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# SEMICLASSICAL RESOLVENT ESTIMATES FOR HÖLDER POTENTIALS

GEORGI VODEV

ABSTRACT. We first prove semiclassical resolvent estimates for the Schrödinger operator in  $\mathbb{R}^d$ ,  $d \geq 3$ , with real-valued potentials which are Hölder with respect to the radial variable. Then we extend these resolvent estimates to exterior domains in  $\mathbb{R}^d$ ,  $d \geq 2$ , and real-valued potentials which are Hölder with respect to the space variable. As an application, we obtain the rate of the decay of the local energy of the solutions to the wave equation with a refraction index which may be Hölder, Lipschitz or just  $L^\infty$ .

Key words: Schrödinger operator, resolvent estimates, Hölder potentials.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we are going to study the resolvent of the Schrödinger operator

$$P(h) = -h^2\Delta + V(x)$$

where  $0 < h \ll 1$  is a semiclassical parameter,  $\Delta$  is the negative Laplacian in  $\mathbb{R}^d$ ,  $d \geq 2$ , and  $V \in L^\infty(\mathbb{R}^d)$  is a real-valued potential satisfying the condition

$$(1.1) \quad V(x) \leq p(|x|)$$

where  $p(r) > 0$ ,  $r \geq 0$ , is a decreasing function such that  $p(r) \rightarrow 0$  as  $r \rightarrow \infty$ . More precisely, we are interested in bounding the quantity

$$g_s^\pm(h, \varepsilon) := \log \left\| (|x| + 1)^{-s} (P(h) - E \pm i\varepsilon)^{-1} (|x| + 1)^{-s} \right\|_{L^2 \rightarrow L^2}$$

from above by an explicit function of  $h$ , independent of  $\varepsilon$ , without imposing extra assumptions on the function  $p$ . Here  $L^2 := L^2(\mathbb{R}^d)$ ,  $0 < \varepsilon < 1$ ,  $s > 1/2$  is independent of  $h$  and  $E > 0$  is a fixed energy level independent of  $h$ . Instead, we impose some regularity on the potential with respect to the radial variable  $r = |x|$ . Note that throughout this paper the space  $C^1$  will denote the Lipschitz functions, that is, the ones with first derivatives belonging to  $L^\infty$  (and not necessarily continuous).

We will first extend Datchev's result [4] to a larger class of potentials. Recall that in [4] the bound

$$(1.2) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-1}$$

is proved when  $d \geq 3$ , with some constant  $C > 0$  independent of  $h$  and  $\varepsilon$ , for potentials  $V \in C^1(\overline{\mathbb{R}^+})$  with respect to the radial variable  $r$  and satisfying (1.1) with  $p(|x|) = C_1(|x| + 1)^{-\delta}$  as well as the condition

$$(1.3) \quad \partial_r V(x) \leq C_2(|x| + 1)^{-\beta}$$

where  $C_1, C_2, \delta > 0$  and  $\beta > 1$  are some constants. We will prove the following

**Theorem 1.1.** *Let  $d \geq 3$  and suppose that the potential  $V$  satisfies the conditions (1.1) and (1.3). Then there exist constants  $C > 0$  and  $h_0 > 0$  independent of  $h$  and  $\varepsilon$  but depending on  $s$ ,  $E$  and the function  $p$ , such that the bound (1.2) holds for all  $0 < h \leq h_0$ .*

Note that the bound (1.2) was first proved for smooth potentials in [2]. A high-frequency analog of (1.2) on Riemannian manifolds was also proved in [1] and [3]. When  $d = 2$  the bound (1.2) is proved in [11] for potentials  $V \in C^1(\mathbb{R}^2)$  satisfying (1.1) with  $p(|x|) = C_1(|x| + 1)^{-\delta}$  as well as the condition

$$(1.4) \quad |\nabla V(x)| \leq C_2(|x| + 1)^{-\beta}$$

where  $C_1, C_2, \delta > 0$  and  $\beta > 1$  are some constants.

On the other hand, for compactly supported  $L^\infty$  potentials without any regularity the following weaker bound

$$(1.5) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-4/3} \log(h^{-1})$$

is proved in [8] and [12] when  $d \geq 2$ . When  $d \geq 3$  the bound (1.5) is extended in [13] to potentials satisfying the condition

$$(1.6) \quad |V(x)| \leq C_3(|x| + 1)^{-\delta}$$

where  $C_3 > 0$  and  $\delta > 3$  are some constants. Moreover, for potentials satisfying (1.6) with  $1 < \delta \leq 3$  the weaker bound

$$(1.7) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-\frac{2\delta+5}{3(\delta-1)}} (\log(h^{-1}))^{\frac{1}{\delta-1}}.$$

is proved in [14].

In the present paper we will show that the bounds (1.5) and (1.7) can be improved if some small regularity of the potential is assumed. To be more precise, given  $0 < \alpha < 1$  and  $\beta > 0$ , we introduce the space  $C_\beta^\alpha(\overline{\mathbb{R}^+})$  of all Hölder functions  $a$  such that

$$\sup_{r' \geq 0: 0 < |r-r'| \leq 1} \frac{|a(r) - a(r')|}{|r - r'|^\alpha} \leq C(r+1)^{-\beta}, \quad \forall r \in \overline{\mathbb{R}^+},$$

for some constant  $C > 0$ . We now suppose that the function  $V(r, w) := V(rw)$  satisfies the condition

$$(1.8) \quad V(\cdot, w) \in C_4^\alpha(\overline{\mathbb{R}^+}), \quad 0 < \alpha < 1,$$

uniformly in  $w \in \mathbb{S}^{d-1}$ . We have the following

**Theorem 1.2.** *Let  $d \geq 3$  and suppose that the potential  $V$  satisfies the conditions (1.1) and (1.8). Then there exist constants  $C > 0$  and  $h_0 > 0$  independent of  $h$  and  $\varepsilon$  but depending on  $s$ ,  $E$  and the function  $p$ , such that the bound*

$$(1.9) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-4/(\alpha+3)} \log(h^{-1})$$

holds for all  $0 < h \leq h_0$ .

The proof of the above theorems is based on the global Carleman estimates proved in [14] but with different phase and weight functions (see Theorem 4.1). In fact, in the case of Hölder or Lipschitz potentials we need to construct better phase functions and hence get better Carleman estimates. Such functions are constructed in Section 2 modifying the construction in [14] in a suitable way. In order that the Carleman estimates (see (4.1) and (4.6) below) hold, the phase and weight functions must satisfy some inequalities (see (2.5), (2.9) and (2.21) below), so most of the proof of the above theorems consists of proving these inequalities.

