z_{n-1}); x_n) [z_{n-1} - z_n] d\gamma d\sigma.$$

For the first addendum inside the integral of the r.h.s. of the equation above we can proceed as follows. Let α such that $\alpha \leq s - s_0 - 1$, we have

$$\partial_{x_0}^\alpha [\partial_x b(\sigma, z_{n-1}; x_{n-1} + \gamma(x_n - x_{n-1}))(x_{n-1} - x_n)] = \\ \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2} \left(\partial_{x_0}^{\alpha_1} \partial_x b(\sigma, z_{n-1}; x_{n-1} + \gamma(x_n - x_{n-1})) \partial_{x_0}^{\alpha_2} (x_{n-1} - x_n) \right), \quad (4.30)$$

Therefore we may deduce, by using (4.22) of $(S2)_n$ and the formula for the derivatives of the composition of functions, that

$$|\partial_x b(\sigma, z_{n-1}; x_{n-1} + \gamma(x_n - x_{n-1}))(x_{n-1} - x_n)|_{\alpha, 0}^{\mathcal{F}, s} \\ \leq |b|_{s-1}^{\mathcal{F}} \|z_{n-1}\|_{s_0} |x_{n-1} - x_n|_{\alpha, 0}^{\mathcal{F}, s} \leq |b|_{s-1}^{\mathcal{F}} Cr 2^{-n},$$

from which the thesis follows if r is chosen small enough in such a way that $rC|b|_s^{\mathcal{F}} \leq 1/2$. The computation is similar for the other addendum of (4.30), therefore (4.22) of $(S2)_{n+1}$ is proved. The proof of (4.23) of $(S2)_{n+2}$ is analogous.

For the proof of (4.24) we define the following quantities and we reason as follows:

$$b_n := b(\tau, z_n; x_0 + \Psi_n^{(x)}(x_0, z_0))(1 + \Psi_n^{(\xi)}(x_0, z_0)), \quad b_{n-1} := b(\tau, z_{n-1}; x_0 + \Psi_{n-1}^{(x)}(x_0, z_0))(1 + \Psi_{n-1}^{(\xi)}(x_0, z_0)), \\ f_n := Op^{\text{BW}}(i(b_n - b_{n-1})\xi_0)z_n, \quad v_{n+1} := z_{n+1} - z_n.$$

Since the function z_{n+1} solves the problem

$$\partial_\tau z_{n+1} = Op^{\text{BW}}\left(i b(\tau, z_n; x_0 + \Psi_n^{(x)}(x_0, z_0))(1 + \Psi_n^{(\xi)}(x_0, z_0))\xi_0\right)[z_{n+1}],$$

(and since the function z_n solves the same problem up to relabelling $n \rightsquigarrow n - 1$) we have that

$$\partial_\tau v_{n+1} = Op^{\text{BW}}(i b_n \xi_0) v_{n+1} + f_n. \quad (4.31)$$

Therefore, by using Theorem 3.5.2 in [9] and the Duhamel principle, we obtain

$$v_{n+1}(\tau) = \Theta_n^\tau \int_0^\tau [\Theta_n^{\tau'}]^{-1} f_n(\tau') d\tau',$$

where Θ_n^τ is the flow of the equation (4.31) with $f_n = 0$. From the equation above one infers that

$$\|v_{n+1}(\tau)\|_{s-1} \leq C \|z_n\|_s \|b_n - b_{n-1}\|_{s_0} \leq Cr \|z_n - z_{n-1}\|_{s_0} \leq Cr 2^{-n} r,$$

where in the last inequality we have used the (4.24) of $(S2)_n$. The thesis is proved if one choses r such that $rC \leq 1/2$. \square

We are ready to prove the existence of the flow of equation (4.2).

Proof of Theorem 4.3. For any $n \geq 1$ we consider the system made by the equations (4.13), (4.14) and (4.15), with initial conditions $z_n(0) = z_0$, $x_n(0) = x_0$, $\xi_n(0) = \xi_0$. By Lemma 4.4 and Remark 2.6 we have, up to extracting a subsequence, a sequence of solutions of the form (4.5) and (4.6) such that

$$\begin{aligned} \Psi_n^{(x)} &\rightharpoonup^* \tilde{\Psi}^{(x)} \text{ in } \mathcal{F}_1^{\mathbb{R}}[r], \quad \partial_x^\alpha \Psi_n^{(x)} \rightarrow \partial_x^\alpha \Psi^{(x)} \text{ uniformly in } x_0 \text{ for } \alpha \leq s - s_0 - 1, \\ \Psi_n^{(\xi)} &\rightharpoonup^* \tilde{\Psi}^{(\xi)} \text{ in } \mathcal{F}_1^{\mathbb{R}}[r], \quad \partial_x^\alpha \Psi_n^{(\xi)} \rightarrow \partial_x^\alpha \Psi^{(\xi)} \text{ uniformly in } x_0 \text{ for } \alpha \leq s - s_0 - 2, \\ z_n &\rightharpoonup^* \tilde{z} \text{ in } L^\infty([0, 1], H^s), \quad z_n \rightarrow z \text{ in } L^\infty([0, 1], H^s). \end{aligned}$$

We deduce that $\tilde{\Psi}^{(x)} = \Psi^{(x)}$, $\tilde{\Psi}^{(\xi)} = \Psi^{(\xi)}$ and $\tilde{z} = z$. We claim that the triple in (4.3) solves respectively the third, the first and the second equation in (4.2). Let us consider the third equation for instance. We want to show that there exists $s' > 0$ such that

$$\|Op^{\text{BW}}(b(\tau, z; x)i\xi)z - Op^{\text{BW}}(b(\tau, z_n; x_n)i\xi_n)z_n\|_{s'} \rightarrow 0,$$

when n goes to infinity. We can estimate from above the preceding inequality by

$$\begin{aligned} &\|Op^{\text{BW}}(b(\tau, z; x)i\xi)[z - z_n]\|_{s'} + \|Op^{\text{BW}}(b(\tau, z; x)i\xi)[z_n] - Op^{\text{BW}}(b(\tau, z; x)i\xi_n)[z_n]\|_{s'} \\ &+ \|Op^{\text{BW}}(b(\tau, z; x)i\xi_n)[z_n] - Op^{\text{BW}}(b(\tau, z; x_n)i\xi_n)[z_n]\|_{s'} \\ &+ \|Op^{\text{BW}}(b(\tau, z; x_n)i\xi_n)[z_n] - Op^{\text{BW}}(b(\tau, z_n; x_n)i\xi_n)[z_n]\|_{s'}. \end{aligned}$$

The first summand is bounded by $C\|z\|_{s_0}\|z - z_n\|_{s'+1}$. The second one may be bounded by $C\|z\|_{s_0}\|\Psi_n^{(\xi)} - \Psi^{(\xi)}\|_{s_0}\|z_n\|_{s'}$, and similarly the others. Therefore it is enough to choose $s' + 1 \leq s - 1$. The regularity of the solution z may be deduced in a classical way by using the third equation in (4.2). One can prove the (4.12) by using the (4.9), (4.10), by differentiating the third equation in (4.2) and reasoning as in Lemma 2.30. \square

Remark 4.5. One can note that the inverse flow $\tilde{\Phi}_b^{(z)}(1, z)$ of (4.2) satisfies estimates similar to (4.12). This follows by differentiating the relation $\tilde{\Phi}_b^{(z)}(1, \Phi_b^{(z)}(1, z_0)) = z_0$.

4.1.2. *Explicit expressions for the flow (4.2).* As shown in subsection 4.1.1, the Theorem 4.3 guarantees the well-posedness of the flow (4.2) for any given symbol $b(\tau, w; x)$ in $\mathcal{F}_1^{\mathbb{R}}[r]$. In particular such a flow has the form (4.3)-(4.6). In order to solve the equation (4.8) and prove the main Theorem 4.2 we shall provide a more explicit expression for the flow (4.2) in the case that $b(\tau, w; x)$ has a special structure (see formulæ (4.32), (4.33)). We have the following.

Theorem 4.6. Let $\beta(W; x)$ be a function in $\mathcal{F}_1^{\mathbb{R}}[r]$. Then if $\|W\|_s \ll 1$ for a sufficiently large s then the following holds.

(i) There is a unique $\gamma(\tau, W; x)$ in $\mathcal{F}_1^{\mathbb{R}}[r]$, $\tau \in [0, 1]$ such that

$$x_0 + \gamma(\tau, W, x_0) + \tau\beta(W, x_0 + \gamma(\tau, W, x_0)) = x_0, \quad \forall \tau \in [0, 1], x_0 \in \mathbb{T}, \quad W \in H^s. \quad (4.32)$$

(ii) There exists a unique $b(\tau, W; x)$ in $\mathcal{F}_1^{\mathbb{R}}[r]$ solution of the following equation

$$b(\tau, W; x) = \frac{\beta(W; x) + \tau d_W \beta(W; x) [Op^{\text{BW}}(b(\tau, W; x)\xi)W]}{1 + \tau(\beta)_x(W, x)}. \quad (4.33)$$

In particular there is a constant $C > 0$ depending only on s and $|\beta|_{s+1}^{\mathcal{F}}$ such that (recall (2.14) and (2.15))

$$|b|_s^{\mathcal{F}} \lesssim C, \quad (4.34)$$

$$|b|_{s,0}^{\mathcal{F}} \lesssim_s |\beta|_{s+1,0}^{\mathcal{F}}. \quad (4.35)$$

(iii) Recalling (4.4)-(4.6) (the flow of (4.2)) we have

$$x(\tau) := x_0 + \Psi_b^{(x)}(\tau, z_0, x_0) = x_0 + \gamma(\tau, \Phi_b^{(z)}(\tau, z_0, x_0)) = x_0 + \gamma(\tau, z(\tau), x_0), \quad (4.36)$$

$$\xi(\tau) = \xi_0(1 + \Psi_b^{(\xi)}(\tau, z_0, x_0)) = \xi_0(1 + \tau(\beta)_x(z(\tau), x(\tau))), \quad (4.37)$$

where $b(\tau, W, y)$ is given by formula (4.33) and $\gamma(\tau; W, y)$ by formula (4.32).

Proof of Item (i) of Theorem 4.6. One can reason as in Section 2.5 in [9]. \square

Proof of Item (ii) of Theorem 4.6. We proceed inductively, we define

$$b_0 = \frac{\beta(W; x)}{1 + \tau\beta_x(W; x)} \quad (4.38)$$

and the n -th problem as

$$\begin{aligned} b_n(\tau, W; x) &= \frac{\beta(W; x) + \tau d_W \beta(W; x) [Op^{\text{BW}}(b_{n-1}(\tau, W; x)\xi)W]}{1 + \tau(\beta)_x(W, x)} \\ &= b_0(W, x) + \tau \frac{d_W \beta(W; x) [Op^{\text{BW}}(b_{n-1}(\tau, W; x)\xi)W]}{1 + \tau(\beta)_x(W, x)}, \end{aligned} \quad (4.39)$$

we shall prove that for any $n \geq 0$ the solution (4.39) is a symbol in $\mathcal{F}_1^{\mathbb{R}}[r]$. More precisely we show that if $\beta(W; x)$ satisfies the estimate (2.11) for a certain s_0 then for any n the symbol b_n defined in (4.39) satisfies the same estimate with $s_0 + 1$.

By Remark 2.8 we deduce that b_0 satisfies the (4.34). Using the smallness of $r > 0$ (see Remark 2.7) one can also check the bound (4.35).

Then let us assume that b_{n-1} satisfies (4.34), (4.35). For simplicity we show that b_n satisfies (2.11), with estimates as in (4.34), (4.35) in the case that $\alpha = 0$ and $k = 0$. Moreover we note that it is enough to prove the claim for $d_W \beta(W; x) [Op^{\text{BW}}(b_{n-1}(\tau, W; x)\xi)W]$. We have

$$|d_W \beta(W; x) [Op^{\text{BW}}(b_{n-1}(\tau, W; x)\xi)W]| \leq C \|Op^{\text{BW}}(b_{n-1}(W; x)\xi)W\|_{s_0} \leq C \|W\|_{s_0} \|W\|_{s_0+1},$$

where we have used in the first estimate the (2.11) for the symbol $\beta(W; x)$ with $\alpha = 0$ and $k = 1$ and in the second we have used the Theorem 2.20.

We now prove that the sequence of b_n converges for any $\tau > 0$ in the space $\mathcal{F}_1^{\mathbb{R}}[r]$, therefore its limit is a symbol and solves the equation (4.33). More precisely we prove, by using that $\|W\|_{s_0+1} \ll 1$, that for any there exists a constant $C > 0$ such that (recall (2.15))

$$|b_{n+1} - b_n|_{s,0}^{\mathcal{F}} \leq C 2^{-(n+1)}, \quad (4.40)$$

for any $n \geq 1$ and $\tau > 0$. We note that

$$(b_1 - b_0)(\tau, W; x) = \tau d_u \beta(W; x) [Op^{\text{BW}}(b_0(\tau, W; x)\xi)W],$$

therefore thanks to the smallness of W and to the fact that b_0 is a function we obtain the (4.40) with $n = 1$. We proceed with the inductive step

$$(b_{n+1} - b_n)(\tau, W; x) = \tau \frac{d_u \beta(W; x) [Op^{\text{BW}}((b_n - b_{n-1})(\tau, W; x)\xi)W]}{1 + \tau(\beta)_x(W; x)},$$

by using formula (2.11), the smallness of $\|W\|_{s_0+1}$ (therefore of $(\beta)_x(W; x)$) and Proposition 2.22 we obtain

$$\begin{aligned} |(b_{n+1} - b_n)(\tau, W; x)| &\leq C \|b_n - b_{n-1}\|_{L^\infty} \|W\|_{s_0+1} \leq C |b_{n+1} - b_n|_s^{\mathcal{F}} \|W\|_{s_0} \|W\|_{s_0+1} \\ &\leq C 2^{-n} \|W\|_{s_0} \|W\|_{s_0+1} \leq C 2^{-n-1} \|W\|_{s_0+1}, \end{aligned}$$

where we have used the inductive hypothesis and the smallness of $\|W\|_{s_0}$. To prove the (4.40) it is enough to apply ∂_x^α for $\alpha \leq s - (s_0 + 1)$ and reason as above by using the chain rule.

Reasoning as done in the proof of Theorem 4.3, one deduces that the sequence $b_n(W; x)$ converges in the space $\mathcal{F}_1^{\mathbb{R}}[r]$ to a symbol $b(W; x)$ solving the equation (4.33). Let us prove that such a solution is unique. Suppose that there exists another solution $\tilde{b}(W; x)$ of (4.33). Then we have

$$(b - \tilde{b})(\tau, W; x) = \tau \frac{d_u \beta(W; x) [Op^{\text{BW}}((b - \tilde{b})(\tau, W; x)\xi)W]}{1 + \tau(\beta)_x(W; x)},$$

which, as before, implies

$$\|b - \tilde{b}\|_{L_x^\infty} \leq \tau C \|W\|_{s_0+1} \|b - \tilde{b}\|_{L_x^\infty},$$

therefore a contradiction if $\|W\|_{s_0+1}$ is small enough. \square

Remark 4.7. Notice that the first order of approximation of our symbol b (see the (4.38)) coincide with the definition of b given in the paper [9].

Proof of Item (iii) of Theorem 4.6. First of all we recall that by Item (ii) we have that $b \in \mathcal{F}_1^{\mathbb{R}}[r]$. Hence the flow in (4.3) of (4.2) is well-posed and unique and satisfies (4.4)-(4.6) by Theorem 4.3. So we just have to show that $x(\tau)$ defined in (4.36), with b given in item (ii), solve the first equation in (4.2). We recall that, by (4.36) and (4.4), we have $x(\tau) = x_0 + \gamma(\tau, \Phi_b^{(z)}(\tau, z_0), x_0)$. By differentiating (4.32) we get

$$\partial_\tau x(\tau) + \beta(z(\tau), x(\tau)) + \tau(d_u \beta)(z(\tau), x(\tau))[\partial_\tau z(\tau)] + \tau(\partial_x \beta)(z(\tau), x(\tau))\partial_\tau x(\tau) = 0.$$

Hence

$$\partial_\tau x(\tau) = - \frac{\beta(z(\tau), x(\tau)) + \tau(d_u \beta)(z(\tau), x(\tau))[\partial_\tau z(\tau)]}{1 + \tau(\partial_x \beta)(z(\tau), x(\tau))}, \quad (4.41)$$

where $u = z(\tau)$. We shall use this notation also in the rest of the proof. Recalling that (see the third equation in (4.2)) one has $\partial_\tau z(\tau) = Op^{\text{BW}}(ib(\tau, z(\tau); x(\tau))\xi(\tau))[z(\tau)]$, and using the definition of b in (4.33) the (4.41) implies

$$\partial_\tau x(\tau) = \partial_\tau (x_0 + \gamma(\tau, z(\tau), x_0)) = -b(\tau, z(\tau), x(\tau)),$$

which is the first equation in (4.2).

Consider now $\xi(\tau)$ in (4.37). Hence

$$\begin{aligned} \partial_\tau \xi(\tau) &= \xi_0(\beta)_x(z(\tau), x(\tau)) + \tau \xi_0(\beta)_{xx}(z(\tau), x(\tau))\partial_\tau x(\tau) \\ &\quad + \xi_0 \tau (d_z \beta)_x(z(\tau), x(\tau))[\partial_\tau z(\tau)]. \end{aligned} \quad (4.42)$$

This is true because

$$\partial_x (d_w(\beta)(w, x)[h]) = (d_w(\beta)_x)(w, x)[h]. \quad (4.43)$$

By (4.42), using (4.37), the first and the third equations in (4.2) we also deduce

$$\begin{aligned} \partial_\tau \xi(\tau) &= \xi(\tau) \frac{(\beta)_x(z(\tau), x(\tau))}{1 + \tau(\beta)_x(z(\tau), x(\tau))} - \tau \xi(\tau) \frac{(\beta)_{xx}(z(\tau), x(\tau))\partial_\tau x(\tau)}{1 + \tau(\beta)_x(z(\tau), x(\tau))} b(\tau, z(\tau), x) \\ &\quad + \tau \xi(\tau) \frac{(d_u \beta)_x(z(\tau), x(\tau)) [Op^{\text{BW}}(ib(\tau, z(\tau); x(\tau))\xi(\tau))[z(\tau)]]}{1 + \tau(\beta)_x(z(\tau), x(\tau))}. \end{aligned}$$

Now, using (4.33), we have

$$\begin{aligned} (b)_x(\tau, z(\tau), x(\tau)) &= \\ &= -\frac{\beta(z(\tau), x(\tau)) + \tau(d_u\beta)(z(\tau), x(\tau))[Op^{BW}(ib(\tau, z(\tau), x(\tau))\xi(\tau))[z(\tau)]]}{(1 + \tau(\beta)_x(w, x))^2} (\beta)_{xx}(z(\tau), x(\tau)) \\ &+ \frac{(\beta)_x(z(\tau), x(\tau))}{1 + \tau(\beta)_x(z(\tau), x(\tau))} + \frac{\tau(d_u\beta)_x(z(\tau), x(\tau))[Op^{BW}(ib(\tau, z(\tau); x(\tau))\xi(\tau))[z(\tau)]]}{1 + \tau(\beta)_x(z(\tau), x(\tau))}, \end{aligned}$$

again recalling (4.43). Therefore

$$\partial_\tau \xi(\tau) = \xi(\tau)(b)_x(\tau, z(\tau), x(\tau)).$$

This means that (4.37) solves the second equation in (4.2). \square

Remark 4.8. Notice that, by Taylor expanding in z the flow $\Phi_b^{(z)}(\tau, z)$ in (4.36), (4.37), one obtains the formula (3.5.31) in [9] up to higher order homogeneity terms.

4.1.3. *Proof of Theorem 4.2.* In this subsection we conclude the proof of Theorem 4.2. The key point is to understand how to choose the function $b(\tau, W; y)$ in such a way the equation (4.8) is satisfied.

Notice that by Theorem 4.6 we also deduce information on the inverse flow (4.2). In particular, using (4.36), (4.37) we can rewrite the equation (4.8). It is easy to check that

$$F(b) := (1 + \tilde{a}_m(z_0, y))(1 + \gamma_y(1, z, y))\Big|_{y=x+\beta(z,x)}^m = m_b \quad (4.44)$$

where γ, b are given (in terms of β) by (4.32) and (4.33) and

$$z = \Phi_b^{(z)}(1, z_0), \quad \longleftrightarrow \quad z_0 = \tilde{\Phi}_b^{(z)}(1, z).$$

The equation (4.44) is non linear in the symbol $b(\tau, W; y)$. So, roughly speaking, we shall construct the solution $b = b_\infty$ of (4.44) as limit of a sequence of approximate solutions $b_n(\tau, W; y) \in \mathcal{F}_1^{\mathbb{R}}[r]$. More precisely, let $\gamma_0 = \beta_0 = b_0 = 0$ and for any $n \geq 1$ consider the problem

$$\partial_\tau x_n(\tau) = -b_n(\tau, z_n(\tau); x_n(\tau)), \quad (4.45)$$

$$\partial_\tau \xi_n(\tau) = (b_n)_x(\tau, z_n(\tau); x_n(\tau))\xi_n(\tau), \quad (4.46)$$

$$\partial_\tau z_n(\tau) = Op^{BW}(ib_n(\tau, z_n(\tau); x_n(\tau))\xi_n(\tau))[z_n(\tau)], \quad (4.47)$$

where $b_n(\tau, w, x)$ is the symbol defined by

$$b_n(\tau, w, x) := \frac{\beta_n(w, x) + \tau(d_u\beta_n)(w, x)[Op^{BW}(ib_n(\tau, w, x)\xi)[w]]}{1 + \tau(\beta_n)_x(w, x)} \quad (4.48)$$

and $\beta_n(w, x)$ is defined by

$$x_0 + \gamma_n(\tau, w, x_0) + \tau\beta_n(w, x_0 + \gamma_n(\tau, w, x_0)) = x_0, \quad \forall \tau \in [0, 1], x_0 \in \mathbb{T}, \quad w \in H^s, \quad (4.49)$$

and

$$\gamma_n(1, w, x) := \partial_x^{-1} \left(\left(\frac{m_n(w)}{1 + \tilde{a}_m(\tilde{\Phi}_{b_{n-1}}^{(z)}(1, w), x)} \right)^{\frac{1}{m}} - 1 \right), \quad (4.50)$$

$$m_n(w) := \left[2\pi \left(\int_{\mathbb{T}} \frac{1}{(1 + \tilde{a}_m(\tilde{\Phi}_{b_{n-1}}^{(z)}(1, w), x))^{\frac{1}{m}}} dy \right)^{-1} \right]^m - 1. \quad (4.51)$$

We shall prove inductively the following: for any $n \geq 1$

(S1)_n One has that m_n in (4.51) belongs to $\mathcal{F}_0^{\mathbb{R}}[r]$ and it is independent of x ; the functions γ_n, β_n, b_n given respectively by (4.50), (4.49) and (4.48) belong to $\mathcal{F}_1^{\mathbb{R}}$. In particular we have that there exists a constant $C > 0$ depending only on $|\tilde{a}_d|_s^{\mathcal{F}}$ such that

$$|m_n|_s^{\mathcal{F}} + |\gamma_n|_s^{\mathcal{F}} + |\beta_n|_s^{\mathcal{F}} + |b_n|_s^{\mathcal{F}} \lesssim_s C. \quad (4.52)$$

Moreover

$$|m_n|_{s,0}^{\mathcal{F}} + |\gamma_n|_{s+1,0}^{\mathcal{F}} + |\beta_n|_{s+1,0}^{\mathcal{F}} + |b_n|_{s,0}^{\mathcal{F}} \lesssim_s |\tilde{a}_d|_{s,0}^{\mathcal{F}}. \quad (4.53)$$

(S2)_n the flow of (4.45)-(4.47) with b_n given by (4.48) is well-posed, has the form

$$\begin{aligned} z_n(\tau) &= \Phi_{b_n}^{(z)}(\tau, z_0) \in \cap_{k=0}^K C^k([0, 1]; H^{s-k}), \\ x_n(\tau) &= x_0 + \gamma_n(\tau, z_n(\tau), x_0), \\ \xi_n(\tau) &= \xi_0(1 + \tau(\beta_n)_x(z_n(\tau), x_n(\tau))). \end{aligned} \quad (4.54)$$

We argue by induction.

Initialization. The (S1)₀, (S2)₀ are trivial.

So we assume that (Sk)_j, for $k = 1, 2$, hold true with $0 \leq j \leq n - 1$.

Proof of (S1)_n. By the inductive hypothesis b_{n-1} is a symbol in $\mathcal{F}_1^{\mathbb{R}}[r]$, hence, by Theorem 4.3, the flow of (4.2) with $b \rightsquigarrow b_{n-1}$ is well-posed. Therefore, using the formula of Faa di Bruno and (4.12), one can check that (reasoning as in (4.26)-(4.29))

$$|\tilde{a}_d(\tilde{\Phi}_{b_{n-1}}^{(z)}(1, w), x)|_s^{\mathcal{F}} \lesssim_s |\tilde{a}_d|_s^{\mathcal{F}}.$$

Hence, using the (4.51) and Remark 2.8, one deduces the (4.52) for m_n . Recalling Remark 2.7 one obtains the (4.53) for m_n . The estimates (4.52), (4.53) for γ_n follow in the same way using (4.50) and the smoothing effect of the Fourier multiplier ∂_x^{-1} .

Using the relation (4.49) at $\tau = 1$ one construct the function β_n (independent of τ) as the inverse diffeomorphism of $y = x_0 + \gamma_n(1, w, x_0)$. One can check that $|\beta_n|_s^{\mathcal{F}} \lesssim_s C(|\gamma_n|_s^{\mathcal{F}})$ and $|\beta_n|_{s,0}^{\mathcal{F}} \lesssim_s |\gamma_n|_{s,0}^{\mathcal{F}}$. Hence the (4.52), (4.53) hold for β_n . The family $[0, 1] \ni \tau \rightarrow \gamma_n(\tau, w; x_0)$ is given by item (i) of Theorem 4.6. Since β_n satisfies (4.52), (4.53) then, item (ii) of Theorem 4.6 implies that the function b_n in (4.48) is well-posed and satisfies (4.52), (4.53). \square

Proof of (S2)_n. Thanks to (4.52), (4.53). we can apply Theorems 4.3 and 4.6 (see item (iii)) which imply the (4.54). \square

In order to conclude the proof of Theorem 4.2 we have to check the (4.8). We need some preliminary results which are consequences of (S1)_n, (S2)_n.

Lemma 4.9. For $r > 0$ satisfying

$$r C_s \sup_{\tau \in [0,1]} (|b_n|_s^{\mathcal{F}} + |b_{n-1}|_s^{\mathcal{F}}) \ll 1, \quad (4.55)$$

for some $C_s \gg 1$ we have

$$\sup_{\tau \in [0,1]} \|\Phi_{b_n}^{(z)} - \Phi_{b_{n-1}}^{(z)}\|_{H^{s-1}} \lesssim_s r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s-1,0}^{\mathcal{F}}, \quad (4.56)$$

where $\Phi_{b_j}^{(z)}$, $j = n, n - 1$ is the solution of (4.47).

We postpone the proof of Lemma 4.9 and we first show some consequences.

Remark 4.10. If $\tilde{\Phi}_{b_n}^{(z)}$ and $\tilde{\Phi}_{b_{n-1}}^{(z)}$ are respectively the inverse flows of $\Phi_{b_n}^{(z)}$ and $\Phi_{b_{n-1}}^{(z)}$, then one can write

$$\tilde{\Phi}_{b_n}^{(z)} - \tilde{\Phi}_{b_{n-1}}^{(z)} = -\tilde{\Phi}_{b_n}^{(z)} \left(\Phi_{b_n}^{(z)} - \Phi_{b_{n-1}}^{(z)} \right) \tilde{\Phi}_{b_{n-1}}^{(z)}.$$

Then, since $\|\tilde{\Phi}_{b_j}^{(z)}\|_{H^s} \lesssim_s \|z_0\|_{H^s}$, $j = n, n-1$ (it satisfies the same estimates of the flow $\Phi_{b_j}^{(z)}$ in (4.11)), and using the (4.56) we deduce

$$\sup_{\tau \in [0,1]} \|\tilde{\Phi}_{b_n}^{(z)} - \tilde{\Phi}_{b_{n-1}}^{(z)}\|_{H^{s-1}} \lesssim_s r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s-1,0}^{\mathcal{F}}. \quad (4.57)$$

Notice that, by (4.52), we have

$$b_n \rightarrow^* \tilde{b} \text{ in } \mathcal{F}_1^{\mathbb{R}}[r],$$

with \tilde{b} still satisfying (4.52). Moreover, by Ascoli-Arzelà theorem, we deduce that

$$b_n \rightarrow b \quad \text{in} \quad C^k(\mathbb{T}, \mathbb{R}), \quad (4.58)$$

for $0 \leq k \leq s - s_0 - 1$ uniformly in $x_0 \in \mathbb{T}$. Hence we deduce $\tilde{b} \equiv b$, which means that $b \in \mathcal{F}_1^{\mathbb{R}}[r]$ with bounded $|\cdot|_{s-1,0}^{\mathcal{F}}$ -norm. In the same way one deduces that $m_n(w)$ converges in norm $|\cdot|_{s,0}^{\mathcal{F}}$ to another constant function $m_b(w)$. We are now ready to prove (4.8).

Lemma 4.11. *One has that $F(b_n) \rightarrow m_b(w)$ in the space $C^k(\mathbb{T}, \mathbb{R})$ with $0 \leq k \leq s - s_0 - 1$.*

Proof. We set

$$z = z_n(1) = \Phi_{b_n}^{(z)}(1, z_0), \quad x = x_n(1) = \Phi_{b_n}^{(x)}(1, x_0).$$

Then, using the (4.54) in $(\mathbf{S2})_n$, we have that

$$\begin{aligned} F(b_n) &= \left(1 + \tilde{a}_m(\tilde{\Phi}_{b_n}^{(z)}(1, z), \tilde{\Phi}_{b_n}^{(x)}(1, z, x))\right) \left(1 + \tilde{\Psi}_{b_n}^{(\xi)}(1, z, x)\right)^m \\ &= \left(1 + \tilde{a}_m(\tilde{\Phi}_{b_n}^{(z)}(1, z), y)\right) \left(1 + (\gamma_n)_y(1, z, y)\right)_{|y=x+\beta_n(z,x)}^m \\ &= \underbrace{\left(1 + \tilde{a}_m(\tilde{\Phi}_{b_{n-1}}^{(z)}(1, z), y)\right) \left(1 + (\gamma_n)_y(1, z, y)\right)_{|y=x+\beta_n(z,x)}^m}_{\stackrel{(4.50)}{=} m_n} \\ &\quad + \underbrace{\left[\tilde{a}_m(\tilde{\Phi}_{b_n}^{(z)}(1, z), y) - \tilde{a}_m(\tilde{\Phi}_{b_{n-1}}^{(z)}(1, z), y)\right] \left(1 + (\gamma_n)_y(1, z, y)\right)_{|y=x+\beta_n(z,x)}^m}_{\text{to be bounded}} \end{aligned} \quad (4.59)$$

To bound the last term in (4.59) we reason as follows. Recalling that $\tilde{a}_m \in \mathcal{F}_1^{\mathbb{R}}$ we have

$$\begin{aligned} \|(d_u \tilde{a}_m)(w) [\tilde{\Phi}_{b_n}^{(z)}(1, z) - \tilde{\Phi}_{b_{n-1}}^{(z)}(1, z)]\|_{L_x^\infty} &\lesssim_s \|\tilde{\Phi}_{b_n}^{(z)}(1, z) - \tilde{\Phi}_{b_{n-1}}^{(z)}(1, z)\|_{H^{s_0}} \\ &\stackrel{(4.57)}{\lesssim_s} r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s-1,0}^{\mathcal{F}}. \end{aligned}$$

By (4.58) we have that the estimate above implies $F(b_n) \rightarrow m_b(w)$ in $C^0(\mathbb{T}, \mathbb{R})$. For the derivatives in x_0 one can reason similarly. \square

In order to prove Lemma 4.9, we need the following result.

Lemma 4.12. *Let $f \in \mathcal{F}_1[r]$ and define*

$$G(\tau, z_0; x_0) := f(z_j(\tau), x_j(\tau)) \xi_j(\tau) \frac{1}{\xi_0}$$

where $(z_j(\tau), x_j(\tau), \xi_j(\tau))$ is the flow of (4.45)-(4.47) generated by b_j . Then, for r small enough, one has

$$\sup_{\tau \in [0,1]} |G|_{s,0}^{\mathcal{F}} \lesssim_s \sup_{\tau \in [0,1]} |f|_{s,0}^{\mathcal{F}} \left(1 + r \sup_{\tau \in [0,1]} |b_j|_{s,0}^{\mathcal{F}}\right).$$

Proof. First of all we have

$$\partial_{x_0} G = f(z_j, x_j) \partial_{x_0} \Psi_{b_j}^{(\xi)} + \xi_j(\tau) \frac{1}{\xi_0} (\partial_x f)(z_j, x_j) \partial_{x_0} x_j$$

Hence (recall (2.11))

$$\begin{aligned} \|\partial_{x_0} G\|_{L_x^\infty} &\lesssim_s \|f(z_j, x_j)\|_{L_x^\infty} \|\partial_{x_0} \Psi_{b_j}^{(\xi)}\|_{L_x^\infty} + (1 + \|\Psi_{b_j}^{(\xi)}\|_{L_x^\infty}) \|(\partial_x f)(z_j, x_j)\|_{L_x^\infty} (1 + \|\partial_{x_0} \Psi_{b_j}^{(x)}\|_{L_x^\infty}) \\ &\stackrel{(4.9), (4.10)}{\lesssim_s} |f|_{0,0}^{\mathcal{F}} |b_j|_{1,0}^{\mathcal{F}} \|z_j\|_{H^{s_0}} \|z_0\|_{H^{s_0+1}} \\ &\quad + (1 + |\Psi_{b_j}^{(\xi)}|_{0,0}^{\mathcal{F}}) |f|_{1,0}^{\mathcal{F}} (1 + |\Psi_{b_j}^{(x)}|_{1,0}^{\mathcal{F}} \|z_0\|_{H^{s_0+1}}) \|z_j\|_{H^{s_0+1}}. \end{aligned} \quad (4.60)$$

Using (4.11) we obtain

$$\sup_{\tau \in [0,1]} |G|_{1,0}^{\mathcal{F}} \lesssim_s \sup_{\tau \in [0,1]} |f|_{1,0}^{\mathcal{F}} (1 + r \sup_{\tau \in [0,1]} |b_j|_{1,0}^{\mathcal{F}}).$$

To estimate $\|\partial_{x_0}^\alpha G\|_{L_x^\infty}$ one can apply the formula of Faa di Bruno and reasoning as done in (4.60). \square

Proof of Lemma 4.9. Consider the equations (4.45)-(4.47) and set, for any $n \geq 1$,

$$X_n(\tau) := x_n(\tau) - x_{n-1}(\tau), \quad (4.61)$$

$$Y_n(\tau) := \xi_n(\tau) - \xi_{n-1}(\tau), \quad (4.62)$$

$$Z_n(\tau) := z_n(\tau) - z_{n-1}(\tau). \quad (4.63)$$

We claim that if (4.55) holds then, for $s \geq s_0 + 2$,

$$\sup_{\tau \in [0,1]} \left(|X_n|_{s-1,0}^{\mathcal{F}} + |Y_n \xi_0^{-1}|_{s-2,0}^{\mathcal{F}} + \|Z_n\|_{H^{s-1}} \right) \lesssim_s r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s-1,0}^{\mathcal{F}}. \quad (4.64)$$

The (4.64) implies the (4.56).

