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► **To cite this version:**

Céline Grandmont, Fabien Vergnet. Existence and uniqueness for a quasi-static interaction problem between a viscous fluid and an active structure. *Journal of Mathematical Fluid Mechanics*, 2021, 23 (45), 10.1007/s00021-020-00552-0 . hal-02493384v2

HAL Id: hal-02493384

<https://hal.science/hal-02493384v2>

Submitted on 9 Jan 2023

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Existence for a quasi-static interaction problem between a viscous fluid and an active structure

Céline Grandmont and Fabien Vergnet

Abstract. We consider a quasi-static fluid-structure interaction problem where the fluid is modeled by the Stokes equations and the structure is an active elastic material. More precisely, the displacement of the structure satisfies the equations of elasticity with an additional active stress, which models the presence of internal biological motors in the structure. Under smallness assumptions on the data, we prove the existence of a solution for this strongly coupled system.

Mathematics Subject Classification (2010). 35A01, 35A02, 35B65, 35D35, 74B99, 74F10, 76D07.

Keywords. Fluid-structure interaction, active structure, strong solution, existence and uniqueness.

1. Introduction

Many biological and physiological phenomena involve active structures interacting with fluids. For example, the mammalian reproduction process relies on the ability of spermatozoa to swim using their flagellum [18]. In the lungs, cilia are responsible for the transport of mucus in the mucociliary clearance process, which protects bronchus from pathogens proliferation [28]. More generally, the universal role played by cilia and flagella in biological transport mechanisms motivates the present study of a fluid-structure interaction problem involving an active material.

Cilia and flagella are soft media that deform themselves thanks to internal motors and induce a flow within the surrounding fluid. Because of their microscopic sizes and their low velocities, these structures evolve at a low Reynolds number regime [25]. Indeed, the typical Reynolds number for fluids, in which microscopic biological organisms (such as flagella and cilia) evolve, lies between 10^{-6} and 10^{-4} . From a modeling point of view, the inertial effects are negligible within the fluid. Moreover, inertia effects within the structure

can also be neglected and we can thus assume an instantaneous answer of the structure to external and internal loads.

In the present article, we consider a fluid-structure interaction problem involving an elastic material, subjected to an internal time-dependent stress, and a Newtonian homogeneous viscous incompressible fluid, modeled by the Stokes equations. The system is consequently time-dependent but non-inertial and we end up with a quasi-static problem for which both the fluid domain and the structure domain depend on the unknown displacement of the structure. Moreover, the fluid and the solid are coupled at the interface and satisfy transmission conditions, namely the continuity of velocities and the continuity of normal stresses. For this coupled problem we prove that, for small enough data (time, external and internal loads), there exists at least one smooth solution and that any small enough solution is unique.

Numerous works have been devoted to the study of the existence of solutions to coupled problems between a fluid and an elastic structure. For problems involving a homogeneous viscous incompressible fluid and a passive elastic structure, several models have been studied. In the case of a 3D linear elastic structure evolving in a 3D viscous incompressible Newtonian flow, we refer the reader to [17] and [8], where the structure is described by a finite number of eigenmodes, or to [5] and [6] for an artificially damped elastic structure. The considered models lead to parabolic-ODE or parabolic-parabolic coupling which avoid the regularity gap induced by the parabolic-hyperbolic coupling of standard fluid-structure interaction problems.

For systems coupling the incompressible Navier-Stokes equations with the equations of linearized elasticity, existence and uniqueness of strong solutions locally in time have been successively obtained in [15], [22], [26] and more recently in [7]. In [15], the authors prove the existence of local strong solutions for small enough data. To that aim, they require high regularity assumptions on the initial data compared to the regularity obtained on the solution. Uniqueness of the solution is obtained under even stronger regularity assumptions. In [22], [26], the existence of local-in-time strong solutions is proven in the case where the fluid-structure interface is flat and the initial displacement of the structure is equal to zero. Once again, there exists a gap between the regularity assumptions on the initial data and the regularity of the solution. One of the main difficulties of such problems is to fill the gap between the fluid (parabolic) and structure (hyperbolic) regularities. This has been partially solved in [7] thanks to the obtention of an hidden regularity property on the structure displacement, in the case of an immersed structure which is initially at rest.

Finally, when the structure is nonlinear, very few results are known on the well-posedness of the fluid-structure system. In [16], the Navier-Stokes equations are coupled to the equations of elastodynamics and the existence of a unique strong solution is proven locally in time under compatibility conditions on the initial data. In [20], the steady state case is studied and the

existence of strong solutions is derived for sufficiently small data, considering either the Stokes or the steady state Navier-Stokes equations.

Regarding the well-posedness of fluid-structure problems involving active structures and homogeneous viscous incompressible fluids, only few results are available. In [19], the steady self-propelled motion of a rigid body in a non-inertial fluid, modeled by either the Stokes or steady state Navier-Stokes equations, is studied. The solid velocity on the fluid-structure interface is divided in two parts: the first one is imposed and represents the self-propelled velocity of the solid, whereas the second one is unknown and depends on the interaction with the surrounding fluid. Existence of solutions is proven. Moreover, conditions under which an imposed distribution of velocities on the interface is able to propel the solid are investigated. In [27], [23], [14] boundary value problems are considered for the swimming of a fish-like deformable structure. Once again, the motion of the structure is split in two parts: the rigid part of the displacement results from the fluid-solid interaction, whereas the elastic deformations are imposed. The resulting system that couples the Navier-Stokes equations for the fluid and Newton's laws for the structure is proven to be well-posed. Existence of weak solutions is derived in [23], whereas [27] (resp. [14]) deals with the existence of a unique local-in-time strong solution in the two-dimensional case (resp. three-dimensional case).

In all these works the strategy is the same: decompose the motion of the structure in two parts. The first part is given and describes the internal activity of the structure, while the second part is an unknown rigid motion which satisfies the Newton's law and is driven by the interaction with the fluid. The originality of the present work lies in the fact that the activity of the structure is modeled by a given internal active stress, so that the whole motion of the structure results from strong interactions between the fluid, the elastic properties of the structure and its internal activity.

Let us now introduce more precisely the fluid-structure problem that we will consider from now on. Let n be the space dimension ($n = 2$ or 3) and Ω be a regular open connected bounded subset of \mathbb{R}^n (satisfying assumptions (\mathbf{H}_1) to (\mathbf{H}_4) below). We assume that the domain Ω is the union of two subdomains: $\Omega = \Omega_s \cup \Omega_f$, with $\Omega_s \cap \Omega_f = \emptyset$. The interface between Ω_s and Ω_f is denoted by Γ and we define the remaining boundaries of Ω_s and Ω_f by $\Gamma_s = \partial\Omega_s \setminus \Gamma$ and $\Gamma_f = \partial\Omega_f \setminus \Gamma$. Examples of such domains are given in Figure 1. Figure 1 (a) shows a sperm cell embedded in a fluid in the case where $|\Gamma_s| = 0$. On the contrary, the case where $|\Gamma_s| \neq 0$ is illustrated in Figure 1 (b), representing cilia attached to a wall.

The domain Ω_s is filled with an active elastic medium. At time t , the behavior of the structure is described by the non-inertial equations of elasticity set in the reference configuration Ω_s . Denoting by d_s the displacement of the structure, the equilibrium equations write

$$\begin{aligned} -\operatorname{div}(\Pi_s(d_s(t), t)) &= f_s(t) && \text{in } \Omega_s, \\ d_s(t) &= 0 && \text{on } \Gamma_s, \end{aligned} \tag{1.1}$$

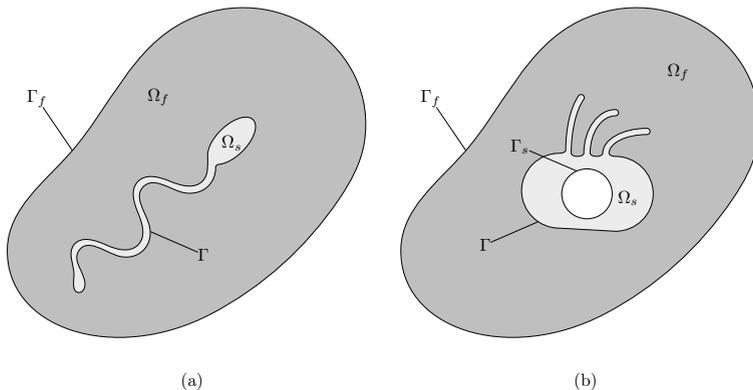


FIGURE 1. Two-dimensional examples of the geometry of the fluid-structure problem, where $|\Gamma_s| = 0$ (a) and $|\Gamma_s| \neq 0$ (b).

where f_s denotes an exterior body force applied on the structure. In the present study, the tensor Π_s , known as the first Piola-Kirchhoff stress tensor, includes the active component. In the framework of continuum mechanics, two popular approaches are used to model contractility of solid materials, namely the active-stress method and the active-strain method (see [2]). The former consists in adding an active component to the passive stress tensor, while in the latter, the activation is modeled by a pre-strain in the multiplicative decomposition of the deformation gradient tensor. Both approaches have been used to describe active biological structures, in particular in the context of myocardium and arteries studies. Yet, to our knowledge, they have never been used to model cilia and flagella (we refer to [24, Chapter 2] for more details on models of active organs). Here, we choose the active-stress formalism to describe the presence of internal biological motors in cilia and we denote by Σ^* the active stress tensor added to the passive component of the stress tensor. Moreover, we do not suppose the tensor Σ^* to have any particular structure. Instead, we consider the general case where it depends both on the time t and the position x . If we consider the Saint Venant-Kirchhoff behavior law to describe the passive component of the elastic medium, the constitutive equations at time t then write

$$\begin{aligned}
 \Pi_s(d_s(t), t) &= (I + \nabla d_s(t))(\Sigma_s(d_s(t)) - \Sigma^*(t)), \\
 \Sigma_s(d_s) &= 2\mu_s E(d_s(t)) + \lambda_s \text{tr}(E(d_s(t)))I, \\
 E(d_s) &= \frac{1}{2}(\nabla d_s(t) + \nabla d_s(t)^T + \nabla d_s(t)^T \cdot \nabla d_s(t)),
 \end{aligned} \tag{1.2}$$

where μ_s and λ_s are Lamé's parameters ($\lambda_s > 0$, $\mu_s > 0$) and I stands for the identity matrix of \mathbb{R}^n . The present active elastic model is particularly interesting for the study of biological suspensions in a viscous fluid since the activity of the structure is an internal activity. In particular, it allows to fully

take into account the fluid-structure interaction, which is not the case when one imposes the velocity or the deformation of the interface as in [19] or [27].

The structure moves under the influence of its internal activity and the action of the surrounding fluid. Then, the deformation of the fluid domain at time t , denoted by $\Phi(d_s(t))$, depends on the displacement of the structure on the interface Γ and satisfies

$$\Phi(d_s(t)) = \mathcal{I} + d_s(t) \quad \text{on } \Gamma,$$

where \mathcal{I} is the identity mapping in \mathbb{R}^n . Moreover, let γ_Γ be the trace operator from Ω_s onto Γ and \mathcal{R} be a continuous linear lifting from Γ into Ω_f (in spaces made precise later on). Then, we extend the definition of the fluid domain deformation $\Phi(d_s(t))$ to the whole domain Ω_f as follows:

$$\Phi(d_s(t)) = \mathcal{I} + \mathcal{R}(\gamma_\Gamma(d_s(t))) \quad \text{in } \Omega_f,$$

assuming that the deformation is equal to the identity on the fluid boundary Γ_f . Thus, $\Phi(d_s(t))$ maps the reference fluid domain Ω_f into the deformed fluid domain at time t denoted by $\Phi(d_s(t))(\Omega_f)$.

The domain Ω_f is filled with a Newtonian fluid of viscosity μ_f . At each time t , the fluid velocity u_f and the fluid pressure p_f satisfy the Stokes equations in the deformed configuration $\Phi(d_s(t))(\Omega_f)$, which write

$$\begin{aligned} -\operatorname{div}(\sigma_f(u_f(t), p_f(t))) &= 0 & \text{in } \Phi(d_s(t))(\Omega_f), \\ \operatorname{div}(u_f(t)) &= 0 & \text{in } \Phi(d_s(t))(\Omega_f), \\ u_f(t) &= 0 & \text{on } \Phi(d_s(t))(\Gamma_f) = \Gamma_f, \end{aligned} \quad (1.3)$$

where

$$\begin{aligned} \sigma_f(u_f(t), p_f(t)) &= 2\mu_f D(u_f(t)) - p_f(t)I, \\ D(u_f(t)) &= \frac{1}{2} (\nabla u_f(t) + \nabla u_f(t)^T). \end{aligned}$$

The tensor $\sigma_f(u_f(t), p_f(t))$ is the fluid stress tensor written in the deformed configuration $\Phi(d_s(t))(\Omega_f)$ and $D(u_f(t))$ denotes the symmetric part of the gradient of $u_f(t)$. Moreover, we assume that no external force is applied to the fluid.

We complete the system of equations (1.1), (1.2) and (1.3), with the usual coupling conditions on the fluid-structure interface Γ , namely the continuity conditions on the velocities:

$$w_f(t) = \frac{\partial d_s}{\partial t}(t) \quad \text{on } \Gamma, \quad (1.4)$$

and the continuity on the normal component of the stress tensors, which stands for the action-reaction principle:

$$\Pi_s(d_s(t), t)n_f = \Pi_f(w_f(t), q_f(t))n_f \quad \text{on } \Gamma. \quad (1.5)$$

The vector n_f is the exterior unit normal vector of $\partial\Omega_f$, while w_f and q_f are respectively the velocity and the pressure of the fluid written in the reference configuration, defined at time t by

$$w_f(t, \cdot) = u_f(t, \Phi(d_s(t)(\cdot))) \quad \text{and} \quad q_f(t, \cdot) = p_f(t, \Phi(d_s(t)(\cdot))) \quad \text{in } \Omega_f.$$

In (1.5), $\Pi_f(w_f(t), q_f(t))$ denotes the fluid stress tensor written in the reference configuration, which will be specified below.

Even though the fluid-structure interaction problem that we consider in the present study is non-inertial, it requires initial data for the displacement of the structure on the interface Γ . This is due to the continuity condition (1.4) on velocities. For the sake of simplicity, as in [7] and [26] for instance, we assume that the fluid-structure interface is initially at rest, i.e. that

$$d_s(0) = 0 \text{ on } \Gamma, \quad (1.6)$$

so that the kinematic boundary condition (1.4) writes

$$d_s(t) = \int_0^t w_f(s) ds \text{ on } \Gamma. \quad (1.7)$$

Note that, even if the fluid-structure interface is initially at rest, it does not imply that the internal structure displacement is equal to zero. Moreover, provided that $d_s(0) = 0$, let us point out that (1.4) and (1.7) are just two different ways to express the same boundary condition, namely the equality of the velocities at the interface. Nevertheless, condition (1.4) can be seen as a Dirichlet boundary condition for the fluid problem (with a possible loss of regularity in time due to the derivation of the structure displacement) whereas condition (1.7) can be seen as a Dirichlet boundary condition for the structure problem. Then, in the decoupling procedure that we shall consider in order to prove the existence of a solution and due to the loss of time regularity just mentioned, we will split the fluid-structure coupled problem by considering the structure part with the Dirichlet boundary condition (1.7) and the fluid part with the Neumann boundary condition (1.5).