We next extend the above results to arbitrary obstacles, all dimensions  $d \geq 2$  and all  $0 < h \leq 1$ . To do so, we need to replace the conditions (1.3) and (1.8) by stronger ones. To be more precise, we let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a connected domain with smooth boundary  $\partial\Omega$  such that  $\mathbb{R}^d \setminus \Omega$  is compact. Let  $r_0 > 0$  be such that  $\mathbb{R}^d \setminus \Omega \subset \{x \in \mathbb{R}^d : |x| \leq r_0\}$ . Given a real-valued potential

$V \in L^\infty(\Omega)$  satisfying (1.1) for  $|x| \geq r_0$ , we denote by  $P(h)$  the Dirichlet self-adjoint realisation of the operator  $-h^2\Delta + V(x)$  on the Hilbert space  $L^2(\Omega)$ . We define the quantity  $g_s^\pm$  in the same way as above with  $L^2 = L^2(\Omega)$ . Given  $0 < \alpha \leq 1$  and  $\beta > 0$ , we introduce the space  $C_\beta^\alpha(\overline{\Omega})$  of all Hölder functions  $a$  such that

$$\sup_{x' \in \overline{\Omega}: 0 < |x-x'| \leq 1} \frac{|a(x) - a(x')|}{|x - x'|^\alpha} \leq C(|x| + 1)^{-\beta}, \quad \forall x \in \overline{\Omega},$$

for some constant  $C > 0$ . Note that the case  $\alpha = 1$  corresponds to the Lipschitz functions. We suppose that

$$(1.10) \quad V \in C_\beta^\alpha(\overline{\Omega}), \quad 0 < \alpha \leq 1, \beta > 1.$$

We have the following

**Theorem 1.3.** *Let  $d \geq 2$  and suppose that the potential  $V \in L^\infty(\Omega)$  satisfies (1.1) for  $|x| \geq r_0$ . If  $V$  satisfies (1.10) with  $\alpha = 1$  and  $\beta > 1$ , then the bound (1.2) holds for all  $0 < h \leq 1$ . If  $V$  satisfies (1.10) with  $0 < \alpha < 1$  and  $\beta = 4$ , then the bound (1.9) holds for all  $0 < h \leq 1$ .*

To prove this theorem we follow the same strategy as in [15], where the bound (1.5) is proved in all dimensions  $d \geq 2$  for potentials  $V \in L^\infty(\Omega)$  satisfying (1.6). It consists of gluing up two different types of estimates - one in a compact set coming from the local Carleman estimates proved in [9] (see Theorem 3.1) with a global Carleman estimate outside a sufficiently big compact (see Theorem 4.2). This is carried out in Section 4.

Theorem 1.3 together with Theorem 1.1 of [15] allow us to get uniform bounds for the resolvent of the Dirichlet self-adjoint realisation,  $G$ , of the operator  $-n(x)^{-1}\Delta$  in the Hilbert space  $H = L^2(\Omega, n(x)dx)$ , where  $n \in L^\infty(\Omega)$  is a real-valued function called refraction index satisfying the conditions

$$(1.11) \quad n_1 \leq n(x) \leq n_2 \quad \text{in } \Omega,$$

with some constants  $n_1, n_2 > 0$ , and

$$(1.12) \quad |n(x) - 1| \leq C(|x| + 1)^{-\delta} \quad \text{in } \Omega,$$

with some constants  $C, \delta > 0$ . More precisely, we have the following

**Corollary 1.4.** *Suppose that the function  $n$  satisfies the conditions (1.11) and (1.12). Then, given any  $s > 1/2$  and  $\lambda_0 > 0$  there is a constant  $C > 0$  depending on  $s$  and  $\lambda_0$  such that the estimate*

$$(1.13) \quad \|\langle x \rangle^{-s} (G - \lambda^2 \pm i\varepsilon)^{-1} \langle x \rangle^{-s}\|_{H \rightarrow H} \leq e^{C\psi(\lambda)}$$

holds for all  $\lambda \geq \lambda_0$  uniformly in  $\varepsilon$ , where  $\psi(\lambda) = \lambda^{4/3} \log(\lambda + 1)$  if  $n \in L^\infty(\Omega)$  satisfies (1.12) with  $\delta > 3$ ,  $\psi(\lambda) = \lambda^{4/(\alpha+3)} \log(\lambda + 1)$  if  $n \in C_4^\alpha(\overline{\Omega})$  with  $0 < \alpha < 1$ ,  $\psi(\lambda) = \lambda$  if  $n \in C_\beta^1(\overline{\Omega})$  with  $\beta > 1$ .

To get (1.13) we apply the theorems mentioned above with  $h = \lambda_0/\lambda$ ,  $V = \lambda_0^2(1 - n)$ ,  $E = \lambda_0^2$  and  $\varepsilon$  replaced by  $\varepsilon h^2 n$ .

Using Corollary 1.4 one can extend Shapiro's result [10] on the local energy decay of the solutions of the following wave equation

$$(1.14) \quad \begin{cases} (n(x)\partial_t^2 - \Delta)u(t, x) = 0 & \text{in } \mathbb{R} \times \Omega, \\ u(t, x) = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ u(0, x) = f_1(x), \partial_t u(0, x) = f_2(x) & \text{in } \Omega. \end{cases}$$

Given any  $r_0 \gg 1$ , denote  $\Omega_{r_0} = \{x \in \Omega : |x| \leq r_0\}$ . We have the following

**Corollary 1.5.** *Suppose that the function  $n$  satisfies (1.11) and that  $n = 1$  outside some compact. Then, the solution  $u(t, x)$  to the equation (1.14) with compactly supported initial data  $(f_1, f_2) \in H_0^2(\Omega) \times H_0^1(\Omega)$  satisfies the estimate*

$$(1.15) \quad \|\nabla u(t, \cdot)\|_{L^2(\Omega_{r_0})} + \|\partial_t u(t, \cdot)\|_{L^2(\Omega_{r_0})} \leq C\omega(t) (\|f_1\|_{H^2(\Omega)} + \|f_2\|_{H^1(\Omega)})$$

for  $t \gg 1$ , where

$$\omega(t) = \left( \frac{\log \log t}{\log t} \right)^{3/4}.$$

Suppose in addition that  $n \in C^\alpha(\bar{\Omega})$  with  $0 < \alpha \leq 1$ . Then the estimate (1.15) holds with

$$\omega(t) = \left( \frac{\log \log t}{\log t} \right)^{(\alpha+3)/4}$$

if  $0 < \alpha < 1$ , and with  $\omega(t) = (\log t)^{-1}$  if  $\alpha = 1$ . The estimate (1.15) remains valid when  $\Omega = \mathbb{R}^d$ .

Note that estimates similar to (1.15) were first proved by Burq [1] in the case  $n \equiv 1$ . Note also that an analog of the above theorem is proved by Shapiro [10] in the case  $\Omega = \mathbb{R}^d$ . Then an estimate similar to (1.15) is proved with  $\omega(t)$  replaced by  $(\log t)^{-3/4+\epsilon}$ ,  $\epsilon > 0$  being arbitrary. Moreover, if in addition the function  $n$  is supposed Lipschitz, then the decay rate is improved to  $\omega(t) = (\log t)^{-1}$ . The proof in [10] is based on the resolvent estimates obtained in [4], [11] and [12].