Let us prove the (4.64). Notice that, for some $\sigma_i \in [0, 1]$, $i = 1, \dots, 6$,

$$\begin{aligned} -\partial_\tau X_n &\stackrel{(4.45)}{=} (\partial_x b_n)(\tau, z_n, x_{n-1} + \sigma_1 X_n) [X_n] + (d_u b_n)(\tau, z_{n-1} + \sigma_2 Z_n, x_{n-1}) [Z_n] \\ &\quad + (b_n - b_{n-1})(\tau, z_{n-1}, x_{n-1}). \end{aligned} \quad (4.65)$$

$$\begin{aligned} \partial_\tau Y_n &\stackrel{(4.46)}{=} (b_n)_x(\tau, z_n, x_n) Y_n + \xi_{n-1} (b_n)_{xx}(\tau, z_n, x_{n-1} + \sigma_3 X_n) [X_n] \\ &\quad + \xi_{n-1} (d_u b_n)_x(\tau, z_{n-1} + \sigma_4 Z_n, x_{n-1}) [Z_n] + \xi_{n-1} (b_n - b_{n-1})_x(\tau, z_{n-1}, x_{n-1}). \end{aligned} \quad (4.66)$$

$$\begin{aligned} \partial_\tau Z_n &\stackrel{(4.47)}{=} Op^{BW}(i b_n(\tau, z_n; x_n) \xi_n) [Z_n] + Op^{BW}(i b_n(\tau, z_n; x_n) Y_n) [z_{n-1}] \\ &\quad + Op^{BW}(i (\partial_x b_n)(\tau, z_n; x_{n-1} + \sigma_5 X_n) [X_n] \xi_{n-1}) [z_{n-1}] \\ &\quad + Op^{BW}(i (d_u b_n)(\tau, z_{n-1} + \sigma_6 Z_n; x_{n-1}) [Z_n] \xi_{n-1}) [z_{n-1}] \\ &\quad + Op^{BW}((b_n - b_{n-1})(\tau, z_{n-1}, x_{n-1}) \xi_{n-1}) [z_{n-1}]. \end{aligned} \quad (4.67)$$

We now estimate $X_n(\tau)$ in (4.61). First we note that, by (4.65),

$$\begin{aligned} X_n(\tau) &= \int_0^\tau (\partial_x b_n)(t, z_n(t), x_{n-1}(t) + \sigma_1 X_n(t)) [X_n(t)] dt \\ &\quad + \int_0^\tau (d_u b_n)(t, z_{n-1}(t) + \sigma_2 Z_n(t), x_{n-1}(t)) [Z_n(t)] dt \\ &\quad + \int_0^\tau (b_n - b_{n-1})(t, z_{n-1}(t), x_{n-1}(t)) dt. \end{aligned} \quad (4.68)$$

Notice that

$$x_{n-1}(t) + \sigma_1 X_n(t) = x_{n-1}(t) + \sigma_1 (x_n - x_{n-1})(t)$$

satisfies, by using (4.9)-(4.10) (with $b \rightsquigarrow b_{n-1}$ and $b \rightsquigarrow b_n$),

$$\sup_{\tau \in I} |x_{n-1} + \sigma_1 X_n|_{\alpha,0}^{\mathcal{F},s} \lesssim_s |b_n|_{s,0}^{\mathcal{F}} + |b_{n-1}|_{s,0}^{\mathcal{F}} \quad (4.69)$$

Similarly we can get bounds on $\xi_{n-1}(t) + \sigma Y_n(t)$ and $z_{n-1}(t) + \sigma Z_n(t)$ for some $\sigma \in [0, 1]$. Then, by differentiating the (4.68), using Lemma 4.12 (see also (4.69)) and recalling Remark 2.7, we get

$$\sup_{\tau \in [0,1]} |X_n|_{s,0}^{\mathcal{F}} \lesssim_s r A_n |X_n|_{s,0}^{\mathcal{F}} + C_n \|Z_n\|_{H^s} + r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s,0}^{\mathcal{F}} (1 + r \sup_{\tau \in [0,1]} |b_{n-1}|_{s,0}^{\mathcal{F}}), \quad (4.70)$$

where

$$A_n := \sup_{\tau \in [0,1]} |b_n|_{s+1,0}^{\mathcal{F}} \left(1 + r |b_n|_{s,0}^{\mathcal{F}} + r |b_{n-1}|_{s,0}^{\mathcal{F}}\right), \quad (4.71)$$

$$C_n := \sup_{\tau \in [0,1]} |b_n|_{s+1}^{\mathcal{F}}. \quad (4.72)$$

Therefore, for $r > 0$ such that

$$r A_n \leq 1/2 \quad (4.73)$$

we get we obtain

$$\sup_{\tau \in [0,1]} |X_n|_{s,0}^{\mathcal{F}} \lesssim_s C_n \|Z_n\|_{H^s} + r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s,0}^{\mathcal{F}} \quad (4.74)$$

We recall that, by (4.6), the symbol $Y_n \xi_0^{-1}$ is actually a function in $\mathcal{F}_1^{\mathbb{R}}[r]$. Then reasoning as done for X_n and using equation (4.66), we deduce

$$\begin{aligned} \sup_{\tau \in [0,1]} |\xi_0^{-1} Y_n|_{s,0}^{\mathcal{F}} &\lesssim_s r \sup_{\tau \in [0,1]} |b_n|_{s+2,0}^{\mathcal{F}} \sup_{\tau \in [0,1]} |X_n|_s^{\mathcal{F}} + \sup_{\tau \in [0,1]} |b_n|_{s+2}^{\mathcal{F}} \sup_{\tau \in [0,1]} \|Z_n\|_{H^s} \\ &\quad + r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s+1,0}^{\mathcal{F}} (1 + r \sup_{\tau \in [0,1]} |b_{n-1}|_s^{\mathcal{F}}) \\ &\stackrel{(4.74)}{\lesssim_s} \sup_{\tau \in [0,1]} |b_n|_{s+2}^{\mathcal{F}} \sup_{\tau \in [0,1]} \|Z_n\|_{H^s} + r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s+1,0}^{\mathcal{F}} \end{aligned} \quad (4.75)$$

where we used the smallness condition

$$r \sup_{\tau \in [0,1]} \left(|b_n|_{s+2,0}^{\mathcal{F}} + |b_{n-1}|_{s+2,0}^{\mathcal{F}} \right) \ll 1. \quad (4.76)$$

We now consider the equation (4.67). Using Proposition 2.22 (and $r > 0$ small enough) we deduce

$$\begin{aligned} \|Op^{\text{BW}}(ib_n(\tau, z_n; x_n) Y_n)[z_{n-1}]\|_{H^s} &\lesssim_s \|z_{n-1}\|_{H^{s+1}} \|b_n(\tau, z_n; x_n) Y_n \xi_0^{-1}\|_{L_x^\infty} \\ &\stackrel{\text{Lem.4.12}}{\lesssim_s} \|z_{n-1}\|_{H^{s+1}} \|z_n\|_{H^{s_0}} \sup_{\tau \in [0,1]} |b_n|_{0,0}^{\mathcal{F}} 2 \sup_{\tau \in [0,1]} |\xi_0^{-1} Y_n|_{0,0}^{\mathcal{F}} \\ &\stackrel{(4.75)}{\lesssim_s} \|z_{n-1}\|_{H^{s+1}} \sup_{\tau \in [0,1]} |b_n|_{s_0+2}^{\mathcal{F}} \sup_{\tau \in [0,1]} \|Z_n\|_{H^{s_0}} + \|z_{n-1}\|_{H^{s+1}} r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s_0+1,0}^{\mathcal{F}} \end{aligned} \quad (4.77)$$

Reasoning in the same way and using (4.74), (4.72), we get

$$\begin{aligned} &\|Op^{\text{BW}}\left(i(\partial_x b_n)(\tau, z_n; x_{n-1} + \sigma_5 X_n)[X_n] \xi_{n-1}\right)[z_{n-1}]\|_{H^s} + \\ &\|Op^{\text{BW}}\left(i(d_u b_n)(\tau, z_{n-1} + \sigma_6 Z_n; x_{n-1})[Z_n] \xi_{n-1}\right)[z_{n-1}]\|_{H^s} \\ &\lesssim_s \|z_{n-1}\|_{H^{s+1}} \sup_{\tau \in [0,1]} |b_n|_{s_0+2}^{\mathcal{F}} \sup_{\tau \in [0,1]} \|Z_n\|_{H^{s_0}} + \|z_{n-1}\|_{H^{s+1}} r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s_0+1,0}^{\mathcal{F}} \end{aligned} \quad (4.78)$$

and

$$\|Op^{\text{BW}}\left((b_n - b_{n-1})(\tau, z_{n-1}, x_{n-1}) \xi_{n-1}\right)[z_{n-1}]\|_{H^s} \lesssim_s \|z_{n-1}\|_{H^{s+1}} \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{0,0}^{\mathcal{F}} r. \quad (4.79)$$

Finally, recalling (2.5) and using symbolic calculus, one can check that

$$\text{Re}\left(\langle D \rangle^s Op^{\text{BW}}(ib_n(\tau, z_n; x_n) \xi_n)[Z_n], \langle D \rangle^s Z_n\right)_{L^2} \lesssim_s \|Z_n\|_{H^s}^s \sup_{\tau \in [0,1]} |b_n|_{s,0}^{\mathcal{F}} r. \quad (4.80)$$

Therefore, putting together the (4.77), (4.78), (4.79) and (4.80), by equation (4.67) we deduce

$$\begin{aligned} \partial_\tau \|Z_n\|_{H^s}^2 &\lesssim_s \|Z_n\|_{H^s}^2 \sup_{\tau \in [0,1]} |b_n|_{s,0}^{\mathcal{F}} r + \|z_{n-1}\|_{H^{s+1}} \|Z_n\|_{H^s} \sup_{\tau \in [0,1]} |b_n|_{s_0+2}^{\mathcal{F}} \sup_{\tau \in [0,1]} \|Z_n\|_{H^{s_0}} \\ &\quad + \|z_{n-1}\|_{H^{s+1}} \|Z_n\|_{H^s} r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s_0+1,0}^{\mathcal{F}}. \end{aligned} \quad (4.81)$$

We now recall that $\|z_{n-1}\|_{H^s} \lesssim_s r$. We use formula (4.81) with $s \rightsquigarrow s-1$. Then, for $r > 0$ (small enough) such that $r \sup_{\tau} |b_n|_{s,0}^{\mathcal{F}} \ll 1$, we get

$$\begin{aligned} \sup_{\tau \in [0,1]} \|Z_n\|_{H^{s-1}}^2 &\lesssim_s \sup_{\tau \in [0,1]} \|z_{n-1}\|_{H^{s+1}} \sup_{\tau \in [0,1]} \|Z_n\|_{H^s} r \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s_0+1,0}^{\mathcal{F}}, \quad \Rightarrow \\ \sup_{\tau \in [0,1]} \|Z_n\|_{H^{s-1}} &\lesssim_s r^2 \sup_{\tau \in [0,1]} |b_n - b_{n-1}|_{s_0+1,0}^{\mathcal{F}}. \end{aligned} \quad (4.82)$$

Since condition (4.55) implies (4.73)-(4.71) with $s \rightsquigarrow s-1$, and the (4.76) with $s \rightsquigarrow s-2$, then by (4.82), (4.75) with $s \rightsquigarrow s-2$ and (4.74) with $s \rightsquigarrow s-1$ we obtain the (4.64).

In order to conclude the proof of Theorem 4.1.3 we need to prove that actually the functions $b(w; x)$ and $m_b(w)$ belong to the class $\Sigma \mathcal{F}_1^{\mathbb{R}}[r, N]$. In order to prove such a fact one makes the following ansatz: $b = \sum_{j=1}^N b_j$ with $b_j \in \widetilde{\mathcal{F}}_1^{\mathbb{R}}$ and $b_N \in \mathcal{F}_N^{\mathbb{R}}[r]$. Then one may prove, by using Taylor expansions, that the flows in (4.5) and (4.6) are verified, while the flow $z(\tau, z_0) = z_0 + M(\tau, z_0)z_0$ in (4.4) with M in the class $\Sigma \mathcal{M}_1[r, N]$. One find, by using (4.8), some recursive equation on the terms b_j , which may be expressed at each depending only on b_k with $k < j$. \square

4.2. Off-diagonal terms at highest order. In this subsection we study a non-linear problem we need to solve in order to conjugate system (3.4) to another one whose matrix of symbols is diagonal at the highest order. A similar problem has been solved in [26]. In such a paper the matrix is diagonalized by means of a parametrix generated by the matrix of eigenvectors of the original matrix. Unfortunately it is not easy to transform such parametrix in a change of coordinates of the phase space. At a linear level the diagonalization problem has achieved in [25] by means of auxiliary linear flows. We adapt the last strategy to the non-linear problem.

The main result of this subsection is Theorem 4.13. This result shows that the equations (4.86) and (4.87) have solutions in the our classes of symbols. Furthermore it guarantees that it is possible to choose the generator of the flow in (4.85) in such a way the new off-diagonal symbol b^+ equals 0. The proof is quite long and it is divided in several steps. In Section 4.2.1 we study the well-posedness of equations (4.86) and (4.87). In Section 4.2.2 we solve (4.89), this is a non-linear problem and the unknown is the generator C of the flow (4.85), the techniques used are similar to the one used in Section 4.1.3.

Consider functions

$$a(U; x) \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N], \quad b(U; x) \in \Sigma \mathcal{F}_1[r, N]. \quad (4.83)$$

Let

$$C(\tau, U; x) \in \Sigma \mathcal{F}_1[r, N], \quad \tau \in [0, 1], \quad (4.84)$$

and consider the equation

$$\begin{cases} \partial_\tau Z(\tau) = Op^{\text{BW}}(\mathbf{C}(\tau, Z(\tau); x))[Z(\tau)], \\ Z(0) = Z_0 = \begin{bmatrix} z_0 \\ \bar{z}_0 \end{bmatrix}, \end{cases} \quad \mathbf{C}(\tau, U; x) := \begin{pmatrix} 0 & C(\tau, U; x) \\ C(\tau, U; x) & 0 \end{pmatrix}. \quad (4.85)$$

In the following we shall write d_U to denote the differential of a symbol $a(U; x, \xi)$ with respect to the variable $U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$, i.e.

$$(d_U a)(U; x, \xi)[H] = (d_u a)(U; x, \xi)[h] + (d_{\bar{u}} a)(U; x, \xi)[\bar{h}], \quad H = \begin{bmatrix} h \\ \bar{h} \end{bmatrix}.$$

We consider the two problems

$$\begin{cases} \partial_\tau a^+(\tau, Z; x) = -2\operatorname{Re}\left(C(\tau, Z; x)\overline{b^+(\tau, Z; x)}\right) - (d_U a^+)(\tau, Z; x)[Op^{\text{BW}}(\mathbf{C}(Z; x))[Z]], \\ 1 + a^+(0, Z; x) = 1 + a(Z; x), \end{cases} \quad (4.86)$$

$$\begin{cases} \partial_\tau b^+(\tau, Z; x) = -2(1 + a^+(\tau, Z; x))C(\tau, Z; x) - (d_U b^+)(\tau, Z; x)[Op^{\text{BW}}(\mathbf{C}(Z; x))[Z]], \\ b^+(0, Z; x) = b(Z; x). \end{cases} \quad (4.87)$$

The following holds true.

Theorem 4.13. *Assume (4.83). For $r > 0$ small enough there exists a symbol $C(\tau, U; x)$ as in (4.84) such that the symbols defined by (4.86), (4.87) are such that*

$$a^+(\tau, U; x) \in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N], \quad b^+(\tau, U; x) \in \Sigma\mathcal{F}_1[r, N] \quad (4.88)$$

with estimates uniform in $\tau \in [0, 1]$. Moreover one has

$$F(C) := b^+(1, Z; x) \equiv 0. \quad (4.89)$$

The rest of the section is devoted to the proof of Theorem 4.13. At the beginning we shall work with non-homogeneous symbols in the class $\mathcal{F}_1[r]$.

4.2.1. *Solutions of (4.86), (4.87).* First of all we note that, if the symbol C belongs to $\mathcal{F}_1[r]$, the flow in (4.85) is well-posed by Theorem 2.31 applied with generator as in (2.54). We denote by Ψ_C^τ , $(\Psi_C^\tau)^{-1}$ respectively the flow and the inverse flow of (4.85). Set

$$g_1(\tau, x) := a^+(\tau, Z(\tau); x), \quad g_2(\tau, x) := b^+(\tau, Z(\tau); x), \quad Z(\tau) := \Psi_C^\tau(U). \quad (4.90)$$

We have that (recall (4.86), (4.87))

$$\begin{aligned} \partial_\tau g_1(\tau, x) &= -2\operatorname{Re}\left(g_2(\tau, x)\overline{C(\tau, Z(\tau); x)}\right), & g_1(0) &= a(U; x), \\ \partial_\tau g_2(\tau, x) &= -2(1 + g_1(\tau))C(\tau, Z(\tau); x), & g_2(0) &= b(U; x), \end{aligned} \quad (4.91)$$

which implies

$$\begin{aligned} g_1(\tau) &:= a(U; x) - 2 \int_0^\tau \operatorname{Re}\left(g_2(\sigma, x)\overline{C(\sigma, Z(\sigma); x)}\right) d\sigma, \\ g_2(\tau) &:= b(U; x) - 2 \int_0^\tau (1 + g_1(\sigma, x))C(\sigma, Z(\sigma); x) d\sigma. \end{aligned} \quad (4.92)$$

By (4.90) we write $Z(\sigma) = \Psi_C^\sigma(U) = \Psi_C^\sigma \circ (\Psi_C^\tau)^{-1}(Z)$ and by (4.92) we deduce that equations (4.86), (4.87) are equivalent to

$$a^+(\tau, Z; x) := a((\Psi_C^\tau)^{-1}(Z); x) - 2 \int_0^\tau \operatorname{Re}\left(b^+(\sigma, \Psi_C^\sigma((\Psi_C^\tau)^{-1}(Z)); x)\overline{C(\sigma, \Psi_C^\sigma((\Psi_C^\tau)^{-1}(Z)); x)}\right) d\sigma, \quad (4.93)$$

$$b^+(\tau, Z; x) := b((\Psi_C^\tau)^{-1}(Z); x) - 2 \int_0^\tau (1 + a^+(\sigma, \Psi_C^\sigma((\Psi_C^\tau)^{-1}(Z)); x))C(\sigma, \Psi_C^\sigma((\Psi_C^\tau)^{-1}(Z)); x) d\sigma. \quad (4.94)$$

To solves the (4.93), (4.94) we reason as follows. Let $a_{-1}^+ = b_{-1}^+ \equiv 0$ and consider, for $n \geq 0$, the following problems:

$$a_n^+(\tau, Z; x) := a((\Psi_C^\tau)^{-1}(Z); x) - 2 \int_0^\tau \operatorname{Re}\left(b_{n-1}^+(\sigma, \Psi_C^\sigma((\Psi_C^\tau)^{-1}(Z)); x)\overline{C(\sigma, \Psi_C^\sigma((\Psi_C^\tau)^{-1}(Z)); x)}\right) d\sigma, \quad (4.95)$$

$$b_n^+(\tau, Z; x) := b((\Psi_C^\tau)^{-1}(Z); x) - 2 \int_0^\tau (1 + a_{n-1}^+(\sigma, \Psi_C^\sigma((\Psi_C^\tau)^{-1}(Z)); x))C(\sigma, \Psi_C^\sigma((\Psi_C^\tau)^{-1}(Z)); x) d\sigma. \quad (4.96)$$

We have the following.

Lemma 4.14. (Iterative Lemma). *Let $C(\tau, U; x) \in \mathcal{F}_1[r]$ and assume that (recall (2.14))*

$$|C|_s^{\mathcal{F}} = \sum_{\alpha + mk \leq s - s_0} |C|_{\alpha, k}^{\mathcal{F}, s} \leq C,$$

for some $C > 0$ (depending on s) and $s_0 \gg 1$. Then for $r > 0$ small enough one has the following for any $n \geq 0$:

(S1)_n one has that (recall (4.95), (4.96)) $a_n^+ \in \mathcal{F}_1^{\mathbb{R}}[r]$, $b_n^+ \in \mathcal{F}_1[r]$. Moreover there are constants $C_{\alpha, k}$, independent of n , increasing in k such that

$$|a_n^+|_{\alpha, k}^{\mathcal{F}, s} + |b_n^+|_{\alpha, k}^{\mathcal{F}, s} \leq C_{\alpha, k}, \quad \alpha + mk \leq s - s_0. \quad (4.97)$$

Moreover the constants $C_{\alpha, k}$ depends on s and on the norms $|a|_s^{\mathcal{F}}$, $|b|_s^{\mathcal{F}}$, $|c|_s^{\mathcal{F}}$.

(S2)_n We have that

$$|a_n^+ - a_{n-1}^+|_{\alpha, 0}^{\mathcal{F}, s-1} + |b_n^+ - b_{n-1}^+|_{\alpha, 0}^{\mathcal{F}, s-1} \leq 2^{-n}, \quad \alpha \leq s - s_0 - 1. \quad (4.98)$$

Proof. Assume inductively that **(S1)_{n-1}**, **(S2)_{n-1}** hold. We prove the **(S1)_n**, **(S2)_n** for the symbol a_n^+ . The proof of the properties for the symbol b_n^+ is the same.

(S1)_n. Using the estimates (2.59) on the flow $(\Psi_C^\tau)^{-1}$ one can prove (using Faa di Bruno's formula) that the term $a((\Psi_C^\tau)^{-1}(Z); x)$ in (4.95) satisfies the (4.97). So we study the term

$$d(\tau, \sigma, Z; x) := b_{n-1}^+(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x) \overline{C(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x)},$$

providing estimates independent of σ and τ . We prove the result only for $\alpha = 0$, $mk \leq s - s_0$. First of all notice that

$$|b_{n-1}^+(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x)|_{0, p}^{\mathcal{F}, s} \lesssim_s \tilde{C} |b_{n-1}^+|_{0, p} \lesssim_s \tilde{C} C_{0, p}, \quad p \leq k, \quad (4.99)$$

where in the last inequality we used the inductive hypothesis. Here the constant \tilde{C} depends also on the constant appearing in (2.59). The first inequality can be deduced by reasoning exactly as in (4.26)-(4.29) using (2.59) and recalling Remarks 2.7, 2.8. In the same way we have

$$|C(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x)|_{0, p}^{\mathcal{F}, s} \lesssim_s \tilde{C} |C|_{0, p}, \quad p \leq k.$$

Therefore we have

$$|d|_{0, k}^{\mathcal{F}, s} \lesssim_s \tilde{C} C_{0, k} |C|_{0, 0}^{\mathcal{F}, s} r + \tilde{C} C_{0, k-1} |C|_{0, k}^{\mathcal{F}, s} \leq C_{0, k}$$

for $C_{0, k}$ suitably chosen and r small enough. This implies the (4.97) with $\alpha = 0$. For $\alpha > 0$ one can reason in the same way.

(S2)_n. We prove the result for a_n^+ . By (4.95) we have

$$a_n^+ - a_{n-1}^+ = -2 \int_0^\tau \operatorname{Re} \left(\left(b_{n-1}^+ - b_{n-2}^+ \right) (\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x) \overline{C(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x)} \right) d\sigma.$$

Reasoning as in (4.99) one gets

$$\left| \left(b_{n-1}^+ - b_{n-2}^+ \right) (\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x) \right|_{\alpha, 0}^{\mathcal{F}, s} \lesssim_s |b_{n-1}^+ - b_{n-2}^+|_{\alpha, 0}^{\mathcal{F}, s} \stackrel{(4.98)}{\lesssim_s} 2^{-(n-1)}.$$

Therefore (recall Remark 2.8)

$$|a_n^+ - a_{n-1}^+|_{\alpha, 0}^{\mathcal{F}, s} \lesssim_s |C|_{\alpha, 0}^{\mathcal{F}, s} r 2^{-(n-1)} \leq 2^{-n}$$

for r small enough. This is the (4.98) at the step n . \square

By Lemma 4.14 and Remark 2.6 we have, up to extracting a subsequence, a sequence of solutions of (4.95), (4.96) such that

$$\begin{aligned} a_n^+ &\rightharpoonup^* \widetilde{a}^+ \text{ in } \mathcal{F}_1[r], \quad \partial_x^\alpha a_n^+ \rightarrow \partial_x^\alpha a^+ \text{ uniformly in } x_0 \text{ for } \alpha \leq s - s_0 - 1, \\ b_n^+ &\rightharpoonup^* \widetilde{b}^+ \text{ in } \mathcal{F}_1[r], \quad \partial_x^\alpha b_n^+ \rightarrow \partial_x^\alpha b^+ \text{ uniformly in } x_0 \text{ for } \alpha \leq s - s_0 - 1. \end{aligned}$$

Therefore we deduce $\widetilde{a}^+ = a^+$, $\widetilde{b}^+ = b^+$.

4.2.2. *The choice of the generator.* We want to exhibit a symbol $C(\tau, U; x)$ in such a way (4.89) holds true. We define the following

$$\begin{aligned} \mathcal{G}_C(\tau, U) &:= b(U; x) - 2 \int_0^\tau (1 + a(U; x)) C(\sigma, \Psi_C^\sigma(U); x) d\sigma \\ &\quad - 4 \int_0^\tau \int_0^\sigma \operatorname{Re} \left(b^+(\theta, \Psi_C^\theta(U); x) \overline{C(\theta, \Psi_C^\theta(U); x)} \right) C(\sigma, \Psi_C^\sigma(U); x) d\theta d\sigma. \end{aligned}$$

Then, by using (4.93) and (4.94), we note that

$$b^+(\tau, Z; x) = \mathcal{G}_C(\tau, (\Psi_C^\tau)^{-1}(Z)).$$

We look for $C(\tau, U; x)$ such that (see (4.89))

$$b^+(1, Z; x) = \mathcal{G}_C(1, (\Psi_C^1)^{-1}(Z)) = 0.$$

We reason as follows. Given functions $f, g, h \in \mathcal{F}_1[r]$ we define the functional

$$\begin{aligned} \mathcal{F}(f, g, h) &:= b(U; x) - 2 \int_0^1 (1 + a(U; x)) f(\sigma, \Psi_g^\sigma(U); x) d\sigma \\ &\quad - 4 \int_0^1 \int_0^\sigma \operatorname{Re} \left(h(\theta, \Psi_g^\theta(U); x) \overline{g(\theta, \Psi_g^\theta(U); x)} \right) f(\sigma, \Psi_g^\sigma(U); x) d\theta d\sigma. \end{aligned} \tag{4.100}$$

According to this notation we have

$$\mathcal{G}_C(1, U) \equiv \mathcal{F}(C, C, b^+).$$

We set $C_0 = 0$ and for any $n \geq 1$ we consider the symbol $C_n(\tau, U; x, \xi)$ as the solution of

$$\mathcal{F}(C_n, C_{n-1}, b_{n-1}^+) = 0 \tag{4.101}$$

where b_{n-1}^+ is defined as

$$b_{n-1}^+(\tau, Z; x) = \mathcal{G}_{C_{n-1}}(\tau, (\Psi_{C_{n-1}}^\tau)^{-1}(Z)). \tag{4.102}$$

We shall prove the following lemma.

Lemma 4.15. *For $r > 0$ small enough one has the following for any $n \geq 1$:*

(S1)_n *The symbols C_n defined by (4.101) belong to $\mathcal{F}_1[r]$. In particular there are constants $C_{\alpha, k}$, independent of n , increasing in k such that*

$$|C_n|_{\alpha, k}^{\mathcal{F}, s} \leq C_{\alpha, k}, \quad \alpha + mk \leq s - s_0. \tag{4.103}$$

Moreover the constants $C_{\alpha, k}$ depends on s and on the norms $|a|_s^{\mathcal{F}}$, $|b|_s^{\mathcal{F}}$.

(S2)_n *We have that*

$$|C_n - C_{n-1}|_{\alpha, 0}^{\mathcal{F}, s-1} \leq 2^{-n}, \quad \alpha \leq s - s_0 - 1. \tag{4.104}$$

In order to prove the above Lemma we need some preliminary results.

Lemma 4.16. *Assume that $C_n, C_{n-1} \in \mathcal{F}_1[r]$. For $r > 0$ small enough*

$$\sup_{\tau \in [0, 1]} \|\Psi_{C_n}^\tau - \Psi_{C_{n-1}}^\tau\|_{H^s} \lesssim_s r \sup_{\tau \in [0, 1]} |C_n - C_{n-1}|_{0, 0}^{\mathcal{F}, s},$$

where $\Psi_{C_j}^\tau$, $j = n, n-1$ is the solution of (4.85) with C_j in place of C .

Proof. We set $v_n(\tau) := Z_n(\tau) - Z_{n-1}(\tau)$. We get

$$\begin{aligned} \partial_\tau v_n &= iOp^{\text{BW}}(\mathbf{C}_n(Z_n; x))[v_n] + iOp^{\text{BW}}(\mathbf{C}_n(Z_n; x) - \mathbf{C}_n(Z_{n-1}; x))[Z_{n-1}] \\ &\quad + iOp^{\text{BW}}((\mathbf{C}_n - \mathbf{C}_{n-1})(Z_{n-1}; x))[Z_{n-1}], \end{aligned}$$

where \mathbf{C}_n is the matrix in (4.85) with $C \rightsquigarrow C_n$. Since the generator is bounded one can reason essentially as in the proof of the estimates for X_n, Y_n in the proof of Lemma 4.9. \square

Proof of Lemma 4.15. We argue by induction. Assume that $(\mathbf{S1})_{n-1}$, $(\mathbf{S2})_{n-1}$ hold.

Proof of $(\mathbf{S1})_n$. Since C_{n-1} is a symbol satisfying (4.103), then by subsection 4.2.1 we have that the symbol $b_{n-1}^+(\tau, U; x)$ defined by (4.102) is in $\mathcal{F}_1[r]$ with estimates

$$|b_{n-1}^+|_{\alpha, k}^{\mathcal{F}, s} \leq C_{\alpha, k}, \quad \alpha + mk \leq s - s_0,$$

for some constants $C_{\alpha, k}$ depending only on the norms $|a|_s^{\mathcal{F}}$, $|b|_s^{\mathcal{F}}$ and $|C_{n-1}|_s^{\mathcal{F}}$. Then, by reasoning as in the proof of (4.97) and using (4.101), we obtain the (4.103) for C_n .

Proof of $(\mathbf{S2})_n$. By (4.100) we have

$$\begin{aligned} 0 &= \mathcal{F}(C_n, C_{n-1}, b_{n-1}^+) - \mathcal{F}(C_{n-1}, C_{n-2}, b_{n-2}^+) = \\ &= -2 \int_0^1 (1 + a(U; x)) \left(C_n(\sigma, \Psi_{C_{n-1}}^\sigma(U); x) - C_{n-1}(\sigma, \Psi_{C_{n-1}}^\sigma(U); x) \right) d\sigma \\ &\quad - 2 \int_0^1 (1 + a(U; x)) \left(C_{n-1}(\sigma, \Psi_{C_{n-1}}^\sigma(U); x) - C_{n-1}(\sigma, \Psi_{C_{n-2}}^\sigma(U); x) \right) d\sigma \\ &\quad - 4 \int_0^1 \int_0^\sigma \operatorname{Re} \left(b_{n-1}^+(\theta, \Psi_{C_{n-1}}^\theta(U); x) \overline{C_{n-1}(\theta, \Psi_{C_{n-1}}^\theta(U); x)} \right) C_n(\sigma, \Psi_{C_{n-1}}^\sigma(U); x) d\theta \\ &\quad + 4 \int_0^1 \int_0^\sigma \operatorname{Re} \left(b_{n-2}^+(\theta, \Psi_{C_{n-2}}^\theta(U); x) \overline{C_{n-2}(\theta, \Psi_{C_{n-2}}^\theta(U); x)} \right) C_{n-1}(\sigma, \Psi_{C_{n-2}}^\sigma(U); x) d\theta. \end{aligned} \tag{4.105}$$

First of all notice that, using Lemma 4.16 and estimates (2.59) on the flow $\Psi_{C_j}^\tau$,

$$\begin{aligned} |C_{n-1}(\sigma, \Psi_{C_{n-1}}^\sigma(U); x) - C_{n-1}(\sigma, \Psi_{C_{n-2}}^\sigma(U); x)|_{\alpha, 0}^{\mathcal{F}, s-1} &\lesssim_s |C_{n-1}|_s^{\mathcal{F}} r |C_{n-1} - C_{n-2}|_{s-1, 0}^{\mathcal{F}}, \\ |C_{n-1}(\sigma, \Psi_{C_{n-1}}^\sigma(U); x) - C_{n-2}(\sigma, \Psi_{C_{n-1}}^\sigma(U); x)|_{\alpha, 0}^{\mathcal{F}, s-1} &\lesssim_s |C_{n-1}|_s^{\mathcal{F}} r |C_{n-1} - C_{n-2}|_{s-1, 0}^{\mathcal{F}}. \end{aligned} \tag{4.106}$$

One can prove the following inequality

$$|b_{n-1}^+ - b_{n-2}^+|_{\alpha, 0}^{\mathcal{F}} \lesssim_s \mathbb{C} |C_{n-1}|_s^{\mathcal{F}} r |C_{n-1} - C_{n-2}|_{s-1, 0}^{\mathcal{F}}, \tag{4.107}$$

for some $\mathbb{C} > 0$ depending on $|a|_s^{\mathcal{F}}$, $|b|_s^{\mathcal{F}}$. In order to prove this fact one has to use the definition (4.102), triangular inequality, (4.106) and the smallness of r . By (4.105), (4.106), (4.107), for $r > 0$ small enough we deduce the (4.104). \square

We have the following.

Lemma 4.17. *One has $\mathcal{F}(C_n, C_n, b_n^+) \rightarrow 0$ as $n \rightarrow \infty$ in the norm $|\cdot|_{\alpha, 0}^{\mathcal{F}, s-1}$.*

Proof. We write

$$\begin{aligned} \mathcal{F}(C_n, C_n, b_n^+) &= \mathcal{F}(C_n, C_{n-1}, b_{n-1}^+) + \mathcal{F}(C_n, C_n, b_n^+) - \mathcal{F}(C_n, C_{n-1}, b_{n-1}^+) \\ &\stackrel{(4.101)}{=} \mathcal{F}(C_n, C_n, b_n^+) - \mathcal{F}(C_n, C_{n-1}, b_{n-1}^+). \end{aligned}$$

By reasoning as in the proof of $(\mathbf{S2})_n$ in Lemma 4.15 one can check

$$|\mathcal{F}(C_n, C_n, b_n^+) - \mathcal{F}(C_n, C_{n-1}, b_{n-1}^+)|_{\alpha, 0}^{\mathcal{F}, s-1} \lesssim_s 2^{-n}$$

for $r > 0$ small enough. Hence the result follows. \square

The discussion above provides symbols $a^+(\tau, U; x)$, $b^+(\tau, U; x)$, $C(\tau, U; x)$, in the class $\mathcal{F}_1[r]$ such that the (4.88), (4.89) hold. Actually, reasoning as in the conclusion of the proof of Theorem 4.2, one can prove that such symbols belong to the class $\Sigma_{\mathcal{F}_1}[r, N]$.

