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Leading hadronic contribution to the muon magnetic anomaly from lattice QCD

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Abstract — Anomalous magnetic moments have guided the evolution of quantum field theory ever since its earliest stages, serving both as a stringent test of the theory at increasingly higher levels of precision and as a possible window to new physics. After decades of perfect agreement, the measured muon magnetic moment (which is known to a precision of about 0.5 parts per million both from theory and experiment) now deviates from the theoretical expectation by around 3.5σ. In order to accentuate or resolve this discrepancy, an experiment at Fermilab is currently underway and is aiming to improve the precision of the measurement to 0.14 ppm. But the theoretical calculation has to be improved as well. The largest source of error are low energy hadronic contributions, that can be evaluated either via a phenomenological approach or ab initio, directly from the standard model Lagrangian, using lattice QCD. We will see how lattice QCD can be used to compute the leading hadronic contribution to the muon magnetic moment: the one induced by hadronic vacuum polarization.

Introduction

The anomalous magnetic moment of the muon, a_μ, is one of the most precisely measured quantities in particle physics. The comparison between its measurement and theoretical expectation provides, at the same time, a severe test of the mathematical framework underlying the Standard Model (SM) and a possible path to new physics.

Short history of a_μ [1, 2, 3] – By definition, the anomalous magnetic moment is $a_\ell = (g_\ell - 2)/2$, where g_ℓ is the g-factor (or gyromagnetic ratio) of a lepton $\ell = e, \mu, \tau$. The g factor appeared in physics around 1925[1]. Earlier that year, Goudsmit and Uhlenbeck [4] postulated that an electron has an intrinsic angular momentum $\mathbf{S}$ with $S_z = \pm\hbar/2$. In analogy with classical electromagnetism where a circulating current due to an orbiting particle of electric charge e and mass m induces a magnetic dipole moment $\boldsymbol{\mu}_L = \frac{e}{2mc}\mathbf{L}$, with $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$ the orbital angular momentum, they associated to the spin $\mathbf{S}$ an intrinsic angular momentum $\boldsymbol{\mu}_S = \frac{e}{2mc}\mathbf{S}$. However, some inconsistencies in the experimental results led Back and Landé to question whether $\boldsymbol{\mu}_S$ couples to a magnetic field in the same way as $\boldsymbol{\mu}_L$. In other words, rewriting $\boldsymbol{\mu}_S = g\frac{e}{2mc}\mathbf{S}$, the question was: does g actually equal 1 as for the orbital gyromagnetic ratio or not? The experimental analysis was inconclusive, so when Pauli formulated his quantum mechanical treatment of the electron spin in 1927, g was left as a free parameter. Just a year later Dirac presented his quantum relativistic theory of the electron, making the unexpected prediction $g = 2$. The first unambiguous experimental confirmation of Dirac's prediction was given by Kinster and Houston in 1934 [5]. Still, the error was quite large and it took 20 more years to establish that g actually exceeds Dirac's expectation by about 0.12%. In 1948 Kusch and Foley [6] published the first precision determination of the electron magnetic moment, $g_e = 2.00238(10)$ (i.e. $a_e \simeq 0.00119$). Meanwhile, with the development of renormalization techniques, Dirac's theory was evolving into quantum electrodynamics (QED). In 1948 Schwinger showed [7] that the anomaly comes from loop corrections to the QED-vertex and computed the leading-order contribution to a_ℓ:

$$a_\ell^{\mathrm{QED}(1)} = \frac{\alpha}{2\pi} \simeq 0.00116 \tag{1}$$

The agreement between Schwinger's evaluation and Kusch and Foley's result was a great success for QED and a key step to establish QFT as the correct framework on which to build a more comprehensive model of fundamental interactions. Today, the whole structure of the SM has to be taken into account in a prediction of a_μ in order to match the current experimental precision.

The reason why a_e and a_μ are still drawing many physicists' attention is that – as theoretical and experimental techniques improved – a tension between SM prediction and measurement started to appear for both a_e and a_μ. Today there is a -2.4σ discrepancy between a_e^{SM} and a_e^{exp} and a 3.5σ discrepancy between a_μ^{SM} and a_μ^{exp}. The latter is one of the most promising signals for new physics and deserves to be investigated further.