The assumption that  $n = 1$  outside some compact is only necessary to study the low-frequency behavior of the cut-off resolvent of  $G$ . Indeed, under this assumption one can easily see that this behavior is exactly the same as in the case when  $n \equiv 1$ , which in turn is well-known (e.g. see Appendix B.2 of [1]). Therefore, in this case the low-frequency analysis can be carried out in precisely the same way as in [10]. Most probably, the condition (1.12) with  $\delta > 3$  would be enough. The high-frequency analysis in our case is also very similar to that one in [10] with some slight modifications allowing to deduce from (1.13) the sharp decay rate  $\omega(t)$  (instead of  $(\log t)^{-3/4+\epsilon}$ ).

## 2. CONSTRUCTION OF THE PHASE AND WEIGHT FUNCTIONS

Let  $\rho \in C_0^\infty([0, 1])$ ,  $\rho \geq 0$ , be a real-valued function independent of  $h$  such that  $\int_0^\infty \rho(\sigma) d\sigma = 1$ . If  $V$  satisfies (1.8), we approximate it by the function

$$V_\theta(r, w) = \theta^{-1} \int_0^\infty \rho((r - r')/\theta) V(r', w) dr' = \int_0^\infty \rho(\sigma) V(r + \theta\sigma, w) d\sigma$$

where  $\theta = h^{2/(\alpha+3)}$ . Indeed, we have

$$(2.1) \quad \begin{aligned} |V(r, w) - V_\theta(r, w)| &\leq \int_0^\infty \rho(\sigma) |V(r + \theta\sigma, w) - V(r, w)| d\sigma \\ &\lesssim \theta^\alpha (r+1)^{-4} \int_0^\infty \sigma^\alpha \rho(\sigma) d\sigma \lesssim \theta^\alpha (r+1)^{-4}. \end{aligned}$$

This bound together with (1.1) imply

$$(2.2) \quad V_\theta(r, w) \leq p(r) + \mathcal{O}((r+1)^{-4}).$$

Clearly,  $V_\theta$  is  $C^1$  with respect to the variable  $r$  and its first derivative  $V_\theta'$  is given by

$$V_\theta'(r, w) = \theta^{-2} \int_0^\infty \rho'((r - r')/\theta) V(r', w) dr'$$

$$= \theta^{-1} \int_0^\infty \rho'(\sigma) V(r + \theta\sigma, w) d\sigma = \theta^{-1} \int_0^\infty \rho'(\sigma) (V(r + \theta\sigma, w) - V(r, w)) d\sigma$$

where we have used that  $\int_0^\infty \rho'(\sigma) d\sigma = 0$ . Hence

$$(2.3) \quad |V'_\theta(r, w)| \lesssim \theta^{-1+\alpha} (r+1)^{-4} \int_0^\infty \sigma^\alpha |\rho'(\sigma)| d\sigma \lesssim \theta^{-1+\alpha} (r+1)^{-4}.$$

We now construct the weight function  $\mu$  as follows:

$$\mu(r) = \begin{cases} (r+1)^{2k} - (r+1)^{2k_0} & \text{for } 0 \leq r \leq a, \\ (a+1)^{2k} - (a+1)^{2k_0} + (a+1)^{-2s+1} - (r+1)^{-2s+1} & \text{for } r \geq a, \end{cases}$$

where  $a = a_0 h^{-m}$  with  $a_0 \gg 1$  independent of  $h$ ,  $m = 0$  if  $V$  satisfies (1.3) and  $m = 2$  if  $V$  satisfies (1.8). We choose  $k = \min\{1, \frac{\beta-1}{4}\}$ ,  $k_0 = 0$  if  $V$  satisfies (1.3), and  $k = 1$ ,  $k_0 = 1/2$  if  $V$  satisfies (1.8). Furthermore,  $s$  is independent of  $h$  such that

$$(2.4) \quad \frac{1}{2} < s < \begin{cases} \frac{\beta+1}{4} & \text{if } V \text{ satisfies (1.3),} \\ \frac{3}{4} & \text{if } V \text{ satisfies (1.8).} \end{cases}$$

Clearly, the first derivative of  $\mu$  is given by

$$\mu'(r) = \begin{cases} 2k(r+1)^{2k-1} - 2k_0(r+1)^{2k_0-1} & \text{for } 0 \leq r < a, \\ (2s-1)(r+1)^{-2s} & \text{for } r > a. \end{cases}$$

We have the following

**Lemma 2.1.** *For all  $r > 0$ ,  $r \neq a$ , we have the inequalities*

$$(2.5) \quad 2r^{-1}\mu(r) - \mu'(r) \geq 0,$$

$$(2.6) \quad \frac{\mu(r)^\ell}{\mu'(r)} \lesssim a^{2k\ell} (r+1)^{2s},$$

for every  $\ell \geq 0$ .

*Proof.* It is shown in Section 2 of [14] that when  $k_0 = 0$  the inequality (2.5) holds for all  $0 < k \leq 1$ . Here we will prove it when  $\nu := 2k - 2k_0 \geq 1$  and  $0 < k \leq 1$ . For  $r < a$  we have

$$\begin{aligned} & 2\mu(r) - r\mu'(r) \\ &= 2(1-k)(r+1)^{2k} - 2(1-k_0)(r+1)^{2k_0} + 2k(r+1)^{2k-1} - 2k_0(r+1)^{2k_0-1} \\ &= 2(r+1)^{2k_0-1} ((1-k)(r+1)^{\nu+1} - (1-k_0)(r+1) + k(r+1)^\nu - k_0) \\ &= 2(r+1)^{2k_0-1} ((1-k)r((r+1)^\nu - 1) + (r+1)^\nu - \nu r/2 - 1) \\ &\geq 2(r+1)^{2k_0-1} ((r+1)^\nu - \nu r/2 - 1) \geq \nu r (r+1)^{2k_0-1} > 0 \end{aligned}$$

where we have used the well-known inequality

$$(r+1)^\nu \geq \nu r + 1$$

as long as  $\nu \geq 1$ . For  $r > a$  the left-hand side of (2.5) is bounded from below by

$$2r^{-1}((a+1)^{2k} - (a+1)^{2k_0} - s) > 0$$

provided  $a$  is taken large enough. To prove (2.6) observe that for  $r < a$  we have

$$\mu'(r) \geq 2(k - k_0)(r+1)^{2k-1} \geq 2(k - k_0)(r+1)^{-1} \geq 2(k - k_0)(r+1)^{-2s}$$

which clearly implies the bound (2.6) with  $\ell = 0$ . This together with the fact that  $\mu = \mathcal{O}(a^{2k})$  implies the bound (2.6) with any  $\ell > 0$ .  $\square$