At first, we also assume that the internal forcing of the structure initially counterbalances the external load, which means that $f_s(0) = \operatorname{div}(\Sigma^*(0))$. Then, together with the initial condition (1.6), it implies that $d_s(0) = 0$ in Ω_s . Section 7 is devoted to the more general case where $f_s(0) \neq \operatorname{div}(\Sigma^*(0))$.

In order to define the fluid stress tensor Π_f we write the Stokes system (1.3) in the reference configuration. It is standard when analyzing fluid-structure interaction problems since the original Stokes equations are set in an unknown domain $\Phi(d_s(t))(\Omega_f)$. To that aim, we introduce the mappings F , G and H defined by

$$\begin{aligned} F(d_s(t)) &= (\nabla(\Phi(d_s(t))))^{-1}, \\ G(d_s(t)) &= \operatorname{cof}(\nabla(\Phi(d_s(t)))) , \\ H(d_s(t)) &= F(d_s(t))G(d_s(t)). \end{aligned} \quad (1.8)$$

The matrix $F(d_s(t))$ (and *a fortiori* the matrix $H(d_s(t))$) is well defined whenever $\Phi(d_s(t))$ is, for instance, a C^1 -diffeomorphism. As we shall argue, it is the case for small enough displacements in well chosen spaces. Then, using the definitions of the mappings F , G and H , the Stokes equations, written in

the reference configuration, are

$$\begin{aligned}
 -\mu_f \operatorname{div} \left((H(d_s(t))\nabla)w_f(t) + F(d_s(t))^T \nabla w_f(t)^T G(d_s(t)) \right) \\
 + G(d_s(t))\nabla q_f(t) &= 0 && \text{in } \Omega_f, \\
 \operatorname{div}(G(d_s(t))^T w_f(t)) &= 0 && \text{in } \Omega_f, \\
 w_f(t) &= 0 && \text{on } \Gamma_f,
 \end{aligned} \tag{1.9}$$

and the fluid stress tensor, written in the reference configuration at time t , is defined by

$$\begin{aligned}
 \Pi_f(w_f(t), q_f(t)) &= \mu_f \left((H(d_s(t))\nabla)w_f(t) \right. \\
 &\quad \left. + F(d_s(t))^T \nabla w_f(t)^T G(d_s(t)) \right) \\
 &\quad - q_f(t)G(d_s(t)).
 \end{aligned} \tag{1.10}$$

Remark 1.1. The tensor Π_f is the image of the fluid stress tensor σ_f by the Piola transformation.

We are going to prove the existence of strong solutions for the latter coupled problem. To that purpose, in Section 2, we introduce useful notations and prove some technical lemmas. In particular, we show that, for sufficiently small displacements of the structure, the mappings F and H , defined by (1.8), are well-defined, so that the fluid problem (1.9) is also well-defined. In Section 3, we state our main result, namely the existence and the uniqueness (locally in time) of a regular strong solution to equations (1.1), (1.2), (1.9), (1.4), (1.5), (1.6) under smallness assumptions on the data and if the structure is initially at rest. As in [20], the proof is based on a fixed point argument applied to a mapping that iterates between the resolution of the elastic problem and the fluid problem. The difference between the present quasi-static problem compared to the steady state problems, studied in [20], relies on the kinematic condition (1.4) on the interface Γ , which is an unsteady condition. In order to prove the convergence of the iterative process, we thus have to split the fluid-structure problem by solving the structure problem with the Dirichlet boundary condition (1.7) and the fluid problem with the Neumann boundary condition (1.5). By doing so we ensure compactness in time, whereas if one solves the structure problem with the Neumann boundary condition (1.5) and the fluid problem with the Dirichlet boundary condition (1.4) (as in [20]), one loses time regularity in the iterative process. In Section 4 and in Section 5, we study the structure and fluid problems independently. In Section 6, the existence of a unique fixed point procedure is derived showing that, under smallness assumptions on the data, the above mapping goes from a ball into itself and is a contraction. Finally, in Section 7, we extend our result to the case where the internal forcing does not initially counterbalance the exterior load, i.e. where $f_s(0) \neq \operatorname{div}(\Sigma^*(0))$.

2. Notations and preliminaries

2.1. Technical lemma

In this subsection, Ω denotes an open connected bounded subset of \mathbb{R}^n of class $\mathcal{C}^{k-1,1}$, for an integer $k \geq 1$. For $r \geq 0$, the space $H^r(\Omega)$ denotes the standard Sobolev space. Note that, the same notation is used whenever it is a space of real-valued functions or vector-valued functions.

If Γ is a part of the boundary of Ω , we say that Γ is a disjoint part of $\partial\Omega$ if Γ is non empty and $\bar{\Gamma} \cap (\partial\Omega \setminus \bar{\Gamma}) = \emptyset$. Then, if Γ is a disjoint part of $\partial\Omega$, we denote by γ_Γ the trace operator on Γ , which is continuous and surjective from $H^r(\Omega)$ onto $H^{r-1/2}(\Gamma)$ for all $\frac{1}{2} < r \leq k$ (see [9, Chap. 3, Sec. 2.5.3]). In particular, there exists a continuous lifting operator from $H^{r-1/2}(\Gamma)$ into $H^r(\Omega)$. Using these notations, we denote by $H_\Gamma^1(\Omega)$ the space defined by

$$H_\Gamma^1(\Omega) = \{u \in H^1(\Omega); \gamma_\Gamma(u) = 0\}.$$

For $s \geq 0$, the space $H^s(0, T; H^r(\Omega))$ denotes the space of Sobolev-valued functions in the time interval $(0, T)$, with $T > 0$. For $s = 0$ we denote this space by $L^2(0, T; H^r(\Omega))$.

The constant C that appears in all that follows always denotes a positive constant that may change from line to line. However, its dependencies on the fluid and structure domains or on the various parameters will always be made clear.

We start by giving a technical lemma.

Lemma 2.1. *Let Ω be a Lipschitz open connected bounded subset of \mathbb{R}^n , with $n \geq 1$, and consider Γ , a part of its boundary.*

- i) *Let $r > \frac{n}{2}$. If u and v belong to $H^r(\Omega)$, then the product uv belongs to $H^r(\Omega)$ and there exists a constant $C(\Omega)$ which depends on the domain Ω such that,*

$$\|uv\|_{H^r(\Omega)} \leq C(\Omega)\|u\|_{H^r(\Omega)}\|v\|_{H^r(\Omega)}.$$

- ii) *Let $r \geq 0$ and $s > \max(\frac{n}{2}, r)$. If u belongs to $H^s(\Omega)$ and v belongs to $H^r(\Omega)$, then the product uv belongs to $H^r(\Omega)$ and there exists a constant $C(\Omega)$ which depends on the domain Ω such that,*

$$\|uv\|_{H^r(\Omega)} \leq C(\Omega)\|u\|_{H^s(\Omega)}\|v\|_{H^r(\Omega)}.$$

In this case $H^s(\Omega)$ will be designated as a multiplier space of $H^r(\Omega)$.

- iii) *Let $k \in \mathbb{N}^*$ and $T > 0$. Moreover, assume that Ω is of class $\mathcal{C}^{k-1,1}$. If w belongs to $L^2(0, T; H^k(\Omega))$, then the function δ defined by*

$$\delta(t) = \int_0^t \gamma_\Gamma(w(s))ds, \quad \forall t \in [0, T],$$

belongs to $H^1(0, T; H^{k-1/2}(\Gamma))$, and there exists a constant $C(\Omega)$ which depends on the domain Ω such that,

$$\|\delta\|_{L^\infty(0, T; H^{k-1/2}(\Gamma))} \leq C(\Omega)T^{1/2}\|w\|_{L^2(0, T; H^k(\Omega))}.$$

Proof. The proof of point *i*) relies on Sobolev injections and we refer to [1] for a detailed proof. The essential point here is that the space $H^r(\Omega)$ is a Banach algebra, because $2r$ is greater than the dimension n .

Point *ii*) is a consequence of [3, Theorem 7.5] and also relies on embedding theorems for Sobolev spaces.

For point *iii*), a straightforward computation gives us that, for all t in $[0, T]$,

$$\|\delta(t)\|_{H^{k-1/2}(\Gamma)} = \left\| \int_0^t \gamma_\Gamma(\omega(s)) ds \right\|_{H^{k-1/2}(\Gamma)} \leq C(\Omega) t^{1/2} \|\omega\|_{L^2(0, T; H^k(\Omega))},$$

where $C(\Omega)$ is the continuity constant of the trace operator from $H^k(\Omega)$ onto $H^{k-1/2}(\Gamma)$. It follows that

$$\|\delta\|_{L^\infty(0, T; H^{k-1/2}(\Gamma))} \leq C(\Omega) T^{1/2} \|\omega\|_{L^2(0, T; H^k(\Omega))}.$$

Moreover, since $\frac{\partial \delta}{\partial t} = \gamma_\Gamma(\omega)$ belongs to $L^2(0, T; H^{k-1/2}(\Gamma))$, we can conclude that δ belongs to $H^1(0, T; H^{k-1/2}(\Gamma))$. \square

2.2. Assumptions and preliminary results

In the remainder of the present article, we assume that the following assumptions hold true.

- (H₁) The domain Ω is an open connected bounded subset of \mathbb{R}^n ($n \in \{2, 3\}$) divided in two open connected bounded sets Ω_f and Ω_s by an interface Γ .
- (H₂) The interface Γ is of class $\mathcal{C}^{3,1}$, is non empty and does not encounter the boundary of Ω , i.e. $\bar{\Gamma} \cap \bar{\partial\Omega} = \emptyset$.
- (H₃) The remaining boundaries are denoted $\Gamma_f = \partial\Omega_f \setminus \Gamma$ and $\Gamma_s = \partial\Omega_s \setminus \Gamma$ and are of class $\mathcal{C}^{2,1}$.
- (H₄) The boundary Γ_f is such that $|\Gamma_f| \neq 0$, whereas the boundary Γ_s satisfies $|\Gamma_s| \geq 0$.

Remark 2.1. Note that at many steps one could assume a $\mathcal{C}^{2,1}$ regularity of the domains only. Yet the required $\mathcal{C}^{3,1}$ regularity on Γ is used to prove the elliptic regularity of the solution to the fluid problem with mixed Dirichlet and Neumann boundary conditions, which requires more regularity than when considering a pure Dirichlet problem for the Stokes equations (see [9]).

Remark 2.2. Under assumptions (H₁)-(H₄), we see that boundaries Γ , Γ_f and Γ_s are all disjoint parts of $\partial\Omega_f$ and $\partial\Omega_s$. An example of such a domain is given in Figure 1.

Let $T > 0$. The fluid problem written in the reference configuration, defined by (1.9), is well-defined if, for almost every t in $(0, T)$, the displacement of the structure at time t , $d_s(t)$, is sufficiently regular and if the deformation $\Phi(d_s(t))$ is a \mathcal{C}^1 -diffeomorphism that maps Ω_f into $\Phi(d_s(t))(\Omega_f)$. In the next lemma, we show that this is true if the displacement of the structure at time t belongs to the ball $\mathcal{B}_{\mathcal{M}_0}^S$ of $H^3(\Omega_s)$, defined by

$$\mathcal{B}_{\mathcal{M}_0}^S = \{b \in H^3(\Omega_s); \|b\|_{H^3(\Omega_s)} \leq \mathcal{M}_0\}, \quad (2.1)$$

with a small enough constant \mathcal{M}_0 .

Let \mathcal{R} be a linear lifting operator from $H^{5/2}(\Gamma)$ to $H^3(\Omega_f) \cap H_{\Gamma_f}^1(\Omega_f)$ and we recall that γ_Γ is the trace operator on the interface Γ . Since the domains are of class $\mathcal{C}^{2,1}$, they are both continuous operators. We then have the following result

Lemma 2.2. *There exists a constant $\mathcal{M}_0 > 0$ such that for all b in $\mathcal{B}_{\mathcal{M}_0}^S$, we have*

- i) $\nabla(\mathcal{I} + \mathcal{R}(\gamma_\Gamma(b))) = I + \nabla(\mathcal{R}(\gamma_\Gamma(b)))$ is an invertible matrix in $H^2(\Omega_f)$,
- ii) $\Phi(b) = \mathcal{I} + \mathcal{R}(\gamma_\Gamma(b))$ is one to one on $\bar{\Omega}_f$,
- iii) $\Phi(b)$ is a \mathcal{C}^1 -diffeomorphism from Ω_f onto $\Phi(b)(\Omega_f)$,

Proof. It is clear that $\Phi(b) = \mathcal{I} + \mathcal{R}(\gamma_\Gamma(b))$ belongs to $H^3(\Omega_f)$ for all b in $H^3(\Omega_s)$. From Lemma 2.1, point i), we know that $H^2(\Omega_f)$ is a Banach algebra. Thus, if \mathcal{M}_0 is chosen such that

$$\|b\|_{H^3(\Omega_s)} \leq \mathcal{M}_0 \implies \|\nabla(\mathcal{R}(\gamma_\Gamma(b)))\|_{H^2(\Omega_f)} < \frac{1}{C(\Omega_f)},$$

where $C(\Omega_f)$ is defined in Lemma 2.1, point i), then $I + \nabla(\mathcal{R}(\gamma_\Gamma(b)))$ is an invertible matrix in $H^2(\Omega_f)$ and point i) is proven.

For point ii), we know from [13, Theorem 5.5-1] that there exists a constant $C > 0$ such that for all ϕ in $\mathcal{C}^1(\bar{\Omega}_f)$,

$$\|\nabla\phi\|_{\mathcal{C}^0(\bar{\Omega}_f)} \leq C \implies \begin{cases} \det(\nabla(\mathcal{I} + \phi))(x) > 0, \forall x \in \bar{\Omega}_f, \\ \mathcal{I} + \phi \text{ is injective on } \bar{\Omega}_f. \end{cases}$$

Then, because of the continuous embedding of $H^3(\Omega_f)$ into $\mathcal{C}^1(\bar{\Omega}_f)$ (see [1, Theorem 6.3, part III]), and if \mathcal{M}_0 is chosen small enough, this result can be applied to $\mathcal{R}(\gamma_\Gamma(b))$ and we obtain point ii).