Why is a_μ special – e, μ, τ have identical properties, except for their masses and lifetimes. This poses some experimental limitations. For example tau leptons are too short-lived ($\tau_\tau = 3 \times 10^{-15} s$) for a_τ to be measured with present technologies.

• Electrons have infinite lifetime and exist in atoms, so a_e can be measured via spectroscopy of atoms in magnetic fields. An alternative approach, developed by Crane et al. [8], inspired the method later used to

[1] The subscript e is implicit until 1936 (discovery of muons).

measure a_μ. The most recent measurement is [9]

$$a_e^{\text{exp}} = 1159652180.73(28) \times 10^{-12}. \tag{2}$$

• The muon's lifetime is small ($\tau_\mu = 2 \times 10^{-6} s$), but large enough to allow relativistic muons to be stored in a magnetic ring for a sufficiently long time. Also, muons are easy to polarize and it is easy to measure their polarization at the moment they decay. So a_μ can be measured with high precision. The latest measurement dates back to 2006 and was performed by experiment Muon E821 at BNL[10]:

$$a_\mu^{\text{exp}} = 11659208.9(6.3) \times 10^{-10} \tag{3}$$

We see that a_e is about 2200 times more precisely measured than a_μ. However, as pointed out in [11], a_ℓ has a m_ℓ^2/Λ^2 dependence on Λ characterizing the scale of new physics. The masses of charged leptons are $m_e = 0.511$ MeV, $m_\mu = 105.66$ MeV, and $m_\tau = 1776.86$ MeV. This implies that a_μ is about $m_\mu^2/m_e^2 \sim 4 \times 10^4$ more sensitive to new physics than a_e. Ultimately, a_μ turns out to be a better monitor for new physics than a_e.

Measuring a_μ – The new Muon g-2 (E989) experiment is currently taking data at Fermilab. It aims to reduce the experimental uncertainty down to 140 ppb (improving E821's precision by a factor 4). A second experiment is planned at J-PARC that aims to bring uncertainty below 100 ppb. The former exploits the same method used in BNL's experiment (the magnetic ring is actually the same) and it is expected to publish first results in the next months. Muon g-2/EDM at J-PARC will use a new method, with a ultra-cold muon beam stored in a compact magnet. In the standard method (Figure 1) a proton beam is directed on a target to produce pions, which then decay as $\pi^+ \to \mu^+ + \nu_\mu$. Because of parity violation and helicity conservation, muons are highly polarized, with magnetic moment directed along the direction of the flight axis. Muons are then injected into a uniform magnetic field $\mathbf{B}$ where they execute relativistic cyclotron motion with angular frequency $\boldsymbol{\omega}_c = \frac{e\mathbf{B}}{m_\mu c\gamma}$. At the same time, the muons' spins undergo Larmor precession with angular frequency $\boldsymbol{\omega}_s$. The observable we are interested in is

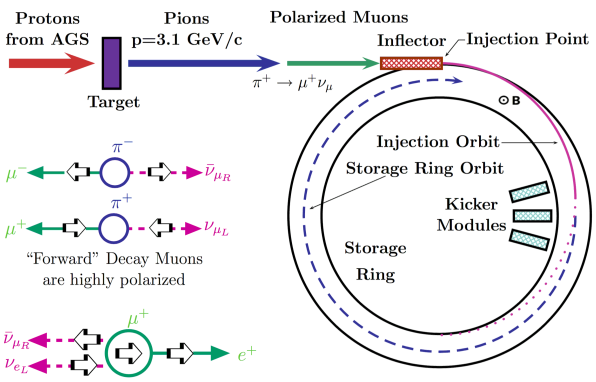

Figure 1: Sketch of BNL E821 experiment

$\omega_a = \omega_s - \omega_c$, that is

$$\boldsymbol{\omega}_a = \frac{e}{m_\mu c}\left[a_\mu \mathbf{B} - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\frac{\mathbf{v} \times \mathbf{E}}{c^2}\right] \tag{4}$$