4.2.3. *Non-homogeneous problem.* In section 5.2.1 we shall deal with the following problem. We fix $m' < m$ and we consider symbols

$$a_{m'}(U; x, \xi), b_{m'}(U; x, \xi), q^{(1)}(\tau, U; x, \xi), q^{(2)}(\tau, U; x, \xi) \in \Sigma\Gamma_1^{m'}[r, N], \quad \tau \in [0, 1] \quad (4.108)$$

and the two problems

$$\begin{aligned} \partial_\tau a_{m'}^+(\tau, Z; x, \xi) &= -2\operatorname{Re}\left(C(\tau, Z; x) \overline{b_{m'}^+(\tau, Z; x, \xi)}\right) + q^{(1)}(\tau, Z; x, \xi) \\ &\quad - (d_U a_{m'}^+)(\tau, Z; x, \xi) [Op^{\text{BW}}(\mathbf{C}(Z; x))][Z], \end{aligned} \quad (4.109)$$

$$a_{m'}^+(0, Z; x, \xi) = a_{m'}(Z; x, \xi),$$

$$\begin{aligned} \partial_\tau b_{m'}^+(\tau, Z; x, \xi) &= -\left(a_{m'}^+(\tau, Z; x, \xi) + \overline{a_{m'}^+(\tau, Z; x, -\xi)}\right) C(\tau, Z; x) + q^{(2)}(\tau, Z; x, \xi) \\ &\quad - (d_U b_{m'}^+)(\tau, Z; x, \xi) [Op^{\text{BW}}(\mathbf{C}(Z; x))][Z], \end{aligned} \quad (4.110)$$

$$b_{m'}^+(0, Z; x, \xi) = b_{m'}(Z; x, \xi),$$

where C is in (4.84). The following holds true.

Proposition 4.18. *Assume (4.108) and (4.84). For $r > 0$ small enough the symbols defined by (4.109), (4.110) are such that*

$$a_{m'}^+(\tau, U; x, \xi), b_{m'}^+(\tau, U; x, \xi) \in \Sigma\Gamma_1^{m'}[r, N],$$

with estimates uniform in $\tau \in [0, 1]$.

Proof. We denote by $\Psi_C^\tau, (\Psi_C^\tau)^{-1}$ respectively the flow and the inverse flow of (4.85) with symbol $C(\tau, U; x)$ in (4.84) and we set

$$g_1(\tau, x, \xi) := a_{m'}^+(\tau, Z(\tau); x, \xi), \quad g_2(\tau, x, \xi) := b_{m'}^+(\tau, Z(\tau); x, \xi), \quad Z(\tau) := \Psi_C^\tau(U).$$

We have that (recall (4.109), (4.110))

$$\begin{aligned} \partial_\tau g_1(\tau, x, \xi) &= -2\operatorname{Re}\left(g_2(\tau, x, \xi) \overline{C(\tau, Z(\tau); x)}\right) + q^{(1)}(\tau, Z(\tau); x, \xi), \\ \partial_\tau g_2(\tau, x, \xi) &= -(g_1(\tau, x, \xi) + \overline{g_1(\tau, x, -\xi)}) C(\tau, Z(\tau); x) + q^{(2)}(\tau, Z(\tau); x, \xi), \\ g_1(0, x, \xi) &= a_{m'}(U; x, \xi), \quad g_2(0, x, \xi) = b_{m'}(U; x, \xi), \end{aligned}$$

which implies

$$\begin{aligned} g_1(\tau, x, \xi) &:= a(U; x, \xi) - \int_0^\tau 2\operatorname{Re}\left(g_2(\sigma, x, \xi) \overline{C(\sigma, Z(\sigma); x)}\right) - q^{(1)}(\sigma, Z(\sigma); x, \xi) d\sigma, \\ g_2(\tau, x, \xi) &:= b(U; x, \xi) - \int_0^\tau \left(g_1(\sigma, x, \xi) + \overline{g_1(\sigma, x, -\xi)}\right) C(\sigma, Z(\sigma); x) - q^{(2)}(\sigma, Z(\sigma); x, \xi) d\sigma. \end{aligned} \quad (4.111)$$

By (4.111) we deduce that equations (4.109), (4.110) are equivalent to

$$\begin{aligned} a_{m'}^+(\tau, Z; x) &:= a_{m'}\left((\Psi_C^\tau)^{-1}(Z); x, \xi\right) - 2 \int_0^\tau \operatorname{Re}\left(b_{m'}^+(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x, \xi) \overline{C(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x)}\right) d\sigma \\ &\quad + \int_0^\tau q^{(1)}(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x, \xi) d\sigma, \end{aligned} \quad (4.112)$$

$$\begin{aligned} b_{m'}^+(\tau, Z; x, \xi) &:= b_{m'}\left((\Psi_C^\tau)^{-1}(Z); x, \xi\right) \\ &\quad - \int_0^\tau \left(a_{m'}^+(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x, \xi) + \overline{a_{m'}^+(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x, -\xi)}\right) C(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x) d\sigma \\ &\quad + \int_0^\tau q^{(2)}(\sigma, \Psi_C^\sigma(\Psi_C^\tau)^{-1}(Z); x, \xi) d\sigma. \end{aligned} \quad (4.113)$$

Equations (4.112), (4.113) differs from (4.93), (4.94) for the non-homogeneous terms depending on $q^{(i)}$, $i = 1, 2$. In order to conclude the proof of the proposition it is sufficient to follow almost word by word the proof of Lemma 4.14 in subsection 4.2.1. \square

4.3. Diagonal terms at lower orders. In the following we consider two non-linear equations which will appear in sections 5.2.2, 5.1.2. Consider symbols

$$f(\xi) \in \Gamma_0^m, \quad m > 1, \quad f(\xi) \in \mathbb{R} \quad a(u; x, \xi) \in \Sigma \Gamma_1^{m'}[r, N], \quad m' < m, \quad (4.114)$$

$$\mathfrak{m}(\tau, u) \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N], \quad \text{independent of } x \in \mathbb{T}, \quad (4.115)$$

$$a(u; x, \xi) - \overline{a(u; x, \xi)} \in \Sigma \Gamma_1^0[r, N]. \quad (4.116)$$

Assume also that $\mathfrak{m}(\tau, u)$ satisfies the estimates (2.8)-(2.11) uniformly in $\tau \in [0, 1]$. Let

$$c(\tau, w; x, \xi) \in \Sigma \Gamma_1^\delta[r, N], \quad \tau \in [0, 1], \quad \delta := m' - m + 1, \quad c(u; x, \xi) - \overline{c(u; x, \xi)} \in \Sigma \Gamma_1^0[r, N]. \quad (4.117)$$

Consider the equation

$$\begin{cases} \partial_\tau z(\tau) = Op^{\text{BW}}(\mathfrak{ic}(\tau, z(\tau); x, \xi))[z(\tau)], \\ z(0) = z_0. \end{cases} \quad (4.118)$$

The aim of the section is to prove the following.

Theorem 4.19. *Assume (4.114), (4.115), (4.116). For $r > 0$ small enough there exist a symbol $c(\tau, w; x, \xi)$ as in (4.117) and a symbol $\mathfrak{C}(w, \xi) \in \Sigma \Gamma_1^{m'}[r, N]$, independent of $x \in \mathbb{R}$, such that the following holds. One has*

$$\begin{aligned} F(c) &:= a(u; x, \xi) - \int_0^1 (\partial_\xi f)(\xi) (1 + \mathfrak{m}(\sigma, \Phi_c^\sigma(u))) (\partial_x c)(\sigma, \Phi_c^\sigma(u); x, \xi) d\sigma \\ &= \mathfrak{C}(u, \xi) := \frac{1}{2\pi} \int_{\mathbb{T}} a(u; x, \xi) dx, \end{aligned} \quad (4.119)$$

where $\Phi^\tau(u)$, $\tau \in [0, 1]$ is the flow of (4.118). Moreover the symbol $\mathfrak{C}(w, \xi)$ satisfy the same condition of $a(u; x, \xi)$ in (4.116).

By following the strategy adopted in the previous sections, it is sufficient to prove the result in the case of non-homogeneous symbols. We reason inductively. Let $c_0 = 0$ and for any $n \geq 1$ consider the problem

$$\partial_\tau z_n(\tau) = iOp^{\text{BW}}(c_n(\tau, z_n(\tau); x, \xi))[z_n(\tau)], \quad z_n(0) = z_0, \quad z_n(\tau) := \Phi_{c_n}^\tau(z_0), \quad (4.120)$$

where the symbol $c_n(\tau, w; x, \xi)$ is defined as

$$c_n(\tau, w; x, \xi) := \partial_x^{-1} \left(\frac{a((\Phi_{c_{n-1}}^\tau)^{-1}(w); x, \xi) - \mathfrak{C}((\Phi_{c_{n-1}}^\tau)^{-1}(w); \xi)}{(1 + \mathfrak{m}(\tau, w))(\partial_\xi f)(\xi)} \right). \quad (4.121)$$

First of all notice that the constant $\mathfrak{C}(u; \xi)$ defined in (4.119) belongs to $\Gamma_1^{m'}[r]$ and

$$|\mathfrak{C}|_{0, \beta, k}^{\Gamma^{m', s}} \lesssim_s |a|_{0, \beta, k}^{\Gamma^{m', s}}, \quad mk \leq s - s_0. \quad (4.122)$$

For any $n \geq 1$ we shall prove inductively that the following conditions hold.

(S1)_n One has that the symbol c_n in (4.121) belongs to $\Gamma_1^{m'-m+1}$. In particular (see (2.13)) there exists constant C (independent of n) depending only on s , $|a|_{\alpha, \beta, k}^{\Gamma^{m', s}}$ and $|\mathfrak{m}|_{0, \beta, k}^{\Gamma^0, s}$ such that

$$|c_n|_{\alpha, \beta, k}^{\Gamma^{\delta, s}} \lesssim_s C, \quad \alpha + mk \leq s - s_0, \quad (4.123)$$

$$|c_n|_{\alpha, \beta, 0}^{\Gamma^{\delta, s+1}} \lesssim_s |a|_{\alpha, \beta, 0}^{\Gamma^{m', s}}, \quad \alpha \leq s + 1 - s_0. \quad (4.124)$$

The symbol c_n satisfies the same condition of $a(u; x, \xi)$ in (4.116).

(S2)_n The flow of (4.120) with c_n given by (4.121) is well-posed, has the form

$$z_n(\tau) = \Phi_{c_n}^{(z)}(\tau, z_0) \in \cap_{k=0}^K C^k([0, 1]; H^{s-k}). \quad (4.125)$$

We argue by induction. So we assume that $(\mathbf{Sk})_j$, for $k = 1, 2$, hold with $0 \leq j \leq n-1$.

Proof of $(\mathbf{S1})_n$. By the inductive hypothesis c_{n-1} is a symbol satisfying (4.117). Then we can apply Theorem 2.31 (with generator as in (2.53) or (2.54)). So the flow of (4.118) with $c \rightsquigarrow c_{n-1}$ is well-posed. Therefore, using the formula of Faa di Bruno and the (2.59), one can check

$$|a((\Phi^{\tau})_{c_{n-1}}^{-1}(w), x, \xi)|_{\alpha, \beta, k}^{\Gamma_{m'}, s} \lesssim_s |a|_{\alpha, \beta, k}^{\Gamma_{m'}, s}.$$

Moreover

$$a((\Phi^{\tau}_{c_{n-1}})^{-1}(w); x, \xi) - \mathfrak{C}((\Phi^{\tau}_{c_{n-1}})^{-1}(w); \xi) \quad (4.126)$$

has zero average in $x \in \mathbb{T}$. Indeed the symbol in (4.126) is nothing but the symbol

$$a(u; x, \xi) - \mathfrak{C}(u; \xi) = a(u; x, \xi) - \frac{1}{2\pi} \int_{\mathbb{T}} a(u; x, \xi) dx$$

evaluated at $u = (\Phi^{\tau}_{c_{n-1}})^{-1}(w)$. Therefore ∂_x^{-1} in (4.121) is well defined. Finally, using (4.122), (4.121) and Remark 2.8, we get the (4.123). Taking into account the smoothing effect of the Fourier multiplier ∂_x^{-1} , Remarks 2.8, 2.7, we obtain the (4.124) for r small enough. By formula (4.121) and hypothesis (4.116) we deduce that c_n satisfies the same condition of a in (4.116). \square

Proof of $(\mathbf{S2})_n$. Thanks to (4.123), (4.124) the (4.125) follows by Theorem 2.31. \square

Inizialization. The $(\mathbf{S1})_0$, $(\mathbf{S2})_0$ are trivial.

In order to conclude the proof of Theorem 4.19 we have check the (4.119). We need some preliminary results which are consequences of $(\mathbf{S1})_n$, $(\mathbf{S2})_n$.

Lemma 4.20. *For $r > 0$ small enough we have*

$$\sup_{\tau \in [0,1]} \|\Phi_{c_n}^{\tau} - \Phi_{c_{n-1}}^{\tau}\|_{H^{s-1}} \lesssim_s r \sup_{\tau \in [0,1]} |c_n - c_{n-1}|_{0,0,0}^{\Gamma_{\delta}}, \quad (4.127)$$

where $\Phi_{c_j}^{\tau}$, $j = n, n-1$ is the solution of (4.120).

Proof. We set $v_n(\tau) := z_n(\tau) - z_{n-1}(\tau)$. By (4.120) we get

$$\begin{aligned} \partial_{\tau} v_n &= iOp^{\text{BW}}(c_n(z_n; x, \xi))[v_n] + iOp^{\text{BW}}(c_n(z_n; x, \xi) - c_n(z_{n-1}; x, \xi))[z_{n-1}] \\ &\quad + iOp^{\text{BW}}((c_n - c_{n-1})(z_{n-1}; x, \xi))[z_{n-1}]. \end{aligned}$$

Let us consider the case $\delta > 0$ which is the most difficult one. If $\delta \leq 0$ one can reason as in the proof of the estimates for X_n, Y_n in the proof of Lemma 4.9 (see (4.64)).

Using that $c_n(z_n; x, \xi)$ is real valued, Propositions 2.22, 2.25, the (2.31), Remark 2.7, and (2.5), we get, for any $s \in \mathbb{R}$,

$$\begin{aligned} \partial_{\tau} \|v_n\|_{H^s}^2 &\lesssim_s \|v_n\|_{H^s}^2 \sup_{\tau \in [0,1]} \sum_{\alpha \leq s-s_0} |c_n|_{\alpha,0,0}^{\Gamma_{\delta}, s} r \\ &\quad + \|z_{n-1}\|_{H^{s+1}} \|v_n\|_{H^s} \left(|c_n|_{s_0,0,1}^{\Gamma_{\delta}, s} \|v_n\|_{H^{s_0}} + |c_n - c_{n-1}|_{0,0,0}^{\Gamma_{\delta}, s} \right). \end{aligned} \quad (4.128)$$

We now recall that $\|z_{n-1}\|_{H^s} \lesssim_s r$. We use formula (4.128) with $s \rightsquigarrow s-1$. Then, reasoning as in (4.82), we deduce that if

$$r \sup_{\tau} \left(|c_n|_{s-s_0,0,0}^{\Gamma_{\delta}} + |c_n|_{s_0,0,1}^{\Gamma_{\delta}} \right) \ll 1,$$

then

$$\sup_{\tau \in [0,1]} \|v_n\|_{H^{s-1}} \lesssim_s r \sup_{\tau \in [0,1]} |c_n - c_{n-1}|_{0,0,0}^{\mathcal{F}}.$$

Then we get the (4.127). \square

We now study the convergence of the symbols c_n in (4.121). We claim that

$$\sup_{\tau \in [0,1]} |c_n - c_{n-1}|_{\alpha, \beta, 0}^{\Gamma^\delta, s} \lesssim_\beta \frac{1}{2^n}, \quad \forall n \geq 1, \alpha \leq s-1-s_0. \quad (4.129)$$

For $n=1$ it is trivial. Assume that (4.129) holds for $j \leq n-1$. By (4.121) we can write

$$\begin{aligned} \partial_x(c_n - c_{n-1}) &:= \frac{1}{(1 + m(\tau, w))(\partial_\xi f)(\xi)} \left(a((\Phi_{c_{n-1}}^\tau)^{-1}(w); x, \xi) - a((\Phi_{c_{n-2}}^\tau)^{-1}(w); x, \xi) \right) \\ &+ \frac{1}{(1 + m(\tau, w))(\partial_\xi f)(\xi)} \left(\mathfrak{C}((\Phi_{c_{n-2}}^\tau)^{-1}(w); \xi) - \mathfrak{C}((\Phi_{c_{n-1}}^\tau)^{-1}(w); \xi) \right). \end{aligned}$$

Define $d_n(\tau) := a((\Phi_{c_{n-1}}^\tau)^{-1}(w); x, \xi) - a((\Phi_{c_{n-2}}^\tau)^{-1}(w); x, \xi)$. We have that

$$d_n(\tau) = (d_u a) \left((\Phi_{c_{n-2}}^\tau)^{-1}(w) + \sigma((\Phi_{c_{n-1}}^\tau)^{-1}(w) - (\Phi_{c_{n-2}}^\tau)^{-1}(w)) \right) [(\Phi_{c_{n-1}}^\tau)^{-1}(w) - (\Phi_{c_{n-2}}^\tau)^{-1}(w)],$$

for some $\sigma \in [0, 1]$. Hence, for any $\alpha_1 \leq s-s_0$ and $\beta_1 \in \mathbb{N}$, we get (see (2.11))

$$\begin{aligned} |\partial_x^{\alpha_1} \partial_\xi^{\beta_1} d_n(\tau)| &\lesssim_s C \langle \xi \rangle^{m'-\beta_1} \|(\Phi_{c_{n-1}}^\tau)^{-1}(w) - (\Phi_{c_{n-2}}^\tau)^{-1}(w)\|_{H^{s_0+\alpha_1}} \\ &\stackrel{(4.127)}{\lesssim_s} C \langle \xi \rangle^{m'-\beta_1} r \sup_{\tau \in [0,1]} |c_{n-1} - c_{n-2}|_{0,0,0}^{\Gamma^\delta}, \end{aligned}$$

where $C > 0$ is some constant depending only on s and $|a|_{\alpha_1, \beta_1, 1}^{\Gamma^{m', s}}$. Let us define $g_n(\tau) := \mathfrak{C}((\Phi_{c_{n-2}}^\tau)^{-1}(w); \xi) - \mathfrak{C}((\Phi_{c_{n-1}}^\tau)^{-1}(w); \xi)$. For any $\alpha \leq s-s_0$ and $\beta \in \mathbb{N}$ we have

$$\begin{aligned} |\partial_x^{\alpha+1} \partial_\xi^\beta (c_n - c_{n-1})| &= \left| \sum_{\beta_1 + \beta_2 = \beta} \partial_\xi^{\beta_1} \left(\frac{1}{(1 + m(\tau, w))(\partial_\xi f)(\xi)} \right) \partial_x^{\alpha+1} \partial_\xi^{\beta_2} (d_n(\tau) - g_n(\tau)) \right| \\ &\lesssim_s K \langle \xi \rangle^{m'-m+1-\beta} r \sup_{\tau \in [0,1]} |c_{n-1} - c_{n-2}|_{0,0,0}^{\Gamma^\delta}, \end{aligned}$$

where K is some constant depending only on s and $|a|_{\alpha, \beta, 1}^{\Gamma^{m', s}}$ and $|m|_{0, \beta, 0}^{\Gamma^{m', s}}$. For $r > 0$ small enough and by the inductive assumption we obtain the (4.129). By (4.129) we deduce that c_n converges to a symbol c in the Fréchet space defined by the semi-norms $|\cdot|_{\alpha, \beta, 0}^{\Gamma^{m'-m+1}}$ with $\alpha \leq s-s_0$. In order to prove that the symbol c admits differentials with respect to the variable w (see formula (2.11)) we reason as follows. For fixed $N > 0$, consider the Banach space defined by the norm

$$\sum_{\substack{\beta \leq N \\ \alpha + m \cdot k \leq s-s_0}} |\cdot|_{\alpha, \beta, k}^{\Gamma^{m'-m+1}}.$$

For any $N > 0$, the sequence c_n is bounded, uniformly in n , in such a space thanks to estimate (4.123). Therefore, up to subsequences, c_n converges weak- $*$ to a function \tilde{c} in the above Banach space. This implies that \tilde{c} admits estimates on the differentials in w . Since \tilde{c} coincides with the symbol c almost everywhere we conclude the proof.

We are now ready to prove (4.119).

Lemma 4.21. *One has that*

$$\lim_{n \rightarrow \infty} \sup_{\tau \in [0,1]} \langle \xi \rangle^{m'} \|F(c_n) - \mathfrak{C}\|_{L^\infty} = 0 \quad (4.130)$$

where \mathfrak{C} is given in (4.119).

Proof. By (4.121) with $w = \Phi_{c_{n-1}}^\tau(u)$ we can check that

$$a(u; x, \xi) - \int_0^1 (\partial_\xi f)(\xi) (1 + m(\sigma, \Phi_{c_{n-1}}^\sigma(u))) (\partial_x c_n)(\sigma, \Phi_{c_{n-1}}^\sigma(u); x, \xi) d\sigma = \mathfrak{C}(u, \xi). \quad (4.131)$$

By (4.119) (reasoning as in (4.59)) we have that

$$\begin{aligned}
F(c_n) &= a(u; x, \xi) - \int_0^1 (\partial_\xi f)(\xi) (1 + \mathfrak{m}(\sigma, \Phi_{c_n}^\sigma(u))) (\partial_x c_n)(\sigma, \Phi_{c_n}^\sigma(u); x, \xi) d\sigma \\
&= a(u; x, \xi) - \int_0^1 (\partial_\xi f)(\xi) (1 + \mathfrak{m}(\sigma, \Phi_{c_{n-1}}^\sigma(u))) (\partial_x c_n)(\sigma, \Phi_{c_{n-1}}^\sigma(u); x, \xi) d\sigma \\
&\quad + \int_0^1 (\partial_\xi f)(\xi) A_n(\sigma) d\sigma, \\
&\stackrel{(4.131)}{=} \mathfrak{C}(u; \xi) + \int_0^1 (\partial_\xi f)(\xi) A_n(\sigma) d\sigma,
\end{aligned} \tag{4.132}$$

where

$$A_n(\sigma) := (1 + \mathfrak{m}(\sigma, \Phi_{c_{n-1}}^\sigma(u))) (\partial_x c_n)(\sigma, \Phi_{c_{n-1}}^\sigma(u); x, \xi) - (1 + \mathfrak{m}(\sigma, \Phi_{c_n}^\sigma(u))) (\partial_x c_n)(\sigma, \Phi_{c_n}^\sigma(u); x, \xi).$$

We now show that $A_n(\sigma)$ goes to zero in norm $\|\cdot\|_{L^\infty}$ uniformly in $\sigma \in [0, 1]$. Notice that

$$A_n(\sigma) = (1 + \mathfrak{m}(\sigma, \Phi_{c_{n-1}}^\sigma(u))) \left[(\partial_x c_n)(\sigma, \Phi_{c_{n-1}}^\sigma(u); x, \xi) - (\partial_x c_n)(\sigma, \Phi_{c_n}^\sigma(u); x, \xi) \right] \tag{4.133}$$

$$+ \left[\mathfrak{m}(\sigma, \Phi_{c_{n-1}}^\sigma(u)) - \mathfrak{m}(\sigma, \Phi_{c_n}^\sigma(u)) \right] (\partial_x c_n)(\sigma, \Phi_{c_n}^\sigma(u); x, \xi). \tag{4.134}$$

Since $\mathfrak{m} \in \mathcal{F}_1^{\mathbb{R}}[r]$ and setting $h = \Phi_{c_n}^\sigma(u) + \sigma'(\Phi_{c_{n-1}}^\sigma(u) - \Phi_{c_n}^\sigma(u))$ for some $\sigma' \in [0, 1]$, we have

$$|\mathfrak{m}(\sigma, \Phi_{c_{n-1}}^\sigma(u)) - \mathfrak{m}(\sigma, \Phi_{c_n}^\sigma(u))| \lesssim |(d_u \mathfrak{m})(h)[\Phi_{c_{n-1}}^\sigma(u) - \Phi_{c_n}^\sigma(u)]| \stackrel{(2.11), (4.127), (4.129)}{\lesssim_s} 2^{-n},$$

for r small enough. Reasoning in the same way on the term in (4.133) we get the claim on $A_n(\sigma)$. Hence, by (4.132), we deduce (4.130). \square

This concludes the proof of Theorem 4.19.

4.4. Off-diagonal terms at lower orders. Consider symbols

$$f(\xi) \in \Gamma_0^m, \quad m > 1, \quad f(\xi) \in \mathbb{R} \quad a_m(U; x) \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N], \tag{4.135}$$

$$a_{m'}(U; x, \xi) \in \Sigma \Gamma_1^{m'}[r, N], \quad m' < m. \tag{4.136}$$

Let

$$C(\tau, U; x, \xi) \in \Sigma \Gamma_1^\delta[r, N], \quad \tau \in [0, 1], \quad \delta := m' - m.$$

and consider the equation

$$\begin{cases} \partial_\tau Z(\tau) = Op^{\text{BW}}(\mathbf{iEC}(\tau, z(\tau); x, \xi))[z(\tau)], \\ Z(0) = Z_0 = \begin{bmatrix} z_0 \\ \bar{z}_0 \end{bmatrix}, \end{cases} \quad \mathbf{C}(\tau, U; x, \xi) := \begin{pmatrix} 0 & C(\tau, U; x, \xi) \\ C(\tau, U; x, -\xi) & 0 \end{pmatrix}. \tag{4.137}$$

The following holds true.

Theorem 4.22. *Assume (4.135), (4.136). For $r > 0$ small enough there exists a symbol $C(\tau, U; x, \xi)$ belonging to $\Sigma \Gamma_1^\delta[r, N]$, $\tau \in [0, 1]$, $\delta = m' - m$ such that*

$$F(c) := a_{m'}(U; x, \xi) - 2 \int_0^1 (1 + a_m(\sigma, \Phi_C^\sigma(u))) f(\xi) C(\sigma, \Phi_C^\sigma(u); x, \xi) d\sigma = 0, \tag{4.138}$$

where $\Phi_C^\tau(u)$, $\tau \in [0, 1]$ is the flow of (4.137).

Proof. One can reason by following almost word by word the proof of Theorem 4.19. \square

5. CONJUGATIONS

In this section we prove four abstract theorems which will be used in order to prove Theorem 3.1. In all the theorems we shall consider a system of the form

$$\begin{cases} \dot{U} = X(U) \\ U(0) = U_0 \in H^s \times H^s \end{cases}, \quad (5.1)$$

where the matrix of operator $X(U)$ is defined as follows

$$\begin{aligned} X(U) &:= iEOp^{\text{BW}}(A(U; x, \xi))[U] + R(U)[U], \quad E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ R &\in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \quad A(U; x, \xi) \in \Sigma \Gamma_0^m[r, N] \otimes \mathcal{M}^2(\mathbb{C}). \end{aligned} \quad (5.2)$$

In Theorem 5.1 we shall consider the system (5.1), where the matrix of operators in r.h.s. is defined in (5.2). Here the matrix of para-differential operators is assumed to be diagonal (its off diagonal symbols equal 0). In such a Theorem we exhibit a change of coordinates of the phase space $H^s \times H^s$ which removes the dependence on the space variable x from the symbols on the diagonal at the highest order.

The main theorem in Section 5.1.2 is the 5.2. In this case the matrix of symbols $A(U; x, \xi)$ is assumed to be diagonal as before with following additional assumption. We assume that symbol on the diagonal a admits an expansion in decreasing orders which is equal to a Fourier multiplier at the highest order plus a lower order term which may depend on x (see equation (5.44)). We provide a change of coordinates which removes the dependence on x from such a lower order term.

In the same spirit, in Section 5.2 we explain how diagonalize the matrix $A(U; x, \xi)$ by means of changes of coordinates. The diagonalization of the highest order is the content of Theorem 5.3, the lower order terms are treated in Theorem 5.4.

The general strategy that we adopt in the proofs of all the theorems of this section is the following. We shall consider changes of coordinates defined through non-linear flows of the form

$$\begin{cases} \partial_\tau \Psi^\tau = G^\tau(\Psi^\tau) \\ \Psi^0 = \text{Id}, \end{cases} \quad (5.3)$$

where G^τ is some non-linear vector field possibly depending explicitly on $\tau \in [0, 1]$. In the application the operator G^τ is para-differential. All the issues of well posedness of (5.3) have been analyzed in Section 2.5. We consider as a new variable the function $Z := \Psi(U) := \Psi_{|\tau=1}^\tau(U)$. The system (5.1) in the new coordinates reads

$$\begin{cases} \dot{Z} = d\Psi(\Psi^{-1}(Z))[X(\Psi^{-1}(Z))] =: (\Psi)^* X(Z) =: P^1(Z) \\ Z(0) = \Psi(U_0), \end{cases} \quad (5.4)$$

where $P^1(Z) = (P^\tau(Z))_{\tau=1}$ with P^τ defined, for $\tau \in [0, 1]$, as

$$P^\tau(Z) := d\Psi^\tau((\Psi^\tau)^{-1}(Z))[X((\Psi^\tau)^{-1}(Z))]. \quad (5.5)$$

On can prove that P^τ satisfies the non-linear Heisenberg equation

$$\begin{cases} \partial_\tau P^\tau(Z) = [G^\tau(Z), P^\tau(Z)] \\ P^0(Z) = X(Z), \end{cases} \quad (5.6)$$

where the non-linear commutator is defined in (1.7). We choose this Heisenberg approach since we start from a para-differential system and we are interested in preserving such structure. The Heisenberg approach provides a systematic way to prove this fact at each step. This Heisenberg strategy has been already used in [9], [26], [25], [11]. As already said in other part of the manuscript the authors in those papers do not attempt to find non-linear changes of coordinates but only some modified energies. As a consequence they face a linear version of system (5.6), where the non linear commutators are replaced by the linear ones and the operator G^τ is linear. In this case the commutator between two para-differential operators is still a para-differential one modulo smoothing remainders, this is a consequence of standard

para-differential calculus. In our case the non-linear commutators involves also the differential of the symbols with respect to the "non-linear" variable U , this is coherent with our definition of symbols and operators given in Section 2.

5.1. Reduction to constant coefficients. In this section we show how to conjugate to constant coefficients para-differential systems which are diagonal up to smoothing remainders, we shall adopt the Notation 4.1.

5.1.1. *Reduction at the highest orders.* We consider system (5.1) with operator $X(U)$ defined in (5.2). We shall assume that the matrix $A(U; x, \xi)$ is as follows

$$A(U; x, \xi) := \begin{pmatrix} a(U; x, \xi) & 0 \\ 0 & a(U; x, -\xi) \end{pmatrix}. \quad (5.7)$$

The symbol $a(U; x, \xi)$ has the form

$$\begin{aligned} a(U; x, \xi) &= (1 + a_m(U; x))f_m(\xi) + a_{m'}(U; x, \xi), \quad m > 1, \quad m' = m - \frac{1}{2}, \\ a_m(U, x) &\in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N], \quad a_{m'}(U; x, \xi) \in \Sigma \Gamma_1^{m'}[r, N], \\ a_{m'}(U; x, \xi) - \overline{a_{m'}(U; x, \xi)} &\in \Sigma \Gamma_1^0[r, N]. \end{aligned} \quad (5.8)$$

where m satisfies (3.1), $f_m \in \Gamma_0^m$ is an even in x classical symbol. In particular f_m admits the expansion

$$f_m(\xi) = f_0(\xi) + \tilde{f}_{m-1}(\xi), \quad \tilde{f}_{m-1} \in \Gamma_0^{m-1}, \quad (5.9)$$

and $f_0(\xi) \in \Gamma_0^m$ is a m -homogeneous function. The symbol in (5.8) (recall also (5.9)) satisfies the assumption (4.1), therefore Theorem 4.2 applies and provides a symbol b such that (4.8) holds true with $\tilde{a}_m \rightsquigarrow a_m$. Having such b we now consider the matrix of symbols

$$\begin{aligned} \mathbf{B}(\tau, U; x, \xi) &:= \begin{pmatrix} B(\tau, U; x, \xi) & 0 \\ 0 & B(\tau, U; x, -\xi) \end{pmatrix} = \begin{pmatrix} B(\tau, U; x, \xi) & 0 \\ 0 & -B(\tau, U; x, \xi) \end{pmatrix} \\ B(\tau, U; x, \xi) &:= b(\tau, U; x)\xi, \quad b \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N], \end{aligned} \quad (5.10)$$

and let Ψ^τ be the flow of (5.3) with

$$G^\tau(U) := iEOp^{\text{BW}}(\mathbf{B}(\tau, U; x, \xi))U. \quad (5.11)$$

Notice that the flow of system (5.3) with generator (5.11) is well-posed by Theorem 2.31 applied with generator as in (2.52). We define

$$Z := \begin{bmatrix} z \\ \bar{z} \end{bmatrix} := \Psi^\tau(U)|_{\tau=1}. \quad (5.12)$$

Notice that, by (5.10),

$$\Psi^\tau(U)|_{\tau=1} := \begin{pmatrix} \Phi^\tau(u) \\ \overline{\Phi^\tau(u)} \end{pmatrix}_{|\tau=1}, \quad \partial_\tau \Phi^\tau(u) = Op^{\text{BW}}(iB(\tau, \Phi^\tau(u); x, \xi))[\Phi^\tau(u)].$$

The main result of this section is the following.

Theorem 5.1. (Non-linear Egorov). *For $r > 0$ small enough the conjugate of X in (5.1) with generator (5.11) has the form (see (5.12))*

$$\dot{Z} = iEOp^{\text{BW}}(A^+(Z; x, \xi))[Z] + R^+(Z)[Z], \quad R^+ \in R \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \quad (5.13)$$

and the matrix of symbols $A^+ \in \Sigma\Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ has the form

$$\begin{aligned} A^+(Z; x, \xi) &:= \begin{pmatrix} a^+(Z; x, \xi) & 0 \\ 0 & \overline{a^+(Z; x, -\xi)} \end{pmatrix}, \\ a^+(Z; x, \xi) &= \mathfrak{m}(Z) f_m(\xi) + a_{m'}^+(Z; x, \xi), \quad \mathfrak{m} \in \Sigma\mathcal{F}_0^{\mathbb{R}}[r, N], \quad m' = m - \frac{1}{2}, \\ a_{m'}^+(Z; x, \xi) &\in \Sigma\Gamma_1^{m'}[r, N], \quad a_{m'}^+(U; x, \xi) - \overline{a_{m'}^+(U; x, \xi)} \in \Sigma\Gamma_1^0[r, N]. \end{aligned} \quad (5.14)$$

with $\mathfrak{m}(Z)$ is independent of $x \in \mathbb{T}$.