We will now construct a phase function  $\varphi \in C^1([0, +\infty))$  such that  $\varphi(0) = 0$  and  $\varphi(r) > 0$  for  $r > 0$ . We define the first derivative of  $\varphi$  by

$$\varphi'(r) = \begin{cases} \tau(r+1)^{-k} - \tau(a+1)^{-k} & \text{for } 0 \leq r \leq a, \\ 0 & \text{for } r \geq a, \end{cases}$$

where

$$(2.7) \quad \tau = \begin{cases} \tau_0 & \text{if } V \text{ satisfies (1.3),} \\ \tau_0 \theta^{2\alpha/3} h^{-1/3} & \text{if } V \text{ satisfies (1.8),} \end{cases}$$

with some parameter  $\tau_0 \gg 1$  independent of  $h$  to be fixed later on. Clearly, the first derivative of  $\varphi'$  satisfies

$$\varphi''(r) = \begin{cases} -k\tau(r+1)^{-k-1} & \text{for } 0 \leq r < a, \\ 0 & \text{for } r > a. \end{cases}$$

**Lemma 2.2.** *For all  $r \geq 0$  we have the bounds*

$$(2.8) \quad h^{-1}\varphi(r) \lesssim \begin{cases} h^{-1} & \text{if } V \text{ satisfies (1.3),} \\ h^{-4/(\alpha+3)} \log(h^{-1}) & \text{if } V \text{ satisfies (1.8),} \end{cases}$$

*Proof.* The lemma follows from the bounds

$$\max \varphi = \int_0^a \varphi'(r) dr \leq \tau \int_0^a (r+1)^{-k} dr \lesssim \begin{cases} \tau a^{1-k} & \text{if } k < 1, \\ \tau \log a & \text{if } k = 1. \end{cases}$$

□

For  $r > 0$ ,  $r \neq a$ , set

$$\begin{aligned} A(r) &= (\mu\varphi'^2)'(r), \\ B(r) &= B_1(r) + B_2(r), \end{aligned}$$

where

$$\begin{aligned} B_1(r) &= (r+1)^{-\beta} \mu(r) + p(r) \mu'(r), \\ B_2(r) &= \frac{(\mu(r)\varphi''(r))^2}{h^{-1}\varphi'(r)\mu(r) + \mu'(r)}, \end{aligned}$$

with  $\beta > 1$ , if  $V$  satisfies (1.3), and

$$\begin{aligned} B_1(r) &= \theta^{-1+\alpha} (r+1)^{-\beta} \mu(r) + (p(r) + (r+1)^{-\beta}) \mu'(r), \\ B_2(r) &= \frac{(\mu(r) (h^{-1}\theta^\alpha (r+1)^{-\beta} + |\varphi''(r)|))^2}{h^{-1}\varphi'(r)\mu(r) + \mu'(r)}, \end{aligned}$$

with  $\beta = 4$ , if  $V$  satisfies (1.8). The following lemma will play a crucial role in the proof of the Carleman estimates (4.1) and (4.6) in the case  $d \geq 3$ .

**Lemma 2.3.** *Given any constant  $C > 0$  there exist positive constants  $a_0 = a_0(C, E)$ ,  $\tau_0 = \tau_0(C, E)$  and  $h_0 = h_0(C, E)$  so that for  $\tau$  satisfying (2.7) and for all  $0 < h \leq h_0$  we have the inequality*

$$(2.9) \quad A(r) - CB(r) \geq -\frac{E}{2} \mu'(r)$$

for all  $r > 0$ ,  $r \neq a$ . Moreover, there is a constant  $r_\# > 0$  such that (2.9) holds for all  $r \geq r_\#$ ,  $r \neq a$ , and all  $0 < h \leq 1$ .

*Proof.* For  $r < a$  we have

$$\begin{aligned}
 A(r) &= - \left( (r+1)^{2k_0} \varphi'^2 \right)' + \tau^2 \partial_r \left( 1 - (r+1)^k (a+1)^{-k} \right)^2 \\
 &= -2(r+1)^{2k_0} \varphi'(r) \varphi''(r) - 2k_0 (r+1)^{2k_0-1} \varphi'(r)^2 \\
 &\quad - 2k\tau^2 (r+1)^{k-1} (a+1)^{-k} \left( 1 - (r+1)^k (a+1)^{-k} \right) \\
 &\geq 2\tau(k-k_0)(r+1)^{2k_0-k-1} \varphi'(r) - 2k\tau^2 (r+1)^{k-1} (a+1)^{-k} \\
 &\quad \geq 2\tau(k-k_0)(r+1)^{2k_0-k-1} \varphi'(r) - \tau^2 a^{-k} \mu'(r) \\
 &\quad \geq 2\tau(k-k_0)(r+1)^{2k_0-k-1} \varphi'(r) - \tau_0^2 a_0^{-k} \mu'(r).
 \end{aligned}$$

Hence, we can arrange the inequality

$$(2.10) \quad A(r) \geq 2\tau(k-k_0)(r+1)^{2k_0-k-1} \varphi'(r) - \frac{E}{4} \mu'(r)$$

for all  $r < a$ , provided  $\tau_0^2 a_0^{-k} \leq E/4$ . Observe now that if  $0 < r \leq a/2$ , then

$$(2.11) \quad \varphi'(r) \geq \gamma\tau(r+1)^{-k}$$

with some constant  $\gamma > 0$ . By (2.10) and (2.11) we conclude

$$(2.12) \quad A(r) \geq \tilde{\gamma}\tau^2(r+1)^{-2(k-k_0)-1} - \frac{E}{4} \mu'(r)$$

for all  $r \leq a/2$  with some constant  $\tilde{\gamma} > 0$ , and

$$(2.13) \quad A(r) \geq -\frac{E}{4} \mu'(r)$$

for all  $r \neq a$ .