Finally, using the continuous embedding of $H^3(\Omega_f)$ into $\mathcal{C}^1(\bar{\Omega}_f)$, the fact that $\det(\nabla\Phi(b))(x) > 0$ for all $x \in \bar{\Omega}_f$ and point ii), we can apply the inverse function theorem and prove point iii). \square

With the previous lemma we know that F and H , defined by (1.8), are well-defined in $H^2(\Omega_f)$ for all d_s in $\mathcal{B}_{\mathcal{M}_0}^S$. We now state another lemma, dealing with the mappings F , G and H .

Lemma 2.3. *The mapping G defined from $H^3(\Omega_s)$ into $H^2(\Omega_f)$ is of class \mathcal{C}^∞ . The mappings F and H are defined from $\mathcal{B}_{\mathcal{M}_0}^S$ into $H^2(\Omega_f)$ and are infinitely differentiable everywhere in $\mathcal{B}_{\mathcal{M}_0}^S$.*

Proof. Let b be in $\mathcal{B}_{\mathcal{M}_0}^S$, then $G(b)$ belongs to $H^2(\Omega_f)$ because $H^2(\Omega_f)$ is a Banach algebra according to Lemma 2.1. The fact that $F(b)$ and $H(d)$ belong to $H^2(\Omega_f)$ is also due to the fact that $H^2(\Omega_f)$ is a Banach algebra, along with the invertible property of $\nabla\Phi(b)$ in $H^2(\Omega_f)$. The mapping G is of class \mathcal{C}^∞ by composition of \mathcal{C}^∞ mappings (such as γ_Γ , \mathcal{R} , ∇ , \det , cof). For the regularity of the mappings F and H , it is sufficient to use the fact that the mapping

$$\begin{aligned} H^2(\Omega_f) &\rightarrow H^2(\Omega_f) \\ M &\mapsto M^{-1} \end{aligned}$$

is infinitely differentiable for all invertible matrices in $H^2(\Omega_f)$ (see [12, chap. I]). \square

Corollary 2.1. *For all b_1 and b_2 in $\mathcal{B}_{\mathcal{M}_0}^S$, we have the following estimates,*

$$\begin{aligned} \|F(b_1) - F(b_2)\|_{H^2(\Omega_f)} &\leq C(\mathcal{M}_0)\|b_1 - b_2\|_{H^3(\Omega_s)}, \\ \|G(b_1) - G(b_2)\|_{H^2(\Omega_f)} &\leq C(\mathcal{M}_0)\|b_1 - b_2\|_{H^3(\Omega_s)}, \\ \|H(b_1) - H(b_2)\|_{H^2(\Omega_f)} &\leq C(\mathcal{M}_0)\|b_1 - b_2\|_{H^3(\Omega_s)}. \end{aligned}$$

where $C(\mathcal{M}_0)$ denotes a positive constant which depends on \mathcal{M}_0 .

Proof. This result is a straightforward application of Lemma 2.3 and the mean value inequality (see [12, Thm. 3.3.2]). The constants appearing in these inequalities are in fact given by

$$\begin{aligned} &\sup_{b \in \mathcal{B}_{\mathcal{M}_0}^S} \|DF(b)\|_{\mathcal{L}(H^3(\Omega_s), H^2(\Omega_f))}, \\ &\sup_{b \in \mathcal{B}_{\mathcal{M}_0}^S} \|DG(b)\|_{\mathcal{L}(H^3(\Omega_s), H^2(\Omega_f))}, \\ &\sup_{b \in \mathcal{B}_{\mathcal{M}_0}^S} \|DH(b)\|_{\mathcal{L}(H^3(\Omega_s), H^2(\Omega_f))}. \end{aligned}$$

For the sake of simplicity in the upcoming computations, we denote them all by $C(\mathcal{M}_0)$. \square

3. Main result

In this section, we state the existence of a local solution (in time) for the fluid-structure interaction system with an active stress term, for small enough applied forces and a small enough internal activity of the structure.

Theorem 3.1. *Let the domains and boundaries Ω_f , Ω_s , Γ_f , Γ_s and Γ be defined by (\mathbf{H}_1) - (\mathbf{H}_4) and let $T > 0$. Consider the force f_s in $L^\infty(0, T; H^1(\Omega_s))$ and the internal activity Σ^* in $L^\infty(0, T; H^2(\Omega_s))$, the data of the problem, such that $f_s(0) = \operatorname{div}(\Sigma^*(0))$. First, we introduce (w_f^0, q_f^0) solution of the following Stokes problem*

$$\begin{aligned} -\operatorname{div}(\sigma_f(w_f^0, q_f^0)) &= 0 && \text{in } \Omega_f, \\ \operatorname{div}(w_f^0) &= 0 && \text{in } \Omega_f, \\ w_f^0 &= 0 && \text{on } \Gamma_f, \\ \sigma_f(w_f^0, q_f^0) \cdot n_f &= -\Sigma^*(0) \cdot n_f && \text{on } \Gamma. \end{aligned} \tag{3.1}$$

Then, let $\mathcal{M}_1 > 0$ and consider the ball $\mathcal{B}_{\mathcal{M}_1}^F$ defined by

$$\begin{aligned} \mathcal{B}_{\mathcal{M}_1}^F = \left\{ (\omega, \pi) \in L^2(0, T; H^3(\Omega_f) \cap H_{\Gamma_f}^1(\Omega_f)) \times L^2(0, T; H^2(\Omega_f)); \right. \\ \left. \|\omega - w_f^0\|_{L^2(0, T, H^3(\Omega_f))} + \|\pi - q_f^0\|_{L^2(0, T, H^2(\Omega_f))} \leq \mathcal{M}_1 \right\}. \end{aligned} \tag{3.2}$$

If the data f_s and Σ^* , the time T and the constant \mathcal{M}_1 are sufficiently small, then there exists a unique solution (w_f, q_f, d_s) to (1.1), (1.2), (1.9), (1.4) and (1.5), with (w_f, q_f) in $\mathcal{B}_{\mathcal{M}_1}^F$ and d_s in $L^\infty(0, T; \mathcal{B}_{\mathcal{M}_0}^S \cap H_{\Gamma_s}^1(\Omega_s))$.

Remark 3.1. To be precise, the data f_s and Σ^* , the time T and the constant \mathcal{M}_1 are considered to be sufficiently small if they satisfy the following conditions, that will appear in the proof of Theorem 3.1, detailed in Section 6:

$$\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T,H^1(\Omega_s))} + R_0 \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))} + C_1 \mathcal{M}_1 T^{1/2} \leq R_1, \quad (3.3)$$

$$C_s^1 \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))} < 1, \quad (3.4)$$

$$\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T,H^1(\Omega_s))} + (R_0 + C_1 C_f T) \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))} + C_1 \mathcal{M}_1 T^{1/2} \leq \frac{\mathcal{M}_0}{C_s^2}, \quad (3.5)$$

$$\begin{aligned} & C_2 \left((1+T) T^{1/2} \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))}^2 \right. \\ & + T^{1/2} \|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T,H^1(\Omega_s))} \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))} \\ & + (\mathcal{M}_1 + T^{1/2}) \|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T,H^1(\Omega_s))} \\ & + (T^{3/2} + \mathcal{M}_1 T + T^{1/2} + \mathcal{M}_1) \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))} \\ & \left. + (T^{1/2} + \mathcal{M}_1) \mathcal{M}_1 T^{1/2} \right) \leq \mathcal{M}_1, \end{aligned} \quad (3.6)$$

$$\begin{aligned} & C_3 \left(\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T,H^1(\Omega_s))} + (1+T) \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))} \right. \\ & \left. + \mathcal{M}_1 T^{1/2} + \frac{T + \mathcal{M}_1 T^{1/2} + T \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))}}{1 - C_s^1 \|\Sigma^*\|_{L^\infty(0,T,H^2(\Omega_s))}} \right) < 1. \end{aligned} \quad (3.7)$$

where R_0 , R_1 , \mathcal{M}_0 , C_s^1 , C_s^2 , C_f , C_1 , C_2 and C_3 are positive constants which only depend on the domains Ω_f and Ω_s , the viscosity of the fluid μ_f and Lamé's parameters μ_s and λ_s .

Remark 3.2. Theorem 3.1 states the existence of solutions to the quasi-static fluid-structure problem with an active structure. Moreover any solution such that $(w_f, q_f) \in \mathcal{B}_{\mathcal{M}_1}^F$ and $d_s \in L^\infty(0, T; \mathcal{B}_{\mathcal{M}_0}^S \cap H_{\Gamma_s}^1(\Omega_s))$ is unique. Note that, if one assumes further time regularity on the data and, to simplify, that the fluid-structure system is initially at rest, it ensures that the solution stays in a neighborhood of zero locally in time (see Remark 6.2) and thus that the solution is unique.

The proof of Theorem 3.1 is based on Banach's fixed point Theorem (see [10, Theorem V.7]). In this scope, we build a mapping \mathcal{S} defined from $\mathcal{B}_{\mathcal{M}_1}^F$ (the ball defined by equation (3.2)) into the space for the velocity and the pressure of the fluid: $L^2(0, T; H^3(\Omega_f)) \times L^2(0, T; H^2(\Omega_f))$. This mapping is defined through a composition of three mappings. The first one associates to a couple (ω, π) in $\mathcal{B}_{\mathcal{M}_1}^F$ a boundary condition δ in $H^1(0, T; H^{5/2}(\Gamma))$. The second one solves an elastic problem with a Dirichlet boundary condition associated to the data (f_s, Σ^*, δ) . The third one solves a fluid problem with mixed Dirichlet

and Neumann boundary conditions. In sum, \mathcal{S} can be represented as the following composition of mappings:

$$\mathcal{S} = \mathcal{O}_3 \circ \mathcal{O}_2 \circ \mathcal{O}_1, \quad (3.8)$$

where each mapping writes

$$\begin{aligned} \mathcal{O}_1 &: \begin{array}{l} \mathcal{B}_{\mathcal{M}_1}^F \rightarrow H^1(0, T; H^{5/2}(\Gamma)) \\ (\omega, \pi) \mapsto \delta \end{array}, \\ \mathcal{O}_2 &: \begin{array}{l} H^1(0, T; H^{5/2}(\Gamma)) \rightarrow L^\infty(0, T; H^3(\Omega_s)) \\ \delta \mapsto d_s \end{array}, \\ \mathcal{O}_3 &: \begin{array}{l} L^\infty(0, T; H^3(\Omega_s)) \rightarrow L^2(0, T; H^3(\Omega_f)) \times L^2(0, T; H^2(\Omega_f)) \\ d_s \mapsto (w_f, q_f) \end{array}, \end{aligned}$$

and will now be defined.

Given a couple (ω, π) in $\mathcal{B}_{\mathcal{M}_1}^F$, the boundary condition δ is build as follows, for all $t \in [0, T]$,

$$\delta(t) = \int_0^t \gamma_\Gamma(\omega(s)) ds.$$

According to Lemma 2.1, δ belongs to $H^1(0, T; H^{5/2}(\Gamma))$. This defines the mapping \mathcal{O}_1 .

Then, with this boundary condition, we consider the following problem in order to obtain an elastic displacement d_s such that, for almost every t in $(0, T)$,

$$\begin{cases} -\operatorname{div}(\Pi_s(d_s(t), t)) = f_s(t) & \text{in } \Omega_s, \\ d_s(t) = \delta(t) & \text{on } \Gamma, \\ d_s(t) = 0 & \text{on } \Gamma_s. \end{cases} \quad (3.9)$$

This elastic problem with internal activity is studied in Section 4. It admits a unique solution if the data (f_s, Σ^*, δ) are small enough, namely if conditions (3.3) and (3.4) are satisfied. This step defines the mapping \mathcal{O}_2 .

Next, to define the mapping \mathcal{O}_3 , we consider a fluid problem obtained from the fluid equations (1.9) through a perturbation argument which writes: find (w_f, q_f) such that, for almost every t in $(0, T)$,

$$\left\{ \begin{array}{l} -\operatorname{div}(\sigma_f(w_f(t), q_f(t))) = -\mu_f \operatorname{div}(((I - H(d_s(t)))\nabla)\omega(t)) \\ \quad -\mu_f \operatorname{div}(\nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t))) \\ \quad \quad \quad + (I - G(d_s(t)))\nabla\pi(t) & \text{in } \Omega_f, \\ \operatorname{div}(w_f(t)) = -\operatorname{div}((I - G(d_s(t)))^T \omega(t)) & \text{in } \Omega_f, \\ w_f(t) = 0 & \text{on } \Gamma_f, \\ \sigma_f(w_f(t), q_f(t))n_f = \Pi_s(d_s(t), t)n_f \\ \quad \quad \quad + \mu_f((I - H(d_s(t)))\nabla\omega(t))n_f \\ \quad \quad \quad + \mu_f(\nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t)))n_f \\ \quad \quad \quad - (\pi(t)(I - G(d_s(t))))n_f & \text{on } \Gamma. \end{array} \right. \quad (3.10)$$

This Stokes problem with mixed Dirichlet and Neumann boundary conditions is studied in Section 5. In particular, problem (3.10) is well-defined since

condition (3.5) ensures that, for almost all t in $(0, T)$, the displacement $d_s(t)$ belongs to the ball $\mathcal{B}_{\mathcal{M}_0}^S$ (defined by (2.1)).

Furthermore, condition (3.6) ensures that the image of the ball $\mathcal{B}_{\mathcal{M}_1}^F$ by the mapping \mathcal{S} is included in $\mathcal{B}_{\mathcal{M}_1}^F$ (i.e. $\mathcal{S}(\mathcal{B}_{\mathcal{M}_1}^F) \subset \mathcal{B}_{\mathcal{M}_1}^F$) and condition (3.7) ensures that \mathcal{S} is a contraction mapping.

We can then apply Banach's fixed point Theorem and conclude that there exists a solution to the fluid-structure interaction problem since any fixed point of \mathcal{S} in $\mathcal{B}_{\mathcal{M}_1}^F$ is a solution of (1.1), (1.2), (1.9), (1.4) and (1.5).

Remark 3.3. In the steady state case studied in [20], the fixed point procedure is done by solving the fluid problem in a given geometry with a Dirichlet boundary condition and the solid problem with a Neumann boundary condition and performing the fixed point argument on the geometry. However, in the present study, where we consider a quasi-static time dependent model, the fixed point is conducted on the velocity and the pressure of the fluid by solving the structure problem with a Dirichlet boundary condition and the fluid problem with mixed Dirichlet and Neumann boundary conditions. Our choice on the fluid-structure splitting is due to the fact that we need time regularity on the solution. Indeed, one instantly loses time regularity if one applies the kinematic coupling condition (1.4) as a Dirichlet boundary condition to the fluid equations.

The remaining of the present article is organized as follows. In Section 4, we consider the system (3.9) and prove its well-posedness and the regularity of its solution. In Section 5, we study equations (3.10) and prove existence, uniqueness and regularity results. In Section 6, the proof of Theorem 3.1 is done using a fixed point procedure on the mapping \mathcal{S} . Finally, in Section 7, a generalization of Theorem 3.1 with less restrictive assumptions on the data is given.