The second term deflects muons from the horizontal plane of the ring so it has to be eliminated. In Fermilab's experiment, the focusing system requires $\mathbf{E} \neq 0$: the solution is to take $1/(\gamma^2 - 1) = a_\mu$, i.e. $\gamma = 29.3$ or $E_{\text{magic}} = 3.098$GeV. In J-PARC's experiment, the cold beam doesn't need any focusing system, so $\mathbf{E}$ can be set to zero. After some time the muon decays as $\mu^+ \to \nu_e + \bar{\nu}_\mu + e^+$. Each positron keeps its parent-muon's polarization, which can be measured, along with the positron's energy, by calorimethers placed all along the ring. The decay rate follows the model

$$N(E,t) = N_0(E)e^{-\frac{t}{\gamma\tau_\mu}}\left[1 + A(E)\sin(\omega_a t + \phi(E))\right]$$

where $A(E)$ is a known energy-dependent asymmetry factor and the desired ω_a appears as a fit parameter. Finally $a_\mu = \frac{\omega_a m_\mu c}{eB}$.

Computing $\mathbf{a_\mu}$ [1, 12] – We want to understand how a_μ naturally emerges from the SM lagrangian. In particular, we are interested in the motion of a lepton $\ell = e, \mu, \tau$ in a static and homogeneous magnetic field, $A_\mu^{\text{cl}} = (0, \vec{A}^{\text{cl}})$. The S-matrix element for the scattering from A^{cl} is

$$i\mathcal{M}(2\pi)(q^0) = -ie\bar{u}(p')\Gamma^\mu(p',p)u(p) \cdot \tilde{A}^{\text{cl}}(q) \tag{5}$$

where $q = p' - p$, and $\tilde{A}_\mu^{\text{cl}}$ is the Fourier transform of A_μ^{cl}. Poincaré invariance and parity conservation in electromagnetic interactions, and the Dirac equations $\not{p}u(p) = mu(p)$ and $\bar{u}(p)\not{p} = m\bar{u}(p)$ imply that Γ^μ takes the form $\Gamma^\mu(p',p) = A\gamma^\mu + B(p'^\mu + p^\mu) + C(p'^\mu - p^\mu)$, with A, B, and C depending only on scalars (i.e. q^2 and masses). C must be zero or the Ward identity $q_\nu\Gamma^\nu = 0$ wouldn't apply. Finally, using the Gordon identity, Γ^μ can be written as

$$\Gamma^\mu(p,p') = F_1(q^2)\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2) \tag{6}$$

where F_1 and F_2 are two form factors that depend on q^2 and masses. Mind that (6) holds only when both sides are sandwiched between $\bar{u}(p')$ and $u(p)$. Replacing (6) in (5) and taking the non relativistic limit in the specific case of a slowly varying (i.e. $q \to 0$) electrostatic field $A^{\text{cl}}(x) = (\phi(x), 0)$, one can identify $F_1(0)$ with the electric charge in units of e, therefore $F_1(0) = 1$. A similar calculation with a homogeneous magnetic field $A^{\text{cl}}(x) = (0, \mathbf{A}^{\text{cl}}(x))$ proves that

$$g_\ell = 2[F_1(0) + F_2(0)] = 2F_1(0) + 2 \tag{7}$$

$$a_\ell = F_2(q^2 = 0) \tag{8}$$

At tree level, $\Gamma^{\mu,(0)} = \gamma^\mu$, meaning $F_2^{(0)} = 0$. Thus a_ℓ depends only on higher order contributions to Γ^μ. Note that the conservation of electric charge implies $F_1(0) = 1$ at every order in perturbation theory.

State of the art of SM predictions – We split

$a_\mu^{\rm SM}$ into three contributions:

$$a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm Had} \tag{9}$$

• $a_\mu^{\rm QED}$ includes all photonic and leptonic e, μ, τ loops; it is given in perturbation theory by an expansion in α. The coefficients have been computed up to 5 loops [14]:

$$\begin{aligned} a_\mu^{\rm QED} &= \tfrac{\alpha}{2\pi} + 0.765857425(17)\left(\tfrac{\alpha}{\pi}\right)^2 \\ &+ 24.05050996(32)\left(\tfrac{\alpha}{\pi}\right)^3 + 130.8796(63)\left(\tfrac{\alpha}{\pi}\right)^4 + \\ &+ 752.2(1.0)\left(\tfrac{\alpha}{\pi}\right)^5 + ... \\ &= 116584718.92(0.03)\times 10^{-11} \end{aligned} \tag{10}$$