We will now bound the function  $B_1$  from above. Since the function  $p$  is decreasing, there is  $b > 0$  such that

$$p(r) + (r+1)^{-4} \leq \frac{E}{9C} \quad \text{for } r \geq b.$$

Hence, for every  $N > 0$  there is a constant  $C_N > 0$  such that we have

$$(2.14) \quad (p(r) + (r+1)^{-4}) \mu'(r) \leq C_N (r+1)^{-N} + \frac{E}{9C} \mu'(r) \quad \text{for } \forall r \neq a.$$

Let  $0 < r < a$ . Then  $\mu(r) < (r+1)^{2k}$ , and in view of (2.14) with  $N$  big enough, we have

$$B_1(r) \leq \tilde{C} (r+1)^{2k-\beta} + \frac{E}{9C} \mu'(r),$$

if  $V$  satisfies (1.3), and

$$B_1(r) \leq \tilde{C} \theta^{-1+\alpha} (r+1)^{2k-\beta} + \frac{E}{9C} \mu'(r),$$

with  $\beta = 4$ , if  $V$  satisfies (1.8). Observe now that the choice of the parameters  $k, k_0$  and  $\theta$  guarantees that  $\beta - 2k \geq 2(k - k_0) + 1$  and  $\theta^{-1+\alpha} = \theta^{4\alpha/3} h^{-2/3}$ . Therefore, the above inequalities imply

$$(2.15) \quad B_1(r) \leq \mathcal{O}(\tau_0^{-2}) \tau^2 (r+1)^{-2(k-k_0)-1} + \frac{E}{8C} \mu'(r) \quad \text{for } r \leq a/2$$

in both cases. Similarly, we get

$$(2.16) \quad B_1(r) \leq \mathcal{O}(\tau^2 a^{-\beta+1}) \mu'(r) + \frac{E}{9C} \mu'(r) \quad \text{for } a/2 < r < a.$$



and

$$(2.17) \quad B_1(r) \leq \mathcal{O}\left(\tau^2 a^{-\beta+2k+2s}\right) \mu'(r) + \frac{E}{9C} \mu'(r) \quad \text{for } r > a.$$

Since

$$\tau^2 a^{-\beta+1} < \tau^2 a^{-\beta+2k+2s} \leq \tau_0^2 a_0^{-\beta+2k+2s}$$

and  $\beta > 2k + 2s$ , we obtain from (2.16) and (2.17),

$$(2.18) \quad B_1(r) \leq \frac{E}{8C} \mu'(r) \quad \text{for } r > a/2, r \neq a,$$

provided  $a_0$  is taken large enough depending on  $\tau_0$ .

We will now bound the function  $B_2$  from above. We will first consider the case when  $V$  satisfies (1.8). Let  $0 < r \leq a/2$ . Since in this case  $\mu(r)/\mu'(r) = \mathcal{O}(r)$  and in view of (2.11), we have

$$\begin{aligned} B_2(r) &\lesssim \frac{\mu(r) (h^{-2}\theta^{2\alpha}(r+1)^{-2\beta} + \varphi''(r)^2)}{h^{-1}\varphi'(r)} \\ &\lesssim h^{-1}\theta^{2\alpha} \frac{\mu(r)(r+1)^{-2\beta}}{\varphi'(r)} + h \frac{\mu(r)\varphi''(r)^2}{\mu'(r)\varphi'(r)} \mu'(r) \\ &\lesssim \tau^{-1}\theta^{2\alpha} h^{-1}(r+1)^{3k-2\beta} + \tau h(r+1)^{-k-1} \mu'(r) \\ &\lesssim \tau_0^{-3} \tau^2 (r+1)^{-2(k-k_0)-1} + \tau_0 h^{2/3} (r+1)^{-k-1} \mu'(r) \end{aligned}$$

where we have used that  $5k - 2k_0 < 2\beta - 1$ . Taking  $h$  small enough, depending on  $\tau_0$ , we get the bound

$$(2.19) \quad B_2(r) \leq \mathcal{O}\left(\tau_0^{-3}\right) \tau^2 (r+1)^{-2(k-k_0)-1} + \frac{E}{8C} \mu'(r) \quad \text{for } 0 < r \leq a/2.$$

It is clear that there is a constant  $r_{\sharp} > 0$ , depending on  $\tau_0$  and  $E/C$ , such that (2.19) holds for  $r_{\sharp} \leq r \leq a/2$  and all  $0 < h \leq 1$ . Let now see that

$$(2.20) \quad B_2(r) \leq \frac{E}{8C} \mu'(r) \quad \text{for } r > a/2, r \neq a.$$

Let  $\frac{a}{2} < r < a$ . Since in this case  $\mu(r)/\mu'(r) = \mathcal{O}(r)$ , we get the bound

$$\begin{aligned} B_2(r) &\lesssim \left(\frac{\mu(r)}{\mu'(r)}\right)^2 \left(h^{-1}\theta^\alpha(r+1)^{-\beta} + |\varphi''(r)|\right)^2 \mu'(r) \\ &\lesssim \left(h^{-2}\theta^{2\alpha}(r+1)^{2-2\beta} + \tau^2(r+1)^{-2k}\right) \mu'(r) \\ &\lesssim \left(h^{-2}a^{2-2\beta} + \tau^2a^{-2k}\right) \mu'(r) \\ &\lesssim \left(h^{2m(\beta-1)-2}a_0^{2-2\beta} + h^{2m-2/3}\tau_0^2a_0^{-2k}\right) \mu'(r) \\ &\lesssim \left(a_0^{2-2\beta} + \tau_0^2a_0^{-2k}\right) \mu'(r) \end{aligned}$$

which clearly implies (2.20) in this case, provided  $a_0$  is taken big enough, depending on  $\tau_0$ . Let  $r > a$ . Using (2.6) with  $\ell = 1$ , we get

$$\begin{aligned} B_2(r) &\lesssim \left(\frac{\mu(r)}{\mu'(r)}\right)^2 \left(h^{-1}\theta^\alpha(r+1)^{-\beta}\right)^2 \mu'(r) \\ &\lesssim h^{-2}a^{4k}(r+1)^{4s-2\beta} \mu'(r) \\ &\lesssim h^{-2}a^{4k+4s-2\beta} \mu'(r) \\ &\lesssim h^{2m(\beta-2k-2s)-2}a_0^{4k+4s-2\beta} \mu'(r) \end{aligned}$$

$$\lesssim a_0^{4k+4s-2\beta} \mu'(r)$$

which again implies (2.20), provided  $a_0$  is taken big enough. Similarly, in the case when  $V$  satisfies (1.3) one concludes that the inequality (2.20) holds for all  $r > 0$ ,  $r \neq a$ , provided  $a_0$  is taken large enough, independent of  $h$ .

It is easy to see that for  $r \leq a/2$  the estimate (2.9) follows from (2.12), (2.15) and (2.19) by taking  $\tau_0$  big enough, while for  $r \geq a/2$ ,  $r \neq a$ , it follows from (2.13), (2.18) and (2.20).  $\square$

The following lemma will play a crucial role in the proof of the Carleman estimate (4.6) in the case  $d = 2$ .

**Lemma 2.4.** *Given any constants  $C, r_0 > 0$  there exist positive constants  $a_0 = a_0(C, E)$ ,  $\tau_0 = \tau_0(C, E)$  and  $h_0 = h_0(C, r_0, E)$  so that for  $\tau$  satisfying (2.7) and for all  $0 < h \leq h_0$  we have the inequality*

$$(2.21) \quad A(r) - h^2 r^{-3} \mu(r) - CB(r) \geq -\frac{2E}{3} \mu'(r)$$

for all  $r \geq r_0$ ,  $r \neq a$ . Moreover, if  $k = 1 - \epsilon$ ,  $1/2 < s \leq (1 + \epsilon)/2$  with  $0 < \epsilon \ll 1$ , then there is  $r_{\sharp} > 0$  such that (2.21) holds for all  $r \geq r_{\sharp}$  and  $0 < h \leq 1$ .