The rest of the section is devoted to the proof of the result above. The system (5.1) in the new coordinates (5.12) has the form (5.4)-(5.6) with G^τ as in (5.11). Moreover, by the remarks under Definition 2.15 and by Theorem 2.31, we note that

$$P^\tau(Z) = iE\Omega Z + M_1(\tau; Z)[Z], \quad M_1 \in \Sigma\mathcal{M}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \quad (5.15)$$

with estimate uniform in $\tau \in [0, 1]$, where Ω is in (3.2). Actually we shall prove that

$$P^\tau(Z) = iE\Omega Z + M_1(\tau; Z)[Z] = iEOp^{\text{BW}}(A^+(\tau, Z; x, \xi))[Z] + R^+(\tau, Z)[Z], \quad (5.16)$$

with

$$A^+(\tau, Z; x, \xi) := \begin{pmatrix} a^+(\tau, Z; x, \xi) & 0 \\ 0 & \overline{a^+(\tau, Z; x, -\xi)} \end{pmatrix}, \quad R^+ \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}). \quad (5.17)$$

In particular we make the ansatz

$$a^+(\tau, Z; x, \xi) = \sum_{j=0}^{2(\rho+m)} a_{m-j}^+(\tau, Z; x, \xi), \quad a_m^+ \in \Sigma\Gamma_0^m[r, N] \quad a_{m-j}^+ \in \Sigma\Gamma_1^{m-\frac{j}{2}}[r, N], \quad j > 0. \quad (5.18)$$

Let $F(Z) := Op^{\text{BW}}(iE\mathbf{B}(\tau, Z; x, \xi))[Z]$. Then one can note that

$$(d_Z F)(Z)[V] = Op^{\text{BW}}(iE\mathbf{B}(\tau, Z; x, \xi))[V] + Op^{\text{BW}}(iE(d_Z \mathbf{B})(\tau, Z; x, \xi)[V])[Z], \quad (5.19)$$

and we also recall that

$$(d_Z \mathbf{B})(\tau, Z; x, \xi)[V] = (\partial_Z \mathbf{B})(\tau, Z; x, \xi) \cdot V.$$

By the definition of the non-linear commutator in (1.7) we write

$$\begin{aligned} [Op^{\text{BW}}(iE\mathbf{B}(\tau, Z; x, \xi))[Z], P^\tau(Z)] &= Op^{\text{BW}}(iE\mathbf{B}(\tau, Z; x, \xi))[P^\tau(Z)] \\ &\quad + Op^{\text{BW}}(iEd_Z \mathbf{B}(\tau, Z; x, \xi)[P^\tau(Z)])[Z] \\ &\quad - d_Z P^\tau(Z)[Op^{\text{BW}}(iE\mathbf{B}(\tau, Z; x, \xi))[Z]]. \end{aligned} \quad (5.20)$$

By (5.15), (5.10), we deduce that the matrix of symbols in the second summand in the r.h.s. of (5.20) has the form

$$\begin{pmatrix} (d_Z B)(\tau, Z; x, \xi)[P^\tau(Z)] & 0 \\ 0 & -(d_Z B)(\tau, Z; x, \xi)[P^\tau(Z)] \end{pmatrix} \in \Sigma\Gamma_1^1[r, N] \otimes \mathcal{M}_2(\mathbb{C}). \quad (5.21)$$

For simplicity we shall write

$$(\partial_t \mathbf{B})(\tau, Z; x, \xi) := d_Z \mathbf{B}(\tau, Z; x, \xi)[P^\tau(Z)]. \quad (5.22)$$

By (5.16), (5.22) we have that (5.20) becomes

$$[Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi))[Z], P^\tau(Z)] = Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi))[Op^{\text{BW}}(iEA^+(\tau, Z; x, \xi))[Z]] \quad (5.23)$$

$$- Op^{\text{BW}}(iEA^+(\tau, Z; x, \xi))[Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi))[Z]] \quad (5.24)$$

$$+ Op^{\text{BW}}(iE(\partial_t \mathbf{B})(\tau, Z; x, \xi))[Z] \quad (5.25)$$

$$- Op^{\text{BW}}(iE(d_Z A^+)(\tau, Z; x, \xi)[Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi))[Z]])[Z] \quad (5.26)$$

$$+ Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi))[R^+(Z)[Z]] - R^+(Z)[Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi))[Z]] \quad (5.27)$$

$$- (d_Z R^+)(Z)[Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi))[Z]]. \quad (5.28)$$

We now analyze each summand above. By Proposition 2.27 we have

$$(5.27) + (5.28) \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}).$$

By (5.18) we have that (recall (5.26))

$$iE(d_Z A^+)(\tau, Z; x, \xi)[Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi))[Z]] = iE \begin{pmatrix} \tilde{d}(\tau, Z; x, \xi) & 0 \\ 0 & \tilde{d}(\tau, Z; x, -\xi) \end{pmatrix}, \quad (5.29)$$

$$\tilde{d}(\tau, Z; x, \xi) = \sum_{j=0}^{\rho+m} (d_Z a_{m-j}^+)(\tau, Z; x, \xi)[Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi))[Z]].$$

By Proposition 2.25 we also deduce

$$(5.23) + (5.24) = Op^{\text{BW}}(i\mathbf{EB}(\tau, Z; x, \xi) \star_\rho iEA^+(\tau, Z; x, \xi))[Z] + \tilde{R}(\tau, Z)[Z],$$

where $\tilde{R} \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ (with estimates uniform in $\tau \in [0, 1]$) and

$$\begin{aligned} i\mathbf{EB}(\tau, Z; x, \xi) \star_\rho iEA^+(\tau, Z; x, \xi) &:= \\ &:= i\mathbf{EB}(\tau, Z; x, \xi) \#_\rho iEA^+(\tau, Z; x, \xi) - iEA^+(\tau, Z; x, \xi) \#_\rho i\mathbf{EB}(\tau, Z; x, \xi) \\ &\stackrel{(5.10), (5.17)}{=} \begin{pmatrix} d(\tau, Z; x, \xi) & 0 \\ 0 & \tilde{d}(\tau, Z; x, -\xi) \end{pmatrix}, \end{aligned}$$

where

$$d(\tau, Z; x, \xi) := iB(\tau, Z; x, \xi) \#_\rho ia^+(\tau, Z; x, \xi) - ia^+(\tau, Z; x, \xi) \#_\rho iB(\tau, Z; x, \xi). \quad (5.30)$$

By recalling the ansatz (5.18) and using the expansion (2.31) we get

$$d(\tau, Z; x, \xi) = i \sum_{j=0}^{2(m+\rho)} \{b(\tau, Z; x) \xi, a_{m-j}^+(\tau, Z; x, \xi)\} + r_j(\tau, Z; x, \xi), \quad (5.31)$$

where

$$r_0 = r_{-1} \equiv 0, \quad r_j(\tau, Z; x, \xi) \in \Sigma \Gamma_1^{m-\frac{j}{2}}[r, N], \quad j \geq 2,$$

the symbol r_j depends only on $b(\tau, Z; x)\xi$ and on the symbols a_{m-k}^+ with $k < j$. Recalling (5.21), (5.22) and (5.31) we define, for $j = 0, \dots, 2(m+\rho)$,

$$q_j(\tau, Z; x, \xi) := \begin{cases} r_j(\tau, Z; x, \xi), & j \neq 2(m-1), \\ r_j(\tau, Z; x, \xi) + (\partial_t B)(\tau, Z; x, \xi), & j = 2(m-1). \end{cases} \quad (5.32)$$

Our aim is to prove the ansatz (5.16) using (5.6), the expansion (5.23)-(5.28). We shall solve such equation iteratively expanding the symbol in (5.30) in decreasing orders.

Order m . By (5.29), (5.30), (2.31), (5.31) we have that, at the highest order, the equation (5.6) reads

$$\begin{cases} \partial_\tau a_m^+(\tau, Z; x, \xi) = \{b(\tau, Z; x, \xi), a_m^+(\tau, Z; x, \xi)\} - d_Z a_m^+(\tau, Z; x, \xi)[Op^{BW}(iE\mathbf{B}(\tau, Z; x, \xi))][Z]] \\ a_m^+(0, Z; x, \xi) = a(Z; x, \xi), \end{cases} \quad (5.33)$$

where $\{\cdot, \cdot\}$ denotes the $\partial_\xi \partial_x - \partial_x \partial_\xi$. Notice that the function

$$g(\tau) = a_m^+(\tau, Z(\tau); x(\tau), \xi(\tau))$$

is constant along the solution of (4.2). By Theorem 4.2 we have that the flow of (4.2) is well posed and invertible. We denote by

$$\begin{aligned} \Upsilon^\tau(U, x_0, \xi_0) &:= \left(\Upsilon^{(Z)}, \Upsilon^{(x)}, \Upsilon^{(\xi)} \right) (\tau, U, x_0, \xi_0), \\ \tilde{\Upsilon}^\tau(Z, x, \xi) &:= \left(\tilde{\Upsilon}^{(Z)}, \tilde{\Upsilon}^{(x)}, \tilde{\Upsilon}^{(\xi)} \right) (\tau, Z, x, \xi), \end{aligned} \quad (5.34)$$

respectively the flow and the inverse flow of (4.2). In particular we have (recall (4.7))

$$\tilde{\Upsilon}^{(\xi)}(\tau, Z; x, \xi) := (1 + \tilde{\Psi}^{(\xi)}(\tau, Z; x))\xi, \quad \tilde{\Psi}^{(\xi)} \in \Sigma \mathcal{F}_1^{\mathbb{R}}[r, N]. \quad (5.35)$$

By the proof of Theorem 4.2 we also deduce that

$$g(Z; x, \xi) \in \Sigma \Gamma_0^m[r, N] \quad \Rightarrow \quad g(\tilde{\Upsilon}^\tau(Z, x, \xi)) \in \Sigma \Gamma_0^m[r, N].$$

Hence the symbol

$$a_m^+(\tau, Z; x, \xi) = a(\tilde{\Upsilon}^\tau(Z; x, \xi)), \quad (5.36)$$

belongs to $\Sigma \Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and solves the problem (5.33). We now study the properties of the symbol a_m^+ by using the result of Theorem 4.2, in particular equation (4.8). By (5.8), (5.9), and using (5.36), (5.34), (5.35), (we shall omit the dependence on Z, x) we have

$$\begin{aligned} a_m^+(\tau, Z; x, \xi) &= (1 + a_m(\tilde{\Upsilon}^{(z)}, \tilde{\Upsilon}^{(x)})) f_m((1 + \tilde{\Psi}^{(\xi)})\xi) + a_{m'}(\tilde{\Upsilon}^{(z)}, \tilde{\Upsilon}^{(x)}, \tilde{\Upsilon}^{(\xi)}) \\ &\stackrel{(5.9)}{=} (1 + a_m(\tilde{\Upsilon}^{(z)}, \tilde{\Upsilon}^{(x)})) (1 + \tilde{\Psi}^{(\xi)})^m f_m(\xi) + r_1 + r_2 \\ &\stackrel{(4.8)}{=} m f_m(\xi) + r_1 + r_2, \end{aligned} \quad (5.37)$$

where $m \in \Sigma \mathcal{F}_0^{\mathbb{R}}[r, N]$ is independent of $x \in \mathbb{T}$ and

$$\begin{aligned} r_1 &:= m(f_0(\xi) - f_m(\xi)) + m(1 + \tilde{\Psi}^{(\xi)})^{-m} \tilde{f}_{m-1}((1 + \tilde{\Psi}^{(\xi)})\xi), \\ r_2 &:= a_{m'}(\tilde{\Upsilon}^{(z)}, \tilde{\Upsilon}^{(x)}, \tilde{\Upsilon}^{(\xi)}). \end{aligned} \quad (5.38)$$

By (5.37), (5.38), and using the properties of $\tilde{\Upsilon}$ in Theorem 4.2 we deduce that

$$a_m^+(1, Z; x, \xi) = m(Z) f_m(\xi) + r(Z; x, \xi), \quad m \in \Sigma \mathcal{F}_0^{\mathbb{R}}[r, N], \quad r \in \Sigma \Gamma_1^{m'}[r, N],$$

and m is independent of $x \in \mathbb{T}$.

Lower orders. Recalling (5.31), (5.32), (5.29) we have that the equation (5.6), at order $m - j/2$ with $j \geq 1$, reads

$$\begin{aligned} \partial_\tau a_{m-j}^+(\tau, Z; x, \xi) &= \{b(\tau, Z; x, \xi), a_{m-j}^+(\tau, Z; x, \xi)\} \\ &\quad - d_z a_{m-j}^+(\tau, Z; x, \xi)[Op^{BW}(iE\mathbf{B}(\tau, Z; x, \xi))][Z]] - iq_j(\tau, Z; x, \xi) \\ a_0(0, z; x, \xi) &= 0, \end{aligned} \quad (5.39)$$

where the q_j 's are defined in (5.32). Setting $g(\tau) = a_{m-j}^+(\tau, Z(\tau); x(\tau), \xi(\tau))$, with z, x, ξ satisfying (4.2), we note that

$$\partial_\tau g(\tau) = -iq_j(\tau, Z(\tau); x(\tau), \xi(\tau)) \quad \Rightarrow \quad g(\tau) = - \int_0^\tau iq_j(\sigma, Z(\sigma); x(\sigma), \xi(\sigma)) d\sigma.$$

Hence we have that (recall (5.34))

$$a_{m-j}^+(\tau, Z; x, \xi) = - \int_0^\tau i q_j(\sigma, Y^\sigma \circ \tilde{Y}^\sigma(Z; x, \xi)) d\sigma \in \Sigma \Gamma_1^{m-\frac{j}{2}}[r, N]$$

solves the problem (5.39). By iterating the procedure above (by solving the problems (5.39)) we construct a symbol a^+ as in (5.18) such that the following holds. Define

$$Q^\tau(Z) = Op^{\text{BW}}(iEA^+(\tau, Z; x, \xi))[Z]$$

with A^+ of the form (5.17) with a^+ as in (5.18). Then the operator $Q^\tau(Z)$ solves the problem (recall (5.11))

$$\begin{cases} \partial_\tau Q^\tau(Z) = [G^\tau(Z), Q^\tau(Z)] + \mathcal{G}_\rho(\tau; Z) \\ Q^0(Z) = iEOp^{\text{BW}}(A(Z; x, \xi))[Z], \end{cases} \quad (5.40)$$

where $\mathcal{G}_\rho(\tau; Z) := Op^{\text{BW}}(G_\rho(\tau, Z; x, \xi))[Z]$ for some matrix of symbols $G_\rho \in \Sigma \Gamma_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. It remains to prove that the difference $Q^\tau - P^\tau$ is a smoothing remainder in $\Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. First of all we write

$$Q^\tau(Z) - P^\tau(Z) = V^\tau \circ (\Psi^\tau)^{-1}(Z), \quad (5.41)$$

where, recalling (5.5),

$$V^\tau(U) := Q^\tau \circ \Psi^\tau(U) - (d_U \Psi^\tau)(U)[X(U)].$$

Recalling (5.11), (5.40), (5.3), $Z = \Psi^\tau(U)$, we deduce that

$$\begin{aligned} \partial_\tau d_U \Psi^\tau(U)[\cdot] &= (d_Z G^\tau)(\Psi^\tau(U))[d_U \Psi^\tau(U)[\cdot]], \\ [G^\tau(\Psi^\tau(U)), Q^\tau(\Psi^\tau(U))] &= (d_Z G^\tau)(\Psi^\tau(U)) [Q^\tau(\Psi^\tau(U))] - (d_Z Q^\tau)(\Psi^\tau(U)) [G^\tau(\Psi^\tau(U))]. \end{aligned}$$

Therefore we have that

$$\begin{cases} \partial_\tau V^\tau(U) = (d_Z G^\tau)(\Psi^\tau) [V^\tau] + \mathcal{G}_\rho(\Psi^\tau), \\ V^0(U) = -R(U)[U]. \end{cases} \quad (5.42)$$

We claim that $V^\tau \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. We start by studying the flow generated by $(d_Z F^\tau)(\Psi^\tau)[\cdot]$, denoted by Φ_{dG}^τ , by using formula (5.19). The existence of such flow can be deduced reasoning as in Lemma 2.30. We now provide estimates on Sobolev spaces. Let us denote Φ_B^τ the (linear) flow generated by $Op^{\text{BW}}(iEB(\tau, \Psi^\tau(U); x, \xi))[\cdot]$ with $U \in H^s$. We have that

$$\|\Phi_B^\tau h\|_{H^s} \lesssim_s \|h(0)\|_{H^s} (1 + \|\Psi^\tau(U)\|_{H^{s_0}}) \lesssim_s \|h(0)\|_{H^s} (1 + \|U\|_{H^{s_0}}). \quad (5.43)$$

Therefore we write (recall (5.19))

$$\Phi_{dG}^\tau h = \Phi_B^\tau h(0) + \Phi_B^\tau \int_0^\tau (\Phi_B^\sigma)^{-1} Op^{\text{BW}}(iE(d_W \mathbf{B})(\sigma, \Psi^\sigma(U); x, \xi) [\Phi_{dG}^\tau h]) [\Psi^\sigma(U)] d\sigma,$$

where $W = \Psi^\sigma(U)$. Hence, using (5.43) and Proposition 2.20,

$$\|\Phi_{dG}^\tau h\|_{H^{s-1}} \lesssim_s \|h(0)\|_{H^{s-1}} + \|\Phi_{dG}^\tau h\|_{H^{s_0}} \|U\|_{H^s},$$

from which we deduce $\|\Phi_{dG}^\tau h\|_{H^{s-1}} \lesssim_s \|h(0)\|_{H^{s-1}}$, for any s , if $U \in B_r(H^s)$ with $r > 0$ is small enough. Then we obtain, using the Duhamel formula on (5.42),

$$\|V^\tau(U)\|_{H^{s+\rho-1}} \lesssim_s \|\Phi_{dF}^\tau R(U)U\|_{H^{s+\rho-1}} + \|\Phi_{dG}^\tau \int_0^\tau (\Phi_{dG}^\sigma)^{-1} \mathcal{G}_\rho(\sigma; \Psi^\sigma) d\sigma\|_{H^{s+\rho-1}} \lesssim_s \|U\|_{H^{s_0}} \|U\|_{H^s}.$$

This proves the (2.20) for V^τ with $k = 0$. The estimates on the differentials in U (with $k \geq 1$) follow by differentiating the equation (5.42) and reasoning in the same way. As a consequence, using also estimates (2.59) on the flow Ψ^τ , we have that the operator in (5.41) belongs to $\mathcal{R}_1^{-\rho}[r] \otimes \mathcal{M}_2(\mathbb{C})$. The fact that the operator $Q^\tau - P^\tau$ admits an expansion in homogeneous operators, i.e. it is in $\Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$, follows by using the expansion in homogeneous symbols and operators of $\mathbf{B}(\tau, Z; x, \xi)$ and of $\Psi^\tau(U) - U \in \Sigma \mathcal{M}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. This proves the (5.13).

5.1.2. *Reduction at lower orders.* In this section we study how a para-differential vector field conjugate under the flow in (2.51) with $f(\tau, u; x, \xi)$ as in (2.53) or (2.54). Let us consider

$$\begin{aligned} M(U; \xi) &= (1 + \mathfrak{m}(U))f_m(\xi) + \widetilde{M}(U; \xi), \quad \mathfrak{m} \in \mathcal{F}_1^{\mathbb{R}}[r, N], \quad \widetilde{M}(U; \xi) \in \Sigma\Gamma_1^{m-\frac{1}{2}}[r, N], \\ \mathbb{R} \ni f_m(\xi) &\in \Gamma_0^m, \quad m > 1, \quad M(U; \xi) \text{ independent of } x \in \mathbb{T}, \\ M(U; \xi) - \overline{M(U; \xi)} &\in \Sigma\Gamma_1^0[r, N]. \end{aligned}$$

We also assume that (3.1) holds. Consider the operator $X(U)$ as in (5.2) with $A(U; x, \xi)$ as in (5.7) and we assume that the symbol $a(U; x, \xi)$ has the form

$$a(U; x, \xi) = M(U; \xi) + a_{m'}(U; x, \xi), \quad a_{m'}(U; x, \xi) \in \Sigma\Gamma_1^{m'}[r, N], \quad m' = m - \frac{p}{2}, \quad (5.44)$$

for some $p \geq 1$ and

$$a_{m'}(U; x, \xi) - \overline{a_{m'}(U; x, \xi)} \in \Sigma\Gamma_1^0[r, N].$$

The symbol a in (5.44) satisfies the assumptions (4.114), (4.115), (4.116). Hence, by Theorem 4.19, there exists a symbol $c(\tau, U; x, \xi)$ satisfying (4.117), such that equation (4.119) is verified. Let us now consider the matrix of symbols

$$\mathbf{C}(\tau, U; x, \xi) := \begin{pmatrix} c(\tau, U; x, \xi) & 0 \\ 0 & \overline{c(\tau, U; x, -\xi)} \end{pmatrix} \quad (5.45)$$

and let Ψ^τ be the flow of (5.3) with

$$G^\tau(U) := \text{iEOp}^{\text{BW}}(\mathbf{C}(\tau, U; x, \xi))[U]. \quad (5.46)$$

Notice that this flow is well-posed by Theorem 2.31. We define

$$Z := \begin{bmatrix} z \\ \bar{z} \end{bmatrix} := \Psi^\tau(U)|_{\tau=1}. \quad (5.47)$$

By using (5.45) one has

$$\Psi^\tau(U)|_{\tau=1} := \begin{pmatrix} \Phi^\tau(u) \\ \overline{\Phi^\tau(u)} \end{pmatrix}_{|\tau=1}, \quad \partial_\tau \Phi^\tau(u) = \text{Op}^{\text{BW}}(\text{i}c(\tau, \Phi^\tau(u); x, \xi))[\Phi^\tau(u)].$$

The main result of this section is the following.

Theorem 5.2. *For $r > 0$ small enough the conjugate of X in (5.1) (with the assumption (5.44)) has the form (see (5.47))*

$$\dot{Z} = \text{iEOp}^{\text{BW}}(A^+(Z; x, \xi))[Z] + R^+(Z)[Z], \quad R^+ \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}),$$

and the matrix of symbols $A^+(Z; x, \xi)$ has the form

$$\begin{aligned} A^+(Z; x, \xi) &:= \begin{pmatrix} a^+(Z; x, \xi) & 0 \\ 0 & \overline{a^+(Z; x, -\xi)} \end{pmatrix}, \quad m'' := m - \frac{p+1}{2}, \\ a^+(Z; x, \xi) &= M((\Psi^1)^{-1}(Z); \xi) + \widetilde{\mathfrak{m}}(Z; \xi) + a_{m''}^+(Z; x, \xi), \quad a_{m''}^+(Z; x, \xi) \in \Sigma\Gamma_1^{m''}[r, N], \\ \widetilde{\mathfrak{m}}(Z; \xi) - \overline{\widetilde{\mathfrak{m}}(Z; \xi)} &\in \Sigma\Gamma_1^0[r, N], \\ a_{m''}^+(Z; x, \xi) - \overline{a_{m''}^+(Z; x, \xi)} &\in \Sigma\Gamma_1^0[r, N], \end{aligned}$$

and where $\widetilde{\mathfrak{m}} \in \Sigma\Gamma_1^{m'}[r, N]$ is independent of $x \in \mathbb{T}$.

The rest of the section is devoted to the proof of the result above and we will follow the strategy used in section 5.1.1. The system (5.1) in the new coordinates (5.47) has the form (5.4)-(5.6) with G^τ as in (5.46). We recall also that, by the remarks under Definition 2.15 and by Theorem 2.31, we have

$$P^\tau(Z) = \text{iE}\Omega Z + M_1(\tau; Z)[Z], \quad M_1 \in \Sigma\mathcal{M}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C}) \quad (5.48)$$

with estimates uniform in $\tau \in [0, 1]$, where Ω is in (3.2). We shall look for a solution of the Heisenberg equation of the form

$$P^\tau(Z) = iE\Omega Z + M_1(\tau; Z)[Z] = iEOp^{\text{BW}}(A^+(\tau, Z; x, \xi))[Z] + R^+(\tau, Z)[Z], \quad (5.49)$$

with A^+ , R^+ as follows

$$A^+(\tau, Z; x, \xi) := \begin{pmatrix} a^+(\tau, Z; x, \xi) & 0 \\ 0 & a^+(\tau, Z; x, -\xi) \end{pmatrix}, \quad R^+ \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}). \quad (5.50)$$

In particular we make the ansatz (recall (3.1))

$$\begin{aligned} a^+(\tau, Z; x, \xi) &= a_m^+(\tau, Z; \xi) + a_{m-\frac{1}{2}}^+(\tau, Z; \xi) + \sum_{j=0}^{2\rho+m} a_{m-\frac{p+j}{2}}^+(\tau, Z; x, \xi), \\ a_{m-\frac{j}{2}}^+(\tau, Z; x, \xi) &\in \Sigma\Gamma_1^{m-\frac{j}{2}}[r, N] \quad \forall j \in \mathbb{N} \\ a_m^+(Z; \xi) &= (1 + \mathfrak{m}^+(Z))f_m(\xi), \quad \mathfrak{m}^+ \in \mathcal{F}_1^{\mathbb{R}}[r, N], \\ \widetilde{M}^+(Z; \xi) &:= a_{m-\frac{1}{2}}^+(\tau, Z; \xi) \in \Sigma\Gamma_1^{m-\frac{1}{2}}[r, N] \text{ is independent of } x \in \mathbb{T}, \\ \widetilde{M}^+(U; \xi) - \overline{\widetilde{M}^+(U; \xi)} &\in \Sigma\Gamma_1^0[r, N]. \end{aligned} \quad (5.51)$$

Expanding the non-linear commutator as in (5.20)-(5.28) we get

$$[Op^{\text{BW}}(iEC(\tau, Z; x, \xi))[Z], P^\tau(Z)] = iEOp^{\text{BW}}\left(\begin{matrix} d(\tau, Z; x, \xi) & 0 \\ 0 & \overline{d(\tau, Z; x, -\xi)} \end{matrix}\right)[Z] + \mathcal{G}_\rho(Z)[Z] \quad (5.52)$$

where $\mathcal{G}_\rho \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and

$$\begin{aligned} \text{id}(\tau, Z; x, \xi) &= ic(\tau, Z; x, \xi)\#_\rho ia^+(\tau, Z; x, \xi) - ia^+(\tau, Z; x, \xi)\#_\rho ic(\tau, Z; x, \xi) \\ &\quad + i(\partial_t c)(\tau, Z; x, \xi) - i(d_Z a^+)(\tau, Z; x, \xi)[iEOp^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z]] \end{aligned} \quad (5.53)$$

with

$$(\partial_t c)(\tau, Z; x, \xi) := (d_Z c)(\tau, Z; x, \xi)[P^\tau(Z)].$$

In order to lighten the notation we shall sometimes omit the dependence on $(\tau, Z; x, \xi)$ of the symbols. By recalling the ansatz (5.51) and using the expansion (2.31) we get

$$\begin{aligned} d &= \{c, a_m^+\} + \sum_{j=1}^{m+2\rho} q_j - (d_Z a^+)(\tau, Z; x, \xi)[iEOp^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z]], \\ q_j &\in \Sigma\Gamma_1^{m-\frac{p+j}{2}}[r, N], \quad j = 1, \dots, 2\rho + m, \quad m - \frac{p+j}{2} \neq \delta \stackrel{(5.45), (5.44)}{=} 1 - \frac{p}{2}, \\ q_j &:= (\partial_t c) + r_j, \quad m - \frac{p+j}{2} := 1 - \frac{p}{2}, \quad r_j \in \Sigma\Gamma_1^\delta[r, N]. \end{aligned} \quad (5.54)$$

Moreover the symbols q_j depend only on c , $a_{m-j/2}^+$, q_k , with $k < j$. Our aim is to prove the ansatz (5.49) using the expansion (5.53). We shall solve the Heisenberg equation iteratively expanding the symbol in (5.53) in decreasing orders as in (5.54).

Order m . By (5.53), (2.31), we have that, at the highest order, the Heisenberg equation reads

$$\begin{cases} \partial_\tau a_m^+(\tau, Z; \xi) = -d_Z a_m^+(\tau, Z; \xi)[Op^{\text{BW}}(iEC(\tau, Z; x, \xi))[Z]] \\ a_m^+(0, Z; \xi) = (1 + \mathfrak{m}(Z))f_m(\xi). \end{cases} \quad (5.55)$$

Notice that the function

$$g(\tau) = a_m^+(\tau, Z(\tau); \xi) \quad (5.56)$$

is constant along the solution generated by (5.46). By Theorem 2.31 such flow is well posed and invertible. Hence the symbol

$$a_m^+(\tau, Z; \xi) = (1 + \mathfrak{m}((\Psi^\tau)^{-1}(Z)))f_m(\xi) =: (1 + \mathfrak{m}^+(\tau, Z))f_m(\xi), \quad (5.57)$$

belongs to $\Sigma\Gamma_1^m[r, N]$ and solves the problem (5.55). The symbol $a_m^+(\tau, Z; \xi)$ in (5.57) is constant in x . We have verified the ansatz (5.51) at the highest order.

Order $m - 1/2$. By (5.53), (2.31), we have that, at order $m - 1/2$, the Heisenberg equation reads

$$\begin{cases} \partial_\tau a_{m-1/2}^+(\tau, Z; x, \xi) = -d_Z a_{m-1/2}^+(\tau, Z; x, \xi)[Op^{BW}(iEC(\tau, Z; x, \xi))[Z]] \\ a_{m-1/2}^+(0, Z; x, \xi) = \widetilde{M}(Z; \xi). \end{cases} \quad (5.58)$$

Notice that the function $g(\tau) = a_{m-1/2}^+(\tau, Z(\tau); x, \xi)$ is constant along the solution of (5.46), therefore the symbol

$$a_{m-1/2}^+(\tau, Z; x, \xi) = \widetilde{M}((\Psi^\tau)^{-1}(Z); \xi) =: \widetilde{M}^+(\tau, Z; \xi)$$

belongs to $\Sigma\Gamma_1^{m-1/2}[r, N]$ and solves the problem (5.58). The symbol $a_{m-1/2}^+(\tau, Z; x, \xi)$ in (5.57) clearly is constant in x . Hence it verifies the ansatz (5.51).

Order $m - p/2$. By (5.53), (5.54), we have that, at order $m - p/2$, the Heisenberg equation reads

$$\begin{aligned} \partial_\tau a_{m-p/2}^+(\tau, Z; x, \xi) &= \{C(\tau, Z; x, \xi), (1 + \mathfrak{m}^+(\tau, Z))f_m(\xi)\} - d_Z a_{m-p/2}^+(\tau, Z; x, \xi)[Op^{BW}(iEC(\tau, Z; x, \xi))[Z]] \\ a_{m-p/2}^+(0, Z; x, \xi) &= a_{m-p/2}(Z; x, \xi). \end{aligned} \quad (5.59)$$

Notice that the function $g(\tau) = a_{m-p/2}^+(\tau, Z(\tau); x, \xi)$ satisfies

$$\begin{aligned} \partial_\tau g(\tau) &= \{C(\tau, Z; x, \xi), (1 + \mathfrak{m}^+(\tau, Z))f_m(\xi)\} - (1 + \mathfrak{m}^+(\tau, Z))(\partial_\xi f_m)(\xi)(\partial_x c)(\tau, Z; x, \xi), \\ g(0) &= a_{m-p/2}(Z(0); x, \xi) = a_{m-p/2}(U; x, \xi). \end{aligned}$$

Therefore

$$\begin{aligned} a_{m-p/2}^+(\tau, Z, x, \xi) &= a_{m-p/2}((\Psi^\tau)^{-1}(Z); x, \xi) \\ &\quad - \int_0^\tau (1 + \mathfrak{m}^+(\sigma, \Psi^\sigma(\Psi^\tau)^{-1}(Z))) (\partial_\xi f_m)(\xi) (\partial_x c)(\sigma, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi) d\sigma \end{aligned}$$

solves the problem (5.59). Moreover, by Theorem 4.19 (see (4.119)), we have

$$a_{m-p/2}^+(1, Z; x, \xi) = \widetilde{\mathfrak{m}}(Z; \xi) \stackrel{(4.119)}{:=} \frac{1}{2\pi} \int_{\mathbb{T}} a_{m-p/2}((\Psi^1)^{-1}(Z); x, \xi) dx,$$

i.e. $a_{m-p/2}^+(\tau, Z; x, \xi)$ at $\tau = 1$ is constant in $x \in \mathbb{T}$.

Lower orders. Recalling (5.54), we have that the equation (5.6), at order $m - (p + j)/2$, reads

$$\begin{aligned} \partial_\tau a_{m-p+j/2}^+(\tau, Z; x, \xi) &= -d_Z a_{m-p+j/2}^+(\tau, Z; x, \xi)[Op^{BW}(iEC(\tau, Z; x, \xi))[Z]] + q_j(\tau, Z; x, \xi) \\ a_{m-p+j/2}^+(0, z; x, \xi) &= 0. \end{aligned} \quad (5.60)$$

Setting

$$g(\tau) = a_{m-p+j/2}^+(\tau, Z(\tau); x, \xi)$$

with $Z(\tau)$ satisfying (5.46) we note that

$$\partial_\tau g(\tau) = q_j(\tau, Z(\tau); x, \xi) \quad \Rightarrow \quad g(\tau) = \int_0^\tau q_j(\sigma, Z(\sigma); x, \xi) d\sigma.$$

Hence we have that

$$a_{m-p+j/2}^+(\tau, Z; x, \xi) = \int_0^\tau q_j(\sigma, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi) d\sigma \in \Sigma\Gamma_1^j[r, N],$$

solves the problem (5.60). To summarize by iterating the procedure above (by solving the problems (5.60)) we construct a symbol a^+ as in (5.51) such that the following holds. Define

$$Q^\tau(Z) = Op^{\text{BW}}(iEA^+(\tau, Z; x, \xi))[Z]$$

with A^+ of the form (5.50) with a^+ as in (5.51). Then the operator $Q^\tau(Z)$ solves the problem

$$\begin{cases} \partial_\tau Q^\tau(Z) = [Op^{\text{BW}}(iEC(\tau, Z; x, \xi))[Z], Q^\tau(Z)] + \mathcal{G}_\rho(\tau; Z) \\ Q^0(Z) = iEOp^{\text{BW}}(A(Z; x, \xi))[Z], \end{cases}$$

where $\mathcal{G}_\rho(\tau; Z) := Op^{\text{BW}}(G_\rho(\tau, Z; x, \xi))[Z]$ for some matrix of symbols $G_\rho \in \Sigma\Gamma_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. The fact that the difference $Q^\tau - P^\tau$ is a smoothing remainder in $\Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ can be proved following word by word the conclusion of the proof of Theorem 5.1.

5.2. Diagonalization of the para-differential matrix. In this section we show how to diagonalize a matrix of para-differential operators up to smoothing remainders.