The error is dominated by the uncertainty in α, whose best estimate, $\alpha^{-1} = 137.035999046(27)$, was obtained via the recoil frequency of Cs atoms [13].
• $a_\mu^{\rm EW}$ collects all loop contributions involving at least one of $W^\pm, Z$ and the Higgs. At two loops, $a_\mu^{\rm EW}$ is found to be [14].

$$\begin{aligned} a_\mu^{\rm EW} &= \mathcal{O}\left(\tfrac{G_\mu m_\mu^2}{8\sqrt{2}\pi^2}\right)\left(1 + \mathcal{O}\left(\tfrac{\alpha}{\pi}\right) + ...\right) \\ &= (194.81(.01) - 41.2(1.0) + \mathcal{O}(.1))\times 10^{-11} \\ &= 153.6(1.0)\times 10^{-11} \end{aligned} \tag{11}$$

The 3-loops contribution to $a_\mu^{\rm EW}$ hasn't been evaluated completely yet, but we know that is of order $\mathcal{O}(10^{-12})$.
• $a_\mu^{\rm Had}$ can be separated into a leading-order (LO) contribution and higher-order contributions. The first corresponds to the LO hadronic vacuum polarization (HVP) contribution, the others include next-to-leading-order (NLO) HVP contribution, the light-by-light (LbL) contribution, NNLO-HVP, etc. Grouping all higher-order contributions under the label N(N)LO [14],

$$\begin{aligned} a_\mu^{\rm Had} &= a_\mu^{\rm LO\text{-}HVP} + a_\mu^{\rm N(N)LO\text{-}Had} \\ &= [6939(40) + 19(26)]\times 10^{-11} \end{aligned} \tag{12}$$

There are two ways to calculate the HVP contribution: from first principles via lattice QCD, or via dispersion relation, which requires the knowledge of the R-ratio $R = \frac{\sigma(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}$ from experimental data. At the moment the latter (phenomenological method) gives the most precise result for $a_\mu^{\rm LO\text{-}HVP}$ (the one listed above), but lattice QCD is catching up rapidly.
• Summing all the SM contributions we get [14]

$$a_\mu^{\rm SM} = 116591830(1)(40)(26)\times 10^{-11} \tag{13}$$

where the errors are due to the EW, Had-LO, and Had-N(N)LO, respectively.

Tension between theory and experiment – The difference between (3) and (13) amounts to

$$\Delta a_\mu = a_\mu^{\rm exp} - a_\mu^{\rm SM} = 261(63)_{\rm exp}(48)_{\rm th} \tag{14}$$

Experimental and theoretical errors are comparable, therefore the error on $a_\mu^{\rm SM}$ has to be reduced by a factor 4 in order to match the precision expected by E989 and Muon g-2/EDM. In particular this means reducing the error on $a_\mu^{\rm Had}$ which accounts for more than 98% of the total theoretical error. We will see in next section how $a_\mu^{\rm LO\text{-}HVP}$ can be computed using lattice QCD.

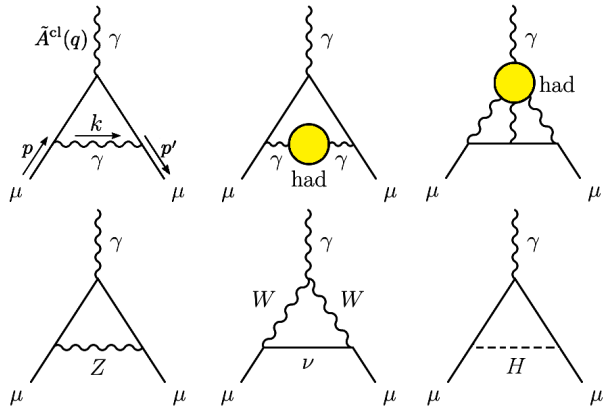

Figure 2: Representative diagrams contributing to $a_\mu^{\rm SM}$. Top: first order QED (Schwinger term), lowest-order hadronic vacuum polarization, hadronic light-by-light. Bottom: the three lowest-order weak contributions.