*Proof.* Given any  $r_0 > 0$  we need to show that for all  $r \geq r_0$ ,  $0 < h \leq h_0$  the inequality

$$(2.22) \quad h^2 r^{-3} \frac{\mu(r)}{\mu'(r)} \leq \frac{E}{6}$$

holds, provided  $h_0$  is taken small enough depending on  $r_0$ . Then (2.21) would follow from (2.9) and (2.22). Since  $\mu(r)/\mu'(r) = \mathcal{O}(r)$  for  $r < a$ , the left-hand side of (2.22) is  $\mathcal{O}_{r_0}(h^2)$ , which implies the desired inequality. When  $r > a$  we have  $\mu(r)/\mu'(r) = \mathcal{O}(a^{2k+2s})$ , and hence in this case the left-hand side of (2.22) is  $\mathcal{O}_{a_0}(h^{2-m(2k+2s-3)}) = \mathcal{O}_{a_0}(h)$ .

To prove the last assertion of the lemma we have to show that (2.22) holds for all  $r \geq r_{\sharp}$  and  $0 < h \leq 1$ . Indeed, the left-hand side of (2.22) is  $\mathcal{O}(r^{-2})$  for  $r < a$  and  $\mathcal{O}(a_0^{-(3-2k-2s)})$  for  $r > a$ . Since  $3 - 2k - 2s \geq \epsilon > 0$ , we can make it as small as we want by taking  $r$  and  $a_0$  large enough. On the other hand, it is easy to see that the inequality (2.9) still holds for  $r \geq r_{\sharp}$  and  $0 < h \leq 1$  with the new values of  $k$  and  $s$ , provided  $\epsilon$  is taken small enough. Thus we get the desired assertion by (2.9) and (2.22) in this case, too.  $\square$

### 3. CARLEMAN ESTIMATES FOR HÖLDER POTENTIALS ON BOUNDED DOMAINS

Throughout this section  $X \subset \mathbb{R}^d$ ,  $d \geq 2$ , will be a bounded, connected domain with a smooth boundary  $\partial X$ . Introduce the operator

$$P(h) = -h^2 \Delta + V(x)$$

where  $0 < h \leq 1$  is a semiclassical parameter and  $V \in L^\infty(X)$  is a real-valued potential. Let  $U \subset X$ ,  $U \neq \emptyset$ , be an arbitrary open domain, independent of  $h$ , such that  $\partial U \cap \partial X = \emptyset$  and let  $z \in \mathbb{C}$ ,  $|z| \leq C_0$ ,  $C_0 > 0$  being a constant independent of  $h$ . We will also denote by  $H_h^1$  the Sobolev space equipped with the semiclassical norm. Given any  $0 < \alpha \leq 1$ , denote by  $C^\alpha(\overline{X})$  the space of all functions  $a$  such that

$$\|a\|_{C^\alpha} := \sup_{x', x \in \overline{X}: 0 < |x-x'| \leq 1} \frac{|a(x) - a(x')|}{|x - x'|^\alpha} < +\infty.$$

We have the following

**Theorem 3.1.** *Let  $V \in C^\alpha(\overline{X})$  with  $0 < \alpha \leq 1$ . Then, there exists a positive constant  $\gamma$  depending on  $U$ ,  $\|V\|_{C^\alpha}$  and  $C_0$  but independent of  $h$  such that for all  $0 < h \leq 1$  we have the estimate*

$$(3.1) \quad \|u\|_{H_h^1(X)} \leq e^{\gamma h^{-4/(\alpha+3)}} \|(P(h) - z)u\|_{L^2(X)} + e^{\gamma h^{-4/(\alpha+3)}} \|u\|_{H_h^1(U)}$$

for every  $u \in H^2(X)$  such that  $u|_{\partial X} = 0$ .

It is proved in Section 2 of [15] that for complex-valued potentials  $V \in L^\infty(X)$  the estimate (3.1) holds with  $\alpha = 0$ . The proof is based on the local Carleman estimates proved in [9]. We will follow the same strategy in the case of Hölder potentials as well. For such potentials we will get new local Carleman estimates by making use of the results of [9]. To be more precise, we let  $W \subset X$  be a small open domain and let  $x$  be local coordinates in  $W$ . If  $\Gamma := \overline{W} \cap \partial X$  is not empty we choose  $x = (x_1, x')$ ,  $x_1 > 0$  being the normal coordinate in  $W$  and  $x'$  the tangential ones. Thus in these coordinates  $\Gamma$  is given by  $\{x_1 = 0\}$ . Let  $\phi, \phi_1 \in C^\infty(\overline{W})$  be real-valued functions such that  $\text{supp } \phi \subset \text{supp } \phi_1 \subset \overline{W}$ ,  $\phi_1 = 1$  on  $\text{supp } \phi$ . When  $V \in C^\alpha(\overline{X})$  with  $0 < \alpha < 1$  we approximate the function  $\phi_1 V$  by the smooth function

$$V_\theta(x) = \theta^{-1} \int_X \varrho((x - x')/\theta) (\phi_1 V)(x') dx'$$

where  $\varrho \in C_0^\infty(|x| \leq 1)$  is a real-valued function such that  $\int_{\mathbb{R}^d} \varrho(x) dx = 1$  and  $0 < \theta < 1$  is a small parameter to be fixed later on. Taking  $\theta$  small enough we can arrange that  $\text{supp } V_\theta \cap \overline{X} \subset \overline{W}$ . The fact that  $V \in C^\alpha(\overline{X})$  implies the bounds

$$(3.2) \quad |(\phi_1 V)(x) - V_\theta(x)| \lesssim \theta^\alpha,$$

$$(3.3) \quad |\partial_x^\beta V_\theta(x)| \lesssim \theta^{\alpha-1},$$

for all multi-indices  $\beta$  such that  $|\beta| = 1$ . Set  $\tilde{V} = \theta^{1-\alpha}(V_\theta - z)$  if  $V \in C^\alpha(\overline{X})$  with  $0 < \alpha < 1$ ,  $\tilde{V} = V - z$  if  $V \in C^1(\overline{X})$ . In view of (3.3) we have  $\partial_x^\beta \tilde{V}(x) = \mathcal{O}(1)$  uniformly in  $\theta$ , for all multi-indices  $\beta$  such that  $|\beta| \leq 1$ .