Remark 3.4. In Section 7, we generalize the existence result states at Theorem 3.1 to the more general case for which $f_s(0) \neq \operatorname{div}(\Sigma^*(0))$. This corresponds to a case where the initial internal displacement of the structure is not equal to zero. The case $\gamma_\Gamma(d_s(0)) \neq 0$, however, is more involved. Indeed, the fact that $\gamma_\Gamma(d_s(0)) = 0$ ensures that the fluid domain is initially "at rest", i.e. that the initial fluid domain is the equilibrium configuration of the fluid. This statement is no longer true when $\gamma_\Gamma(d_s(0)) \neq 0$ and the linearization of the fluid problem (3.10) has to be done around the initial geometrical configuration defined by $d_s(0)$. If $\gamma_\Gamma(d_s(0))$ is regular enough, the same regularity result is obtained for the solution of the fluid equations. However, if $\gamma_\Gamma(d_s(0))$ belongs only to $H^{\frac{5}{2}}(\Gamma)$, it would require regularity results for the Stokes problem with mixed boundary conditions in domains with non standard boundary regularity.

4. Structure equations

In this section we study the two or three-dimensional elasticity equations with a non-homogeneous Dirichlet boundary condition, where the solid is described by the nonlinear Saint Venant-Kirchhoff behavior law with an additional active stress. The domain Ω_s satisfies assumptions (\mathbf{H}_1) - (\mathbf{H}_4) . For a given body force f_s , a given Dirichlet boundary condition δ on Γ and a given active stress tensor Σ^* , the considered structure problem writes:

$$\begin{cases} -\operatorname{div}((I + \nabla d)(\Sigma_s(d) - \Sigma^*)) &= f_s & \text{in } \Omega_s, \\ d &= \delta & \text{on } \Gamma, \\ d &= 0 & \text{on } \Gamma_s, \end{cases} \quad (4.1)$$

where the passive stress tensor Σ_s is defined by (1.2). Then, the following lemma states that problem (4.1) admits a unique solution d in $H^3(\Omega_s)$, for small enough data.

Lemma 4.1. *Let Ω_s , Γ_s and Γ be defined by (\mathbf{H}_1) - (\mathbf{H}_4) and suppose that f_s belongs to $H^1(\Omega_s)$, δ belongs to $H^{5/2}(\Gamma)$ and Σ^* belongs to $H^2(\Omega_s)$. There exist three real positive constants R_0 , R_1 and C_s^1 which only depend on the domain Ω_s and Lamé's parameters μ_s and λ_s such that, if the data satisfy*

$$\|f_s - \operatorname{div}(\Sigma^*)\|_{H^1(\Omega_s)} + R_0\|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)} \leq R_1, \quad (4.2)$$

$$C_s^1\|\Sigma^*\|_{H^2(\Omega_s)} < 1, \quad (4.3)$$

then there exists a unique solution d to (4.1) in a neighborhood of 0 in the space $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$. Moreover, there exists a positive constant C_s^2 which only depends on Ω_s , μ_s and λ_s such that the solution can be estimated with respect to the data:

$$\|d\|_{H^3(\Omega_s)} \leq C_s^2(\|f_s - \operatorname{div}(\Sigma^*)\|_{H^1(\Omega_s)} + R_0\|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)}). \quad (4.4)$$

Proof. The proof is based on Banach's fixed point Theorem. We build a mapping \mathcal{T} defined from $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$ into itself which, to all fonction u in $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$, associates the solution to the following passive elastic problem: find a structure displacement d such that

$$\begin{cases} -\operatorname{div}((I + \nabla d)\Sigma_s(d)) = f_s - \operatorname{div}(\Sigma^*) - \operatorname{div}(\nabla u \Sigma^*) & \text{in } \Omega_s, \\ d = \delta & \text{on } \Gamma, \\ d = 0 & \text{on } \Gamma_s. \end{cases} \quad (4.5)$$

More precisely, \mathcal{T} is the following mapping:

$$\mathcal{T} : \begin{array}{ccc} H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s) & \rightarrow & H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s) \\ u & \mapsto & d. \end{array}$$

The objective is to show that the mapping \mathcal{T} has a unique fixed point in $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$, which in turn solves problem (4.1).

The rest of the proof is divided in three parts. Firstly, we prove that, under condition (4.2) on the data, the mapping \mathcal{T} is well-defined, i.e. that problem (4.5) admits a unique solution in $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$. Secondly, we show that under condition (4.3) the mapping \mathcal{T} is a contraction from a

ball in $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$ into itself. Finally, we apply Banach's fixed point Theorem and obtain the desired estimate.

Let us show that (4.5) admits a unique solution in a neighborhood of 0 in $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$ if the data are sufficiently small. A study of elastic problems similar to problem (4.5) has been done in [13] in Sobolev spaces $W^{2,p}$ with $p > n$ and considering a homogeneous Dirichlet boundary condition; our proof essentially uses the same arguments. We introduce the following nonlinear operator of passive elasticity:

$$\begin{aligned} A : H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s) &\rightarrow H^1(\Omega_s) \times H^{5/2}(\Gamma), \\ d &\mapsto (-\operatorname{div}((I + \nabla d)\Sigma_s(d)), \gamma_\Gamma(d)), \end{aligned}$$

where we recall that γ_Γ is the trace operator on Γ . The mapping A is defined since $H^2(\Omega_s)$ is a Banach algebra in both two or three space dimensions (see Lemma 2.1) and is infinitely differentiable since it is a sum of continuous multilinear mappings. As a consequence, problem (4.5) can be written in terms of operator: find d such that

$$A(d) = (\tilde{f}_s, \delta),$$

where

$$\tilde{f}_s := f_s - \operatorname{div}(\Sigma^*) - \operatorname{div}(\nabla u \Sigma^*).$$

Here, we observe that $d = 0$ is a particular solution corresponding to the data $(\tilde{f}_s, \delta) = (0, 0)$. Thus, a natural idea consists in showing that the mapping A is locally invertible in a neighborhood of this particular solution. In order to prove it, we need to check that the differential of A at 0 is an isomorphism between $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$ and $H^1(\Omega_s) \times H^{5/2}(\Gamma)$, so that the implicit function Theorem can be applied. This differential at point 0 is given by

$$\mathcal{D}A(0) \cdot d = (-\operatorname{div}(2\mu_s D(d) + \lambda_s \operatorname{div}(d)I), \gamma_\Gamma(d)),$$

where $D(d) = \frac{1}{2}(\nabla d + \nabla d^T)$ is the symmetric part of the gradient of d . The operator $\mathcal{D}A(0)$ is the linearized elasticity operator and is an isomorphism if for all \tilde{f}_s in $H^1(\Omega_s)$ and for all δ in $H^{5/2}(\Gamma)$, there exists a unique solution in $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$ to the problem: find d such that

$$\begin{cases} -\operatorname{div}(2\mu_s D(d) + \lambda_s \operatorname{div}(d)I) &= \tilde{f}_s & \text{in } \Omega_s, \\ d &= \delta & \text{on } \Gamma, \\ d &= 0 & \text{on } \Gamma_s. \end{cases} \quad (4.6)$$

Because δ belongs to $H^{5/2}(\Gamma)$ and Ω_s is of class $\mathcal{C}^{2,1}$, there exists a lifting of δ in $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$, denoted by $\tilde{\delta}$, such that $\gamma_\Gamma(\tilde{\delta}) = \delta$. Then, the function \tilde{d} defined by

$$\tilde{d} = d - \tilde{\delta}$$

is solution to the problem of linearized elasticity with a homogeneous Dirichlet boundary condition: find \tilde{d} such that,

$$\begin{cases} -\operatorname{div} \left(2\mu_s D(\tilde{d}) + \lambda_s \operatorname{div}(\tilde{d}) I \right) = \\ \tilde{f}_s + \operatorname{div} \left(2\mu_s D(\tilde{\delta}) + \lambda_s \operatorname{div}(\tilde{\delta}) I \right) & \text{in } \Omega_s, \\ \tilde{d} = 0 & \text{on } \partial\Omega_s. \end{cases} \quad (4.7)$$

Problem (4.7) is known as the *linearized pure displacement problem* and has been studied for instance in [13, Theorem 6.3-6]. Because the boundary of Ω_s is of class $\mathcal{C}^{2,1}$ and the right-hand side in the first equation of (4.7) belongs to $H^1(\Omega_s)$, then problem (4.7) admits a unique solution in $H^3(\Omega_s) \cap H_0^1(\Omega_s)$ (see [21, Theorem 2.5.1.1]), satisfying the following continuity estimate

$$\|\tilde{d}\|_{H^3(\Omega_s)} \leq C(\Omega_s, \mu_s, \lambda_s) (\|\tilde{f}_s\|_{H^1(\Omega_s)} + \|\tilde{\delta}\|_{H^3(\Omega_s)}).$$

Furthermore, problem (4.6) admits $\tilde{d} + \tilde{\delta}$ as unique solution.

Hence, the linear continuous operator

$$\mathcal{D}A(0) : H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s) \rightarrow H^1(\Omega_s) \times H^{5/2}(\Gamma)$$

is bijective and its inverse is also continuous, by the closed graph Theorem. Thus, we can apply the implicit function Theorem. Consequently, there exists \mathcal{V}_0 a neighborhood of 0 in $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$ and \mathcal{V}_1 a neighborhood of $(0, 0)$ in $H^1(\Omega_s) \times H^{5/2}(\Gamma)$ such that the mapping A is a \mathcal{C}^1 -diffeomorphism from \mathcal{V}_0 to \mathcal{V}_1 . In particular, there exist two positive constants R_0 and R_1 such that the ball

$$\mathcal{B}_{R_0}^S = \{u \in H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s); \|u\|_{H^3(\Omega_s)} \leq R_0\},$$

satisfies

$$A^{-1} \left(\left\{ (f, \delta) \in H^1(\Omega_s) \times H^{3/2}(\Gamma); \|f\|_{H^1(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)} \leq R_1 \right\} \right) \subset \mathcal{B}_{R_0}^S.$$

Going back to the nonlinear problem (4.5), for all u in $\mathcal{B}_{R_0}^S$, if the couple $(\tilde{f}_s, \tilde{\delta})$ belongs to the space

$$\left\{ (f, \delta) \in H^1(\Omega_s) \times H^{3/2}(\Gamma); \|f\|_{H^1(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)} \leq R_1 \right\},$$

i.e. if the inequality (4.2) is satisfied, there exists a unique solution d in $\mathcal{B}_{R_0}^S$ to problem (4.5). Moreover, this solution can be estimated with respect to the data:

$$\|d\|_{H^3(\Omega_s)} \leq C_s^2 (\|f_s - \operatorname{div}((I + \nabla u)\Sigma^*)\|_{H^1(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)}), \quad (4.8)$$

where the constant C_s^2 is defined by

$$C_s^2 = \sup_{\|(f, \delta)\| \leq R_1} \|\mathcal{D}A^{-1}(f, \delta)\|_{\mathcal{L}(H^1(\Omega_s) \times H^{5/2}(\Gamma), H^3(\Omega_s))}.$$

Thus, we just proved that the mapping \mathcal{T} is well-defined from $\mathcal{B}_{R_0}^S$ into itself, if the data satisfy condition (4.2).

Now, let us show that condition (4.3) ensures that \mathcal{T} is a contraction from $\mathcal{B}_{R_0}^S$ into itself. Under condition (4.2) on the data we have, for all u_1 and u_2 in $\mathcal{B}_{R_0}^S$,

$$\begin{aligned}\mathcal{T}(u_1) &= A^{-1}(f_s - \operatorname{div}((I + \nabla u_1)\Sigma^*), \delta), \\ \mathcal{T}(u_2) &= A^{-1}(f_s - \operatorname{div}((I + \nabla u_2)\Sigma^*), \delta).\end{aligned}$$

Since the mapping A^{-1} is a \mathcal{C}^1 -diffeomorphism from \mathcal{V}_1 to \mathcal{V}_0 , the mean value inequality can be applied. From the mean value inequality and Lemma 2.1, we obtain the following inequality

$$\begin{aligned}\|\mathcal{T}(u_1) - \mathcal{T}(u_2)\|_{H^3(\Omega_s)} &\leq C_s^2 \|(\operatorname{div}((\nabla u_1 - \nabla u_2)\Sigma^*), 0)\|_{H^1(\Omega_s) \times H^{5/2}(\Gamma)}, \\ &\leq C_s^1 \|\Sigma^*\|_{H^2(\Omega_s)} \|u_1 - u_2\|_{H^3(\Omega_s)},\end{aligned}$$

where $C_s^1 = C_s^2 C(\Omega_s)$. It follows that \mathcal{T} is a contraction from $\mathcal{B}_{R_0}^S$ into itself if the active stress Σ^* satisfies inequality (4.3).

So, under conditions (4.2) and (4.3) the mapping \mathcal{T} is a contraction from $\mathcal{B}_{R_0}^S$ into itself. Thus, Banach's fixed point Theorem implies that \mathcal{T} has a unique fixed point d in $\mathcal{B}_{R_0}^S$, which proves, under smallness assumptions on the data, that there exists a unique solution $d \in H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$ to (4.1). Moreover, replacing the fixed point d in (4.8) and using the fact that d belongs to $\mathcal{B}_{R_0}^S$ we obtain the estimate

$$\|d\|_{H^3(\Omega_s)} \leq C_s^2 (\|f_s - \operatorname{div}(\Sigma^*)\|_{H^1(\Omega_s)} + R_0 \|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta\|_{H^{5/2}(\Gamma)}).$$

□

Remark 4.1. In the particular case where $f_s = \operatorname{div}(\Sigma^*)$ and $\delta = 0$, if Σ^* satisfies conditions (4.2) and (4.3), then Lemma 4.1 implies that $d_s = 0$ is the unique solution to problem (4.1).

Using the notation introduced in the proof of Lemma 4.1, we also state the following corollary, which gives a continuity estimate of the solutions of (4.1) with respect to the boundary data.

Corollary 4.1. *Let f_s be in $H^1(\Omega_s)$, Σ^* be in $H^2(\Omega_s)$, and δ_1 and δ_2 be in $H^{5/2}(\Gamma)$. Moreover, suppose that these data satisfy the following conditions:*

$$\begin{aligned}\|f_s - \operatorname{div}(\Sigma^*)\|_{H^1(\Omega_s)} + R_0 \|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta_1\|_{H^{5/2}(\Gamma)} &\leq R_1, \\ \|f_s - \operatorname{div}(\Sigma^*)\|_{H^1(\Omega_s)} + R_0 \|\Sigma^*\|_{H^2(\Omega_s)} + \|\delta_2\|_{H^{5/2}(\Gamma)} &\leq R_1, \\ C_s^1 \|\Sigma^*\|_{H^2(\Omega_s)} &< 1,\end{aligned}$$

where R_0 , R_1 and C_s^1 have been defined in Lemma 4.1. Then there exists a unique solution d_1 of (4.1) in a neighborhood of 0 in $H^3(\Omega_s) \cap H_{\Gamma_s}^1(\Omega_s)$, associated to the data $(f_s, \Sigma^*, \delta_1)$ and there exists a unique solution d_2 to the same problem associated to the data $(f_s, \Sigma^*, \delta_2)$. Furthermore, we have the following estimate:

$$(1 - C_s^1 \|\Sigma^*\|_{H^2(\Omega_s)}) \|d_1 - d_2\|_{H^3(\Omega_s)} \leq C_s^2 \|\delta_1 - \delta_2\|_{H^{5/2}(\Gamma)},$$

where C_s^1 and C_s^2 have been defined in Lemma 4.1.