LO-HVP on the lattice

Lattice QCD is a regularization of QCD. The regularization is performed by enclosing spacetime in a finite and discrete four-dimensional euclidean box of size $L^3 \times T$ and elementary step (lattice spacing) a, thus getting rid of IR and UV divergences. The euclidean action, $\mathcal{S}_E$, has to be discretized as well. There is no unique way to do so. The important thing is that $\mathcal{S}_E$ is recovered when taking the limit for $a \to 0$. The QCD partition function on the lattice looks like

$$\mathcal{Z} = \int \prod_{\mu,x} dU_\mu(x)\det(D[M])e^{-\frac{\beta}{3}\sum_P \Re\mathrm{Tr}\,(1-U_P)} \tag{15}$$

where $U_\mu(x)$ is the gauge field, $\det(D_x[M])$ contains all the information about fermions, $\frac{\beta}{3}\sum_P \Re\mathrm{Tr}\,(1-U_P)$ is the Wilson gauge action (a lattice version of the Yang-Mills gauge action), and the parameter β is connected to the strong coupling, $\alpha_s = g^2/4\pi$ by $\beta = 6/g^2$. $\mathcal{Z}$ in (15) is similar to the partition function of a statistical, discrete system in the canonical ensemble. Therefore, to compute it we can use all the methods available in statistical mechanics, included Monte-Carlo (MC) simulations. The key point of this approach and the reason it works for QCD, is that – as known from asymptotic freedom – the strong coupling runs to a fixed point where $g^2 = 0$ as the energy scale is increased: picking $1/a$ as our energy scale, it means $\lim_{a\to 0} g^2 = 0$, but also, conversely, $\lim_{g^2\to 0} a = 0$. In short, there is a way to recover the continuum limit $a \to 0$ by simply tuning $\beta \to \infty$ in the MC simulation. Since it is not possible to perform a simulation at $\beta = \infty \leftrightarrow g^2 = 0$ (it would be like expecting a statistical system to evolve at zero temperature), the procedure is to measure a chosen observ-

able O at different values of β (as big as possible in the limit of available CPU time), link each β to its respective value of a, fitting the data $O_i(a_i)$ to a polynomial $O_i = c_0 + c_1 \cdot a^2 + \mathcal{O}(a^4)$ (this behaviour is suggested by the renormalization group), and extracting the continuum limit c_0. To fully recover QCD, finite-volume corrections have to be added to the continuum extrapolation c_0. A final remark: in order to give a physical meaning to lattice prediction, we have to fix the parameters of QCD to their physical value. In our case, all the observables depend on four parameters: a (which is a stand-in for α_s), $m_{ud} = m_u = m_d$ (we work in the chiral limit), m_s and m_c (3rd generation quarks are not included: their contribution is much smaller than our statistical error). a is set by w_0 [15], i.e $a = w_0^\phi/(w_0^{\rm lat}/a)$ where w_0^ϕ is the physical value in fm and $(w_0^{\rm lat}/a)$ is the corresponding lattice value expressed in units of a. The quark masses are set such that the meson masses, m_π, m_K and m_{η_c} take their physical value.

Defining LO-HVP on the lattice [19] –. The lowest-order HVP contribution to the anomalous magnetic moment of the muon can be written as [16, 17]

$$a_\mu^{\text{LO-HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w\left(\frac{Q^2}{m_\mu^2}\right) \hat{\Pi}(Q^2) \tag{16}$$

where $w(r) = (r + 2 - \sqrt{r(r+4)})^2/\sqrt{r(r+4)}$ and $\hat{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0)$ is the renormalized scalar polarization function. We define

$$C_{\mu\nu}(t) = \frac{1}{e^2} \int d^3x \langle j_\mu(x) j_\nu(0)\rangle \tag{17}$$

$$= C_{\mu\nu}^{ud}(t) + C_{\mu\nu}^{s}(t) + C_{\mu\nu}^{c}(t) + C_{\mu\nu}^{\rm disc}(t) \tag{18}$$

where $j_\mu/e = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c$. The four terms in (18) correspond to different observables on the lattice. $C_{ii}^f(t)$ allows us to compute $\hat{\Pi}(Q^2)$; the two are related by (see [18])