5.2.1. Diagonalization of the matrix at the highest order. In this section we study how a para-differential vector field conjugates under the flow in (2.51) with $f(\tau, u; x, \xi)$ as in (2.54). Let us consider $X(U)$ as in (5.2) with the matrix $A(U; x, \xi)$ as follows

$$\begin{aligned} A(U; x, \xi) &:= A_m(U; x, \xi) + A_{m'}(U; x, \xi), \quad m' = m - \frac{1}{2}, \quad A_m(U; x, \xi) := (\mathbb{1} + \tilde{A}_m(U; x))f_m(\xi), \\ \tilde{A}_m(U; x) &:= \begin{pmatrix} a_m(U; x) & b_m(U; x) \\ b_m(U; x) & a_m(U; x) \end{pmatrix} \in \Sigma\mathcal{F}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \quad a_m(U, x) \in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N] \\ A_{m'}(U; x, \xi) &:= \begin{pmatrix} a_{m'}(U; x, \xi) & b_{m'}(U; x, \xi) \\ b_{m'}(U; x, -\xi) & a_{m'}(U; x, -\xi) \end{pmatrix} \in \Sigma\Gamma_1^{m'}[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \\ a_{m'}(U; x, \xi) - \overline{a_{m'}(U; x, \xi)} &\in \Sigma\Gamma_1^0[r, N]. \end{aligned} \tag{5.61}$$

where $f_m \in \Gamma_0^m$ and in particular it is a m -homogeneous $C^\infty(\mathbb{R}^+, \mathbb{R})$. We also assume that $f_m(\xi)$ is even in ξ and the (3.1). Our aim is to conjugate system (5.1) with hypotheses (5.61) with the flow $\Psi_C^\tau(U)$ of (5.3) with

$$G^\tau(U) := Op^{\text{BW}}(\mathbf{C}(\tau, U; x))[U] \quad \mathbf{C}(\tau, U; x) := \begin{pmatrix} 0 & C(\tau, U; x) \\ C(\tau, U; x) & 0 \end{pmatrix} \tag{5.62}$$

where the symbol $C(\tau, U; x)$ in $\Sigma\mathcal{F}_1[r, N]$ is given by Theorem 4.13. Notice that the flow of (5.62) is well-posed by Theorem 2.31 with generator as in (2.54). We define

$$Z := \begin{bmatrix} z \\ \bar{z} \end{bmatrix} := \Psi_C^\tau(U)|_{\tau=1}. \tag{5.63}$$

The main result of this section is the following.

Theorem 5.3. *For $r > 0$ small enough the conjugate of X in (5.1) with assumption (5.61) has the form (see (5.63))*

$$\dot{Z} = iEOp^{\text{BW}}(A^+(Z; x, \xi))[Z] + R^+(Z)[Z], \quad R^+ \in R \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}),$$

where

$$\begin{aligned} A^+(Z; x, \xi) &:= A_m^+(Z; x, \xi) + A_{m'}^+(Z; x, \xi), \quad A_m^+(Z; x, \xi) := (\mathbb{1} + \tilde{A}_m^+(Z; x))f_m(\xi), \\ \tilde{A}_m^+(Z; x) &:= \begin{pmatrix} a_m^+(Z; x) & 0 \\ 0 & a_m^+(Z; x) \end{pmatrix} \in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \\ A_{m'}^+(Z; x, \xi) &:= \begin{pmatrix} a_{m'}^+(Z; x, \xi) & b_{m'}^+(Z; x, \xi) \\ b_{m'}^+(Z; x, -\xi) & a_{m'}^+(Z; x, -\xi) \end{pmatrix} \in \Sigma\Gamma_1^{m'}[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \\ a_{m'}^+(U; x, \xi) - \overline{a_{m'}^+(U; x, \xi)} &\in \Sigma\Gamma_1^0[r, N]. \end{aligned} \tag{5.64}$$

The rest of the section is devoted to the proof of the result above and we will follow the strategy used in section 5.1.1.

The system (5.61) in the new coordinates (5.63) has the form (5.4)-(5.6) with G^τ as in (5.62). We note that because of the remarks under Definition 2.15 and by Theorem 2.31,

$$P^\tau(Z) = iE\Omega Z + M_1(\tau; Z)[Z], \quad M_1 \in \Sigma\mathcal{M}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C}),$$

with estimate uniform in $\tau \in [0, 1]$, where Ω is in (3.2). We shall look for a solution of the Heisenberg equation of the form

$$P^\tau(Z) = iE\Omega Z + M_1(\tau; Z)[Z] = iEOp^{\text{BW}}(A^+(\tau, Z; x, \xi))[Z] + R^+(\tau, Z)[Z], \quad (5.65)$$

with $R^+ \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and A^+ of the form

$$A^+(\tau, Z; x, \xi) := \begin{pmatrix} a^+(\tau, Z; x, \xi) & b^+(\tau, Z; x, \xi) \\ \overline{b^+(\tau, Z; x, -\xi)} & \overline{a^+(\tau, Z; x, -\xi)} \end{pmatrix}. \quad (5.66)$$

In particular we make the ansatz

$$\begin{aligned} a^+(\tau, Z; x, \xi) &= (1 + a_m^+(\tau, Z; x))f_m(\xi) + \sum_{j=1}^{2(m+\rho)} a_{m-\frac{j}{2}}^+(\tau, Z; x, \xi), \\ b^+(\tau, Z; x, \xi) &= b_m^+(\tau, Z; x)f_m(\xi) + \sum_{j=1}^{2(m+\rho)} b_{m-\frac{j}{2}}^+(\tau, Z; x, \xi), \end{aligned} \quad (5.67)$$

$$a_m^+ \in \mathcal{F}_1^{\mathbb{R}}[r, N], \quad b_m^+ \in \mathcal{F}_1[r, N], \quad a_{m-\frac{j}{2}}^+, b_{m-\frac{j}{2}}^+ \in \Sigma\Gamma_1^{m-\frac{j}{2}}[r, N], \quad j = 1, \dots, 2(m+\rho).$$

Expanding the non-linear commutator as in (5.20)-(5.28) we get

$$[Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z], P^\tau(Z)] = iEOp^{\text{BW}}\left(\begin{array}{cc} d_1(\tau, Z; x, \xi) & d_2(\tau, Z; x, \xi) \\ \overline{d_2(\tau, Z; x, -\xi)} & \overline{d_1(\tau, Z; x, -\xi)} \end{array}\right)Z + \mathcal{G}_\rho(Z)[Z]$$

where $\mathcal{G}_\rho \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and

$$\begin{aligned} d_1 &= -C(\tau, Z; x)\#_\rho \overline{b^+(\tau, Z; x, -\xi)} - b^+(\tau, Z; x, \xi)\#_\rho \overline{C(\tau, Z; x)} \\ &\quad - (d_Z a^+)(\tau, Z; x, \xi)[Op^{\text{BW}}(\mathbf{C}(\tau, Z; x))[Z]] \end{aligned} \quad (5.68)$$

$$\begin{aligned} d_2 &= -C(\tau, Z; x)\#_\rho \overline{a^+(\tau, Z; x, -\xi)} - a^+(\tau, Z; x, \xi)\#_\rho C(\tau, Z; x) \\ &\quad + (\partial_t C)(\tau, Z; x) - (d_Z b^+)(\tau, Z; x, \xi)[Op^{\text{BW}}(\mathbf{C}(\tau, Z; x))[Z]] \end{aligned} \quad (5.69)$$

with

$$(\partial_t C)(\tau, Z; x) := (d_Z C)(\tau, Z; x)[P^\tau(Z)].$$

By recalling the ansatz (5.67) and using the expansion (2.31) we get

$$\begin{aligned} d_1 &= -2\text{Re}\left(\overline{C(\tau, Z; x)}b_m^+(\tau, Z; x)\right)f_m(\xi) - 2\sum_{j=1}^{2(m+\rho)} \text{Re}\left(\overline{C(\tau, Z; x)}b_{m-\frac{j}{2}}^+(\tau, Z; x, \xi)\right) \\ &\quad + \sum_{j=1}^{2(m+\rho)} q_j^{(1)}(\tau, Z; x, \xi) - (d_Z a^+)(\tau, Z; x, \xi)[Op^{\text{BW}}(\mathbf{C}(\tau, Z; x))[Z]] \end{aligned} \quad (5.70)$$

$$\begin{aligned} d_2 &= -2C(\tau, Z; x)(1 + a_m^+(\tau, Z; x))f_m(\xi) - \sum_{j=1}^{2(m+\rho)} \left(a_{m-\frac{j}{2}}^+(\tau, Z; x, \xi) + \overline{a_{m-\frac{j}{2}}^+(\tau, Z; x, -\xi)}\right)C(\tau, Z; x) \\ &\quad + \sum_{j=1}^{2(m+\rho)} q_j^{(2)}(\tau, Z; x, \xi) - (d_Z b^+)(\tau, Z; x, \xi)[Op^{\text{BW}}(\mathbf{C}(\tau, Z; x))[Z]] \end{aligned} \quad (5.71)$$

where

$$q_j^{(1)}, q_j^{(2)} \in \Sigma\Gamma_1^{m-\frac{j}{2}}[r, N]. \quad (5.72)$$

Moreover the symbols $q_j^{(k)}$ depends only on the symbols $C, a_m^+, b_m^+, q_p^{(k)}$ with $p < j$. Note that in the notation above the term $\partial_t C$ in (5.69) is contained in $q_{2m}^{(2)}$. In the expansion for d_2 we used that a_m^+ is real valued.

Our aim is to prove the ansatz (5.65), (5.67) using the expansions (5.68), (5.69).

Order m . By (5.68), (5.69), the expansions (5.70), (5.71) and (5.67), we have that, at the highest order, the Heisenberg equation reads

$$\begin{cases} \partial_\tau a_m^+(\tau, Z; x) = -\operatorname{Re}\left(\overline{C(\tau, Z; x)} b_m^+(\tau, Z; x)\right) - d_Z a_m^+(\tau, Z; x)[Op^{\text{BW}}(\mathbf{C}(\tau, Z; x))[Z]] \\ a_m^+(0, Z; x) = a_m(Z; x), \end{cases} \quad (5.73)$$

$$\begin{cases} \partial_\tau b_m^+(\tau, Z; x) = -2(1 + a_m^+(\tau, Z; x))C(\tau, Z; x) - d_Z b_m^+(\tau, Z; x)[Op^{\text{BW}}(\mathbf{C}(\tau, Z; x))[Z]] \\ b_m^+(0, Z; x) = b_m(Z; x). \end{cases} \quad (5.74)$$

Notice that the problems (5.73), (5.74) have the form (4.86), (4.87). Therefore, by Theorem 4.13 we have

$$a_m^+(\tau, Z; x) \in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N], \quad b_m^+(\tau, Z; x) \in \Sigma\mathcal{F}_1[r, N],$$

with estimates uniform in $\tau \in [0, 1]$ and

$$b_m^+(1, Z; x) \equiv 0.$$

Lower orders. Recalling (5.70), (5.71), (5.72) we have that the Heisenberg equation, at order $m - j/2$, for $j \geq 1$, reads

$$\begin{aligned} \partial_\tau a_{m-\frac{j}{2}}^+(\tau, Z; x, \xi) &= -2\operatorname{Re}\left(C(\tau, Z; x) \overline{b_{m-\frac{j}{2}}^+(\tau, Z; x, \xi)}\right) + q_j^{(1)}(\tau, Z; x, \xi) \\ &\quad - (d_Z a_{m-\frac{j}{2}}^+)(\tau, Z; x, \xi)[Op^{\text{BW}}(\mathbf{C}(Z; x))[Z]], \end{aligned} \quad (5.75)$$

$$a_{m-\frac{j}{2}}^+(0, Z; x, \xi) = a_{m-\frac{j}{2}}(Z; x, \xi),$$

$$\begin{aligned} \partial_\tau b_{m-\frac{j}{2}}^+(\tau, Z; x, \xi) &= -\left(a_{m-\frac{j}{2}}^+(\tau, Z; x, \xi) + \overline{a_{m-\frac{j}{2}}^+(\tau, Z; x, -\xi)}\right)C(\tau, Z; x) + q_j^{(2)}(\tau, Z; x, \xi) \\ &\quad - (d_Z b_{m-\frac{j}{2}}^+)(\tau, Z; x)[Op^{\text{BW}}(\mathbf{C}(Z; x))[Z]], \end{aligned} \quad (5.76)$$

$$b_{m-\frac{j}{2}}^+(0, Z; x, \xi) = b_{m-\frac{j}{2}}(Z; x, \xi),$$

where (recall (5.61)) we defined

$$a_{m-\frac{j}{2}}(Z; x, \xi) := 0, \quad b_{m-\frac{j}{2}}(Z; x, \xi) := 0, \quad j \geq 2. \quad (5.77)$$

Notice that the problems (5.75), (5.76) are of the form (4.109), (4.110). Since the symbols $q_j^{(1)}, q_j^{(2)}$ depend only on the symbols $C, a_m^+, b_m^+, q_p^{(k)}$ with $p < j$, we can iteratively solve the problems (5.75), (5.76) and verify that, at the step j , the symbols $q_j^{(1)}, q_j^{(2)}$ satisfy the hypothesis (4.108). Therefore, by Proposition 4.18, we have that the solutions of (5.75), (5.76) satisfy

$$a_{m-\frac{j}{2}}^+(\tau, U; x, \xi), b_{m-\frac{j}{2}}^+(\tau, U; x, \xi) \in \Sigma\Gamma_1^{m-\frac{j}{2}}[r, N].$$

By iterating the procedure above we construct symbols a^+, b^+ as in (5.67) such that the following holds. Define

$$Q^\tau(Z) = Op^{\text{BW}}(iEA^+(\tau, Z; x, \xi))[Z]$$

with A^+ of the form (5.66) with a^+, b^+ as in (5.67). Then the operator $Q^\tau(Z)$ solves the problem

$$\begin{cases} \partial_\tau Q^\tau(Z) = [Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z], Q^\tau(Z)] + \mathcal{G}_\rho(\tau; Z) \\ Q^0(Z) = iEOp^{\text{BW}}(A(Z; x, \xi))[Z], \end{cases}$$

where $\mathcal{G}_\rho(\tau; Z) := Op^{\text{BW}}(G_\rho(\tau, Z; x, \xi))[Z]$ for some matrix of symbols $G_\rho \in \Sigma\Gamma_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. The fact that the difference $Q^\tau - P^\tau$ is a smoothing remainder in $\Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ can be proved following word by word the conclusion of the proof of Theorem 5.1.

5.2.2. *Diagonalization of the matrix at lower orders.* Let us consider the operator $X(U)$ as in (5.2) with

$$A(U; x, \xi) := \begin{pmatrix} a(U; x, \xi) & b(U; x, \xi) \\ b(U; x, -\xi) & a(U; x, -\xi) \end{pmatrix},$$

where

$$\begin{aligned} a(U; x, \xi) &= (1 + a_m(U; x))f_m(\xi) + a_{m'}(U; x, \xi), \quad m > 1, \quad m' = m - \frac{1}{2}, \\ b(U; x, \xi) &= b_{m''}(U; x, \xi) \in \Sigma\Gamma_1^{m''}[r, N], \quad m'' := m' - \frac{p}{2}, \quad \text{for some } p \geq 0 \\ a_m(U, x) &\in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N], \quad a_{m'}(U; x, \xi) \in \Sigma\Gamma_1^{m'}[r, N], \\ a_{m'}(U; x, \xi) - \overline{a_{m'}(U; x, \xi)} &\in \Sigma\Gamma_1^0[r, N]. \end{aligned} \tag{5.78}$$

where $f_m \in \Gamma_0^m$ and in particular is a m -homogeneous $C^\infty(\mathbb{R}^+, \mathbb{R})$ and even in $\xi \in \mathbb{R}$. Assume also that (3.1) holds. Let us now consider the matrix of symbols

$$\mathbf{C}(\tau, U; x, \xi) := \begin{pmatrix} 0 & C(\tau, U; x, \xi) \\ C(\tau, U; x, -\xi) & 0 \end{pmatrix}, \quad C(\tau, U; x, \xi) \in \Sigma\Gamma_1^\delta[r, N], \quad \delta := m'' - m,$$

where $C(\tau, U; x, \xi)$ is the symbol given by Theorem 4.22 with $m' \rightsquigarrow m''$, and let Ψ^τ be the flow of (5.3) with

$$G^\tau(U) := Op^{\text{BW}}(\mathbf{C}(\tau, U; x, \xi))[U]. \tag{5.79}$$

Notice that the flow of (5.3) with generator in (5.79) is well-posed by Theorem 2.31. We define

$$Z := \begin{bmatrix} z \\ \bar{z} \end{bmatrix} := \Psi^\tau(U)|_{\tau=1}. \tag{5.80}$$

The main result of this section is the following.

Theorem 5.4. *For $r > 0$ small enough the conjugate of X in (5.1) with assumption (5.78) has the form (see (5.80))*

$$\dot{Z} = iEOp^{\text{BW}}(A^+(Z; x, \xi))[Z] + R^+(Z)[Z], \quad R^+ \in R \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}),$$

and the matrix of symbols $A^+ \in \Sigma\Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ has the form

$$A^+(Z; x, \xi) := \begin{pmatrix} a^+(Z; x, \xi) & b^+(Z; x, \xi) \\ b^+(Z; x, -\xi) & a^+(Z; x, -\xi) \end{pmatrix}, \tag{5.81}$$

where

$$\begin{aligned} a^+(Z; x, \xi) &= (1 + a_m^+(Z; x))f_m(\xi) + a_{m'}^+(Z; x, \xi), \quad b^+(Z; x, \xi) = b_{m''-\frac{1}{2}}^+(Z; x, \xi), \\ a_m^+(Z, x) &\in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N], \quad a_{m'}^+(Z; x, \xi) \in \Sigma\Gamma_1^{m'}[r, N], \quad b^+(Z; x, \xi) \in \Sigma\Gamma_1^{m''-\frac{1}{2}}[r, N] \\ a_{m'}^+(U; x, \xi) - \overline{a_{m'}^+(U; x, \xi)} &\in \Sigma\Gamma_1^0[r, N]. \end{aligned} \tag{5.82}$$

The rest of the section is devoted to the proof of the result above and we will follow the strategy used in section 5.1.1.

The system in the new coordinates (5.80) has the form (5.4)-(5.6) with G^τ as in (5.79). We recall also that, by the remarks under Definition 2.15 and by Theorem 2.31, we note that

$$P^\tau(Z) = iE\Omega Z + M_1(\tau; Z)[Z], \quad M_1 \in \Sigma\mathcal{M}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C}),$$

with estimate uniform in $\tau \in [0, 1]$, where Ω is in (3.2). We shall look for a solution of the Heisenberg equation of the form

$$P^\tau(Z) = iE\Omega Z + M_1(\tau; Z)[Z] = iEOp^{\text{BW}}(A^+(\tau, Z; x, \xi))[Z] + R^+(\tau, Z)[Z],$$

with A^+, R^+

$$A^+(\tau, Z; x, \xi) := \begin{pmatrix} a^+(\tau, Z; x, \xi) & b^+(\tau, Z; x, \xi) \\ b^+(\tau, Z; x, -\xi) & a^+(\tau, Z; x, -\xi) \end{pmatrix}, \quad R^+ \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}). \quad (5.83)$$

In particular we make the ansatz

$$\begin{aligned} a^+(\tau, Z; x, \xi) &= (1 + a_m^+(\tau, Z; x))f_m(\xi) + a_{m'}^+(\tau, Z; x, \xi) + \sum_{j=0}^{2(m+\rho)} a_{2m''-m-\frac{j}{2}}^+(\tau, Z; x, \xi), \\ b^+(\tau, Z; x, \xi) &= b_{m''}^+(\tau, Z; x, \xi) + \sum_{j=1}^{2(m+\rho)} b_{m''-\frac{j}{2}}^+(\tau, Z; x, \xi), \end{aligned} \quad (5.84)$$

$$a_m^+ \in \Sigma\mathcal{F}_1^{\mathbb{R}}[r, N], \quad a_{m'}^+ \in \Sigma\Gamma_1^{m'}[r, N], \quad a_{2m''-m-\frac{j}{2}}^+ \in \Sigma\Gamma_1^{2m''-m-\frac{j}{2}}[r, N], \quad b_{m''-\frac{j}{2}}^+ \in \Sigma\Gamma_1^{m''-\frac{j}{2}}[r, N],$$

with $j \geq 0$. Expanding the non-linear commutator as in (5.20)-(5.28) we get

$$[Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z], P^\tau(Z)] = iE \begin{pmatrix} d_1(\tau, Z; x, \xi) & d_2(\tau, Z; x, \xi) \\ d_2(\tau, Z; x, -\xi) & d_1(\tau, Z; x, -\xi) \end{pmatrix} Z + \mathcal{G}_\rho(Z)[Z]$$

where $\mathcal{G}_\rho \in \Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and

$$\begin{aligned} d_1 &= -C(\tau, Z; x, \xi) \#_\rho \overline{b^+(\tau, Z; x, -\xi)} - b^+(\tau, Z; x, \xi) \#_\rho \overline{C(\tau, Z; x, -\xi)} \\ &\quad - (d_Z a^+)(\tau, Z; x, \xi) [Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z]] \end{aligned} \quad (5.85)$$

$$\begin{aligned} d_2 &= -C(\tau, Z; x, \xi) \#_\rho \overline{a^+(\tau, Z; x, -\xi)} - a^+(\tau, Z; x, \xi) \#_\rho C(\tau, Z; x, \xi) \\ &\quad + (\partial_t C)(\tau, Z; x, \xi) - (d_Z b^+)(\tau, Z; x, \xi) [Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z]] \end{aligned} \quad (5.86)$$

with

$$(\partial_t C)(\tau, Z; x, \xi) := (d_Z C)(\tau, Z; x, \xi) [P^\tau(Z)].$$

By recalling the ansatz (5.84) and using the expansion (2.31) we get

$$\begin{aligned} d_1 &= -2 \sum_{j=0}^{2(m+\rho)} \text{Re} \left(\overline{C(\tau, Z; x, \xi)} b_{m''-\frac{j}{2}}^+(\tau, Z; x, \xi) \right) + \sum_{j=2}^{2(m+\rho)} q_j^{(1)}(\tau, Z; x, \xi) \\ &\quad - (d_Z a^+)(\tau, Z; x, \xi) [Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z]] \end{aligned} \quad (5.87)$$

$$\begin{aligned} d_2 &= -2(1 + a_m^+(\tau, Z; x))f_m(\xi)C(\tau, Z; x, \xi) \\ &\quad - \left(a_{m'}^+(\tau, Z; x, \xi) + \overline{a_{m'}^+(\tau, Z; x, -\xi)} \right) C(\tau, Z; x, \xi) + \sum_{j=0}^{2(m+\rho)} q_j^{(2)}(\tau, Z; x, \xi) \\ &\quad - (d_Z b^+)(\tau, Z; x, \xi) [Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z]] \end{aligned} \quad (5.88)$$

where

$$q_j^{(1)} \in \Sigma\Gamma_1^{2m''-m-\frac{j}{2}}[r, N], \quad j \geq 2,$$

depends only on the symbols C and $b_{m''-\frac{k}{2}}^+$ with $0 \leq k \leq j-2$, while

$$q_j^{(2)} \in \Sigma\Gamma_1^{m''-\frac{j}{2}}[r, N], \quad j \geq 2, \quad \text{and} \quad \frac{j}{2} \neq m,$$

$$q_j^{(2)} = r_j + (\partial_t C), \quad r_j \in \Sigma\Gamma_1^{m''-\frac{j}{2}}[r, N], \quad \text{for} \quad \frac{j}{2} = m,$$

depends only on the symbols C , a_m^+ , $a_{m'}^+$ and $a_{2m''-m-k/2}^+$ with k such that $2m'' - m - k/2 < m'' - j/2$.

High orders of the symbol a^+ . By (5.85), (2.31), we have that, at the highest order, the Heisenberg equation at orders m and m' reads

$$\begin{cases} \partial_\tau a_m^+(\tau, Z; x) = -d_Z a_m^+(\tau, Z; x)[Op^{BW}(C(\tau, Z; x, \xi))][Z] \\ a_m^+(0, Z; x) = a_m(Z; x)f_m(\xi). \end{cases} \quad (5.89)$$

$$\begin{cases} \partial_\tau a_{m'}^+(\tau, Z; x, \xi) = -d_Z a_{m'}^+(\tau, Z; x, \xi)[Op^{BW}(C(\tau, Z; x, \xi))][Z] \\ a_{m'}^+(0, Z; x, \xi) = a_{m'}(Z; x, \xi). \end{cases} \quad (5.90)$$

Notice that the functions

$$g_1(\tau) = a_m^+(\tau, Z(\tau); x), \quad g_2(\tau) = a_{m'}^+(\tau, Z(\tau); x, \xi)$$

are constant along the solution of (5.79). By Theorem 2.31 such flow is well posed and invertible. Hence the symbols

$$a_m^+(\tau, Z; x, \xi) = a_m((\Psi^\tau)^{-1}(Z); x), \quad a_{m'}^+(\tau, Z; x, \xi) = a_{m'}((\Psi^\tau)^{-1}(Z); x, \xi),$$

belong respectively to $\Sigma\Gamma_1^m[r, N]$, $\Sigma\Gamma_1^{m'}[r, N]$ and solves the problem (5.90).

High orders of the symbol b^+ . By (5.86), (2.31), we have that, at the highest order, the Heisenberg equation at orders m'' for the symbol b^+ reads

$$\begin{aligned} \partial_\tau b_{m''}^+(\tau, Z; x, \xi) &= -2(1 + a_m^+(\tau, Z; x))f_m(\xi)C(\tau, Z; x, \xi) - d_Z b_{m''}^+(\tau, Z; x, \xi)[Op^{BW}(C(\tau, Z; x, \xi))][Z] \\ b_{m''}^+(0, Z; x, \xi) &= b_{m''}(Z; x, \xi). \end{aligned} \quad (5.91)$$

Notice that the function $g(\tau) = b_{m''}^+(\tau, Z(\tau); x, \xi)$ satisfies

$$\begin{aligned} \partial_\tau g(\tau) &= -2(1 + a_m^+(\tau, Z(\tau); x))f_m(\xi)C(\tau, Z(\tau); x, \xi), \\ g(0) &= b_{m''}(Z(0); x, \xi) = b_{m''}(U; x, \xi). \end{aligned}$$

Therefore

$$\begin{aligned} b_{m''}^+(\tau, Z, x, \xi) &= b_{m''}((\Psi^\tau)^{-1}(Z); x, \xi) \\ &\quad - 2 \int_0^\tau (1 + a_m^+(\sigma, \Psi^\sigma(\Psi^\tau)^{-1}(Z)))f_m(\xi)C(\sigma, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi) d\sigma \end{aligned}$$

solves the problem (5.91). Moreover, by Theorem 4.22 (see (4.138)), one has that

$$b_{m''}^+(1, Z; x, \xi) \stackrel{(4.138)}{=} 0.$$

Lower orders. We start by studying the first lower order corrections to the symbols a^+ , b^+ in (5.84) which are respectively of order $2m'' - m$ and $m'' - \frac{1}{2}$. Recalling (5.87), (5.88), we have that the Heisenberg equation, at these orders reads

$$\begin{aligned} \partial_\tau a_{2m''-m}^+(\tau, Z; x) &= -d_Z a_{2m''-m}^+(\tau, Z; x)[Op^{BW}(C(\tau, Z; x, \xi))][Z] \\ &\quad - 2\text{Re}\left(\overline{C(\tau, Z; x, \xi)} b_{m''}^+(\tau, Z; x, \xi)\right), \\ a_{2m''-m}^+(0, Z; x) &= 0, \end{aligned} \quad (5.92)$$

$$\begin{aligned} \partial_\tau b_{m''-\frac{1}{2}}^+(\tau, Z; x, \xi) &= -d_Z b_{m''-\frac{1}{2}}^+(\tau, Z; x, \xi)[Op^{BW}(C(\tau, Z; x, \xi))][Z] \\ &\quad - \left(a_{m'}^+(\tau, Z; x, \xi) + \overline{a_{m'}^+(\tau, Z; x, -\xi)}\right)C(\tau, Z; x, \xi), \\ b_{m''-\frac{1}{2}}^+(0, Z; x, \xi) &= 0. \end{aligned} \quad (5.93)$$

The symbols $b_{m''}^+$ and $a_{m'}^+$ have been already computed at the previous steps. Therefore, one can check that

$$\begin{aligned} a_{2m''-m}^+(\tau, Z; x, \xi) &:= -2 \int_0^\tau \operatorname{Re} \left(\overline{C(\tau, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi)} b_{m''}^+(\tau, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi) \right) d\sigma, \\ b_{m''-\frac{1}{2}}^+(\tau, Z; x, \xi) &:= \int_0^\tau Q(\sigma, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi) d\sigma, \\ Q(\tau, Z; x, \xi) &:= - \left(a_{m'}^+(\tau, Z; x, \xi) + \overline{a_{m'}^+(\tau, Z; x, -\xi)} \right) C(\tau, Z; x, \xi), \end{aligned}$$

solve the problems (5.92), (5.93). In particular, using the estimates (2.59) on the flow Ψ^τ (and reasoning as in section 4), one can prove that $a_{2m''-m}^+ \in \Sigma\Gamma_1^{2m''-m}[r, N]$ and $b_{m''-\frac{1}{2}}^+ \in \Sigma\Gamma_1^{m''-\frac{1}{2}}[r, N]$. For the lower order terms we reason similarly and iteratively. Recalling (5.87), (5.88), we have that the Heisenberg equation, at lower orders reads

$$\begin{aligned} \partial_\tau a_k^+(\tau, Z; x, \xi) &= -2 \operatorname{Re} \left(\overline{C(\tau, Z; x, \xi)} b_{m''-\frac{j}{2}}^+(\tau, Z; x, \xi) \right) \\ &\quad - d_z a_k^+(\tau, Z; x, \xi) [Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z]] + q_j^{(1)}(\tau, Z; x, \xi) \\ a_k^+(0, z; x, \xi) &= 0, \end{aligned} \tag{5.94}$$

with $k := 2m'' - m - j/2$. It is important to note that $2m'' - m - j/2 < m'' - j/2$. Then the corresponding equation for the symbol $b_{m''-\frac{j}{2}}^+$ does not depend on the symbol $a_{2m''-m-j/2}^+$. Actually, recalling (5.88), we have

$$\begin{aligned} \partial_\tau b_k^+(\tau, Z; x, \xi) &= -d_Z b_k^+(\tau, Z; x, \xi) [Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z]] + \tilde{q}_j^{(2)}(\tau, Z; x, \xi) \\ b_k^+(0, z; x, \xi) &= 0, \end{aligned} \tag{5.95}$$

with $k := m'' - \frac{j}{2}$ for some $\tilde{q}_j^{(2)}$ depending on $q_j^{(2)}$ and a_p^+ with $p < 2m'' - m - j/2$. One can easily check that

$$\begin{aligned} a_k^+(\tau, Z; x, \xi) &= -2 \int_0^\tau \operatorname{Re} \left(\overline{C(\tau, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi)} b_{m''-\frac{j}{2}}^+(\tau, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi) \right) d\sigma \\ &\quad + \int_0^\tau q_j^{(1)}(\sigma, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi) d\sigma \in \Sigma\Gamma_1^j[r, N], \quad k = 2m'' - m - \frac{j}{2}, \\ b_k^+(\tau, Z; x, \xi) &= \int_0^\tau \tilde{q}_j^{(2)}(\sigma, \Psi^\sigma(\Psi^\tau)^{-1}(Z); x, \xi) d\sigma \in \Sigma\Gamma_1^k[r, N], \quad k = m'' - \frac{j}{2} \end{aligned}$$

solve the problems (5.94), (5.95). To summarize by iterating the procedure above we construct symbols a^+, b^+ as in (5.84) such that the following holds. Define

$$Q^\tau(Z) = Op^{\text{BW}}(iEA^+(\tau, Z; x, \xi))[Z]$$

with A^+ of the form (5.83) with a^+, b^+ as in (5.84). We define the matrix in (5.81) as $A^+(Z; x, \xi) := A^+(1, Z; x, \xi)$. By the construction above we can note that the conditions in (5.82) are satisfied. Moreover the operator $Q^\tau(Z)$ solves the problem

$$\begin{cases} \partial_\tau Q^\tau(Z) = [Op^{\text{BW}}(\mathbf{C}(\tau, Z; x, \xi))[Z], Q^\tau(Z)] + \mathcal{G}_\rho(\tau; Z) \\ Q^0(Z) = iEOp^{\text{BW}}(A(Z; x, \xi))[Z], \end{cases}$$

where $\mathcal{G}_\rho(\tau; Z) := Op^{\text{BW}}(G_\rho(\tau, Z; x, \xi))[Z]$ for some matrix of symbols $G_\rho \in \Sigma\Gamma_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. The fact that the difference $Q^\tau - P^\tau$ is a smoothing remainder in $\Sigma\mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ can be proved following word by word the conclusion of the proof of Theorem 5.1. This concludes the proof of Theorem 5.4.

6. SYMPLECTIC CONJUGATIONS

In this section we explain how, with mild modifications, the change of coordinates of Section 3 can be adapted to our Hamiltonian contexts. In other words in the case that the equation (3.4) has the Hamiltonian structure introduced in Section 1.3, one may be interested in preserving such a structure for the equation in the new coordinates (3.6).

Para-differential Hamiltonian vector field. Let u be a C^∞ function on the \mathbb{T} with values in \mathbb{C} . We define the following frequency localization:

$$S_\xi u := \sum_{|k| \leq \varepsilon |\xi|} u_k e^{ikx}, \quad (6.1)$$

for some $0 < \varepsilon < 1$. Let $A(U; x, \xi) \in \Sigma \Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ be a matrix of symbols real-to-real and *self-adjoint*, i.e. satisfying (2.36) and (2.40) and consider the Hamiltonian function

$$H(U) := \frac{1}{2} \int_{\mathbb{T}} Op^{\text{BW}}(A(S_\xi U; x, \xi)) U \cdot \bar{U} dx, \quad U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}, \quad (6.2)$$

where $S_\xi U := (S_\xi u, S_\xi \bar{u})^T$. We are going to show that the Hamiltonian vector field of H in (6.2), which by definition is $i \nabla H(U)$, equals $i E Op^{\text{BW}}(A(U; x, \xi)) U$ modulo smoothing remainders. We need some preliminary lemmas.

Lemma 6.1 (Differential of symbols). Fix $p \in \mathbb{N}$, $m \in \mathbb{R}$ and consider a symbol $a \in \tilde{\Gamma}_p^m$. For u_1, \dots, u_{p+1}, h in $C^\infty(\mathbb{T}; \mathbb{C}^2)$ define a linear operator on h as

$$F[h] = F(u_2, \dots, u_{p+1})[h] := Op^{\text{BW}}(a(h, \dots, u_p; \xi))[u_{p+1}].$$

Then the operator F^* , the adjoint with respect to the L^2 scalar product of F , belongs to the class $\tilde{\mathcal{R}}_p^{-\rho}$ for any $\rho > 0$.