Proof. The existence and uniqueness of d_1 and d_2 is a direct consequence of Lemma 4.1. Moreover, using the fact that the mapping A^{-1} is everywhere differentiable in \mathcal{V}_1 and according to Lemma 2.1, we have that

$$\begin{aligned} & \|d_1 - d_2\|_{H^3(\Omega_s)} \\ &= \left\| A^{-1}(f - \operatorname{div}(\Sigma^*) - \operatorname{div}(\nabla d_1 \Sigma^*), \delta_1) \right. \\ &\quad \left. - A^{-1}(f - \operatorname{div}(\Sigma^*) - \operatorname{div}(\nabla d_2 \Sigma^*), \delta_2) \right\|_{H^3(\Omega_s)}, \\ &\leq C_s^2 \|(\operatorname{div}((\nabla d_1 - \nabla d_2)\Sigma^*), \delta_1 - \delta_2)\|_{H^1(\Omega_s) \times H^{5/2}(\Gamma)}, \\ &\leq C_s^1 \|\Sigma^*\|_{H^2(\Omega_s)} \|d_1 - d_2\|_{H^3(\Omega_s)} + C_s^2 \|\delta_1 - \delta_2\|_{H^{5/2}(\Gamma)}. \end{aligned}$$

Then, using (4.3) we obtain the desired inequality

$$0 \leq (1 - C_s^1 \|\Sigma^*\|_{H^2(\Omega_s)}) \|d_1 - d_2\|_{H^3(\Omega_s)} \leq C_s^2 \|\delta_1 - \delta_2\|_{H^{5/2}(\Gamma)}.$$

□

5. Fluid equations

In this section, we study the two or three-dimensional Stokes equations with mixed non-homogeneous Dirichlet and Neumann boundary conditions, where the domain Ω_f is defined by (\mathbf{H}_1) - (\mathbf{H}_4) . The cases of pure Dirichlet or Neumann boundary conditions have been, for example, treated in [9], where existence, uniqueness and regularity results have been obtained for the solution of the Stokes problem, depending on the regularity of the domain and the data.

Let f be a given body force, g be a given divergence constraint and h be a given Neumann boundary condition. We consider the Stokes problem: find (u, p) such that

$$\begin{cases} -\operatorname{div}(\sigma_f(u, p)) &= f & \text{in } \Omega_f, \\ \operatorname{div}(u) &= g & \text{in } \Omega_f, \\ u &= 0 & \text{on } \Gamma_f, \\ \sigma_f(u, p)n_f &= h & \text{on } \Gamma, \end{cases} \quad (5.1)$$

where $\sigma_f(u, p)$ is the fluid stress tensor defined by

$$\begin{aligned} \sigma_f(u, p) &= 2\mu_f D(u) - pI, \\ D(u) &= \frac{1}{2}(\nabla u + \nabla u^T). \end{aligned}$$

In the following lemma, we state an existence, uniqueness and regularity result for the solution of problem (5.1). The key argument here is the fact that the boundary Γ_f , where the homogeneous Dirichlet boundary condition is applied, and the boundary Γ , where the Neumann boundary condition is applied, are disjoint parts of $\partial\Omega_f$, i.e. $\bar{\Gamma}_f \cap \bar{\Gamma} = \emptyset$. Indeed, this assumption on the domain Ω_f allows us to easily obtain the regularity of the solution, using existing regularity results on the solutions of the pure Dirichlet boundary problem and the pure Neumann boundary problem, that can be found in [9].

Lemma 5.1. *Let Ω_f , Γ_f and Γ be defined by (\mathbf{H}_1) - (\mathbf{H}_4) and suppose that f belongs to $H^1(\Omega_f)$, g to $H^2(\Omega_f)$ and h to $H^{3/2}(\Gamma)$. Then, the Stokes problem (5.1) admits a unique solution in $(H^3(\Omega_f) \cap H_{\Gamma_f}^1(\Omega_f)) \times H^2(\Omega_f)$. Moreover, there exists a positive constant C_f depending only on Ω_f and μ_f such that*

$$\|u\|_{H^3(\Omega_f)} + \|p\|_{H^2(\Omega_f)} \leq C_f (\|f\|_{H^1(\Omega_f)} + \|g\|_{H^2(\Omega_f)} + \|h\|_{H^{3/2}(\Gamma)}).$$

To prove this lemma, we start by giving a preliminary result concerning the divergence operator, which is based on Bogovskii's result in [4].

Lemma 5.2. *Let Ω_f and Γ_f be defined by (\mathbf{H}_1) - (\mathbf{H}_4) and suppose that g is a function in $L^2(\Omega_f)$. Then, there exists a function u in $H_{\Gamma_f}^1(\Omega_f)$ such that $\operatorname{div}(u) = g$. Moreover, there exists a constant $C(\Omega_f)$ which depends on Ω_f such that,*

$$\|u\|_{H^1(\Omega_f)} \leq C(\Omega_f) \|g\|_{L^2(\Omega_f)}.$$

Proof. We consider an extension of Ω_f , denoted by Ω^* , which strictly contains Ω_f and such that $\Gamma_f \subset \partial\Omega^*$. We define the function g^* by

$$g^*(x) = \begin{cases} g(x) & \text{if } x \in \Omega_f, \\ \frac{-1}{|\Omega^* \setminus \Omega_f|} \int_{\Omega_f} g & \text{if } x \in \Omega^* \setminus \Omega_f, \end{cases}$$

where $|\Omega_f|$ (resp. $|\Omega^* \setminus \Omega_f|$) is the volume of the domain Ω_f (resp. $\Omega^* \setminus \Omega_f$). Then, g^* belongs to $L^2(\Omega^*)$ and has zero mean value over Ω^* . Consequently, we can apply Bogovskii's result [4], which states that there exists a function u^* in $H_0^1(\Omega^*)$ such that $\operatorname{div}(u^*) = g^*$. Moreover, there exists a constant that only depends on the domain Ω^* (then depends on Ω_f) such that

$$\|u^*\|_{H^1(\Omega^*)} \leq C(\Omega_f) \|g^*\|_{L^2(\Omega^*)}.$$

Now, defining u as the restriction of u^* over Ω_f , we obtain that $\operatorname{div}(u) = g$ in Ω_f . Moreover, we have the following estimate:

$$\|u\|_{H^1(\Omega_f)} \leq C(\Omega_f) \|g\|_{L^2(\Omega_f)}.$$

□

We are now able to prove Lemma 5.1.

Proof of Lemma 5.1. We define the saddle-point formulation of problem (5.1) as follows:

$$\begin{cases} \text{find } (u, p) \text{ in } H_{\Gamma_f}^1(\Omega_f) \times L^2(\Omega_f) \text{ such that,} \\ a(u, v) + (p, Bv)_{L^2(\Omega_f)} = l(v) \quad \forall v \in H_{\Gamma_f}^1(\Omega_f), \\ (q, Bu)_{L^2(\Omega_f)} = (q, g)_{L^2(\Omega_f)} \quad \forall q \in L^2(\Omega_f), \end{cases} \quad (5.2)$$

where (\cdot, \cdot) denotes the scalar product in $L^2(\Omega_f)$. The bilinear and linear forms a and l are defined by

$$\begin{aligned} a(u, v) &= \mu_f \int_{\Omega_f} D(u) : D(v), \\ l(v) &= \int_{\Omega_f} f \cdot v + \int_{\Gamma} h \cdot v. \end{aligned}$$

Moreover, B is the divergence operator defined by

$$\begin{aligned} B &: H_{\Gamma_f}^1(\Omega_f) \rightarrow L^2(\Omega_f) \\ u &\mapsto \operatorname{div}(u). \end{aligned}$$

It is easy to check that a is a symmetric coercive continuous bilinear form on $H_{\Gamma_f}^1(\Omega_f) \times H_{\Gamma_f}^1(\Omega_f)$ and that l is a continuous linear form on $H_{\Gamma_f}^1(\Omega_f)$. Moreover, the operator B is a linear continuous operator from $H_{\Gamma_f}^1(\Omega_f)$ into $L^2(\Omega)$ and Lemma 5.2 ensures that B is surjective. Thus, according to Brezzi's result on saddle-point problems (see [11]), there exists a unique solution (u, p) to problem (5.2). Moreover, there exists a constant which depends on Ω_f such that

$$\|u\|_{H^1(\Omega_f)} + \|p\|_{L^2(\Omega_f)} \leq C(\Omega_f)(\|f\|_{H^1(\Omega_f)} + \|g\|_{H^2(\Omega_f)} + \|h\|_{H^{3/2}(\Gamma)}).$$

Hence, since the boundaries Γ_f and Γ are disjoint and because Γ_f is of class $\mathcal{C}^{2,1}$ and Γ is of class $\mathcal{C}^{3,1}$, the rest of the proof, i.e. the regularity of the solution, follows from the method of translations. For local (interior) regularity we refer to Theorem IV.6.1 in [9]. For tangential regularity on the boundary Γ_f , where a Dirichlet boundary condition is applied, we refer to the proof of Theorem IV.5.8 in [9]. For the tangential regularity on the boundary Γ , where a Neumann boundary condition is applied, we refer to the proof of Theorem IV.7.4 in [9], which is presented in three space dimension but is still true in two space dimension. Then, we use the tangential and normal coordinates in a tubular neighborhood of $\partial\Omega_f$ in Ω_f , to deduce the regularity up to the boundary of the solution from the tangential regularity. The desired estimate follows from these results. \square

Remark 5.1. On one hand, in [9], no assumption on the regularity of the domain is necessary to apply the result on the interior regularity of the solution, stated in Theorem IV.6.1, which is also independent of the chosen boundary conditions. On the other hand, Theorem IV.5.8, dealing with pure Dirichlet boundary condition, and Theorem IV.7.4, dealing with pure Neumann boundary condition, require hypotheses on the regularity of the domain and compatibility conditions on the data due to the particular choice of boundary conditions. Yet, these compatibility conditions only appear in the case of pure Dirichlet or pure Neumann boundary conditions, which is not the case here. Nevertheless, the proofs for the regularity of the solution apply in the same way, since the boundaries where Dirichlet and Neumann boundary conditions are applied are disjoint.

6. Fixed point procedure. Proof of Theorem 3.1

In this section, we go back to the coupled fluid-structure problem and apply the results of Section 4 and Section 5 to prove Theorem 3.1.

We recall that we consider $T > 0$, a positive constant \mathcal{M}_1 , a force f_s in the space $L^\infty(0, T; H^1(\Omega_s))$ and an internal activity of the structure Σ^* in the space $L^\infty(0, T; H^2(\Omega_s))$, that satisfy conditions (3.3), (3.4), (3.5), (3.6)

and (3.7). Moreover, the structure is supposed to be initially at rest, i.e. we suppose that $f_s(0) = \operatorname{div}(\Sigma^*(0))$ and that $d_s(0) = 0$ on Γ , which implies that $d_s(0) = 0$ in Ω_s (see Remark 4.1). However, because the active stress tensor Σ^* is not necessarily equal to zero at $t = 0$, neither are the velocity and the pressure of the fluid. This is the reason why we introduce the initial solution of the fluid problem (3.10) denoted by (w_f^0, q_f^0) , which satisfies equations (3.1), that we recall:

$$\left\{ \begin{array}{ll} -\operatorname{div}(\sigma_f(w_f^0, q_f^0)) = 0 & \text{in } \Omega_f, \\ \operatorname{div}(w_f^0) = 0 & \text{in } \Omega_f, \\ w_f^0 = 0 & \text{on } \Gamma_f, \\ \sigma_f(w_f^0, q_f^0)n_f = -\Sigma^*(0)n_f & \text{on } \Gamma. \end{array} \right.$$

This Stokes problem is similar to the one studied in Lemma 5.1. Then, because the domain Ω_f satisfies assumptions (\mathbf{H}_1) - (\mathbf{H}_4) and $\Sigma^*(0) \cdot n_f$ is in $H^{3/2}(\Gamma)$, it follows that equations (3.1) admit a unique solution denoted (w_f^0, q_f^0) in $(H^3(\Omega_f) \cap H_{\Gamma_f}^1(\Omega_f)) \times H^2(\Omega_f)$, which satisfies the following inequality:

$$\|w_f^0\|_{H^3(\Omega_f)} + \|q_f^0\|_{H^2(\Omega_f)} \leq C_f \|\Sigma^*(0)\|_{H^2(\Omega_s)}.$$

Furthermore, by taking the L^2 -norm in time, between 0 and T , we obtain

$$\|w_f^0\|_{L^2(0,T;H^3(\Omega_f))} + \|q_f^0\|_{L^2(0,T;H^2(\Omega_f))} \leq C_f T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}. \quad (6.1)$$

As explained in Section 3, the fixed-point procedure is done on the velocity and the pressure of the fluid, in a neighborhood of the initial fluid state (w_f^0, q_f^0) . This is the reason why we introduced the ball $\mathcal{B}_{\mathcal{M}_1}^F$, defined by (3.2), for which we recall the definition:

$$\mathcal{B}_{\mathcal{M}_1}^F = \left\{ (\omega, \pi) \in L^2(0, T; H^3(\Omega_f) \cap H_{\Gamma_f}^1(\Omega_f)) \times L^2(0, T; H^2(\Omega_f)); \right. \\ \left. \|\omega - w_f^0\|_{L^2(0,T,H^3(\Omega_f))} + \|\pi - q_f^0\|_{L^2(0,T,H^2(\Omega_f))} \leq \mathcal{M}_1 \right\}.$$

Remark 6.1. From (6.1), we deduce that a given couple (ω, π) in $\mathcal{B}_{\mathcal{M}_1}^F$ can be estimated with respect to the constant \mathcal{M}_1 , the time T and the norm of Σ^* as

$$\|\omega\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi\|_{L^2(0,T;H^2(\Omega_f))} \leq \mathcal{M}_1 + C_f T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}. \quad (6.2)$$

The remaining of the proof is divided in three steps. First, we show that the mapping \mathcal{S} , defined by (3.8), is well-defined from $\mathcal{B}_{\mathcal{M}_1}^F$ into itself under condition (3.6). Then, we prove that \mathcal{S} is a contraction mapping if the data satisfy condition (3.7). Finally, we conclude using Banach's fixed point Theorem.