$$\hat{\Pi}^f(Q^2) = \int_0^\infty dt \left[t^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qt}{2}\right)\right] \frac{\sum_{i=1}^3 \Re C_{ii}^f(t)}{3} \tag{19}$$

where $f = ud, s, c, \text{disc}$. Inserting (19) to (16), performing the integral over Q^2 up to $Q^2_{\max} \ll 2\pi/T$, replacing the integral in t with a sum in steps of a from $t = 0$ to $t = T/2$, replacing $\frac{1}{3}\sum_i C_{ii}^f(t)$ with its discretized, finite-volume version $C_L^f(t) = (a^3/3)\sum_i \sum_{\vec{x}} \langle j_i(x) j_i(0)\rangle$, and averaging $C_L^f(t)$ with $C_L^f(T-t)$ (the lattice has periodic boundary conditions), we get to

$$a_{\mu,f}^{\text{LO-HVP}}[Q < Q^2_{\max}] = \lim_{a\to 0} \lim_{L,T\to\infty} \times$$
$$\times\, \alpha^2 \frac{a}{m_\mu^2} \sum_{t=0}^{T/2} W\left(t m_\mu, Q^2_{\max}/m_\mu^2\right) \Re C_L^f(t) \tag{20}$$

where $W(\tau, x_{\max}) = \int_0^{x_{\max}} dx\, w(x) \left(\tau^2 - \frac{4}{x}\sin^2\frac{\tau\sqrt{x}}{2}\right)$. For each $f = ud, s, c, \text{disc}$, (20) has to be completed by $a_{\mu,f}^{\text{LO-HVP}}[Q \geq Q^2_{\max}]$, evaluated in perturbative QCD.

Analysis and results

Fitting procedure – After computing (20) on a number N of different ensembles (lattices with different $a, L, T, m_{ud}, m_s, m_c$), one has to fit the N measurements and pick the value corresponding to $a = 0$ and physical meson masses m_π, m_K, m_{η_C}. There are many adjustments that can be implemented in the fit model:
- Q-cuts ($Q^2_{\max}$ above which pQCD is used),
- a-cuts (minimum a that an ensemble must have to be included in the fit),
- t-cuts (C_L^f becomes noisy as t increases: the sum of data over t in (20) can be replaced by the average between an upper and a lower bound above a certain t_c),
- choice of functional form for over a-dependence,
- interpolation of results to physical values of the quark masses.

The final result is a weighted average of the extrapolation resulting from all the different procedures (weighted by the procedure's χ^2 via the AIC method).

Additional challenges – The scale setting method has to be carefully developed because the error on the determination of a propagates with a factor ~ 2 in a_μ. Strong-isospin breaking effects have to included. Standard methods to evaluate finite-volume corrections help but are not sufficient to reach the envisaged precision.

Results – Our collaboration (BMW) is now updating the results obtained in [19]. Since the analysis is still still being finalized, only the last published results will be showed. BMW-17 [19] is consistent with both phenomenology and the "no new physics" scenario. Today, the picture from lattice QCD is still not conclusive. Whatever lattice will tell us in the future, the question remains why the $a_\mu^{\text{LO-HVP}}$ obtained via phenomenology is inconsistent with "no new physics", supposing that the upcoming E989 result confirms the previous measurements. In summary, we have to wait just few more months to know if a_μ will be a confirmed hint of new physics or just the umpteenth validation of the SM.

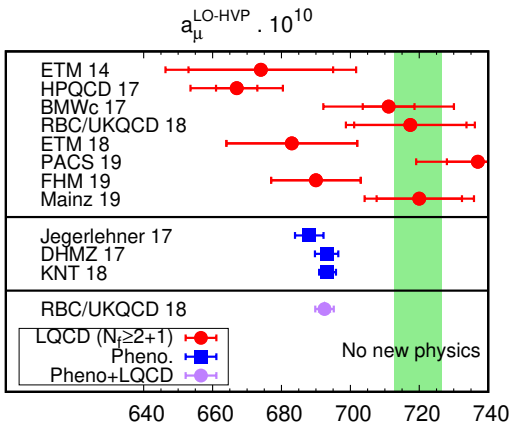

Figure 3: Comparison between lattice and phenomenology predictions of $a_\mu^{\text{LO-HVP}}$ by various collaborations.

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