Proof. First we note that

$$F[h] := \sum_{n_0 \in \mathbb{Z} \sigma_1 n_1 + \sigma_2 n_2 + \dots + \sigma_p n_p = \sigma_0 n_0 - \sigma_{p+1} n_{p+1}} \sum_{C_{n_1, \dots, n_p}^{n_{p+1}}} (\Pi_{n_1} h) (\Pi_{n_2} u_2) \dots (\Pi_{n_p} u_p) (\Pi_{n_{p+1}} u_{p+1})$$

for some coefficients $C_{n_1, \dots, n_p}^{n_{p+1}} \in \mathbb{C}$, $\sigma_i \in \{\pm\}$, which are different from zero only if

$$\sum_{j=1}^p |n_j| \leq \varepsilon \langle n_{p+1} \rangle, \quad 0 < \varepsilon \ll 1.$$

Then one has that

$$\left(F[h], \overline{u_{p+2}} \right)_{L^2} = \int_{\mathbb{T}} F[h] \cdot u_{p+2} = \int_{\mathbb{T}} h \cdot L(u_2, \dots, u_{p+1})[u_{p+2}],$$

where the operator L is defined as

$$L(u_2, \dots, u_{p+1})[u_{p+2}] := \sum_{j \in \mathbb{Z} \sigma_2 n_2 + \sigma_3 n_3 + \dots + \sigma_{p+1} n_{p+1} - n_{p+2} = n_1} C_{-j, \dots, n_p}^{n_{p+1}} (\Pi_{n_2} u_2) \dots (\Pi_{n_p} u_p) (\Pi_{n_{p+1}} u_{p+1}) (\Pi_{-n_{p+2}} u_{p+2}).$$

In order to obtain the thesis it is sufficient to prove that L belongs to the class of remainders, i.e. one has to show that (2.17) holds. First we note that, using (2.8), there exists $\mu > 0$ such that

$$\|\Pi_j L(\Pi_{n_2} u_2, \dots, \Pi_{n_{p+1}} u_{p+1}) \Pi_{n_{p+2}} u_{p+2}\|_{L^2} \leq C \max(\langle n_2 \rangle, \dots, \langle n_{p+1} \rangle, \langle n_{p+2} \rangle)^\mu \prod_{j=2}^{p+2} \|\Pi_{n_j} u_j\|_{L^2}. \quad (6.3)$$

We remind that L is different from zero only if the following two conditions hold:

$$\sum_{i=2}^p |n_i| + |j| \leq \varepsilon \langle n_{p+1} \rangle, \quad \sigma_2 n_2 + \sigma_3 n_3 + \dots + \sigma_{p+1} n_{p+1} - n_{p+2} = -\sigma_1 n_1.$$

Hence there exist constants $c, C > 0$ such that

$$c \max(\langle n_2 \rangle, \dots, \langle n_{p+1} \rangle, \langle n_{p+2} \rangle) \leq \max_2(\langle n_2 \rangle, \dots, \langle n_{p+1} \rangle, \langle n_{p+2} \rangle) \leq C \max(\langle n_2 \rangle, \dots, \langle n_{p+1} \rangle, \langle n_{p+2} \rangle),$$

which, together with (6.3), proves that L satisfies the estimate (2.17) for any $\rho > 0$. \square

In the following lemma we prove that a para-differential operator $Op^{\text{BW}}(a(u; x, \xi))$ equals to the truncated one $Op^{\text{BW}}(a(S_\xi u; x, \xi))$ modulo smoothing remainders.

Lemma 6.2. *Fix $p \in \mathbb{N}$, $m \in \mathbb{R}$, $r > 0$, any $\rho > 0$ and consider $a(u; \xi) \in \Gamma_p^m[r]$ and set*

$$R(u) := Op^{\text{BW}}(a(u; x, \xi) - a(S_\xi u; x, \xi)).$$

Then $R(u)$ belongs to the class $\mathcal{R}_p^{-\rho}[r]$.

Proof. Let $v \in H^s$, expanding we obtain

$$R(u)v = \sum_{k \in \mathbb{Z}} e^{ikx} \sum_{j: |k-j| \leq \varepsilon|j|} (a(u; j) - a(S_j u; j))_{k-j} v_j.$$

Setting $R_j u = u - S_j u$ there exists $\sigma \in (0, 1)$ such that

$$\begin{aligned} R(u)v &= \sum_{k \in \mathbb{Z}} e^{ikx} \sum_{j: |k-j| \leq \varepsilon|j|} (d_u a(u - \sigma R_j u) R_j u)_{k-j} v_j \\ &= \sum_{k \in \mathbb{Z}} e^{ikx} \sum_{j: |k-j| \leq \varepsilon|j|} \sum_{j': |j'| > \varepsilon|j|} (d_u a(u - \sigma R_j u))_{k-j-j'} (R_j u)_{j'} v_j. \end{aligned}$$

Therefore

$$\begin{aligned} \|R(u)v\|_{H^{s+\rho}}^2 &\leq \sum_{k \in \mathbb{Z}} \left(\sum_{j: |k-j| \leq \varepsilon|j|} \sum_{j': |j'| > \varepsilon|j|} \left| (d_u a(u - \sigma R_j u))_{k-j-j'} (R_j u)_{j'} v_j \right| \langle k \rangle^{s+\rho} \right)^2 \\ &\leq C \sum_{k \in \mathbb{Z}} \left(\sum_{j: |k-j| \leq \varepsilon|j|} \sum_{j': |j'| > \varepsilon|j|} \left| (d_u a(u - \sigma R_j u))_{k-j-j'} \right| |(R_j u)_{j'}| \langle j' \rangle^\rho |v_j| \langle j \rangle^s \right)^2 \\ &\leq C \|d_u a(u - \sigma R_j u)\|_{H^{s_0}}^2 \|u\|_{H^{s_0+\rho}}^2 \|v\|_{H^s}^2, \end{aligned}$$

for any $s_0 > 1/2$, $C > 0$ depending on s . This implies the bound (2.20). The bound on the differential in u follows by (2.11) on the symbol a and reasoning similarly. This concludes the proof. \square

We leave to the reader the proof of the following lemma which is similar to the one of Lemma 6.2.

Lemma 6.3. *Fix $p \in \mathbb{N}$, $m \in \mathbb{R}$, $r > 0$, any $\rho > 0$ and consider $a(u; \xi) \in \tilde{\Gamma}_p^m$ and set*

$$R(u) := Op^{\text{BW}}(a(u; x, \xi) - a(S_\xi u; x, \xi)).$$

Then $R(u)$ belongs to the class $\tilde{\mathcal{R}}_p^{-\rho}$.

In the following proposition we give an explicit structure of the Hamiltonian vector field of the Hamiltonian function in (6.2) as a sum of a para-differential operator plus a smoothing term.

Proposition 6.4. *Fix $p \in \mathbb{N}$, $m \in \mathbb{R}$, $r > 0$, any $\rho > 0$ and let $A(U; x, \xi) \in \Sigma \Gamma_p^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ of the form (2.36), consider the Hamiltonian function $H(U)$ in (6.2). Then there exist $R \in \Sigma \mathcal{R}_p^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ such that*

$$X_H(U) = iJ\nabla H(U) = iEOp^{\text{BW}}(A(U; x, \xi))U + R(U)U.$$

Proof. By using (6.2) and (2.36) we have the following

$$\begin{aligned} H(U) = H(u, \bar{u}) &= \frac{1}{2} \int_{\mathbb{T}} \left(Op^{\text{BW}}(a(S_\xi(u, \bar{u}); x, \xi))u \cdot \bar{u} + Op^{\text{BW}}(b(S_\xi(u, \bar{u}); x, \xi))\bar{u} \cdot \bar{u} \right. \\ &\quad \left. + Op^{\text{BW}}(\bar{b}(S_\xi(u, \bar{u}); x, -\xi))u \cdot \bar{u} + Op^{\text{BW}}(\bar{a}(S_\xi(u, \bar{u}); x, -\xi))\bar{u} \cdot u \right) dx. \end{aligned}$$

We prove the theorem for the first addendum, one can deal with the others in the same way and deduce the thesis. Since $a(U; x, \xi) \in \Sigma \Gamma_p^m[r, N]$ we may decompose it as a sum of multilinear terms in $\tilde{\Gamma}_j^m$ and a non homogeneous one in $\Gamma_N^m[r]$. We have

$$\partial_{\bar{u}} \left(\int_{\mathbb{T}} Op^{\text{BW}}(a(S_\xi(u, \bar{u}); x, \xi)) u \cdot \bar{u} dx \right) = Op^{\text{BW}}(a(S_\xi(u, \bar{u}); x, \xi)) u + F^*[\bar{u}],$$

where F^* is the adjoint of the linear operator $F : h \mapsto Op^{\text{BW}}(d_{\bar{u}}a(S_\xi(u, \bar{u}))h)u$. Thanks to Lemma 6.2 it is enough to prove that the addendum $F^*[\bar{u}]$ is a smoothing remainder in the class $\Sigma \mathcal{R}_p^{-\rho}[r, N]$. The multilinear counterpart of this claim is true thanks to Lemma 6.1. In the following we prove that the non homogeneous part of $F^*[\bar{u}]$ is a smoothing remainder. With abuse of notation let a be in $\Gamma_N^m[r]$, we have

$$\begin{aligned} \int_{\mathbb{T}} Op^{\text{BW}}([\partial_u a(S_j u)] S_j h) u \cdot \bar{u} dx &= \sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} (\partial_u a(S_j u) S_j h)_{k-j} u_j \right) (\bar{u}_{-k}) \\ &= \sum_{k \in \mathbb{Z}} \sum_{|k-j| \leq \varepsilon |j|} \sum_{|j'| \leq \varepsilon |j|} (\partial_u a(S_j u))_{k-j-j'} (h)_{j'} u_j (\bar{u})_{-k} \\ &= \sum_{j' \in \mathbb{Z}} h_{j'} \sum_{\varepsilon |j| > |j'|} \sum_{|k-j| \leq \varepsilon |j|} (\partial_u a(S_j u))_{k-j-j'} u_j (\bar{u})_{-k} \\ &:= \sum_{j'} h_{j'} (f)_{-j'}. \end{aligned}$$

Therefore we can continue as follows

$$\begin{aligned} \|f\|_{H^{s+\rho}}^2 &\leq \sum_{m \in \mathbb{Z}} \left(\sum_{|k-j| \leq \varepsilon |j|} \sum_{|m| \leq \varepsilon |j|} |(d_u a(S_j u))_{k-j+m}| |u_j| |(\bar{u})_{-k}| \langle m \rangle^{s+\rho} \right)^2 \\ &\leq \sum_{m \in \mathbb{Z}} \left(\sum_{|k-j| \leq \varepsilon |j|} \sum_{j \in \mathbb{Z}} |(d_u a(S_j u))_{k-j+m}| |u_j| \langle j \rangle^s |(\bar{u})_{-k}| \langle m \rangle^\rho \right)^2 \\ &\leq \sum_{m \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} |(d_u a(S_j u))_{k-j+m}| |u_j| \langle j \rangle^s |(\bar{u})_{-k}| \langle k \rangle^\rho \right)^2 \\ &\leq \|d_u a(S_\xi u)\|_{H^{s_0}} \|u\|_{H^s} \|\bar{u}\|_{H^{s_0+\rho}}. \end{aligned}$$

As done in Lemma 6.2 one concludes that f is a smoothing remainder hence the result follows. \square

Symplectic Flows. In this section we study the symplectic corrections of the flows considered in Section 2.5.2. Consider a symbol $f(\tau, U; x, \xi)$ as in (2.52), (2.53), (2.54), (2.55) a symbol $g(\tau, U; x, \xi) \in \Sigma \Gamma_1^m[r, N]$, $m \leq 0$, and assume that

$$f(\tau, U; x, \xi) = \overline{f(\tau, U; x, \xi)}, \quad g(\tau, U; x, -\xi) = g(\tau, U; x, \xi).$$

We also assume that the symbols $f(\tau, U; x, \xi)$, $g(\tau, U; x, \xi)$ satisfy the estimates (2.8)-(2.11) uniformly in $\tau \in [0, 1]$. Let us define the operator

$$\mathcal{G}(U)[\cdot] := Op^{\text{BW}} \left(\frac{f(\tau, U; x, \xi)}{g(\tau, U; x, -\xi)} \frac{g(\tau, U; x, \xi)}{f(\tau, U; x, -\xi)} \right) [\cdot], \quad (6.4)$$

and the Hamiltonian function $G(U) : H^s(\mathbb{T}; \mathbb{C}) \cap \mathcal{U} \rightarrow \mathbb{R}$ defined as

$$G(U) := \int_{\mathbb{T}} \mathcal{G}(S_\xi U)[U] \cdot \bar{U} dx, \quad (6.5)$$

where S_ξ is defined in (6.1). We have the following result.

Proposition 6.5. (Symplectic flow). *Let us define Ψ_G^τ , $\tau \in [0, 1]$, the flow generated by the Hamiltonian G in (6.5). We have that $U(\tau) := \Psi_G^\tau(U_0)$, $U_0 \in H^s(\mathbb{T}; \mathbb{C}^2) \cap \mathcal{U}$, solves the problem*

$$\partial_\tau U = iJ\nabla G(U) = iE\mathcal{G}(U)[U] + \mathcal{R}(U)U, \quad U(0) = U_0, \quad (6.6)$$

where $\mathcal{R} \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. The map Ψ_G^τ satisfies estimates like (2.57)-(2.59) and it is symplectic, i.e.

$$\lambda(\Psi_G^\tau(U), \Psi_G^\tau(V)) = \lambda(U, V), \quad \forall U, V \in \mathcal{U},$$

where $\lambda(\cdot, \cdot)$ is the symplectic form in (1.19).

Proof. Notice that the operator $\mathcal{G}(U)$ in (6.4) is self-adjoint according to Definition 2.29. Hence formula (6.6) follows by Proposition 6.4. The flow Ψ_G^τ is well-posed by Theorem 2.31. Moreover it is symplectic since it is the flow on an Hamiltonian vector field. \square

Proof of Theorem 3.3 . In order to prove Theorem 3.1 we apply iteratively Theorems 5.3, 5.4, 5.1 and 5.2. The maps provided by such results are constructed as flows of para-differential vector fields. In order to obtain a symplectic correction to such maps one can reason as follows. Instead of considering the flows generated by (5.62), (5.79), (5.11) and (5.46) one can consider their symplectic corrections generated by Hamiltonian functions of the form (6.5). By Proposition 6.5 the maps constructed in this way are symplectic. Moreover the generators of such flows have the form (6.6). In other words the generator of the symplectic correction to the maps (5.62), (5.79), (5.11) and (5.46) has the same form up to a smoothing remainder. Therefore the results of Theorems 5.3, 5.4, 5.1 and 5.2 hold true with some very mild modifications of their proofs.

In order to prove the self-adjointness of the operator $\mathcal{L}(Z)$ is it sufficient to show that the symbol $m(Z; \xi)$ is real valued. We need to enter the proofs of the results in section 5.

First of all, since $X(U)$ in (3.4) is Hamiltonian, then the matrix $A(U; x, \xi)$ in (3.5) is self-adjoint, i.e. it satisfies (2.40). In giving the proof of Theorem 5.3 with this additional assumption on $A(U; x, \xi)$ it is evident that the matrix $A^+(Z; x, \xi)$ in (5.64) still satisfies (2.40). This can be deduced from equations (5.67)-(5.69) and (5.73), (5.74). This gives the self-adjointness at the highest order m . One can check this property at lower orders by using equations (5.75), (5.76). At this point rename $A^+ = A$ and apply Theorem 5.4. In the same way one deduces the self-adjointness of the matrix A^+ in (5.81) by equations (5.84)-(5.86) and (5.89), (5.90), at the highest order. Similarly for the lower orders.

One has to repeat this check in the proofs of Theorems 5.1, 5.2. \square

7. POINCARÉ-BIRKHOFF NORMAL FORMS

The proof of the Theorem 3.7 is based on the iterative procedure which is performed in the following subsections.

7.1. Abstract conjugation results. In this section we provide a conjugation result of a vector field of the form (3.12) under the flow generated either by Fourier multipliers or by smoothing reminders. We shall we consider the system

$$\dot{U} = \mathcal{X}(U) = iE\Omega U + iEOp^{\text{BW}}(\mathfrak{N}(U; \xi))U + \mathcal{R}(U)[U], \quad U(0) = U_0, \quad (7.1)$$

where Ω is in (3.2), $\mathcal{R} \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$, $\mathfrak{N}(Z; \xi) \in \Sigma \Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is independent of $x \in \mathbb{T}$ and has the form

$$\mathfrak{N}(U; \xi) := \begin{bmatrix} n(U; \xi) & 0 \\ 0 & n(U; -\xi) \end{bmatrix},$$

for some $n \in \Sigma \Gamma_1^m[r, N]$ real valued.

7.1.1. *Flows of Fourier multipliers.* Fix $p \in \mathbb{N}$ and consider the matrix of symbols

$$B_p(U; \xi) := \begin{pmatrix} b_p(U; \xi) & 0 \\ 0 & b_p(U; -\xi) \end{pmatrix}, \quad b_p(U; \xi) \in \tilde{\Gamma}_p^m, \quad (7.2)$$

where b_p is real, independent of $x \in \mathbb{T}$ and admits the expansion (2.16). Consider the Hamiltonian function (recall (6.1))

$$\mathcal{G}(U) := \frac{1}{2} \int_{\mathbb{T}} Op^{\text{BW}}(B_p(S_\xi U; \xi)) U \cdot \bar{U} dx. \quad (7.3)$$

We are using the abuse of notation $b_p(U; \xi) := b_p(U, \dots, U; \xi)$. Let $\tilde{\mathcal{A}}_p^\tau$ be the solution of

$$\begin{cases} \partial_\tau \tilde{\mathcal{A}}_p^\tau(U) = X_{\mathcal{G}}(\tilde{\mathcal{A}}_p^\tau(U)) \\ \tilde{\mathcal{A}}_p^0(U) = U, \end{cases} \quad (7.4)$$

where $X_{\mathcal{G}}$ is the Hamiltonian vector field of (7.3) and has the form (see Proposition 6.4 and Lemma 6.3)

$$X_{\mathcal{G}}(U) = iEOp^{\text{BW}}(B_p(U; \xi))U + \mathcal{B}_p(U)U, \quad \mathcal{B}_p \in \tilde{\mathcal{R}}_p^{-\rho}. \quad (7.5)$$

We prove the following.

Proposition 7.1. *For $r > 0$ small enough the following holds. Setting (recall (3.6), (7.1))*

$$W := \tilde{\mathcal{A}}_p(U) := \tilde{\mathcal{A}}_p^1(U), \quad \mathcal{X}^+(W) := P^\tau(W)|_{\tau=1} := d\tilde{\mathcal{A}}_p^\tau(\tilde{\mathcal{A}}_p^{-\tau}(W))[\mathcal{X}(\tilde{\mathcal{A}}_p^{-\tau}(W))]|_{\tau=1}, \quad (7.6)$$

we have that

$$\begin{cases} \dot{W} = \mathcal{X}^+(W) := iE\Omega W + iEOp^{\text{BW}}(\mathfrak{N}^+(W; \xi))[W] + \mathcal{R}^+(W)[W] \\ W(0) = \mathcal{A}(Z_0) \end{cases} \quad (7.7)$$

where $\mathcal{R}^+ \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathfrak{N}^+ \in \Sigma \Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is independent of $x \in \mathbb{T}$, real valued and has the form

$$\begin{aligned} \mathfrak{N}^+(W; \xi) &:= \begin{pmatrix} \mathfrak{n}^+(W; \xi) & 0 \\ 0 & \mathfrak{n}^+(W; -\xi) \end{pmatrix}, \\ \mathfrak{n}^+(W; \xi) &= \mathfrak{n}(\tilde{\mathcal{A}}_p^{-1}(W); \xi) + \int_0^1 (d_Z b_p)(\tilde{\mathcal{A}}_p^\sigma \tilde{\mathcal{A}}_p^{-1}(W); \xi) [P^\sigma(\tilde{\mathcal{A}}_p^\sigma \tilde{\mathcal{A}}_p^{-1}(W))] d\sigma. \end{aligned} \quad (7.8)$$

Moreover, for any $s \geq s_0$, the maps $\tilde{\mathcal{A}}_p^{\pm 1}$ are symplectic and satisfy

$$\|\tilde{\mathcal{A}}_p^{\pm 1}(U)\|_{H^s} \leq \|U\|_{H^s} (1 + C\|U\|_{H^{s_0}}^p), \quad (7.9)$$

for some constant $C > 0$ depending on s .

Proof. The estimates (7.9) on the flow of (7.4) follow by Theorem 2.31. The vector field \mathcal{X} in (7.1) in the new coordinates W has the form (7.6). In particular P^τ satisfies, for $\tau \in [0, 1]$, (recall (7.1), (7.4))

$$\begin{cases} \partial_\tau P^\tau(W) = [X_{\mathcal{G}}(W), P^\tau(W)] \\ P^0(W) = \mathcal{X}(W). \end{cases} \quad (7.10)$$

Moreover, by the remarks under Definition 2.15 and by Theorem 2.31, we note that

$$P^\tau(W) = iE\Omega W + M_1(\tau; W)[W], \quad M_1 \in \Sigma \mathcal{M}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \quad (7.11)$$

with estimate uniform in $\tau \in [0, 1]$. Actually we shall prove that

$$P^\tau(W) = iE\Omega W + iEOp^{\text{BW}}(\mathfrak{N}^+(\tau, W; \xi))[W] + \mathcal{R}^+(\tau, W)[W],$$

with \mathfrak{N}^+ , \mathcal{R}^+ as follows

$$\mathfrak{N}^+(\tau, W; \xi) := \begin{pmatrix} \mathfrak{n}^+(\tau, W; \xi) & 0 \\ 0 & \mathfrak{n}^+(\tau, W; -\xi) \end{pmatrix}, \quad \mathcal{R}^+ \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}). \quad (7.12)$$

In particular we make the ansatz that $\mathfrak{n}^+(\tau, W; \xi)$ is real valued and x -independent. By expanding the non linear commutator as in (5.23)-(5.28), and using that \mathfrak{N}^+, B_p are independent of x we deduce (recall (7.5))

$$[X_{\mathcal{G}}(W), P^\tau(W)] = Op^{\text{BW}}(iE(d_W B_p)(W; \xi)[P^\tau(W)])[W] \quad (7.13)$$

$$- Op^{\text{BW}}(iE(d_W \mathfrak{N}^+)(\tau, W; \xi)[X_{\mathcal{G}}(W)])[W] \quad (7.14)$$

$$+ Op^{\text{BW}}(iE B_p(\tau, W; \xi))[\mathcal{R}^+(\tau, W)[W]] + \mathcal{B}_p(W)\mathcal{R}^+(\tau, W)[W] \quad (7.15)$$

$$- \mathcal{R}^+(\tau, W)[X_{\mathcal{G}}(W)] - (d_W \mathcal{R}^+)(\tau, W)[X_{\mathcal{G}}(W)]. \quad (7.16)$$

Notice that, by (7.11) (recall also (7.2)), we have

$$(\partial_t b_p)(W; \xi) := (d_W b_p)(W; \xi)[P^\tau(W)] \in \Sigma \Gamma_p^m[r, N]. \quad (7.17)$$

Order m . By (7.13)-(7.16) we have that, at the highest order, the equation (7.10) reads

$$\begin{cases} \partial_\tau \mathfrak{n}^+(\tau, W; \xi) = (\partial_t b_p)(W; \xi) - d_W \mathfrak{n}^+(\tau, W; \xi)[X_{\mathcal{G}}(W)] \\ \mathfrak{n}^+(0, W; \xi) = \mathfrak{n}(W; \xi). \end{cases} \quad (7.18)$$

Notice that, if \mathfrak{n}^+ solves (7.18), the function

$$g(\tau) = \mathfrak{n}^+(\tau, W(\tau); \xi)$$

where $W(\tau) = \tilde{\mathcal{A}}_p^\tau(U)$ in (7.4), satisfies

$$\partial_\tau g(\tau) = (\partial_t b_p)(W(\tau); \xi), \quad g(0) := \mathfrak{n}(Z; \xi), \quad \Rightarrow \quad g(\tau) = g(0) + \int_0^\tau (\partial_t b_p)(W(\sigma); \xi) d\sigma.$$

Therefore

$$\mathfrak{n}^+(\tau, W; \xi) := \mathfrak{n}(\tilde{\mathcal{A}}_p^{-\tau}(W); \xi) + \int_0^\tau (\partial_t b_p)(\tilde{\mathcal{A}}_p^\sigma \tilde{\mathcal{A}}_p^{-\tau}(W); \xi) d\sigma, \quad (7.19)$$

solves the problem (7.18). Reasoning as in Section 4 (see for instance the proof of Lemma 4.4) using the estimates of Theorem 2.31, one deduces that $\mathfrak{n}^+(\tau, W; \xi)$ belongs to $\Sigma \Gamma_1^m[r, N]$. By (7.19), (7.17) one gets the (7.8).

Define

$$Q^\tau(W) = iE\Omega W + Op^{\text{BW}}(iE\mathfrak{N}^+(\tau, W; \xi))[W]$$

with \mathfrak{N}^+ of the form (7.12). Then the operator $Q^\tau(W)$ solves the problem (recall (7.4), (7.13)-(7.16), (7.18))

$$\begin{cases} \partial_\tau Q^\tau(W) = [X_{\mathcal{G}}(W), Q^\tau(W)] \\ Q^0(W) = iE\Omega W + iE Op^{\text{BW}}(\mathfrak{N}(W; \xi))[W]. \end{cases} \quad (7.20)$$

It remains to prove that the difference $Q^\tau - P^\tau$ is a smoothing remainder in $\Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. First of all we write

$$Q^\tau(W) - P^\tau(W) = V^\tau \circ (\tilde{\mathcal{A}}_p^\tau)^{-1}(W),$$

where, recalling (7.6),

$$V^\tau(U) := Q^\tau \circ \tilde{\mathcal{A}}_p^\tau(U) - (d_U \tilde{\mathcal{A}}_p^\tau)(U)[X(U)].$$

Reasoning as for the operator in (5.41) we deduce (see (7.1))

$$\begin{cases} \partial_\tau V^\tau(U) = (d_{W(\tau)} X_{\mathcal{G}})(\tilde{\mathcal{A}}_p^\tau)[V^\tau], \\ V^0(U) = -\mathcal{R}(U)U. \end{cases} \quad (7.21)$$

Reasoning as done for (5.42) one can check $V^\tau \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. This implies that $\mathcal{R}^+(\tau, W)[W] := Q^\tau(W) - P^\tau(W)$ (again using Theorem 2.31) belongs to $\Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. Hence we have the (7.7). \square

7.1.2. *Flows of smoothing remainders.* In this section we consider the flow \mathcal{A}_p^τ of

$$\begin{cases} \partial_\tau \mathcal{A}_p^\tau(U) = \mathcal{Q}_{aux}^{(p)}(\mathcal{A}_p^\tau(U))\mathcal{A}_p^\tau(U) \\ \mathcal{A}_p^0(U) = U, \end{cases} \quad (7.22)$$

where $\mathcal{Q}_{aux}^{(p)} \in \widetilde{\mathcal{R}}_p^{-\rho}$, this problem is well-posed by standard theory of ODEs on Banach spaces. Assume also that $\mathcal{Q}_{aux}^{(p)}(U)U$ is Hamiltonian, i.e. there is a map $Q^{(p)} \in \widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$ such that (recall (1.18))

$$\mathcal{Q}_{aux}^{(p)}(U)U = X_{\mathcal{C}}(U), \quad \mathcal{C}(U) := \int_{\mathbb{T}} Q^{(p)}(U)U \cdot \bar{U} dx. \quad (7.23)$$

Recalling the system (7.1) we define

$$W := \mathcal{A}_p(U) := \mathcal{A}_p^1(U), \quad \mathcal{X}^+(W) := P^\tau(W)|_{\tau=1} := d\mathcal{A}_p^\tau(\mathcal{A}_p^{-\tau}(W))[\mathcal{X}(\mathcal{A}_p^{-\tau}(W))]|_{\tau=1}. \quad (7.24)$$

We prove the following.

Proposition 7.2. *For $r > 0$ small enough the following holds. The function W in (7.24) satisfies*

$$\begin{cases} \dot{W} = \mathcal{X}^+(W) := iE\Omega W + iEOp^{\text{BW}}(\mathfrak{N}^+(W; \xi))[W] + \mathcal{R}^+(W)[W] \\ W(0) = \mathcal{A}(Z_0) \end{cases} \quad (7.25)$$

where $\mathcal{R}^+ \in \Sigma \widetilde{\mathcal{R}}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathfrak{N}^+ \in \Sigma \Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is independent of $x \in \mathbb{T}$, real valued and it has the form

$$\mathfrak{N}^+(W; \xi) := \begin{pmatrix} \mathfrak{n}^+(W; \xi) & 0 \\ 0 & \mathfrak{n}^+(W; -\xi) \end{pmatrix}, \quad \mathfrak{n}^+(\tau, W; \xi) := \mathfrak{n}(\mathcal{A}_p^{-\tau}(W); \xi). \quad (7.26)$$

Moreover, for any $s \geq s_0$, the maps $\mathcal{A}_p^{\pm 1}$ are symplectic and satisfy

$$\|\mathcal{A}_p^{\pm 1}(U)\|_{H^s} \leq \|U\|_{H^s} (1 + C\|U\|_{H^{s_0}}^p), \quad (7.27)$$

for some constant $C > 0$ depending on s .

Proof. The field P^τ in (7.24) satisfies, for $\tau \in [0, 1]$, (recall (7.1))

$$\begin{cases} \partial_\tau P^\tau(W) = [\mathcal{Q}_{aux}^{(p)}(W)W, P^\tau(W)] \\ P^0(W) = \mathcal{X}(W). \end{cases} \quad (7.28)$$

Moreover, by the remarks under Definition 2.15 and by Theorem 2.31, we note that

$$P^\tau(W) = iE\Omega W + M_1(\tau; W)[W], \quad M_1 \in \Sigma \mathcal{M}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \quad (7.29)$$

with estimate uniform in $\tau \in [0, 1]$. Actually we shall prove that

$$P^\tau(W) = iE\Omega W + iEOp^{\text{BW}}(\mathfrak{N}^+(\tau, W; \xi))[W] + \mathcal{R}^+(\tau, W)[W],$$

with $\mathfrak{N}^+ \in \Sigma \Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$, $\mathcal{Q}^+ \in \Sigma \widetilde{\mathcal{R}}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and

$$\mathfrak{N}^+(\tau, W; \xi) := \begin{pmatrix} \mathfrak{n}^+(\tau, W; \xi) & 0 \\ 0 & \mathfrak{n}^+(\tau, W; -\xi) \end{pmatrix}. \quad (7.30)$$

Furthermore we make the ansatz that $\mathfrak{n}^+(\tau, W; \xi)$ is real valued and x -independent. By expanding the non linear commutator as in (5.23)-(5.28), we deduce

$$[\mathcal{Q}_{aux}^{(p)}(W)W, P^\tau(W)] = \mathcal{Q}_{aux}^{(p)}(W)[P^\tau(W)] + (d_W \mathcal{Q}_{aux}^{(p)}(W)W)[P^\tau(W)] \quad (7.31)$$

$$- \mathcal{R}^+(W)[\mathcal{Q}_{aux}^{(p)}(W)W] - (d_W \mathcal{R}^+(W)W)[\mathcal{Q}_{aux}^{(p)}(W)W] \quad (7.32)$$

$$- Op^{\text{BW}}(\mathfrak{N}^+(\tau, W; \xi))[\mathcal{Q}_{aux}^{(p)}(W)W] \quad (7.33)$$

$$- Op^{\text{BW}}((d_W \mathfrak{N}^+(\tau, W; \xi))[\mathcal{Q}_{aux}^{(p)}(W)W])[W]. \quad (7.34)$$

Notice that, by (7.29), we have

$$\frac{d}{dt}(\mathcal{Q}_{aux}^{(p)}(W)W) = \mathcal{Q}_{aux}^{(p)}(W)[P^\tau(W)] + (d_W \mathcal{Q}_{aux}^{(p)}(W)W)[P^\tau(W)] \in \Sigma \mathcal{R}_1^{-\rho}[r, N]. \quad (7.35)$$

Order m . By (7.31)-(7.35) we have that, at the highest order, the equation (7.28) reads (recall (7.1), (7.30))

$$\begin{cases} \partial_\tau \mathfrak{n}^+(\tau, W; \xi) = -d_Z \mathfrak{n}^+(\tau, W; \xi) [\mathcal{Q}_{aux}^{(p)}(W)W] \\ \mathfrak{n}^+(0, W; \xi) = \mathfrak{n}(W; \xi). \end{cases} \quad (7.36)$$

Notice that, if \mathfrak{n}^+ solves (7.36), the function $g(\tau) = \mathfrak{n}^+(\tau, W(\tau); \xi)$ is constant along the solution of (7.22). Therefore

$$\mathfrak{n}^+(\tau, W; \xi) := \mathfrak{n}(\mathcal{A}_p^{-\tau}(W); \xi), \quad (7.37)$$

solves the problem (7.36). Reasoning as in Section 4 (see for instance the proof of Lemma 4.4) using the estimates of Theorem 2.31, one deduces that $\mathfrak{n}^+(\tau, W; \xi)$ belongs to $\Sigma \Gamma_1^m[r, N]$. This proves the (7.26).

Let us now define

$$Q^\tau(W) = iE\Omega W + Op^{\text{BW}}(iE\mathfrak{N}^+(\tau, W; \xi))[W]$$

with \mathfrak{N}^+ of the form (7.30), (7.37). Then the operator $Q^\tau(W)$ solves the problem (recall (7.22), (7.31)-(7.34), (7.36))

$$\begin{cases} \partial_\tau Q^\tau(W) = [\mathcal{Q}_{aux}^{(p)}(W)W, Q^\tau(W)] + \mathcal{G}_\rho(W)W \\ Q^0(W) = iE\Omega W + iEOp^{\text{BW}}(\mathfrak{N}(W; \xi))[W], \end{cases} \quad (7.38)$$

for some $\mathcal{G}_\rho \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. It remains to prove that the difference $Q^\tau - P^\tau$ is a smoothing remainder in $\Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. First of all we write

$$Q^\tau(W) - P^\tau(W) = V^\tau \circ (\mathcal{A}_p^\tau)^{-1}(W),$$

where, recalling (7.24),

$$V^\tau(U) := Q^\tau \circ \mathcal{A}_1^\tau(U) - (d_U \mathcal{A}_1^\tau)(U)[X(U)].$$

Reasoning as for the operator in (5.41) we deduce (see (7.1))

$$\begin{cases} \partial_\tau V^\tau(Z) = (d_Z \mathcal{Q}_{aux})(\mathcal{A}_p^\tau)[V^\tau], & \mathcal{Q}_{aux}(Z) := \mathcal{Q}_{aux}^{(p)}(Z)Z, \\ V^0(Z) = -\mathcal{R}(Z)Z. \end{cases} \quad (7.39)$$

Reasoning as done for (5.42) one can check that $V^\tau \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. This implies that the remainder $\mathcal{R}^+(\tau, W)[W] := Q^\tau(W) - P^\tau(W)$ (again using Theorem 2.31) belongs to $\Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. Hence, by the discussion above, we have obtained the (7.25). \square

7.2. Elimination of quadratic terms. Consider the system (3.12). By hypothesis we have (see also (3.8))

$$\begin{aligned} \mathfrak{M}(Z; \xi) &= \mathfrak{M}_1(Z; \xi) + \mathfrak{M}_{\geq 2}(Z; \xi), & \mathfrak{M}_1(Z; \xi) &\in \tilde{\Gamma}_1^m \otimes \mathcal{M}_2(\mathbb{C}), & \mathfrak{M}_{\geq 2}(Z; \xi) &\in \Sigma \Gamma_2^m[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \\ \mathcal{Q}(Z) &= \mathcal{Q}_1(Z) + \mathcal{Q}_{\geq 2}(Z), & \mathcal{Q}_1(Z) &\in \tilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C}) & \mathcal{Q}_{\geq 2}(Z) &\in \Sigma \mathcal{R}_2^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}). \end{aligned}$$

The aim of this section is to eliminate the quadratic vector field

$$\mathcal{Y}_2(Z) := Op^{\text{BW}}(\mathfrak{M}_1(Z; \xi))Z + \mathcal{Q}_1(Z)Z,$$

in (3.12). This will be done into two steps. We shall first reduce the symbol $\mathfrak{M}_1(Z; x)$ (see section 7.2.1) and then the smoothing operator $\mathcal{Q}_1(Z)$ (see section 7.2.2). Before giving the proof we make a comment on the symbol $\mathfrak{m}_1(Z; \xi)$. By the expansion (2.16) (with $p = 1$) and the fact that $\mathfrak{m}_1(Z; \xi)$ is independent of $x \in \mathbb{T}$ we can write

$$\mathfrak{m}_1(U; \xi) = \sum_{j \in \mathbb{Z}} e^{ijx} u_j(\mathfrak{m}_1)_j^+(\xi) + e^{-ijx} \overline{u_j}(\mathfrak{m}_1)_j^-(\xi) = u_0(\mathfrak{m}_1)_0^+(\xi) + \overline{u_0}(\mathfrak{m}_1)_0^-(\xi). \quad (7.40)$$

Notice that $\mathfrak{m}_1(U; \xi)$ depends *only* on the average u_0 of u . We shall assume that (see (3.2)) $\omega_0 \neq 0$.