Step 1. Let us show that the mapping \mathcal{S} is well-defined from $\mathcal{B}_{\mathcal{M}_1}^F$ into itself under condition (3.6). We consider a couple (ω, π) in $\mathcal{B}_{\mathcal{M}_1}^F$ and, for all $t \in [0, T]$,

$$\delta(t) = \int_0^t \gamma_\Gamma(\omega(s)) ds.$$

According to Lemma 2.1, δ belongs to $H^1(0, T; H^{5/2}(\Gamma))$ with the following estimate

$$\|\delta\|_{L^\infty(0, T; H^{5/2}(\Gamma))} \leq C_1 T^{1/2} \|\omega\|_{L^2(0, T; H^3(\Omega_f))}, \quad (6.3)$$

where $C_1 = C(\Omega_f)$.

Now, we consider the elastic problem (3.9) associated to the newly constructed boundary condition δ that writes: find d_s such that, for almost every $t \in (0, T)$,

$$\begin{cases} -\operatorname{div}(\Pi_s(d_s(t), t)) &= f_s(t) & \text{in } \Omega_s, \\ d_s(t) &= \delta(t) & \text{on } \Gamma, \\ d_s(t) &= 0 & \text{on } \Gamma_s. \end{cases}$$

Conditions (3.3) and (3.4) guarantee that f_s , Σ^* and δ satisfy conditions (4.2) and (4.3) for almost every t in $(0, T)$, namely

$$\begin{aligned} \|f_s(t) - \operatorname{div}(\Sigma^*(t))\|_{H^1(\Omega_s)} + R_0 \|\Sigma^*(t)\|_{H^2(\Omega_s)} + \|\delta(t)\|_{H^{5/2}(\Gamma)} &\leq R_1, \\ C_s^1 \|\Sigma^*(t)\|_{H^2(\Omega_s)} &< 1, \end{aligned}$$

where R_0 , R_1 and C_s^1 have been introduced in Lemma 4.1. Then, for almost every t in $(0, T)$, Lemma 4.1 ensures the existence of a unique solution d_s in $\mathcal{B}_{R_0}^S$ to problem (3.9). Moreover, d_s can be estimated with respect to the data as follows

$$\begin{aligned} \|d_s(t)\|_{H^3(\Omega_s)} &\leq C_s^2 \left(\|f_s(t) - \operatorname{div}(\Sigma^*(t))\|_{H^1(\Omega_s)} \right. \\ &\quad \left. + R_0 \|\Sigma^*(t)\|_{H^2(\Omega_s)} + \|\delta(t)\|_{H^{5/2}(\Gamma)} \right), \end{aligned}$$

where C_s^2 has been introduced in Lemma 4.1. Using (6.3) and (6.2), it follows that

$$\begin{aligned} \|d_s\|_{L^\infty(0, T; H^3(\Omega_s))} &\leq C_s^2 \left(\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0, T; H^1(\Omega_s))} \right. \\ &\quad \left. + R_0 \|\Sigma^*\|_{L^\infty(0, T; H^2(\Omega_s))} \right. \\ &\quad \left. + C_1 T^{1/2} \|\omega\|_{L^2(0, T; H^3(\Omega_f))} \right), \\ &\leq C_s^2 \left(\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0, T; H^1(\Omega_s))} \right. \\ &\quad \left. + (R_0 + C_1 C_f T) \|\Sigma^*\|_{L^\infty(0, T; H^2(\Omega_s))} \right. \\ &\quad \left. + C_1 T^{1/2} \mathcal{M}_1 \right). \end{aligned} \quad (6.4)$$

The next step is the study of the fluid problem (3.10), that we recall:

$$\left\{ \begin{array}{ll} -\operatorname{div}(\sigma_f(w_f(t), q_f(t))) = -\mu_f \operatorname{div}(((I - H(d_s(t)))\nabla)\omega(t)) \\ \quad -\mu_f \operatorname{div}(\nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t))) \\ \quad \quad + (I - G(d_s(t)))\nabla\pi(t) & \text{in } \Omega_f, \\ \operatorname{div}(w_f(t)) = -\operatorname{div}((I - G(d_s(t)))^t \omega(t)) & \text{in } \Omega_f, \\ w_f(t) = 0 & \text{on } \Gamma_f, \\ \sigma_f(w_f(t), q_f(t))n_f = \Pi_s(d_s(t), t)n_f \\ \quad + \mu_f (((I - H(d_s(t)))\nabla)\omega(t))n_f \\ \quad + \mu_f (\nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t)))n_f \\ \quad \quad - (\pi(t)(I - G(d_s(t))))n_f & \text{on } \Gamma. \end{array} \right.$$

Problem (3.10) is well-defined if $d_s(t)$ belongs to the ball $\mathcal{B}_{\mathcal{M}_0}^S$ (defined by (2.1)) for almost every t . Indeed, the matrix $F(d_s(t))$ is well-defined under this condition (see Lemma 2.3). From estimate (6.4), we see that condition (3.5) ensures that $d_s(t)$ belongs to $\mathcal{B}_{\mathcal{M}_0}^S$, hence that problem (3.10) is well-defined. Moreover, in order to apply Lemma 5.1, we must show that every term in the right-hand side of problem (3.10) is regular enough. The term

$$\begin{aligned} f(t) = & -\mu_f \operatorname{div}(((I - H(d_s(t)))\nabla)\omega(t)) \\ & -\mu_f \operatorname{div}(\nabla w_f(t)^T - F(d_s(t))^T \nabla w_f(t)^T G(d_s(t))) \\ & +(I - G(d_s(t)))\nabla \pi(t), \end{aligned}$$

belongs to $H^1(\Omega_f)$ a.e. in t because $F(d_s(t))$, $G(d_s(t))$ and $H(d_s(t))$ are in $H^2(\Omega_f)$ and because $H^2(\Omega_f)$ is a Banach algebra and a multiplier space of $H^1(\Omega_f)$ (see Lemma 2.1). Moreover $f \in L^2(0, T; H^1(\Omega_f))$. For the same reasons and because of the $H^2(\Omega_f)$ regularity of $\Sigma^*(t)$ a.e. in t , the term

$$\begin{aligned} h(t) = & \Pi_s(d_s(t), t)n_f + \mu_f((I - H(d_s(t)))\nabla)\omega(t)n_f \\ & +\mu_f(\nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t)))n_f \\ & -(\pi(t)(I - G(d_s(t))))n_f \end{aligned}$$

belongs to $H^{3/2}(\Gamma)$ a.e. in t and $h \in L^2(0, T; H^{3/2}(\Gamma))$. In addition, because the matrix $G(d_s(t)) = \operatorname{cof}(\nabla\Phi(d_s(t)))$, the Piola identity (see [13, Chapter I, p 39]) implies that the term

$$g(t) = -\operatorname{div}((I - G(d_s(t)))^\sharp)\omega(t) = (I - G(d_s(t)))^\sharp : \nabla\omega(t)$$

belongs to $H^2(\Omega_f)$ a.e. in t and $g \in L^2(0, T; H^2(\Omega_f))$. Moreover, the domain Ω_f satisfies assumptions (\mathbf{H}_1) - (\mathbf{H}_4) . As a consequence, and according to Lemma 5.1, problem (3.10) admits a unique solution $(w_f(t), q_f(t))$ in the product space $(H^3(\Omega_f) \cap H_{\Gamma}^1(\Omega_f)) \times H^2(\Omega_f)$ for almost every t in $(0, T)$. Therefore, due to the linearity of the Stokes equations (3.10), the couple $(w_f(t) - w_f^0, q_f(t) - q_f^0)$, is also solution of a Stokes problem. Using Lemma 5.1 again, we obtain the following inequality for almost every t in $(0, T)$,

$$\begin{aligned} & \|w_f(t) - w_f^0\|_{H^3(\Omega_f)} + \|q_f(t) - q_f^0\|_{H^2(\Omega_f)} \\ & \leq C_f \left(\|\mu_f \operatorname{div}(((I - H(d_s(t)))\nabla)\omega(t))\|_{H^1(\Omega_f)} \right. \\ & \quad + \|\mu_f \operatorname{div}(\nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t)))\|_{H^1(\Omega_f)} \\ & \quad + \|(I - G(d_s(t)))\nabla\pi(t)\|_{H^1(\Omega_f)} \\ & \quad + \|(I - G(d_s(t)))^\sharp : \nabla\omega(t)\|_{H^2(\Omega_f)} \\ & \quad + \|(\Pi_s(d_s(t), t) + \Sigma^*(0))n_f\|_{H^{3/2}(\Gamma)} \\ & \quad + \|\mu_f(((I - H(d_s(t)))\nabla)\omega(t))n_f\|_{H^{3/2}(\Gamma)} \\ & \quad + \|\mu_f(\nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t)))n_f\|_{H^{3/2}(\Omega_f)} \\ & \quad \left. + \|(\pi(t)(I - G(d_s(t))))n_f\|_{H^{3/2}(\Gamma)} \right). \end{aligned} \tag{6.5}$$

Now, we estimate each term appearing in the right-hand side of the previous inequality. For the first term in the right-hand side of estimate (6.5), using

Lemma 2.1 and Corollary 2.1 and since $d_s(t)$ belongs to $\mathcal{B}_{\mathcal{M}_0}^S$, we have

$$\begin{aligned} & \|\mu_f \operatorname{div}(((I - H(d_s(t)))\nabla)\omega(t))\|_{H^1(\Omega_f)} \\ & \leq C(\Omega_f, \mu_f) \|\nabla\omega(t)\|_{H^2(\Omega_f)} \|I - H(d_s(t))\|_{H^2(\Omega_f)}, \\ & \leq C(\Omega_f, \mathcal{M}_0, \mu_f) \|d_s(t)\|_{H^3(\Omega_s)} \|\omega(t)\|_{H^3(\Omega_f)}. \end{aligned}$$

For the second term in the right-hand side of estimate (6.5), we remark that

$$\begin{aligned} & \nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t)) \\ & = (I - F(d_s(t))^T) \nabla\omega(t)^T G(d_s(t)) - \nabla\omega(t)^T (G(d_s(t)) - I). \end{aligned} \quad (6.6)$$

Then, using (6.6), Lemma 2.1, Corollary 2.1 and the fact that $G(d_s(t))$ is bounded because $d_s(t)$ belongs to $\mathcal{B}_{\mathcal{M}_0}^S$, it follows that

$$\begin{aligned} & \|\mu_f \operatorname{div}(\nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t)))\|_{H^1(\Omega_f)} \\ & \leq C(\Omega_f, \mathcal{M}_0, \mu_f) \left(\|(I - F(d_s(t))^T) \nabla\omega(t)^T\|_{H^2(\Omega_f)} \right. \\ & \quad \left. + \|\nabla\omega(t)^T (I - G(d_s(t)))\|_{H^2(\Omega_f)} \right) \\ & \leq C(\Omega_f, \mathcal{M}_0, \mu_f) \|d_s(t)\|_{H^3(\Omega_s)} \|\omega(t)\|_{H^3(\Omega_f)}. \end{aligned}$$

Similarly, for the third, fourth, sixth, seventh and eighth terms in the right-hand side of estimate (6.5), we use Lemma 2.1, Corollary 2.1 and equation (6.6) (for the seventh term) and it yields

$$\|(I - G(d_s(t)))\nabla\pi(t)\|_{H^1(\Omega_f)} \leq C(\Omega_f, \mathcal{M}_0) \|d_s(t)\|_{H^3(\Omega_s)} \|\pi(t)\|_{H^2(\Omega_f)},$$

$$\|(I - G(d_s(t)^t)) : \nabla\omega(t)\|_{H^2(\Omega_f)} \leq C(\Omega_f, \mathcal{M}_0) \|d_s(t)\|_{H^3(\Omega_s)} \|\omega(t)\|_{H^3(\Omega_f)},$$

$$\begin{aligned} & \|\mu_f(((I - H(d_s(t)))\nabla)\omega(t))n_f\|_{H^{3/2}(\Gamma)} \\ & \leq C(\Omega_f, \mathcal{M}_0, \mu_f) \|d_s(t)\|_{H^3(\Omega_s)} \|\omega(t)\|_{H^3(\Omega_f)}, \end{aligned}$$

$$\begin{aligned} & \|\mu_f(\nabla\omega(t)^T - F(d_s(t))^T \nabla\omega(t)^T G(d_s(t)))n_f\|_{H^{3/2}(\Omega_f)} \\ & \leq C(\Omega_f, \mathcal{M}_0, \mu_f) \|d_s(t)\|_{H^3(\Omega_s)} \|\omega(t)\|_{H^3(\Omega_f)}, \end{aligned}$$

$$\|(\pi(t)(I - G(d_s(t))))n_f\|_{H^{3/2}(\Gamma)} \leq C(\Omega_f, \mathcal{M}_0) \|d_s(t)\|_{H^3(\Omega_s)} \|\pi(t)\|_{H^2(\Omega_f)}.$$

Finally, for the fifth term in the right-hand side of estimate (6.5), we use the continuity of the trace operator from $H^2(\Omega_s)$ to $H^{3/2}(\Gamma)$, the multilinearity property of the operator A (defined in the proof of Lemma 4.1) and Lemma 2.1. We obtain

$$\begin{aligned} & \|(\Pi_s(d_s(t), t) + \Sigma^*(0))n_f\|_{H^{3/2}(\Gamma)} \\ & \leq C(\Omega_s) \|\Pi_s(d_s(t), t) + \Sigma^*(0)\|_{H^2(\Omega_s)}, \\ & \leq C(\Omega_s) \left(\|(I + \nabla d_s(t))\Sigma_s(d_s(t))\|_{H^2(\Omega_s)} + \|\nabla d_s(t)\Sigma^*(t)\|_{H^2(\Omega_s)} \right. \\ & \quad \left. + \|\Sigma^*(t) - \Sigma^*(0)\|_{H^2(\Omega_s)} \right), \\ & \leq C(\Omega_s, R_0, \mu_s, \lambda_s) \|d_s(t)\|_{H^3(\Omega_s)} + C(\Omega_s) \|\nabla d_s(t)\|_{H^2(\Omega_s)} \|\Sigma^*(t)\|_{H^2(\Omega_s)} \\ & \quad + C(\Omega_s) \|\Sigma^*\|_{L^\infty(0, T; H^2(\Omega_s))}, \\ & \leq C(\Omega_s, R_0, \mu_s, \lambda_s) \|d_s(t)\|_{H^3(\Omega_s)} + C(\Omega_s) \|\Sigma^*\|_{L^\infty(0, T; H^2(\Omega_s))} \|d_s(t)\|_{H^3(\Omega_s)} \\ & \quad + C(\Omega_s) \|\Sigma^*\|_{L^\infty(0, T; H^2(\Omega_s))}. \end{aligned}$$