Let now  $\psi \in C^\infty(\overline{W})$  be a real-valued function independent of  $h$  and  $\theta$  such that

$$(3.4) \quad \nabla \psi \neq 0 \quad \text{in } \overline{W}.$$

If  $\Gamma \neq \emptyset$  we also suppose that

$$(3.5) \quad \frac{\partial \psi}{\partial x_1}(0, x') > 0, \quad \forall x'.$$

We set  $\varphi = e^{\lambda \psi}$ , where  $\lambda > 0$  is a big parameter to be fixed later on, independent of  $h$  and  $\theta$ . Let  $p(x, \xi) \in C^\infty(T^*W)$  be the principal symbol of the operator  $-\Delta$  and let  $0 < \hbar \ll 1$  be a new semiclassical parameter. Then the principal symbol,  $\tilde{p}_\varphi$ , of the operator

$$e^{\varphi/\hbar} (-\hbar^2 \Delta + \tilde{V}) e^{-\varphi/\hbar}$$

is given by the formula

$$\tilde{p}_\varphi(x, \xi) = p(x, \xi + i\nabla \varphi(x)) + \tilde{V}(x).$$

An easy computation shows that given any constant  $C > 0$  there is  $\lambda = \lambda(C)$  such that the condition (3.4) for the function  $\psi$  implies the following condition for the function  $\varphi$ :

$$(3.6) \quad \{\text{Re } \tilde{p}_\varphi, \text{Im } \tilde{p}_\varphi\}(x, \xi) \geq c_1 \quad \text{for } |\xi| \leq C,$$

with some constant  $c_1 > 0$  independent of  $\theta$ . On the other hand, if  $C$  is taken large enough we can arrange the lower bound

$$(3.7) \quad |\tilde{p}_\varphi(x, \xi)| \geq c_2 |\xi|^2 \quad \text{for } |\xi| \geq C,$$

with some constant  $c_2 > 0$  independent of  $\theta$ . If  $\Gamma \neq \emptyset$  the condition (3.5) implies

$$(3.8) \quad \frac{\partial \varphi}{\partial x_1}(0, x') > 0, \quad \forall x'.$$

Now we are in position to use Propositions 1 and 2 of [9], where the proof is based on the properties (3.6), (3.7) and (3.8). We have the following

**Proposition 3.2.** *Let the function  $u$  be as in Theorem 3.1. Then there exist constants  $C_1, \hbar_0 > 0$  such that for all  $0 < \hbar \leq \hbar_0$  we have the estimate*

$$(3.9) \quad \int_X (|\phi u|^2 + |\hbar \nabla(\phi u)|^2) e^{2\varphi/\hbar} dx \leq C_1 \hbar^{-1} \int_X |(-\hbar^2 \Delta + \tilde{V})(\phi u)|^2 e^{2\varphi/\hbar} dx.$$

We take  $\hbar = h\theta^{(1-\alpha)/2}$  when  $\alpha < 1$  and we rewrite the inequality (3.9) as follows

$$\begin{aligned} & \int_X (|\phi u|^2 + \theta^{1-\alpha} |h \nabla(\phi u)|^2) e^{2\varphi/h\theta^{(1-\alpha)/2}} dx \\ & \leq C_1 h^{-1} \theta^{3(1-\alpha)/2} \int_X |(-h^2 \Delta + V_\theta - z)(\phi u)|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}} dx \\ & \leq C_1 h^{-1} \theta^{3(1-\alpha)/2} \int_X |(P(h) - z)(\phi u)|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}} dx \\ & + C_1 h^{-1} \theta^{3(1-\alpha)/2} \sup |\phi_1 V - V_\theta|^2 \int_X |\phi u|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}} dx \\ & \leq C_1 h^{-1} \theta^{3(1-\alpha)/2} \int_X |(P(h) - z)(\phi u)|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}} dx \\ & + C_2 h^{-1} \theta^{(3+\alpha)/2} \int_X |\phi u|^2 e^{2\varphi/h\theta^{(1-\alpha)/2}} dx. \end{aligned}$$

We now take  $\theta = h^{2/(\alpha+3)} \kappa^{2/(1-\alpha)}$ , where  $\kappa > 0$  is a small parameter independent of  $h$ . Thus, taking  $\kappa$  small enough we can absorb the last term in the right-hand side of the above inequality. When  $\alpha = 1$  we take  $\hbar = h\kappa$ . Thus we deduce from Proposition 3.2 the following

**Proposition 3.3.** *Let the function  $u$  be as in Theorem 3.1. Then there exist constants  $\tilde{C}, \kappa_0 > 0$  such that for all  $0 < \kappa \leq \kappa_0$  and all  $0 < h \leq 1$  we have the estimate*

$$(3.10) \quad \begin{aligned} & \int_X (|\phi u|^2 + |h \nabla(\phi u)|^2) e^{2\varphi/\kappa h^{4/(\alpha+3)}} dx \\ & \leq \tilde{C} \kappa h^{-2(\alpha+1)/(\alpha+3)} \int_X |(P(h) - z)(\phi u)|^2 e^{2\varphi/\kappa h^{4/(\alpha+3)}} dx. \end{aligned}$$

Now Theorem 3.1 follows from Proposition 3.3 in precisely the same way as in Section 2 of [15], where the analysis is carried out in the particular case  $\alpha = 0$ . It is an easy observation that the general case requires no changes in the arguments, and therefore we omit the details.

## 4. RESOLVENT ESTIMATES

The following global Carlemann estimate is similar to that one in Section 3 of [14] and can be proved in the same way. To this end, we decompose the potential as  $V = V_L + V_S$  with  $V_L := V_\theta$ ,  $V_S := V - V_\theta$  if  $V$  satisfies (1.8), and  $V_L := V$ ,  $V_S := 0$  if  $V$  satisfies (1.3). We then use the bounds (2.1), (2.2) and (2.3) in the first case as well as Lemma 2.3. The proof will be carried out in Section 5. In what follows we set  $\mathcal{D}_r = -ih\partial_r$ .

**Theorem 4.1.** *Let  $d \geq 3$  and let the potential  $V$  satisfy (1.1). Let also  $V$  satisfy either (1.3) or (1.8) and let  $s$  satisfy (2.4). Then, for all  $0 < h \ll 1$  and for all functions  $f \in H^2(\mathbb{R}^d)$  such that*

$$(|x| + 1)^s (P(h) - E \pm i\varepsilon)f \in L^2(\mathbb{R}^d)$$

we have the estimate

$$(4.1) \quad \begin{aligned} & \|(|x| + 1)^{-s} e^{\varphi/h} f\|_{L^2(\mathbb{R}^d)} + \|(|x| + 1)^{-s} e^{\varphi/h} \mathcal{D}_r f\|_{L^2(\mathbb{R}^d)} \\ & \leq C a^2 h^{-1} \|(|x| + 1)^s e^{\varphi/h} (P(h) - E \pm i\varepsilon)f\|_{L^2(\mathbb{R}^d)} \\ & \quad + C \tau a (\varepsilon/h)^{1/2} \|e^{\varphi/h} f\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

with a constant  $C > 0$  independent of  $h$ ,  $\varepsilon$  and  $f$ .