7.2.1. *Elimination of linear symbols.* Notice that the system (3.12) has the form (7.1) with $\mathfrak{N} \rightsquigarrow \mathfrak{M}$, $\mathcal{R} \rightsquigarrow \mathcal{Q}$, $U \rightsquigarrow Z$. Consider now a real valued, independent of $x \in \mathbb{T}$ symbol $b_1 \in \tilde{\Gamma}_1^m$ and the matrix $B_1(Z; \xi)$ having the form (7.2) with $p = 1$. Let $\tilde{\mathcal{A}}_1^\tau$ be the flow of (7.4) generated by the Hamiltonian (7.3) with $p = 1$. By Proposition 7.1 we have that the variable

$$W := \tilde{\mathcal{A}}_1(Z) := \tilde{\mathcal{A}}_1^1(Z), \quad \tilde{\mathcal{Y}}^{(1)}(W) := P^\tau(W)|_{\tau=1} := d\tilde{\mathcal{A}}_1^\tau(\tilde{\mathcal{A}}_1^{-\tau}(W))[\mathcal{Y}(\tilde{\mathcal{A}}_1^{-\tau}(W))]|_{\tau=1},$$

satisfies (recall (3.12), (7.7))

$$\begin{cases} \dot{W} = \tilde{\mathcal{Y}}^{(1)}(W) := iE\Omega W + iEOp^{\text{BW}}(\tilde{\mathfrak{M}}^{(1)}(W; \xi))[W] + \tilde{\mathcal{Q}}^{(1)}(W)[W] \\ W(0) = \tilde{\mathcal{A}}(Z_0) \end{cases} \quad (7.41)$$

where $\tilde{\mathcal{Q}}^{(1)} \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\tilde{\mathfrak{M}}^{(1)} \in \Sigma \Gamma_1^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is independent of $x \in \mathbb{T}$, real valued and has the form

$$\begin{aligned} \tilde{\mathfrak{M}}^{(1)}(W; \xi) &:= \begin{pmatrix} \tilde{\mathfrak{m}}^{(1)}(W; \xi) & 0 \\ 0 & \tilde{\mathfrak{m}}^{(1)}(W; -\xi) \end{pmatrix}, \\ \tilde{\mathfrak{m}}^{(1)}(W; \xi) &= \mathfrak{m}(\tilde{\mathcal{A}}_1^{-1}(W); \xi) + \int_0^1 (d_Z b_1)(\tilde{\mathcal{A}}_1^\sigma \tilde{\mathcal{A}}_1^{-1}(W); \xi) [P^\sigma(\tilde{\mathcal{A}}_1^\sigma \tilde{\mathcal{A}}_1^{-1}(W))] d\sigma. \end{aligned} \quad (7.42)$$

The homological equation. We look for a linear symbol $b_1 \in \tilde{\Gamma}_1^m$ such that the symbol $\tilde{\mathfrak{m}}^{(1)}(W; \xi)$ in (7.42) is at least quadratic in the variable W . In order to do this we reason as follows. First of all, by Theorem 2.31, we deduce that the flow $\tilde{\mathcal{A}}_1^\tau$ of (7.4) is such that

$$\tilde{\mathcal{A}}_1^\tau(Z) = Z + N_1(\tau, Z)[Z], \quad \tilde{\mathcal{A}}_1^{-\tau}(Z) = Z + N_2(\tau, Z)[Z], \quad N_1, N_2 \in \Sigma \mathcal{M}_1[r, N] \otimes \mathcal{M}_2(\mathbb{C}),$$

with estimates uniform in $\tau \in [0, 1]$. Recall also that, by hypothesis, one has

$$\mathfrak{m}(Z; \xi) = \mathfrak{m}_1(Z; \xi) + \mathfrak{m}_{\geq 2}(Z; \xi), \quad \mathfrak{m}_1 \in \tilde{\Gamma}_1^m, \quad \mathfrak{m}_{\geq 2} \in \Sigma \Gamma_2^m[r, N].$$

Therefore, using also (7.11) and the composition Proposition 2.27, we can write $\tilde{\mathfrak{m}}^{(1)}$ in (7.42) as

$$\begin{aligned} \tilde{\mathfrak{m}}^{(1)}(W; \xi) &= \tilde{\mathfrak{m}}_1^{(1)}(W; \xi) + \tilde{\mathfrak{m}}_{\geq 2}^{(1)}(W; \xi), \quad \tilde{\mathfrak{m}}_{\geq 2}^{(1)} \in \Sigma \Gamma_2^m[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \\ \tilde{\mathfrak{m}}_1^{(1)}(W; \xi) &:= \mathfrak{m}_1(W; \xi) + (d_W b_1)(W; \xi) [i\Omega W]. \end{aligned} \quad (7.43)$$

We prove the following.

Lemma 7.3. (Homological equation). *There exists a symbol $b_1 \in \tilde{\Gamma}_1^m$ such that (see (7.43))*

$$\tilde{\mathfrak{m}}_1^{(1)}(W; \xi) = 0. \quad (7.44)$$

Moreover b_1 is real valued.

Proof. We look for a symbol $b_1 \in \tilde{\Gamma}_1^m$ of the form

$$b_1(W; \xi) = u_0(b_1)_0^+(\xi) + \overline{u_0}(b_1)_0^-(\xi), \quad (7.45)$$

for some coefficients $(b_1)_0^\sigma$, $\sigma \in \{\pm\}$. Let us define (recall (7.40))

$$(b_1)_0^\sigma(\xi) := \frac{(m_1)_0^-(\xi)}{i\sigma\omega_0}, \quad \sigma \in \{\pm\}. \quad (7.46)$$

Using the expansion (2.16), with $p = 1$ (recall also (7.40)) one can check, by an explicit computation, that the function in (7.45) with coefficients in (7.46) solves the equation (7.44). Moreover, since \mathfrak{m}_1 is real valued, one has

$$\overline{(m_1)_0^+(\xi)} = (m_1)_0^-(\xi).$$

Hence, by (7.46), also the coefficients $(b_1)_0^\sigma(\xi)$ have the same property. This implies that b_1 is real valued. \square

7.2.2. *Elimination of linear smoothing operators.* Consider the system (7.41) obtained from the change of coordinates generated (see (7.3), (7.4)) by b_1 given by the Lemma 7.3. We now consider the flow \mathcal{A}_1^τ of

$$\begin{cases} \partial_\tau \mathcal{A}_1^\tau(W) = \mathcal{Q}_{aux}^{(1)}(\mathcal{A}_1^\tau(W))\mathcal{A}_1^\tau(W) \\ \mathcal{A}_1^0(W) = W, \end{cases} \quad (7.47)$$

where $\mathcal{Q}_{aux}^{(1)} \in \tilde{\mathcal{R}}_1^{-\rho}$ which is well-posed by Theorem 2.31. Assume that $\mathcal{Q}_{aux}^{(1)}$ has the form (2.21), (2.22) with $p = 1$ and coefficients (see (2.23))

$$(\mathcal{Q}_{aux}^{(1)}(W_1))_{\sigma,j}^{\sigma',k} = \sum_{\substack{\sigma_1 \in \{\pm\}, j_1 \in \mathbb{Z} \\ \sigma_1 j_1 = \sigma j - \sigma' k}} ((q_{aux}^{(1)})_{j_1}^{\sigma_1})_{\sigma,j}^{\sigma',k} W_{j_1}^{\sigma_1}, \quad ((q_{aux}^{(1)})_{j_1}^{\sigma_1})_{\sigma,j}^{\sigma',k} \in \mathbb{C}. \quad (7.48)$$

Finally assume that $\mathcal{Q}_{aux}^{(1)}$ is Hamiltonian, see (7.23). We define

$$W_1 := \mathcal{A}_1(W) := \mathcal{A}_1^1(W), \quad \mathcal{Y}^{(1)}(W_1) := P^\tau(W_1)|_{\tau=1} := d\mathcal{A}_1^\tau(\mathcal{A}_1^{-\tau}(W_1))[\tilde{\mathcal{Y}}^{(1)}(\mathcal{A}_1^{-\tau}(W_1))]_{|\tau=1}. \quad (7.49)$$

In the following lemma we show that it is possible to choose the coefficients $((q_{aux}^{(1)})_{j_1}^{\sigma_1})_{\sigma,j}^{\sigma',k}$ in (7.48) in such a way the vector field in (7.49) does not contain any *quadratic* monomials.

Lemma 7.4. *For $r > 0$ small enough the following holds. There exists $\mathcal{Q}_{aux}^{(1)} \in \tilde{\mathcal{R}}_1^{-\rho}$, of the form (7.48), such that the function W_1 in (7.49) satisfies*

$$\begin{cases} \dot{W}_1 = \mathcal{Y}^{(1)}(W) := iE\Omega W_1 + iEOp^{\text{BW}}(\mathfrak{M}^{(1)}(W_1; \xi))[W_1] + \mathcal{Q}^{(1)}(W_1)[W_1] \\ W_1(0) = \mathcal{A}_1(W) \end{cases} \quad (7.50)$$

where $\mathcal{Q}^{(1)} \in \Sigma \mathcal{R}_2^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathfrak{M}^{(1)} \in \Sigma \Gamma_2^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is independent of $x \in \mathbb{T}$, real valued and has the form

$$\mathfrak{M}^{(1)}(W; \xi) := \begin{bmatrix} m^{(1)}(W; \xi) & 0 \\ 0 & m^{(1)}(W; -\xi) \end{bmatrix}, \quad (7.51)$$

Moreover, for any $s \geq s_0$, the maps $\mathcal{A}_1^{\pm 1}$ are symplectic and satisfy

$$\|\mathcal{A}_1^{\pm 1}(U)\|_{H^s} \leq \|U\|_{H^s} (1 + C\|U\|_{H^{s_0}}), \quad (7.52)$$

for some constant $C > 0$ depending on s .

Proof. We start by studying the conjugate of (7.41) under the flow (7.47) assuming that $\mathcal{Q}_{aux}^{(1)} \in \tilde{\mathcal{R}}_1^{-\rho}$. Notice that the vector field $\tilde{\mathcal{Y}}^{(1)}$ in (7.41) has the same form of \mathcal{X} in (7.1), with $\mathfrak{N} \rightsquigarrow \tilde{\mathfrak{M}}^{(1)}$, $\mathcal{R} \rightsquigarrow \tilde{\mathcal{Q}}^{(1)}$, $U \rightsquigarrow W$. The generator $\mathcal{Q}_{aux}^{(1)}$ has the same properties of the generator $\mathcal{Q}_{aux}^{(p)}$ in (7.22). Therefore Proposition 7.2 applies. As a consequence we obtain that

$$\dot{W}_1 = iE\Omega W_1 + iEOp^{\text{BW}}(\mathfrak{M}^{(1)}(W_1; \xi))[W_1] + \mathcal{Q}^{(1)}(W_1)[W_1], \quad (7.53)$$

where $\mathcal{Q}^{(1)} \in \Sigma \mathcal{R}_1^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$, $\mathfrak{M}^{(1)}$ has the form (7.51) with (see (7.26))

$$m^{(1)}(\tau, W_1; \xi) := \tilde{m}^{(1)}(\mathcal{A}_1^{-\tau}(W_1); \xi) \in \Sigma \Gamma_2^m[r, N].$$

The symbol above is quadratic since, by (7.43) and Lemma 7.3, the symbol $\tilde{m}^{(1)}(\tau, W; \xi)$ is at least quadratic. Moreover we have that

$$\mathcal{Q}^{(1)}(W_1)[W_1] = \mathcal{Q}_1^{(1)}(W_1)[W_1] + \mathcal{Q}_{\geq 2}^{(1)}(W_1)[W_1], \quad \mathcal{Q}_{\geq 2}^{(1)} \in \Sigma \mathcal{R}_2^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \quad (7.54)$$

with $\mathcal{Q}_1^{(1)} \in \tilde{\mathcal{R}}_1^{-\rho}$. We claim that there exists $\mathcal{Q}_{aux}^{(1)}$ of the form (7.48) such that

$$\mathcal{Q}_1^{(1)}(W_1)W_1 = 0. \quad (7.55)$$

The (7.53) and the (7.55) imply the (7.50).

In order to solve (7.55) we need a more explicit expression of the remainder $\mathcal{Q}_1^{(1)}$ in (7.54) in terms of the generator of the flow $\mathcal{Q}_{aux}^{(1)}$. This could be deduced by developing the computation in the proof

of Proposition 7.2. In particular by calculating explicitly the remainder \mathcal{G}_ρ in (7.38) and using equation (7.39). However the computation is quite involved. Then we reason as follows.

We remark that the flow \mathcal{A}_1^τ in (7.47) is C^k in $\tau \in [0, 1]$ with values in the Banach space $H^s(\mathbb{T}; \mathbb{C}^2)$, because it solves an ODE on $H^s(\mathbb{T}; \mathbb{C}^2)$. Then, by Taylor expanding P^τ in (7.49) at $\tau = 0$ (and using the Heisenberg equation (7.28)), we get

$$\mathcal{Y}^{(1)} = P^1(W_1) = \widetilde{\mathcal{Y}}^{(1)}(W_1) + [\mathcal{Q}_{aux}^{(1)}(W_1)W_1, \widetilde{\mathcal{Y}}^{(1)}(W_1)] + \int_0^1 (1-\sigma)\partial_\sigma^2 P^\sigma(W_1) d\sigma. \quad (7.56)$$

Recall (7.43) and that the operator $\widetilde{\mathcal{Q}}^{(1)}$ has the form

$$\widetilde{\mathcal{Q}}^{(1)}(W_1) = \widetilde{\mathcal{Q}}_1^{(1)}(W_1) + \widetilde{\mathcal{Q}}_{\geq 2}^{(1)}(W_1), \quad \widetilde{\mathcal{Q}}_{\geq 2}^{(1)} \in \Sigma \mathcal{R}_2^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}),$$

with $\widetilde{\mathcal{Q}}_1^{(1)} \in \widetilde{\mathcal{R}}_1^{-\rho}$. Then by (7.56) we deduce

$$\mathcal{Y}^{(1)}(W_1) = i\Omega W_1 + \underbrace{\widetilde{\mathcal{Q}}_1^{(1)}(W_1)W_1 + [\mathcal{Q}_{aux}^{(1)}(W_1)W_1, iE\Omega W_1]}_{quadratic} + \underbrace{M_{\geq 2}(W_1)W_1}_{cubic}, \quad (7.57)$$

for some maps $M_{\geq 2} \in \Sigma \mathcal{M}_2[r, N] \otimes \mathcal{M}_2(\mathbb{C})$. Since \mathcal{A}_1^τ is regular, the quadratic terms in (7.57) must coincide with the quadratic terms in (7.53). Therefore we have (recall (7.54))

$$\mathcal{Q}_1^{(1)}(W_1)W_1 = \widetilde{\mathcal{Q}}_1^{(1)}(W_1)W_1 + [\mathcal{Q}_{aux}^{(1)}(W_1)W_1, iE\Omega W_1]. \quad (7.58)$$

Recall the notation (2.21) and the definition of the non-linear commutator (1.7). Then equation (7.55), using (7.58), is equivalent to

$$(\widetilde{\mathcal{Q}}_1^{(1)}(W_1))_\sigma^{\sigma'} w_1^{\sigma'} - i\sigma\Omega(\mathcal{Q}_{aux}^{(1)}(W_1))_\sigma^{\sigma'} w_1^{\sigma'} + i\sigma'(\mathcal{Q}_{aux}^{(1)}(W_1))_\sigma^{\sigma'} \Omega w_1^{\sigma'} + (\mathcal{Q}_{aux}^{(1)}(iE\Omega W_1))_\sigma^{\sigma'} w_1^{\sigma'} = 0, \quad (7.59)$$

for $\sigma, \sigma' \in \{\pm\}$ and where $W_1 = [\frac{w_1}{w_1}]$ and $w_1^+ = w_1$, $w_1^- = \overline{w_1}$. Passing to the Fourier representation (see (2.22), (3.2)) we have that equation (7.59) reads

$$(\widetilde{\mathcal{Q}}_1^{(1)}(W_1))_{\sigma, j}^{\sigma', k} + (i\sigma' \omega_k - i\sigma \omega_j)(\mathcal{Q}_{aux}^{(1)}(W_1))_{\sigma, j}^{\sigma', k} + (\mathcal{Q}_{aux}^{(1)}(iE\Omega W_1))_{\sigma, j}^{\sigma', k} = 0, \quad (7.60)$$

for any $j, k \in \mathbb{Z}$. Recall that, since $\widetilde{\mathcal{Q}}_1^{(1)}(W_1) \in \widetilde{\mathcal{R}}_1^{-\rho}$, the coefficients $(\widetilde{\mathcal{Q}}_1^{(1)}(W_1))_{\sigma, j}^{\sigma', k}$ have the form (see (2.23))

$$(\widetilde{\mathcal{Q}}_1^{(1)}(W_1))_{\sigma, j}^{\sigma', k} = \sum_{\substack{\sigma_1 \in \{\pm\}, j_1 \in \mathbb{Z} \\ \sigma_1 j_1 = \sigma j - \sigma' k}} ((q_1^{(1)})_{j_1}^{\sigma_1, \sigma', k} w_{j_1}^{\sigma_1}), \quad ((q_1^{(1)})_{j_1}^{\sigma_1, \sigma', k}) \in \mathbb{C}. \quad (7.61)$$

We look for the operator $\mathcal{Q}_{aux}^{(1)}$ as in (2.21), (2.22) with coefficients as in (7.48). Using (7.61), (7.48), we rewrite (7.60) as

$$((q_1^{(1)})_{j_1}^{\sigma_1, \sigma', k}) + (i\sigma' \omega_k - i\sigma \omega_j + i\sigma_1 j_1)((q_{aux}^{(1)})_{j_1}^{\sigma_1, \sigma', k}) = 0,$$

and hence we define

$$((q_{aux}^{(1)})_{j_1}^{\sigma_1, \sigma', k}) := \frac{-((q_1^{(1)})_{j_1}^{\sigma_1, \sigma', k})}{(i\sigma' \omega_k - i\sigma \omega_j + i\sigma_1 j_1)}, \quad \forall \sigma_1 j_1 + \sigma' k = \sigma j, \quad j_1, j, k \in \mathbb{Z}, \quad \sigma, \sigma', \sigma_1 \in \{\pm\}. \quad (7.62)$$

Notice that, by assumption, the frequencies ω_j are not resonant according to Definition 3.4. By Lemma 6.5 in [11] we have that, since $\widetilde{\mathcal{Q}}_1^{(1)} \in \widetilde{\mathcal{R}}_1^{-\rho}$ and using the bounds (3.14), the operator $\mathcal{Q}_{aux}^{(1)}$ with coefficients as in (7.62) belongs to $\widetilde{\mathcal{R}}_1^{-\rho}$.

To conclude the proof of the lemma it remains to show that the flow \mathcal{A}_1^τ with generators defined above is symplectic. We show that $\mathcal{Q}_{aux}^{(1)}(W_1)W_1$ is an Hamiltonian vector field. We reason as follows. First of all we have that (recall (1.18))

$$iE\Omega W_1 := X_{H_2}(W_1), \quad H_2(W_1) := \frac{1}{2} \int_{\mathbb{T}} \Omega W_1 \cdot \overline{W_1} dx = \sum_{j \in \mathbb{Z}} \omega_j |w_1(j)|^2$$

is the Hamiltonian vector field of the Hamiltonian H_2 . Moreover, the field $\tilde{\mathcal{Y}}^{(1)}$ in (7.41) is Hamiltonian and hence its quadratic terms are Hamiltonian. Then

$$\begin{aligned} \tilde{\mathcal{Q}}_1^{(1)}(W_1)(W_1) &= X_A(W_1), \\ A(W_1) &= \int_{\mathbb{T}} \tilde{\mathcal{Q}}^{(1)}(W_1) W_1 \cdot \overline{W_1} dx = \sum_{\sigma_1 j_1 + \sigma' k = \sigma j} (\tilde{\mathcal{Q}}^{(1)})_{j_1, k, j}^{\sigma_1, \sigma', \sigma} (w_1)_{j_1}^{\sigma_1} (w_1)_k^{\sigma'} (w_1)_j^{-\sigma}, \end{aligned}$$

for some multilinear map $\tilde{\mathcal{Q}}^{(1)} \in \tilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$. Since $\mathcal{Q}_1^{(1)}(W_1)W_1$ solves (7.55) (see also (7.58)), then one can check that

$$\mathcal{Q}_{aux}^{(1)}(W_1)W_1 = X_{\text{ad}_{H_2}^{-1}A}(W_1), \quad (\text{ad}_{H_2}^{-1}A)(W_1) := \sum_{\sigma_1 j_1 + \sigma' k = \sigma j} \frac{(\tilde{\mathcal{Q}}^{(1)})_{j_1, k, j}^{\sigma_1, \sigma', \sigma}}{i\sigma' \omega_k - i\sigma \omega_j + i\sigma_1 j_1} (w_1)_{j_1}^{\sigma_1} (w_1)_k^{\sigma'} (w_1)_j^{-\sigma},$$

and hence it is Hamiltonian. This implies that the flow (7.47) is symplectic. The bounds (7.52) follow by Theorem 2.31. This concludes the proof. \square

7.3. Elimination of (j+1)-homogeneous terms. Let $j \in \mathbb{N}$, $j \geq 2$ and consider a para-differential system of the form

$$\dot{W}_j = \mathcal{Y}^{(j)}(W_j) := iE\Omega W_j + iEOp^{\text{BW}}(\mathfrak{M}^{(j)}(W_j; \xi))W_j + \mathcal{Q}^{(j)}(W_j)W_j, \quad W_j := \begin{bmatrix} w_j \\ \overline{w_j} \end{bmatrix} \quad (7.63)$$

and assume the following. The matrix of symbols $\mathfrak{M}^{(j)}$ as the form

$$\mathfrak{M}^{(j)}(W_j; \xi) := \begin{pmatrix} \mathfrak{m}^{(j)}(W_j; \xi) & 0 \\ 0 & \mathfrak{m}^{(j)}(W_j; -\xi) \end{pmatrix}, \quad \mathfrak{m}^{(j)} \in \Sigma\Gamma_2^m[r, N],$$

and $\mathfrak{m}^{(j)}$ is real valued and independent of $x \in \mathbb{T}$. Moreover we assume that it has the expansion (recall Definition 3.5)

$$\begin{aligned} \mathfrak{m}^{(j)}(W_j; \xi) &:= \sum_{k=2}^{j-1} \llbracket \mathfrak{m}_k^{(j)} \rrbracket(W_j; \xi) + \mathfrak{m}_j^{(j)}(W_j; \xi) + \mathfrak{m}_{\geq j+1}^{(j)}(W_j; \xi), \\ \mathfrak{m}_k^{(j)} &\in \tilde{\Gamma}_k^m, \quad k = 2, \dots, j, \quad \mathfrak{m}_{\geq j+1}^{(j)} \in \Sigma\Gamma_{j+1}^m[r, N]. \end{aligned} \quad (7.64)$$

The smoothing remainder $\mathcal{Q}^{(j)}(W_j)$ admits the expansion

$$\begin{aligned} \mathcal{Q}^{(j)}(W_j) &= \sum_{k=2}^{j-1} \llbracket \mathcal{Q}_k^{(j)} \rrbracket(W_j) + \mathcal{Q}_j^{(j)}(W_j) + \mathcal{Q}_{\geq j+1}^{(j)}(W_j), \\ \mathcal{Q}_k^{(j)} &\in \tilde{\mathcal{R}}_k^{-\rho}, \quad k = 2, \dots, j, \quad \mathcal{Q}_{\geq j+1}^{(j)} \in \Sigma\tilde{\mathcal{R}}_{j+1}^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}). \end{aligned} \quad (7.65)$$

The aim of the section is to eliminate all the non resonant terms of degree of homogeneity $j+1$ appearing in the vector field $\mathcal{Y}^{(j)}$ in (7.63), i.e. the non resonant terms of

$$\mathcal{Y}_j^{(j)}(W_j) := iEOp^{\text{BW}} \begin{pmatrix} \mathfrak{m}_j^{(j)}(W_j; \xi) & 0 \\ 0 & \mathfrak{m}_j^{(j)}(W_j; -\xi) \end{pmatrix} W_j + \mathcal{Q}_j^{(j)}(W_j)W_j.$$

As done in section 7.2 we eliminate $\mathcal{Y}_j^{(j)}$ in two steps. In section 7.3.1 we reduce the symbol $\mathfrak{m}_j^{(j)}$. In subsection 7.3.2 we deal with the j -homogeneous smoothing remainders.

7.3.1. Elimination of j -homogeneous symbols. Consider a real valued, independent of $x \in \mathbb{T}$ symbol $b_j \in \tilde{\Gamma}_j^m$ and let $B_j(W_j; \xi)$ be a matrix of symbols of the form (7.2) with $p = j$. Let $\tilde{\mathcal{A}}_j^r$ be the solution of

$$\begin{cases} \partial_\tau \tilde{\mathcal{A}}_j^r(W_j) = X_{\mathcal{G}_j}(\tilde{\mathcal{A}}_j^r(W_j)) \\ \tilde{\mathcal{A}}_j^0(W_j) = W_j, \end{cases} \quad (7.66)$$

where

$$X_{\mathcal{G}_j}(W_j) = iEOp^{\text{BW}}(B_j(W_j; \xi))W_j + \mathcal{B}_j(W_j)W_j, \quad \mathcal{B}_j \in \tilde{\mathcal{R}}_j^{-\rho}, \quad (7.67)$$

is the Hamiltonian vector field of an Hamiltonian of the form (7.3) with $p = j$. In the following lemma we conjugate the field (7.63) under the flow (7.66).

Lemma 7.5. *For $r > 0$ small enough there exists a symbol $b_j \in \Gamma_j^m$ such that the following holds. Setting (recall (7.63))*

$$\tilde{W}_j := \tilde{\mathcal{A}}_j(W_j) := \tilde{\mathcal{A}}_j^1(W_j), \quad \tilde{\mathcal{Y}}^{(j)}(\tilde{W}_j) := P^\tau(\tilde{W}_j)|_{\tau=1} := d\tilde{\mathcal{A}}_p^\tau(\tilde{\mathcal{A}}_p^{-\tau}(\tilde{W}_j))[\mathcal{Y}^{(j)}(\tilde{\mathcal{A}}_p^{-\tau}(\tilde{W}_j))]|_{\tau=1}, \quad (7.68)$$

we have that

$$\begin{cases} \dot{\tilde{W}}_j = \tilde{\mathcal{Y}}^{(j)}(\tilde{W}_j) := iE\Omega\tilde{W}_j + iEOp^{\text{BW}}(\tilde{\mathfrak{M}}^{(j)}(\tilde{W}_j; \xi))[\tilde{W}_j] + \tilde{\mathcal{Q}}^{(j)}(\tilde{W}_j)[\tilde{W}_j] \\ \tilde{W}_j(0) = \tilde{\mathcal{A}}_j(W_j(0)) \end{cases} \quad (7.69)$$

where $\tilde{\mathcal{Q}}^{(j)} \in \Sigma\mathcal{R}_2^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\tilde{\mathfrak{M}}^{(j)} \in \Sigma\Gamma_2^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is independent of $x \in \mathbb{T}$, real valued and has the form

$$\tilde{\mathfrak{M}}^{(j)}(\tilde{W}_j; \xi) := \begin{pmatrix} \tilde{\mathfrak{m}}^{(j)}(\tilde{W}_j; \xi) & 0 \\ 0 & \tilde{\mathfrak{m}}^{(j)}(\tilde{W}_j; -\xi) \end{pmatrix}, \quad (7.70)$$

$$\tilde{\mathfrak{m}}^{(j)}(\tilde{W}_j; \xi) := \sum_{k=2}^j \llbracket \tilde{\mathfrak{m}}_k^{(j)} \rrbracket(\tilde{W}_j; \xi) + \tilde{\mathfrak{m}}_{j+1}^{(j)}(\tilde{W}_j; \xi) + \tilde{\mathfrak{m}}_{\geq j+2}^{(j)}(\tilde{W}_j; \xi), \quad (7.71)$$

$$\tilde{\mathfrak{m}}_k^{(j)} \in \tilde{\Gamma}_k^m, \quad k = 2, \dots, j+1, \quad \tilde{\mathfrak{m}}_{\geq j+2}^{(j)} \in \Sigma\Gamma_{j+2}^m[r, N].$$

Moreover the remainder $\tilde{\mathcal{Q}}^{(j)}$ has the form

$$\tilde{\mathcal{Q}}^{(j)}(\tilde{W}_j) = \sum_{k=2}^{j-1} \llbracket \tilde{\mathcal{Q}}_k^{(j)} \rrbracket(\tilde{W}_j) + \tilde{\mathcal{Q}}_j^{(j)}(\tilde{W}_j) + \tilde{\mathcal{Q}}_{\geq j+1}^{(j)}(\tilde{W}_j), \quad (7.72)$$

$$\tilde{\mathcal{Q}}_k^{(j)} \in \tilde{\mathcal{R}}_k^{-\rho} \otimes \mathcal{M}_2(\mathbb{C}), \quad k = 2, \dots, j, \quad \tilde{\mathcal{Q}}_{\geq j+1}^{(j)} \in \Sigma\mathcal{R}_{j+1}^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}).$$

Finally, for any $s \geq s_0$, the maps $\tilde{\mathcal{A}}_j^{\pm 1}$ are symplectic and satisfy

$$\|\tilde{\mathcal{A}}_j^{\pm 1}(U)\|_{H^s} \leq \|U\|_{H^s} (1 + C\|U\|_{H^{s_0}}^p), \quad (7.73)$$

for some constant $C > 0$ depending on s .

Proof. The map $\tilde{\mathcal{A}}_j$ is symplectic by (7.66), (7.67) and satisfies (7.73) by Theorem 2.31. Notice that the vector field $\tilde{\mathcal{Y}}^{(j)}$ in (7.63) has the same form of (7.1) with $\mathfrak{N} \rightsquigarrow \tilde{\mathfrak{M}}^{(j)}$, $\mathcal{R} \rightsquigarrow \tilde{\mathcal{Q}}^{(j)}$, $U \rightsquigarrow \tilde{W}_j$. Then Proposition 7.1 applies to $\tilde{\mathcal{Y}}^{(j)}$. We obtain (recall (7.68) and (7.7))

$$\tilde{\mathcal{Y}}^{(j)}(\tilde{W}_j) := iE\Omega\tilde{W}_j + iEOp^{\text{BW}}(\tilde{\mathfrak{M}}^{(j)}(\tilde{W}_j; \xi))[\tilde{W}_j] + \tilde{\mathcal{Q}}^{(j)}(\tilde{W}_j)[\tilde{W}_j]$$

where $\tilde{\mathcal{Q}}^{(j)} \in \Sigma\mathcal{R}_2^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\tilde{\mathfrak{M}}^{(j)} \in \Sigma\Gamma_2^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ has the form (7.70) with (see formula (7.8))

$$\tilde{\mathfrak{m}}^{(j)}(\tilde{W}_j; \xi) = \mathfrak{m}^{(j)}(\tilde{\mathcal{A}}_j^{-1}(\tilde{W}_j); \xi) + \int_0^1 (d_Z b_j)(\tilde{\mathcal{A}}_j^\sigma \tilde{\mathcal{A}}_j^{-1}(\tilde{W}_j); \xi) [P^\sigma(\tilde{\mathcal{A}}_j^\sigma \tilde{\mathcal{A}}_j^{-1}(\tilde{W}_j))] d\sigma. \quad (7.74)$$

To conclude the proof we need to show the expansions (7.71) and (7.72).

The homological equation. We look for a symbol $b_j \in \tilde{\Gamma}_j^m$ such that $\tilde{\mathfrak{m}}^{(j)}(\tilde{W}_j; \xi)$ in (7.74) satisfies (7.71). First of all, by Theorem 2.31, we deduce that the flow $\tilde{\mathcal{A}}_j^\tau$ of (7.66) is such that

$$\begin{aligned} \tilde{\mathcal{A}}_j^\tau(Z) &= Z + N_j^{(1)}(\tau, Z)[Z], & \tilde{\mathcal{A}}_j^{-\tau}(Z) &= Z + N_j^{(2)}(\tau, Z)[Z], \\ N_j^{(1)}, N_j^{(2)} &\in \Sigma\mathcal{M}_j[r, N] \otimes \mathcal{M}_2(\mathbb{C}), \end{aligned} \quad (7.75)$$

with estimates uniform in $\tau \in [0, 1]$ since the generator is a map in $\tilde{\mathcal{M}}_j$. We also recall that the symbol $\mathfrak{m}^{(j)}$ admits the expansion (7.64), i.e. it is resonant (see Def. 3.5) up to degree of homogeneity j . Finally, notice that, for any $k \leq j-1$,

$$\llbracket \mathfrak{m}_k^{(j)} \rrbracket (\tilde{\mathcal{A}}_j^{-\tau}(\tilde{W}_j); \xi) \stackrel{(7.75)}{=} \llbracket \mathfrak{m}_k^{(j)} \rrbracket (\tilde{W}_j; \xi) + \mathfrak{r}_{j+k}(\tilde{W}_j; \xi), \quad (7.76)$$

for some real valued and independent of $x \in \mathbb{T}$ symbol $\mathfrak{r}_{j+k} \in \Sigma \Gamma_{j+k}^m[r, N]$. We now expand in degree of homogeneity the symbol in (7.74). By the discussion above we obtain

$$\tilde{\mathfrak{m}}^{(j)}(\tilde{W}_j; \xi) := \sum_{k=2}^{j-1} \llbracket \mathfrak{m}_k^{(j)} \rrbracket (\tilde{W}_j; \xi) + \tilde{\mathfrak{m}}_j^{(j)}(\tilde{W}_j; \xi) + \tilde{\mathfrak{m}}_{\geq j+1}^{(j)}(\tilde{W}_j; \xi), \quad (7.77)$$

for some real, independent of x symbol $\tilde{\mathfrak{m}}_{\geq j+1}^{(j)} \in \Sigma \Gamma_{j+1}^m[r, N]$ depending on $\mathfrak{m}^{(j)}$ and b_j , and where

$$\tilde{\mathfrak{m}}_j^{(j)}(\tilde{W}_j; \xi) := \mathfrak{m}^{(j)}(\tilde{W}_j; \xi) + (d_Z b_j)(\tilde{W}_j; \xi) [iE\Omega \tilde{W}_j]. \quad (7.78)$$

We prove the following.