Then, replacing each term in (6.5), it yields

$$\begin{aligned} & \|w_f(t) - w_f^0\|_{H^3(\Omega_f)} + \|q_f(t) - q_f^0\|_{H^2(\Omega_f)} \\ & \leq C \left((\|\omega(t)\|_{H^3(\Omega_f)} + \|\pi(t)\|_{H^2(\Omega_f)}) \|d_s(t)\|_{H^3(\Omega_s)} \right. \\ & \quad \left. + (1 + \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}) \|d_s(t)\|_{H^3(\Omega_s)} \right. \\ & \quad \left. + \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right), \end{aligned}$$

where $C = C(\Omega_f, \Omega_s, R_0, \mathcal{M}_0, \mu_f, \mu_s, \lambda_s)$. Taking the L^2 -norm in time, we obtain

$$\begin{aligned} & \|w_f - w_f^0\|_{L^2(0,T;H^3(\Omega_f))} + \|q_f - q_f^0\|_{L^2(0,T;H^2(\Omega_f))} \\ & \leq C \left((\|\omega\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi\|_{L^2(0,T;H^2(\Omega_f))}) \|d_s\|_{L^\infty(0,T;H^3(\Omega_s))} \right. \\ & \quad \left. + T^{1/2} (1 + \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}) \|d_s\|_{L^\infty(0,T;H^3(\Omega_s))} \right. \\ & \quad \left. + T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right). \end{aligned}$$

Then, using estimate (6.2), we find the following inequality

$$\begin{aligned} & \|w_f - w_f^0\|_{L^2(0,T;H^3(\Omega_f))} + \|q_f - q_f^0\|_{L^2(0,T;H^2(\Omega_f))} \\ & \leq C \left((T^{1/2} + \mathcal{M}_1 + (1 + C_f)T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}) \|d_s\|_{L^\infty(0,T;H^3(\Omega_s))} \right. \\ & \quad \left. + T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right). \end{aligned} \tag{6.7}$$

Now, we estimate the first term in the right-hand side of (6.7), using (6.4):

$$\begin{aligned} & (T^{1/2} + \mathcal{M}_1 + (1 + C_f)T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}) \|d_s\|_{L^\infty(0,T;H^3(\Omega_s))} \\ & \leq C_s^2 \left((1 + C_f)(R_0 + C_1 C_f T) T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}^2 \right. \\ & \quad \left. + (1 + C_f) T^{1/2} \|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right. \\ & \quad \left. + (\mathcal{M}_1 + T^{1/2}) \|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} \right) \\ & + (C_1 C_f T^{3/2} + C_1(1 + 2C_f)\mathcal{M}_1 T + R_0 T^{1/2} + R_0 \mathcal{M}_1) \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\ & \quad + C_1(T^{1/2} + \mathcal{M}_1)\mathcal{M}_1 T^{1/2}, \end{aligned}$$

which rewrites

$$\begin{aligned} & (T^{1/2} + \mathcal{M}_1 + (1 + C_f)T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}) \|d_s\|_{L^\infty(0,T;H^3(\Omega_s))} \\ & \leq C \left((1 + T) T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}^2 \right. \\ & \quad \left. + T^{1/2} \|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right. \\ & \quad \left. + (\mathcal{M}_1 + T^{1/2}) \|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} \right. \\ & \quad \left. + (T^{3/2} + \mathcal{M}_1 T + T^{1/2} + \mathcal{M}_1) \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right. \\ & \quad \left. + (T^{1/2} + \mathcal{M}_1)\mathcal{M}_1 T^{1/2} \right), \end{aligned} \tag{6.8}$$

where $C = C(R_0, C_1, C_f, C_s^2)$. Replacing (6.8) in (6.7) we find

$$\begin{aligned} & \|w_f - w_f^0\|_{L^2(0,T;H^3(\Omega_f))} + \|q_f - q_f^0\|_{L^2(0,T;H^2(\Omega_f))} \\ & \leq C_2 \left((1+T)T^{1/2} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}^2 \right. \\ & + T^{1/2} \|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\ & \quad + (\mathcal{M}_1 + T^{1/2}) \|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} \\ & \quad + (T^{3/2} + \mathcal{M}_1 T + T^{1/2} + \mathcal{M}_1) \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \\ & \quad \left. + (T^{1/2} + \mathcal{M}_1) \mathcal{M}_1 T^{1/2} \right), \end{aligned}$$

where $C_2 = C(\Omega_f, \Omega_s, R_0, \mathcal{M}_0, \mu_f, \mu_s, \lambda_s)$. Therefore, condition (3.6) guarantees that the solution (w_f, q_f) to the fluid problem belongs to $\mathcal{B}_{\mathcal{M}_1}^F$. Hence, the mapping \mathcal{S} is well-defined from $\mathcal{B}_{\mathcal{M}_1}^F$ into $\mathcal{B}_{\mathcal{M}_1}^F$ if the data Σ^* and f_s , the time T and the constant \mathcal{M}_1 satisfy condition (3.6).

Step 2. Now, let us show that \mathcal{S} is a contraction mapping. Let (ω_1, π_1) and (ω_2, π_2) be given in $\mathcal{B}_{\mathcal{M}_1}^F$. We built δ_1 and δ_2 such that, for all $t \in [0, T]$,

$$\begin{aligned} \delta_1(t) &= \int_0^t \gamma_\Gamma(\omega_1(s)) ds, \\ \delta_2(t) &= \int_0^t \gamma_\Gamma(\omega_2(s)) ds. \end{aligned}$$

Applying Lemma 2.1 to the difference $\delta_1 - \delta_2$, it yields

$$\|\delta_1 - \delta_2\|_{L^\infty(0,T;H^{5/2}(\Gamma))} \leq C_1 T^{1/2} \|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))}, \quad (6.9)$$

where C_1 has been introduced in (6.3) and depends on the domain Ω_f .

As before, conditions (3.3) and (3.4) ensure that the data $(f_s, \Sigma^*, \delta_1)$ and $(f_s, \Sigma^*, \delta_2)$ are sufficiently small to apply Lemma 4.1. Thus, there exists a unique solution $d_{s,1}(t)$ to problem (3.9) associated to the data $(f_s, \Sigma^*, \delta_1)$ and a unique solution $d_{s,2}(t)$ associated to the data $(f_s, \Sigma^*, \delta_2)$. Moreover, according to Corollary 4.1 and using (6.9), we also have the estimate:

$$\begin{aligned} & \|d_{s,1} - d_{s,2}\|_{L^\infty([0,T],H^3(\Omega_s))} \\ & \leq \frac{C_s^2 C_1 T^{1/2}}{1 - C_s^1 \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}} \|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))}. \end{aligned} \quad (6.10)$$

Furthermore, condition (3.5) ensures that $d_{s,1}(t)$ and $d_{s,2}(t)$ belong to $\mathcal{B}_{\mathcal{M}_0}^S$ for almost every t in $(0, T)$ and, as before, the two fluid problems of type (3.10), associated with the data $(d_{s,1}, \omega_1, \pi_1)$ and $(d_{s,2}, \omega_2, \pi_2)$ are well-defined for almost every t in $(0, T)$. According to Lemma 5.1, it follows that they both admit a unique solution in $(H^3(\Omega_f) \cap H_{\Gamma_f}^1(\Omega_f)) \times H^2(\Omega_f)$ denoted by $(w_{f,1}(t), q_{f,1}(t))$ and $(w_{f,2}(t), q_{f,2}(t))$ respectively. By linearity of the Stokes problem (3.10), the couple $(w_{f,1}(t) - w_{f,2}(t), q_{f,1}(t) - q_{f,2}(t))$ is

also solution to a Stokes problem, which writes

$$\left\{ \begin{array}{ll} -\mu_f \Delta(w_{f,1}(t) - w_{f,2}(t)) + \nabla(q_{f,1}(t) - q_{f,2}(t)) & = \bar{f}(t) \quad \text{in } \Omega_f, \\ \operatorname{div}(w_{f,1}(t) - w_{f,2}(t)) & = \bar{g}(t) \quad \text{in } \Omega_f, \\ w_{f,1}(t) - w_{f,2}(t) & = 0 \quad \text{on } \Gamma_f, \\ \sigma_f(w_{f,1}(t) - w_{f,2}(t), q_{f,1}(t) - q_{f,2}(t))n_f & = \bar{h}(t) \quad \text{on } \Gamma, \end{array} \right. \quad (6.11)$$

where, \bar{f} , \bar{g} and \bar{h} are defined, for almost every t in $(0, T)$ by

$$\begin{aligned} \bar{f}(t) &= -\mu_f \operatorname{div}(((I - H(d_{s,1}(t)))\nabla)(\omega_1(t) - \omega_2(t))) \\ &\quad + \mu_f \operatorname{div}(((H(d_{s,1}(t)) - H(d_{s,2}(t)))\nabla)\omega_2(t)) \\ &\quad - \mu_f \operatorname{div}(\nabla\omega_1(t)^T - F(d_{s,1}(t))^T \nabla\omega_1(t)^T G(d_{s,1}(t))) \\ &\quad + \mu_f \operatorname{div}(\nabla\omega_2(t)^T - F(d_{s,2}(t))^T \nabla\omega_2(t)^T G(d_{s,2}(t))) \\ &\quad + (I - G(d_{s,1}(t)))\nabla(\pi_1(t) - \pi_2(t)) \\ &\quad - (G(d_{s,1}(t)) - G(d_{s,2}(t)))\nabla\pi_2(t), \end{aligned} \quad (6.12)$$

$$\begin{aligned} \bar{g}(t) &= -\operatorname{div}((I - G(d_{s,1}(t)))^T(\omega_1(t) - \omega_2(t))) \\ &\quad + \operatorname{div}((G(d_{s,1}(t)))^T - G(d_{s,2}(t))^T)\omega_2(t)), \end{aligned}$$

$$\begin{aligned} \bar{h}(t) &= (\Pi_s(d_{s,1}(t)) - \Pi_s(d_{s,2}(t)))n_f \\ &\quad + \mu_f((I - H(d_{s,1}(t)))\nabla(\omega_1(t) - \omega_2(t)))n_f \\ &\quad - \mu_f((H(d_{s,1}(t)) - H(d_{s,2}(t)))\nabla\omega_2(t))n_f \\ &\quad + \mu_f(\nabla\omega_1(t)^T - F(d_{s,1}(t))^T \nabla\omega_1(t)^T G(d_{s,1}(t)))n_f \\ &\quad - \mu_f(\nabla\omega_2(t)^T - F(d_{s,2}(t))^T \nabla\omega_2(t)^T G(d_{s,2}(t)))n_f \\ &\quad - (\pi_1(t) - \pi_2(t))(I - G(d_{s,1}(t)))n_f \\ &\quad + \pi_2(t)(G(d_{s,1}(t)) - G(d_{s,2}(t)))n_f. \end{aligned}$$

We recall that matrices $F(d_{s,1}(t))$, $F(d_{s,2}(t))$, $G(d_{s,1}(t))$, $G(d_{s,2}(t))$, $H(d_{s,1}(t))$ and $H(d_{s,2}(t))$ are in $H^2(\Omega_f)$ for a.e. $t \in (0, T)$. Moreover the space $H^2(\Omega_f)$ is a Banach algebra and a multiplier space of $H^1(\Omega_f)$. Therefore we can show that $\bar{f}(t)$ belongs to $H^1(\Omega_f)$, $\bar{g}(t)$ belongs to $H^2(\Omega_f)$ and the function $\bar{h}(t)$ belongs to $H^{5/2}(\Gamma)$ for a.e. $t \in (0, T)$. Thus, applying Lemma 5.1, the solution to problem (6.11) is unique and satisfies the estimate

$$\begin{aligned} &\|w_{f,1}(t) - w_{f,2}(t)\|_{H^3(\Omega_f)} + \|q_{f,1}(t) - q_{f,2}(t)\|_{H^2(\Omega_f)} \\ &\leq C_f(\|\bar{f}(t)\|_{H^1(\Omega_f)} + \|\bar{g}(t)\|_{H^2(\Omega_f)} + \|\bar{h}(t)\|_{H^{3/2}(\Gamma)}). \end{aligned} \quad (6.13)$$

Let us now estimate each term in the right-hand side of (6.13). First, we remark that

$$\begin{aligned} &\nabla\omega_1(t)^T - F(d_{s,1}(t))^T \nabla\omega_1(t)^T G(d_{s,1}(t)) \\ &\quad - \nabla\omega_2(t)^T + F(d_{s,2}(t))^T \nabla\omega_2(t)^T G(d_{s,2}(t)) \\ &= (I - F(d_{s,1}(t))^T)(\nabla\omega_1(t)^T - \nabla\omega_2(t)^T)G(d_{s,1}(t)) \\ &\quad + (F(d_{s,2}(t))^T - F(d_{s,1}(t))^T)\nabla\omega_2(t)^T G(d_{s,1}(t)) \\ &\quad - F(d_{s,2}(t))^T \nabla\omega_2(t)^T (G(d_{s,1}(t)) - G(d_{s,2}(t))) \\ &\quad + (\nabla\omega_1(t)^T - \nabla\omega_2(t)^T)(I - G(d_{s,1}(t))). \end{aligned} \quad (6.14)$$

Then, injecting (6.14) in (6.12) and using Lemma 2.1, Corollary 2.1 and the fact that $F(d_{s,2}(t))$ and $G(d_{s,1}(t))$ are bounded since $d_{s,1}(t)$ and $d_{s,2}(t)$ belong to $\mathcal{B}_{\mathcal{M}_0}^S$, \bar{f} satisfies

$$\begin{aligned}
 & \|\bar{f}(t)\|_{H^1(\Omega_f)} \leq \\
 C & \left(\|d_{s,1}(t)\|_{H^3(\Omega_s)} (\|\omega_1(t) - \omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_1(t) - \pi_2(t)\|_{H^2(\Omega_f)}) \right. \\
 & \left. + \|d_{s,1}(t) - d_{s,2}(t)\|_{H^3(\Omega_s)} (\|\omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_2(t)\|_{H^2(\Omega_f)}) \right),
 \end{aligned}$$

where $C = C(\Omega_f, \mathcal{M}_0, \mu_f)$. Using Lemma 2.1 and Corollary 2.1, \bar{g} satisfies the inequality

$$\begin{aligned}
 \|\bar{g}(t)\|_{H^2(\Omega_f)} \leq & C(\Omega_f, \mathcal{M}_0) \left(\|d_{s,1}(t)\|_{H^3(\Omega_s)} \|\omega_1(t) - \omega_2(t)\|_{H^3(\Omega_f)} \right. \\
 & \left. + \|d_{s,1}(t) - d_{s,2}(t)\|_{H^3(\Omega_s)} \|\omega_2(t)\|_{H^3(\Omega_f)} \right).
 \end{aligned}$$

Then, using (6.14), the continuity property of the trace operator from $H^2(\Omega_s)$ into $H^{3/2}(\Gamma)$ and from $H^2(\Omega_f)$ into $H^{3/2}(\Gamma)$, Lemma 2.1 and Corollary 2.1, \bar{h} can be bounded by

$$\begin{aligned}
 & \|\bar{h}(t)\|_{H^{3/2}(\Gamma)} \leq \\
 & C \left(1 + \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right) \|d_{s,1}(t) - d_{s,2}(t)\|_{H^3(\Omega_s)} \\
 + C' & \left(\|d_{s,1}(t)\|_{H^3(\Omega_s)} (\|\omega_1(t) - \omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_1(t) - \pi_2(t)\|_{H^2(\Omega_f)}) \right. \\
 & \left. + \|d_{s,1}(t) - d_{s,2}(t)\|_{H^3(\Omega_s)} (\|\omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_2(t)\|_{H^2(\Omega_f)}) \right),
 \end{aligned}$$

where $C = C(\Omega_s, R_0, \mu_s, \lambda_s)$ and $C' = C(\Omega_f, \mathcal{M}_0, \mu_f)$. Then, replacing all three inequalities in (6.13), yields

$$\begin{aligned}
 & \|w_{f,1}(t) - w_{f,2}(t)\|_{H^3(\Omega_f)} + \|q_{f,1}(t) - q_{f,2}(t)\|_{H^2(\Omega_f)} \\
 & \leq C \left(1 + \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right) \|d_{s,1}(t) - d_{s,2}(t)\|_{H^3(\Omega_s)} \\
 + C' & \left(\|d_{s,1}(t)\|_{H^3(\Omega_s)} (\|\omega_1(t) - \omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_1(t) - \pi_2(t)\|_{H^2(\Omega_f)}) \right. \\
 & \left. + \|d_{s,1}(t) - d_{s,2}(t)\|_{H^3(\Omega_s)} (\|\omega_2(t)\|_{H^3(\Omega_f)} + \|\pi_2(t)\|_{H^2(\Omega_f)}) \right),
 \end{aligned}$$

where $C = C_f C(\Omega_s, R_0, \mu_s, \lambda_s)$ and $C' = C_f C(\Omega_f, \mathcal{M}_0, \mu_f)$. Taking the L^2 -norm in time we obtain

$$\begin{aligned}
 & \|w_{f,1} - w_{f,2}\|_{L^2(0,T;H^3(\Omega_f))} + \|q_{f,1} - q_{f,2}\|_{L^2(0,T;H^2(\Omega_f))} \\
 & \leq C \left(T^{1/2} \left(1 + \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right) \|d_{s,1} - d_{s,2}\|_{L^\infty(0,T;H^3(\Omega_s))} \right. \\
 + \|d_{s,1}\|_{L^\infty(0,T;H^3(\Omega_s))} & \left(\|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_1 - \pi_2\|_{L^2(0,T;H^2(\Omega_f))} \right) \\
 + \|d_{s,1} - d_{s,2}\|_{L^\infty(0,T;H^3(\Omega_s))} & \left(\|\omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_2\|_{L^2(0,T;H^2(\Omega_f))} \right) \Big),
 \end{aligned}$$

where $C = C(\Omega_f, \Omega_s, \mathcal{M}_0, R_0, \mu_f, \mu_s, \lambda_s)$. Then, using estimate (6.2) on the couple (ω_2, π_2) which belongs to $\mathcal{B}_{\mathcal{M}_1}^F$,

$$\begin{aligned}
 & \|w_{f,1} - w_{f,2}\|_{L^2(0,T;H^3(\Omega_f))} + \|q_{f,1} - q_{f,2}\|_{L^2(0,T;H^2(\Omega_f))} \\
 & \leq C \left(T^{1/2} \left(1 + \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right) \|d_{s,1} - d_{s,2}\|_{L^\infty(0,T;H^3(\Omega_s))} \right. \\
 + \|d_{s,1}\|_{L^\infty(0,T;H^3(\Omega_s))} & \left(\|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_1 - \pi_2\|_{L^2(0,T;H^2(\Omega_f))} \right) \\
 + \|d_{s,1} - d_{s,2}\|_{L^\infty(0,T;H^3(\Omega_s))} & \left(\mathcal{M}_1 + C_f \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} T^{1/2} \right) \Big),
 \end{aligned}$$

where $C = C(\Omega_f, \Omega_s, \mathcal{M}_0, R_0, \mu_f, \mu_s, \lambda_s)$. Finally, using estimate (6.10) applied to the difference $d_{s,1} - d_{s,2}$ and estimate (6.4) to the displacement $d_{s,1}$,

we obtain

$$\begin{aligned} & \|w_{f,1} - w_{f,2}\|_{L^2(0,T;H^3(\Omega_f))} + \|q_{f,1} - q_{f,2}\|_{L^2(0,T;H^2(\Omega_f))} \\ & \leq C \left((\|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_1 - \pi_2\|_{L^2(0,T;H^2(\Omega_f))}) \right. \\ & \quad \times C_s^2 \left(\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} + (R_0 + C_1 C_f T) \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} \right. \\ & \quad \left. \left. + C_1 \mathcal{M}_1 T^{1/2} \right) \right. \\ & \quad \left. + C_s^2 C_1 \frac{T + \mathcal{M}_1 T^{1/2} + (1 + C_f) T \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}}{1 - C_s^1 \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}} \|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))} \right), \end{aligned}$$

which rewrites

$$\begin{aligned} & \|w_{f,1} - w_{f,2}\|_{L^2(0,T;H^3(\Omega_f))} + \|q_{f,1} - q_{f,2}\|_{L^2(0,T;H^2(\Omega_f))} \\ & \leq C_3 \left((\|\omega_1 - \omega_2\|_{L^2(0,T;H^3(\Omega_f))} + \|\pi_1 - \pi_2\|_{L^2(0,T;H^2(\Omega_f))}) \right. \\ & \quad \times \left(\|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0,T;H^1(\Omega_s))} + (1 + T) \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))} + \mathcal{M}_1 T^{1/2} \right. \\ & \quad \left. \left. + \frac{T + \mathcal{M}_1 T^{1/2} + T \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}}{1 - C_s^1 \|\Sigma^*\|_{L^\infty(0,T;H^2(\Omega_s))}} \right) \right), \end{aligned}$$

where $C_3 = C(\Omega_f, \Omega_s, R_0, \mathcal{M}_0, \mu_f, \mu_s, \lambda_s)$. Therefore, we see that condition (3.7) guarantees that the mapping \mathcal{S} is a contraction.

Step 3. To conclude, we have proved that, *i*) under conditions (3.3), (3.4) and (3.5), the mapping \mathcal{S} is well-defined from the ball $\mathcal{B}_{\mathcal{M}_1}^F$ into the product space $L^2(0, T; H^3(\Omega_f)) \times L^2(0, T; H^2(\Omega_f))$, *ii*) under condition (3.6), the image of $\mathcal{B}_{\mathcal{M}_1}^F$ by \mathcal{S} is included in $\mathcal{B}_{\mathcal{M}_1}^F$ and *iii*) under condition (3.7), \mathcal{S} is a contraction mapping. Moreover, $\mathcal{B}_{\mathcal{M}_1}^F$ is a bounded closed subset of a Banach space. Consequently, we apply Banach's fixed point Theorem and conclude that the mapping \mathcal{S} has a unique fixed point in $\mathcal{B}_{\mathcal{M}_1}^F$. This proves the existence, for small enough forces and a small enough time, of a regular solution to the fluid-structure interaction system (1.1), (1.2), (1.9), (1.4) and (1.5). Moreover any solution such that $(w_f, q_f) \in \mathcal{B}_{\mathcal{M}_1}^F$ and $d_s \in L^\infty(0, T; \mathcal{B}_{\mathcal{M}_0}^S \cap H_{\Gamma_s}^1(\Omega_s))$ is unique. \square

Remark 6.2. Note that here we have both smallness conditions on the time and on the amplitude of the applied forces. This is due to the quasi-static nature of the problem, which involves elliptic problems coupled through a kinematic condition at the interface. Nevertheless, assuming further time regularity on the data, namely assuming that $f_s \in \mathcal{C}^{0,\alpha}([0, T]; H^1(\Omega_s))$ and $\Sigma^* \in \mathcal{C}^{0,\alpha}([0, T]; H^2(\Omega_s))$ with, to simplify, $f_s(0) = 0$ and $\Sigma^*(0) = 0$, one could prove, following the very same lines as in the previous proof, a local-in-time existence result and ensure that the solution stays in a neighborhood of zero for T small enough. Thus, this solution would be unique.

7. More general conditions on the data

In this section, we relax the assumption on the initial configuration of the system, $f_s(0) = \operatorname{div}(\Sigma^*(0))$. This means that the structure is not initially at

rest. Then, we show that the result stated in Theorem 3.1 holds true, i.e. that, under some smallness conditions on the data, the time T and the constant \mathcal{M}_1 , there exists a solution (w_f, q_f, d_s) to the coupled fluid-structure system (1.1), (1.2), (1.9), (1.4) and (1.5).

Theorem 7.1. *Let the domains Ω_f and Ω_s and the boundaries Γ_f, Γ_s and Γ be defined by assumptions (\mathbf{H}_1) - (\mathbf{H}_4) and let $T > 0$ and $0 < \varepsilon < T$. Consider f_s in $L^\infty(0, T; H^1(\Omega_s))$ and Σ^* in the space $L^\infty(0, T; H^2(\Omega_s))$, the data of the problem. Let $\mathcal{M}_1 > 0$ and consider the ball $\mathcal{B}_{\mathcal{M}_1}^{F, \varepsilon}$ defined by*

$$\mathcal{B}_{\mathcal{M}_1}^{F, \varepsilon} = \{(\omega, \pi) \in L^2(0, T - \varepsilon; H^3(\Omega_f) \cap H_{\Gamma_f}^1(\Omega_f)) \times L^2(0, T - \varepsilon; H^2(\Omega_f)); \\ \|\omega - w_f^0\|_{L^2(0, T - \varepsilon, H^3(\Omega_f))} + \|\pi - q_f^0\|_{L^2(0, T - \varepsilon, H^2(\Omega_f))} \leq \mathcal{M}_1\},$$

where (w_f^0, q_f^0) is the initial state of the fluid, solution to (3.1).

There exists positive constants $R_0, R_1, \mathcal{M}_0, C_s^1, C_s^2, C_f, C_1, C_2$ and C_3 , which only depend on the domains Ω_f and Ω_s , the viscosity of the fluid μ_f and Lamé's parameters μ_s and λ_s such that, if the data f_s and Σ^* , the time T and the constant \mathcal{M}_1 satisfy conditions (3.3), (3.4), (3.5), (3.6) and (3.7), then, there exists a unique solution (w_f, q_f, d_s) to (1.1), (1.2), (1.9), (1.4) and (1.5), with (w_f, q_f) in $\mathcal{B}_{\mathcal{M}_1}^{F, \varepsilon}$ and d_s in the space $L^\infty(0, T - \varepsilon; \mathcal{B}_{\mathcal{M}_0}^S \cap H_{\Gamma_s}^1(\Omega_s))$.

The idea of the proof of Theorem 7.1 is inspired by the *incremental method*, often used for the numerical resolution of elastic problems involving large deformations. In [13, sec. 6.10], Ciarlet describes it as a method which consists in “letting the forces vary by small increments from zero to the given ones and to compute corresponding approximate solutions by successive linearization”. Actually, in the context of numerical simulation, it allows to compute the displacement of a structure whose equilibrium position is “far” from its reference position, and that could not be obtained directly. Here, this trick is used to apply Theorem 3.1 on a slightly different problem whose data, f_s^ε and Σ_ε^* , satisfy the condition $f_s^\varepsilon(0) = \operatorname{div}(\Sigma_\varepsilon^*(0))$. Then, we recover the solution associated with the true data, f_s and Σ^* .

Proof. Let us introduce a body force f_s^ε and an internal activity Σ_ε^* , defined by

$$f_s^\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} f_s(0)t & \text{if } t \leq \varepsilon, \\ f_s(t - \varepsilon) & \text{for almost every } t \text{ in } (\varepsilon, T), \end{cases}$$

$$\Sigma_\varepsilon^*(t) = \begin{cases} \frac{1}{\varepsilon} \Sigma^*(0)t & \text{if } t \leq \varepsilon, \\ \Sigma^*(t - \varepsilon) & \text{for almost every } t \text{ in } (\varepsilon, T). \end{cases}$$

With these definitions, f_s^ε belongs to the space $L^\infty(0, T; H^1(\Omega_s))$, Σ_ε^* belongs to the space $L^\infty(0, T; H^2(\Omega_s))$ and the condition $f_s^\varepsilon(0) = \operatorname{div}(\Sigma_\varepsilon^*(0)) = 0$ is satisfied. Moreover,

$$\|f_s^\varepsilon - \operatorname{div}(\Sigma_\varepsilon^*)\|_{L^\infty(0, T; H^1(\Omega_s))} \leq \|f_s - \operatorname{div}(\Sigma^*)\|_{L^\infty(0, T; H^1(\Omega_s))}, \\ \|\Sigma_\varepsilon^*\|_{L^\infty(0, T; H^2(\Omega_s))} \leq \|\Sigma^*\|_{L^\infty(0, T; H^2(\Omega_s))},$$

then the data f_s^ε and Σ_ε^* , the time T and the constant \mathcal{M}_1 also satisfy the conditions (3.3), (3.4), (3.6) and (3.7) of Theorem 3.1. Thus, we can apply Theorem 3.1 to the fluid-structure system (1.1), (1.2), (1.9), (1.4) and (1.5) associated to the data f_s^ε and Σ_ε^* . Consequently, there exists a unique solution $(w_f^\varepsilon, q_f^\varepsilon, d_s^\varepsilon)$, with $(w_f^\varepsilon, q_f^\varepsilon)$ in $\mathcal{B}_{\mathcal{M}_1}^F$ and d_s^ε in $L^\infty(0, T; \mathcal{B}_{\mathcal{M}_0}^S \cap H_{\Gamma_s}^1(\Omega_s))$. Furthermore, as we choose $\varepsilon < T$, the triplet (w_f, q_f, d_s) defined almost everywhere in $(0, T - \varepsilon)$ by,

$$\begin{aligned} w_f(t) &= w_f^\varepsilon(t + \varepsilon), \\ q_f(t) &= q_f^\varepsilon(t + \varepsilon), \\ d_s(t) &= d_s^\varepsilon(t + \varepsilon), \end{aligned}$$

is solution to the fluid-structure system (1.1), (1.2), (1.9), (1.4) and (1.5), associated to the data f_s and Σ^* , but only almost everywhere in $(0, T - \varepsilon)$:

$$(w_f, q_f) \in \mathcal{B}_{\mathcal{M}_1}^{F,\varepsilon} \quad \text{and} \quad d_s \in L^\infty(0, T - \varepsilon; \mathcal{B}_{\mathcal{M}_0}^S \cap H_{\Gamma_s}^1(\Omega_s)).$$

□

Conflict of Interest Statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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