Lemma 7.6. (Homological equation). *There exists a symbol $b_j \in \tilde{\Gamma}_j^m$ such that (see (7.78))*

$$\tilde{\mathfrak{m}}_j^{(j)}(W; \xi) = \mathfrak{m}_j^{(j)}(W; \xi) + (d_Z b_j)(W; \xi) [iE\Omega W] = \llbracket \mathfrak{m}_j^{(j)} \rrbracket (W; \xi)$$

Moreover $b_j(W; \xi)$ is real valued and independent of $x \in \mathbb{T}$.

Proof. We recall that, by (2.16), the symbol $\mathfrak{m}_j^{(j)}$ has the form

$$\mathfrak{m}_j^{(j)}(W; \xi) = \sum_{\substack{\sigma_i \in \{\pm\}, i=1, \dots, j \\ n_i \in \mathbb{Z}, \\ \sum_{i=1}^j \sigma_i n_i = 0}} (\mathfrak{m}_j^{(j)})_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j}(\xi) w_{n_1}^{\sigma_1} \dots w_{n_j}^{\sigma_j}. \quad (7.79)$$

Recall (3.16), (3.2) and define

$$(b_j)_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j}(\xi) := \frac{-(\mathfrak{m}_j^{(j)})_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j}(\xi)}{i(\sigma_1 \omega_{n_1} + \dots + \sigma_j \omega_{n_j})}, \quad \begin{array}{l} \sigma_1 n_1 + \dots + \sigma_j n_j = 0, \\ (\sigma_1, \dots, \sigma_j, n_1, \dots, n_j) \notin \mathcal{S}_j \end{array} \quad (7.80)$$

and $(b_j)_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j}(\xi) := 0$ otherwise. One can check, by an explicit computation, that the symbol

$$b_j(W; \xi) = \sum_{\substack{\sigma_i \in \{\pm\}, i=1, \dots, j \\ n_i \in \mathbb{Z}, \\ \sum_{i=1}^j \sigma_i n_i = 0}} (b_j)_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j}(\xi) w_{n_1}^{\sigma_1} \dots w_{n_j}^{\sigma_j}, \quad (7.81)$$

with $(b_j)_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j}(\xi)$ in (7.80) solves the equation (7.79) where the r.h.s. $\llbracket \mathfrak{m}^{(j)} \rrbracket (W; \xi)$ is defined as in (3.17).

Since the symbol $\mathfrak{m}_j^{(j)}$ is real valued, its coefficients satisfies

$$\overline{(\mathfrak{m}_j^{(j)})_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j}(\xi)} = (\mathfrak{m}_j^{(j)})_{n_1, \dots, n_j}^{-\sigma_1, \dots, -\sigma_n}(\xi). \quad (7.82)$$

By formula (7.80) one can check that $(b_j)_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j}(\xi)$ satisfies the same property as in (7.82). Therefore the symbol in (7.81) is real valued. \square

In view of Lemma 7.6 we have that formula (7.77) implies the (7.71) by setting $\tilde{\mathfrak{m}}_k^{(j)} = \mathfrak{m}_k^{(j)}$ for $0 \leq k \leq j$. In order to prove the (7.72) we reason as follows. Recalling (7.70), (7.71), we define the operator

$$Q^\tau(W) := iE\Omega W + iEOp^{\text{BW}}(\tilde{\mathfrak{M}}^{(j)}(W; \xi))W.$$

By the proof of Proposition 7.1 (see (7.20)-(7.21)), we deduce that the smoothing remainder in (7.69) has the form

$$\tilde{\mathcal{Q}}^{(j)}(W)W := V^\tau \circ \tilde{\mathcal{A}}_j^{-\tau}(W)|_{\tau=1} \quad (7.83)$$

where V^τ solves the problem (see (7.67))

$$\begin{cases} \partial_\tau V^\tau(U) = (d_Z X_{\mathcal{G}_j})(\mathcal{A}_j^\tau)[V^\tau], \\ V^0(U) = -\mathcal{Q}^{(j)}(U)U. \end{cases}$$

Since the generator $X_{\mathcal{G}_j}$ has degree of homogeneity equals to $j+1$, the Taylor expansion of the remainder $\tilde{\mathcal{Q}}^{(j)}$ in (7.83) coincides with the expansion in degree of homogeneity of the initial remainder $\mathcal{Q}^{(j)}$. Therefore, recalling (7.65), we have that (7.72) holds by setting $\tilde{\mathcal{Q}}_k^{(j)} = \mathcal{Q}_k^{(j)}$ for $2 \leq k \leq j$. \square

7.3.2. Elimination of j -homogeneous smoothing operators. Consider the vector field $\tilde{\mathcal{Y}}^{(j)}$ in (7.69). The aim of this section is to eliminate the non resonant terms of degree of homogeneity $j+1$ of $\tilde{\mathcal{Y}}^{(j)}$. In view of Lemma 7.5 we have that such terms appear only in the smoothing remainder $\tilde{\mathcal{Q}}^{(j)}$ (see (7.71), (7.72)).

We now consider the flow \mathcal{A}_j^τ of

$$\begin{cases} \partial_\tau \mathcal{A}_j^\tau(W) = \mathcal{Q}_{aux}^{(j)}(\mathcal{A}_j^\tau(W))\mathcal{A}_1^\tau(W) \\ \mathcal{A}_1^0(W) = W, \end{cases} \quad (7.84)$$

where $\mathcal{Q}_{aux}^{(j)} \in \tilde{\mathcal{R}}_j^{-p}$. Assume also that the vector field $\mathcal{Q}_{aux}^{(j)}(W)W$ is Hamiltonian, i.e. there is a map $\mathcal{Q}^{(j)} \in \tilde{\mathcal{M}}_j \otimes \mathcal{M}_2(\mathbb{C})$ such that (recall (1.18))

$$\mathcal{Q}_{aux}^{(j)}(W)W = X_{\mathcal{C}}(W), \quad \mathcal{C}(W) := \int_{\mathbb{T}} \mathcal{Q}^{(j)}(W)W \cdot \bar{W} dx.$$

Finally assume that $\mathcal{Q}_{aux}^{(j)}$ has the form (2.21), (2.22) with $p = j$ and coefficients (see (2.23))

$$(\mathcal{Q}_{aux}^{(j)}(W))_{\sigma,k}^{\sigma',k'} = \sum_{\substack{\sigma_i \in \{\pm\}, n_i \in \mathbb{Z} \\ \sum_{i=1}^j \sigma_i n_i = \sigma k - \sigma' k'}} ((\mathcal{q}_{aux}^{(j)})_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j})_{\sigma,k}^{\sigma',k'} w_{n_1}^{\sigma_1} \dots w_{n_j}^{\sigma_j}, \quad ((\mathcal{q}_{aux}^{(j)})_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j})_{\sigma,k}^{\sigma',k'} \in \mathbb{C}. \quad (7.85)$$

In the following lemma we show that it is possible to choose the coefficients $((\mathcal{q}_{aux}^{(j)})_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j})_{\sigma,k}^{\sigma',k'}$ in (7.85) in such a way that the vector field in (7.69) does not contain any non resonant monomials of degree $j+1$.

Lemma 7.7. *For $r > 0$ small enough there exists $\mathcal{Q}_{aux}^{(j)} \in \tilde{\mathcal{R}}_j^{-p}$, of the form (7.85), such that the following holds. Setting (recall (7.68))*

$$\begin{aligned} W_{j+1} &:= \mathcal{A}_j(\tilde{W}_j) := \mathcal{A}_j^1(\tilde{W}_j), \\ \mathcal{Y}^{(j+1)}(W_{j+1}) &:= P^\tau(W_{j+1})|_{\tau=1} := d\mathcal{A}_j^\tau(\mathcal{A}_j^{-\tau}(W_{j+1}))[\tilde{\mathcal{Y}}^{(j)}(\mathcal{A}_j^{-\tau}(W_{j+1}))]|_{\tau=1}, \end{aligned} \quad (7.86)$$

we have that

$$\begin{cases} \dot{W}_{j+1} = \mathcal{Y}^{(j+1)}(W_{j+1}) := iE\Omega W_{j+1} + iEOp^{\text{BW}}(\mathfrak{M}^{(j+1)}(W_{j+1}; \xi))[W_{j+1}] + \mathcal{Q}^{(j+1)}(W_{j+1})[W_{j+1}] \\ W_{j+1}(0) = \mathcal{A}_1(\tilde{W}_j(0)) \end{cases}$$

where $\mathcal{Q}^{(j+1)} \in \Sigma \tilde{\mathcal{R}}_2^{-p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathfrak{M}^{(j+1)} \in \Sigma \Gamma_2^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is independent of $x \in \mathbb{T}$, real valued and has the form

$$\begin{aligned} \mathfrak{M}^{(j+1)}(W_{j+1}; \xi) &:= \begin{bmatrix} \mathfrak{m}^{(j+1)}(W_{j+1}; \xi) & 0 \\ 0 & \mathfrak{m}^{(j+1)}(W_{j+1}; -\xi) \end{bmatrix}, \\ \mathfrak{m}^{(j+1)}(W_{j+1}; \xi) &:= \sum_{k=2}^j \llbracket \mathfrak{m}_k^{(j+1)} \rrbracket(W_j; \xi) + \mathfrak{m}_{j+1}^{(j+1)}(W_j; \xi) + \mathfrak{m}_{\geq j+2}^{(j+1)}(W_j; \xi), \\ \mathfrak{m}_k^{(j+1)} &\in \tilde{\Gamma}_k^m, \quad k = 2, \dots, j+1, \quad \mathfrak{m}_{\geq j+2}^{(j+1)} \in \Sigma \Gamma_{j+2}^m[r, N]. \end{aligned} \quad (7.87)$$

Moreover the remainder $\mathcal{Q}^{(j+1)}$ has the form

$$\begin{aligned} \mathcal{Q}^{(j+1)}(W_{j+1}) &= \sum_{k=2}^j \llbracket \mathcal{Q}_k^{(j+1)} \rrbracket(W_{j+1}) + \tilde{\mathcal{Q}}_{j+1}^{(j+1)}(W_{j+1}) + \mathcal{Q}_{\geq j+2}^{(j+1)}(W_{j+1}), \\ \mathcal{Q}_k^{(j+1)} &\in \tilde{\mathcal{R}}_k^{-\rho} \otimes \mathcal{M}_2(\mathbb{C}), \quad k=2, \dots, j+1, \quad \mathcal{Q}_{\geq j+2}^{(j+1)} \in \Sigma \mathcal{R}_{j+1}^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C}). \end{aligned} \quad (7.88)$$

Finally, for any $s \geq s_0$, the maps $\mathcal{A}_j^{\pm 1}$ are symplectic and satisfy

$$\|\mathcal{A}_j^{\pm 1}(U)\|_{H^s} \leq \|U\|_{H^s} (1 + C\|U\|_{H^{s_0}}), \quad (7.89)$$

for some constant $C > 0$ depending on s .

Proof. Notice that the vector field $\tilde{\mathcal{Y}}^{(j)}$ in (7.68), (7.69) has the same form of \mathcal{X} in (7.1), with $\mathfrak{N} \rightsquigarrow \tilde{\mathfrak{M}}^{(j)}$, $\mathcal{R} \rightsquigarrow \tilde{\mathcal{Q}}^{(j)}$, $U \rightsquigarrow \tilde{W}_j$. The generator $\mathcal{Q}_{aux}^{(j)}$ in (7.84) has the same properties of the generator $\mathcal{Q}_{aux}^{(p)}$, $p = j$, in (7.22). Therefore Proposition 7.2 applies. As a consequence we obtain that

$$\dot{W}_{j+1} = iE\Omega W_{j+1} + iEOp^{\text{BW}}(\mathfrak{M}^{(j+1)}(W_{j+1}; \xi)) [W_{j+1}] + \mathcal{Q}^{(j+1)}(W_{j+1}) [W_{j+1}], \quad (7.90)$$

where $\mathcal{Q}^{(j+1)} \in \Sigma \mathcal{R}_2^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$, $\mathfrak{M}^{(j+1)}$ has the form (7.87) with (see (7.26))

$$\mathfrak{m}^{(j+1)}(\tau, W_{j+1}; \xi) := \tilde{\mathfrak{m}}^{(j+1)}(\mathcal{A}_j^{-\tau}(W_{j+1}); \xi) \in \Sigma \Gamma_2^m[r, N]. \quad (7.91)$$

Reasoning as in (7.75), since the generator in (7.84) has homogeneity $j+1$, we deduce that the flow \mathcal{A}_j^τ of (7.66) is such that

$$\begin{aligned} \mathcal{A}_j^\tau(Z) &= Z + N_j^{(3)}(\tau, Z)[Z], \quad \mathcal{A}_j^{-\tau}(Z) = Z + N_j^{(4)}(\tau, Z)[Z], \\ N_j^{(3)}, N_j^{(4)} &\in \Sigma \mathcal{M}_j[r, N] \otimes \mathcal{M}_2(\mathbb{C}). \end{aligned} \quad (7.92)$$

Then, using (7.91), (7.92) and reasoning as in (7.76), we deduce that the expansion (7.87) holds by setting $\mathfrak{m}_k^{(j+1)} = \tilde{\mathfrak{m}}_k^{(j)}$ for $2 \leq k \leq j$. Let us check the (7.88). In order to provide an explicit expression of the terms of homogeneity smaller than j in the remainder $\mathcal{Q}^{(j+1)}$ we reason as in (7.56). We Taylor expand the vector field $\mathcal{Y}^{(j+1)}$ by using formula (7.86). We get

$$\mathcal{Y}^{(j+1)}(W_{j+1}) = \tilde{\mathcal{Y}}^{(j)}(W_{j+1}) + [\mathcal{Q}_{aux}^{(j)}(W_{j+1})W_{j+1}, \tilde{\mathcal{Y}}^{(j)}(W_{j+1})] + \int_0^1 (1-\sigma) \partial_\sigma^2 P^\sigma(W_{j+1}) d\sigma.$$

Recalling (7.69)-(7.72) we obtain the expansion

$$\begin{aligned} \mathcal{Y}^{(j+1)}(W_{j+1}) &= iE\Omega W_{j+1} + \sum_{k=1}^j Op^{\text{BW}} \left(\begin{bmatrix} \llbracket \tilde{\mathfrak{m}}_k^{(j)} \rrbracket(W_{j+1}; \xi) & 0 \\ 0 & \llbracket \tilde{\mathfrak{m}}_k^{(j)} \rrbracket(W_{j+1}; -\xi) \end{bmatrix} \right) W_{j+1} + \sum_{k=1}^{j-1} \llbracket \tilde{\mathcal{Q}}_k^{(j)} \rrbracket(W_{j+1}) W_{j+1} \\ &\quad + \mathcal{Q}_j^{(j)}(W_{j+1}) W_{j+1} + \mathbb{M}_{>j}(W_{j+1}) W_{j+1}, \end{aligned} \quad (7.93)$$

where $\mathbb{M}_{>j}$ is some map in $\Sigma \mathcal{M}_{j+1}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and

$$\mathcal{Q}_j^{(j+1)}(W_{j+1}) W_{j+1} := \tilde{\mathcal{Q}}_j^{(j)}(W_{j+1}) W_{j+1} + [\mathcal{Q}_{aux}^{(j)}(W_{j+1}) W_{j+1}, iE\Omega W_{j+1}]. \quad (7.94)$$

The expansion (7.93) coincide with the expansion of the vector field in (7.90) in multilinear maps. Notice that the terms of homogeneity $j+1$

$$Op^{\text{BW}} \left(\begin{bmatrix} \llbracket \tilde{\mathfrak{m}}_j^{(j)} \rrbracket(W_{j+1}; \xi) & 0 \\ 0 & \llbracket \tilde{\mathfrak{m}}_j^{(j)} \rrbracket(W_{j+1}; -\xi) \end{bmatrix} \right) W_{j+1}$$

are *resonant* (see Def. 3.5), i.e. they are already in Birkhoff normal form. Hence the only non-resonant terms of degree $j+1$ belongs to $\mathcal{Q}_j^{(j+1)}(W_{j+1}) W_{j+1}$ in (7.94). By the definition of the non-linear commutator (1.7) one can easily note that the term $[\mathcal{Q}_{aux}^{(j)}(W_{j+1}) W_{j+1}, iE\Omega W_{j+1}]$ does not contain any *resonant* monomials. Then we define

$$\tilde{\mathcal{Q}}_j^{(j), \perp}(W_{j+1}) W_{j+1} := \tilde{\mathcal{Q}}_j^{(j)}(W_{j+1}) W_{j+1} - \llbracket \tilde{\mathcal{Q}}_j^{(j)} \rrbracket(W_{j+1}) W_{j+1} \quad (7.95)$$

the non-resonant part of (7.94). We claim that the vector field $\tilde{\mathcal{Q}}_j^{(j),\perp}(W_{j+1})W_{j+1}$ is Hamiltonian. Indeed we have that $\tilde{\mathcal{Q}}_j^{(j)}$ is Hamiltonian. Hence also its terms of homogeneity $(j+1)$, i.e. (see (7.95))

$$\tilde{\mathcal{Q}}_j^{(j)}(W_{j+1}) := Op^{BW} \left(\begin{bmatrix} [[\tilde{m}_j^{(j)}]](W_{j+1};\xi) & 0 \\ 0 & [[\tilde{m}_k^{(j)}]](W_{j+1};-\xi) \end{bmatrix} \right) W_{j+1} + [[\tilde{\mathcal{Q}}_j^{(j)}]](W_{j+1})W_{j+1} + \tilde{\mathcal{Q}}_j^{(j),\perp}(W_{j+1})W_{j+1} \quad (7.96)$$

are Hamiltonian. This means that there is a multilinear map $M_j \in \tilde{\mathcal{M}}_j$ such that $\tilde{\mathcal{Q}}_j^{(j)}(W_{j+1})$ is the Hamiltonian vector field of the Hamiltonian function $A(W_{j+1}) := A_1(W_{j+1}) + A_2(W_{j+1})$ with

$$\begin{aligned} A_1(W_{j+1}) &:= \frac{1}{2} \int_{\mathbb{T}} Op^{BW} \left(\begin{bmatrix} [[\tilde{m}_j^{(j)}]](W_{j+1};\xi) & 0 \\ 0 & [[\tilde{m}_k^{(j)}]](W_{j+1};-\xi) \end{bmatrix} \right) W_{j+1} \cdot \overline{W}_{j+1} dx, \\ A_2(W_{j+1}) &:= \int_{\mathbb{T}} M_j(W_{j+1}) W_{j+1} \cdot \overline{W}_{j+1} dx. \end{aligned}$$

By Proposition 6.4 and Lemma 6.3 the vector field of $A_1(W_{j+1})$ has the form

$$X_{A_1}(W_{j+1}) = Op^{BW} \left(\begin{bmatrix} [[\tilde{m}_j^{(j)}]](W_{j+1};\xi) & 0 \\ 0 & [[\tilde{m}_k^{(j)}]](W_{j+1};-\xi) \end{bmatrix} \right) W_{j+1} + R_1(W_{j+1})W_{j+1}, \quad R_1(W_{j+1}) \equiv [[R_1]](W_{j+1}),$$

for some $R_1 \in \tilde{\mathcal{R}}_j^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$. On the other hand the vector field of $A_2(W_{j+1})$ has the form

$$X_{A_2}(W_{j+1}) = [[R_2]](W_{j+1})W_{j+1} + \left(R_2(W_{j+1})W_{j+1} - [[R_2]](W_{j+1})W_{j+1} \right),$$

for some multilinear map $R_2 \in \tilde{\mathcal{M}}_j \otimes \mathcal{M}_2(\mathbb{C})$. Then, recalling (7.96), we must have

$$\begin{aligned} [[\tilde{\mathcal{Q}}_j^{(j)}]](W_{j+1})W_{j+1} &\equiv [[R_1]](W_j)W_{j+1} + [[R_2]](W_{j+1})W_{j+1}, \\ \tilde{\mathcal{Q}}_j^{(j),\perp}(W_{j+1})W_{j+1} &\equiv R_2(W_{j+1})W_{j+1} - [[R_2]](W_{j+1})W_{j+1}, \end{aligned} \quad (7.97)$$

implying that $\tilde{\mathcal{Q}}_j^{(j),\perp}(W_{j+1})W_{j+1}$ is Hamiltonian (see Remark 3.6). In order to conclude the proof of Lemma 7.7 we need the following.

Lemma 7.8. (Homological equation). *There is an Hamiltonian vector field of the form $\mathcal{Q}_{aux}^{(j)}(W)W$ with $\mathcal{Q}_{aux}^{(j)} \in \tilde{\mathcal{R}}_j^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ such that (see (7.95), (7.94))*

$$\tilde{\mathcal{Q}}_j^{(j),\perp}(W)W + [\mathcal{Q}_{aux}^{(j)}(W)W, iE\Omega W] = 0. \quad (7.98)$$

Proof. We look for a solution $\mathcal{Q}_{aux}^{(j)}$ in the class of multilinear operators $\tilde{\mathcal{R}}_j^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ of the form (2.21), (2.22) with coefficients

$$(\mathcal{Q}_{aux}^{(j)}(W))_{\sigma,k}^{\sigma',k'} := \frac{1}{(2\pi)^j} \sum_{\substack{\sigma_i \in \{\pm\}, n_i \in \mathbb{Z} \\ \sum_{i=1}^j \sigma_i n_i = \sigma k - \sigma' k'}} ((q_{aux}^{(j)})_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j})_{\sigma,k}^{\sigma',k'} w_{n_1}^{\sigma_1} \dots w_{n_j}^{\sigma_j}, \quad k, k' \in \mathbb{Z}, \quad (7.99)$$

for some $((q_{aux}^{(j)})_{n_1, \dots, n_j}^{\sigma_1, \dots, \sigma_j})_{\sigma,k}^{\sigma',k'} \in \mathbb{C}$. For convenience we write

$$\mathcal{Q}_{aux}^{(j)}(W)W = \mathcal{Q}_{aux}^{(j)}(\underbrace{W, \dots, W}_{j\text{-times}})W.$$

With this notation we have, for any $Y = \left[\frac{Y}{Y} \right]$,

$$d_W \left(\mathcal{Q}_{aux}^{(j)}(W)W \right) [Y] = \mathcal{Q}_{aux}^{(j)}(W, \dots, W)Y + \sum_{k=1}^j \mathcal{Q}_{aux}^{(j)}(W, \dots, \underbrace{Y}_{k\text{-th}}, \dots, W)W.$$

Therefore (recall (1.7)) the equation (7.98) reads

$$\begin{aligned} & (\tilde{\mathcal{Q}}_j^{(j),\perp}(W))_\sigma^{\sigma'} w_{\sigma'} - i\sigma\Omega(\tilde{\mathcal{Q}}_j^{(j),\perp}(W))_\sigma^{\sigma'} w_{\sigma'} + i\sigma'(\tilde{\mathcal{Q}}_j^{(j),\perp}(W))_\sigma^{\sigma'} \Omega w_{\sigma'} \\ & + \sum_{k=1}^j (\mathcal{Q}_{aux}^{(j)}(W, \dots, iE\Omega W, \dots, W))_\sigma^{\sigma'} w_{\sigma'} = 0. \end{aligned} \quad (7.100)$$

for any $\sigma, \sigma' \in \{\pm\}$. Recall that $(\tilde{\mathcal{Q}}_j^{(j),\perp}(W))_\sigma^{\sigma'}$ has the form (2.22) with coefficients

$$(\tilde{\mathcal{Q}}_j^{(j),\perp}(W))_{\sigma,k}^{\sigma',k'} := \frac{1}{(2\pi)^j} \sum_{\substack{\sigma_i \in \{\pm\}, n_i \in \mathbb{Z} \\ \sum_{i=1}^j \sigma_i n_i = \sigma k - \sigma' k'}} ((\tilde{q}_j^{(j)})_{n_1, \dots, n_j}^{\sigma_1 \dots \sigma_j})_{\sigma,k}^{\sigma',k'} w_{n_1}^{\sigma_1} \dots w_{n_j}^{\sigma_j}, \quad k, k' \in \mathbb{Z},$$

for $((\tilde{q}_j^{(j)})_{n_1, \dots, n_j}^{\sigma_1 \dots \sigma_j})_{\sigma,k}^{\sigma',k'} \in \mathbb{C}$ and where the sum is restricted to indexes outside the set \mathcal{S}_j (see (3.16)). Passing to the Fourier coefficients the equation (7.100) becomes (recall (7.99))

$$((\tilde{q}_j^{(j)})_{n_1, \dots, n_j}^{\sigma_1 \dots \sigma_j})_{\sigma,k}^{\sigma',k'} + i \left(\sum_{i=1}^j \sigma_i \omega_{n_i} - \sigma k + \sigma' k' \right) ((\mathcal{Q}_{aux}^{(j)})_{n_1, \dots, n_j}^{\sigma_1 \dots \sigma_j})_{\sigma,k}^{\sigma',k'} = 0,$$

for any indexes satisfying

$$\sum_{i=1}^j \sigma_i n_i = \sigma k - \sigma' k', \quad (\sigma_1, \dots, \sigma_j, \sigma, \sigma', n_1, \dots, k, k') \notin \mathcal{S}_j. \quad (7.101)$$

Therefore we define the operator $\mathcal{Q}_{aux}^{(j)}(W)$ as in (7.99) with coefficients

$$((\mathcal{Q}_{aux}^{(j)})_{n_1, \dots, n_j}^{\sigma_1 \dots \sigma_j})_{\sigma,k}^{\sigma',k'} = \frac{-((\tilde{q}_j^{(j)})_{n_1, \dots, n_j}^{\sigma_1 \dots \sigma_j})_{\sigma,k}^{\sigma',k'}}{i(\sigma_1 n_1 + \dots + \sigma_j n_j + \sigma' k' - \sigma k)} \quad (7.102)$$

for indexes satisfying (7.101) and 0 otherwise. Thanks to (7.102), the bound (3.14) and reasoning as in Lemma 6.5 in [11] one can check that $\mathcal{Q}_{aux}^{(j)} \in \tilde{\mathcal{R}}_j^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$. Finally the vector field $\mathcal{Q}_{aux}^{(j)}(W)W$ is Hamiltonian since it solves (7.98) and the field $\tilde{\mathcal{Q}}_j^{(j),\perp}(W)W$ is Hamiltonian (see (7.97)). \square

We conclude the proof of Lemma 7.7. Thanks to Lemma 7.8 we have that the operator $\mathcal{Q}_j^{(j+1)}(W)$ in (7.94) is equal to $[[\tilde{\mathcal{Q}}_j^{(j)}]](W)$. Hence the expansion (7.88) follows by (7.90), (7.93) setting $\mathcal{Q}_k^{(j+1)} = \tilde{\mathcal{Q}}_k^{(j)}$ for $2 \leq k \leq j$. Since, by Lemma 7.8, $\mathcal{Q}_{aux}^{(j)}(W)W$ is Hamiltonian then the flow in (7.84) is symplectic. The estimates (7.89) follow by Theorem 2.31. This concludes the proof. \square

Proof of Corollary 3.8. Consider the system (3.22). Since $\mathcal{Q}_N \in \Sigma \mathcal{R}_2^{-\rho}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathfrak{M}^{(N)}$ belongs to $\Sigma \Gamma_2^m[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ we write

$$\begin{aligned} \mathfrak{M}^{(N)}(W; \xi) &= \sum_{j=2}^{N-1} \mathfrak{M}_j^{(N)}(W; \xi) + \mathfrak{M}_N^{(N)}(W; \xi), \quad \mathfrak{M}_j^{(N)} \in \tilde{\Gamma}_j^m \otimes \mathcal{M}_2(\mathbb{C}), \quad \mathfrak{M}_N^{(N)} \in \Gamma_N^m[r] \otimes \mathcal{M}_2(\mathbb{C}) \\ \mathcal{Q}_N(W) &= \sum_{j=2}^{N-1} \mathcal{Q}_{N,j}(W) + \mathcal{Q}_{N,N}(W), \quad \mathcal{Q}_{N,j} \in \tilde{\mathcal{R}}_j^{-\rho} \otimes \mathcal{M}_2(\mathbb{C}), \quad \mathcal{Q}_{N,N} \in \mathcal{R}_N^{-\rho}[r] \otimes \mathcal{M}_2(\mathbb{C}). \end{aligned} \quad (7.103)$$

In particular (see (3.23)) we have

$$\mathfrak{M}_N^{(N)}(W; \xi) := \begin{pmatrix} \mathfrak{m}_N^{(N)}(W; \xi) & 0 \\ 0 & \mathfrak{m}_N^{(N)}(W; -\xi) \end{pmatrix}, \quad \mathfrak{m}_N^{(N)}(W; \xi) = \overline{\mathfrak{m}_N^{(N)}(W; \xi)}, \quad (7.104)$$

with $\mathfrak{m}_N^{(N)} \in \Gamma_N^m[r]$ independent of $x \in \mathbb{T}$. We also recall that $\mathcal{Q}_{N,N}$ is a real-to-real matrix of operators, hence we can write (recall (2.21), (2.34), (2.35))

$$\mathcal{Q}_{N,N} = \left((\mathcal{Q}_{N,N})_{\sigma}^{\sigma'} \right)_{\sigma, \sigma' \in \{\pm\}}, \quad (\mathcal{Q}_{N,N})_{\sigma}^{\sigma'} = \overline{(\mathcal{Q}_{N,N})_{-\sigma}^{-\sigma'}}. \quad (7.105)$$

By Theorem 3.7 we know that the vector field \mathcal{Y}_N in (3.22) is Hamiltonian. Hence, recalling the expansions (7.103) and Definition 3.5, we have (see (1.18))

$$\begin{aligned} \mathcal{Y}_N(W) &= iJ\nabla H^{(N)}(W), \quad H^{(N)}(W) = \frac{1}{2} \int_{\mathbb{T}} \Omega W \cdot \overline{W} + H_{<N}^{(N)}(W) + H_{\geq N}^{(N)}(W), \\ H_{<N}^{(N)}(W) &= \sum_{j=2}^{N-1} H_j^{(N)}(W), \quad H_j^{(N)}(W) := \int_{\mathbb{T}} M_j(W) W \cdot \overline{W}, \quad H_{\geq N}^{(N)}(W) := \int_{\mathbb{T}} M_N(W) W \cdot \overline{W} \end{aligned} \quad (7.106)$$

for some multilinear maps $M_j \in \widetilde{\mathcal{M}}_j \otimes \mathcal{M}_2(\mathbb{C})$ and a non-homogeneous map $M_N \in \mathcal{M}_N[r] \otimes \mathcal{M}_2(\mathbb{C})$. We deduce that the term with highest homogeneity in \mathcal{Y}_N is

$$iEOp^{\text{BW}}(\mathfrak{M}_N^{(N)}(W; \xi))W + \mathcal{Q}_{N,N}(W)W = iJ\nabla H_{\geq N}^{(N)}(W), \quad (7.107)$$

while at lower orders we have

$$iE\Omega W + \sum_{j=2}^{N-1} \left(iEOp^{\text{BW}}([\mathfrak{M}_j^{(N)}](W; \xi))W + [\mathcal{Q}_{N,j}](W)W \right) = iE\Omega W + \sum_{j=2}^{N-2} iJ\nabla H_j^{(N)}(W).$$

Moreover (recall Def. 3.5) these terms having lower degree of homogeneity are *resonant*. Therefore we must have that the Hamiltonians $H_j^{(N)}$ have the following form. For $j = 2p$ (recall (3.16)) we have

$$H_j^{(N)}(W) = \sum_{\substack{n_i \in \mathbb{Z}, i=0, \dots, j+1 \\ \{|n_0|, \dots, |n_p|\} = \{|n_{p+1}|, \dots, |n_{j+1}|\}}} h_{n_0, \dots, n_{j+1}} w_{n_0} \cdots w_{n_p} \overline{w_{n_{p+1}}} \cdots \overline{w_{n_{j+1}}}.$$

If j is odd then $H_j^{(N)} \equiv 0$. In particular the Hamiltonians $H_j^{(N)}$ are real valued and the coefficients $h_{n_0, \dots, n_{j+1}} \in \mathbb{C}$ are invariant under permutations of the indexes n_0, \dots, n_{j+1} . Recalling (2.5) we define

$$G(W) := \|w\|_{H^s}^2 = \int_{\mathbb{T}} \langle D \rangle^s w \cdot \langle D \rangle^s \overline{w} dx = \sum_{j \in \mathbb{Z}} \langle j \rangle^{2s} |w_j|^2.$$

It is a straightforward computation to check that (recall (1.20))

$$\{G, H_j^{(N)}\} \equiv 0. \quad (7.108)$$

Then

$$\begin{aligned} \partial_t \|w\|_{H^s}^2 &\stackrel{(2.5)}{=} 2\text{Re}(\langle D \rangle^s \dot{w}, \langle D \rangle^s w)_{L^2} \stackrel{(3.22), (7.106), (1.20)}{=} \{G, H^{(N)}\} \stackrel{(7.108)}{=} \{G, H_{\geq N}^{(N)}\} \\ &\stackrel{(7.107), (7.104), (7.105)}{=} 2\text{Re}(\langle D \rangle^s iOp^{\text{BW}}(\mathfrak{m}_N^{(N)}(W; \xi))w, \langle D \rangle^s w)_{L^2} \\ &\quad + 2\text{Re}(\langle D \rangle^s ((\mathcal{Q}_{N,N}(W))_+^\dagger w + (\mathcal{Q}_{N,N}(W))_+^- \overline{w}), \langle D \rangle^s w)_{L^2}. \end{aligned}$$

Since the symbol $\mathfrak{m}_N^{(N)}(W; \xi)$ is real-valued and independent of $x \in \mathbb{T}$ we have

$$\text{Re}(\langle D \rangle^s iOp^{\text{BW}}(\mathfrak{m}_N^{(N)}(W; \xi))w, \langle D \rangle^s w)_{L^2} = 0.$$

By estimate (2.20) on the smoothing remainders and using the Cauchy-Schwarz inequality we get

$$\text{Re}(\langle D \rangle^s ((\mathcal{Q}_{N,N}(W))_+^\dagger w + (\mathcal{Q}_{N,N}(W))_+^- \overline{w}), \langle D \rangle^s w)_{L^2} \lesssim_s \|W\|_{H^s}^{N+2}.$$

Therefore we have obtained $\partial_t \|w\|_{H^s}^2 \lesssim_s \|w\|_{H^s}^{N+2}$, by integrating in t we get the (3.25). \square

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