Theorems 1.1 and 1.2 can be obtained from Theorem 4.1 in the same way as in Section 4 of [14]. We will sketch the proof for the sake of completeness. It follows from the estimate (4.1) and Lemma 2.2 that for  $0 < h \ll 1$  and  $s$  satisfying (2.4) we have the estimate

$$(4.2) \quad \|(|x| + 1)^{-s} f\|_{L^2} \leq M \|(|x| + 1)^s (P(h) - E \pm i\varepsilon)f\|_{L^2} + M \varepsilon^{1/2} \|f\|_{L^2}$$

where  $M > 0$  is given by

$$\log M = \begin{cases} Ch^{-1} & \text{if } V \text{ satisfies (1.3),} \\ Ch^{-4/(\alpha+3)} \log(h^{-1}) & \text{if } V \text{ satisfies (1.8),} \end{cases}$$

with a constant  $C > 0$  independent of  $h$  and  $\varepsilon$ . On the other hand, since the operator  $P(h)$  is symmetric, we have

$$\begin{aligned} & \varepsilon \|f\|_{L^2}^2 = \pm \operatorname{Im} \langle (P(h) - E \pm i\varepsilon)f, f \rangle_{L^2} \\ & \leq (2M)^{-2} \|(|x| + 1)^{-s} f\|_{L^2}^2 + (2M)^2 \|(|x| + 1)^s (P(h) - E \pm i\varepsilon)f\|_{L^2}^2 \end{aligned}$$

which yields

$$(4.3) \quad M \varepsilon^{1/2} \|f\|_{L^2} \leq \frac{1}{2} \|(|x| + 1)^{-s} f\|_{L^2} + 2M^2 \|(|x| + 1)^s (P(h) - E \pm i\varepsilon)f\|_{L^2}.$$

By (4.2) and (4.3) we get

$$(4.4) \quad \|(|x| + 1)^{-s} f\|_{L^2} \leq 4M^2 \|(|x| + 1)^s (P(h) - E \pm i\varepsilon)f\|_{L^2}.$$

It follows from (4.4) that the resolvent estimate

$$(4.5) \quad \|(|x| + 1)^{-s} (P(h) - E \pm i\varepsilon)^{-1} (|x| + 1)^{-s}\|_{L^2 \rightarrow L^2} \leq 4M^2$$

holds for all  $0 < h \ll 1$  and  $s$  satisfying (2.4), and hence for all  $s > 1/2$  independent of  $h$ . Clearly, (4.5) implies the desired bounds for  $g_s^\pm$ .

Given any  $r_0 > 0$  we denote  $Y_{r_0} := \{x \in \mathbb{R}^d : |x| \geq r_0\}$  and we let  $\eta_{r_0} \in C^\infty(\mathbb{R})$  be such that  $\eta_{r_0}(r) = 0$  for  $r \leq r_0/3$ ,  $\eta_{r_0}(r) = 1$  for  $r \geq r_0/2$ . We set  $V_\eta(x) := \eta_{r_0}(|x|)V(x)$ . To prove Theorem 1.3 we need the following

**Theorem 4.2.** *Let  $d \geq 2$  and let the potential  $V$  satisfy (1.1) for  $|x| \geq r_0$ . Let also  $V_\eta$  satisfy either (1.3) or (1.8) and let  $s$  satisfy (2.4). Then, for all  $0 < h \ll 1$  and for all functions  $f \in H^2(Y_{r_0})$  such that  $f = \partial_r f = 0$  on  $\partial Y_{r_0}$  and*

$$(|x| + 1)^s (P(h) - E \pm i\varepsilon) f \in L^2(Y_{r_0})$$

we have the estimate

$$\begin{aligned} & \|(|x| + 1)^{-s} e^{\varphi/h} f\|_{L^2(Y_{r_0})} + \|(|x| + 1)^{-s} e^{\varphi/h} \mathcal{D}_r f\|_{L^2(Y_{r_0})} \\ & \leq C a^2 h^{-1} \|(|x| + 1)^s e^{\varphi/h} (P(h) - E \pm i\varepsilon) f\|_{L^2(Y_{r_0})} \\ (4.6) \quad & + C \tau a (\varepsilon/h)^{1/2} \|e^{\varphi/h} f\|_{L^2(Y_{r_0})} \end{aligned}$$

with a constant  $C > 0$  independent of  $h$ ,  $\varepsilon$  and  $f$ . Moreover, there is a constant  $r_\# > 0$  such that if  $r_0 \geq r_\#$ , the estimate (4.6) holds for all  $0 < h \leq 1$ .

The proof of Theorem 4.2 is similar to that one of Theorem 4.1 with some suitable modifications when  $d = 2$  and will be carried out in Section 5.

Theorem 1.3 can be derived from Theorems 3.1 and 4.2 in a way similar to the one developed in Section 5 of [15]. Let  $r_\#$  be as in Theorem 4.2 and let  $r_0 > r_\#$  be such that  $Y_{r_0/3} \subset \Omega$ . Fix  $r_j$ ,  $j = 1, 2, 3, 4$ , such that  $r_0 < r_1 < r_2 < r_3 < r_4$ . Choose functions  $\psi_1, \psi_2 \in C^\infty(\mathbb{R}^d)$ , depending only on the radial variable  $r$ , such that  $\psi_1 = 1$  in  $\mathbb{R}^d \setminus Y_{r_1}$ ,  $\psi_1 = 0$  in  $Y_{r_2}$ ,  $\psi_2 = 1$  in  $\mathbb{R}^d \setminus Y_{r_3}$ ,  $\psi_2 = 0$  in  $Y_{r_4}$ . If  $s$  satisfies (2.4), we choose a function  $\chi_s \in C^\infty(\overline{\Omega})$ ,  $\chi_s > 0$ , such that  $\chi_s(x) = |x|^{-s}$  on  $Y_{r_0}$ . Let  $f \in H^2(\Omega)$  be such that  $\chi_s^{-1} (P(h) - E \pm i\varepsilon) f \in L^2(\Omega)$  and  $f|_{\partial\Omega} = 0$ . Set

$$\begin{aligned} \mathcal{Q}_0 &= \|\chi_s^{-1} (P(h) - E \pm i\varepsilon) f\|_{L^2(\Omega)}, \\ \mathcal{Q}_1 &= \|f\|_{L^2(Y_{r_1} \setminus Y_{r_2})} + \|\mathcal{D}_r f\|_{L^2(Y_{r_1} \setminus Y_{r_2})}, \\ \mathcal{Q}_2 &= \|f\|_{L^2(Y_{r_3} \setminus Y_{r_4})} + \|\mathcal{D}_r f\|_{L^2(Y_{r_3} \setminus Y_{r_4})}, \end{aligned}$$

and observe that

$$\|[P(h), \psi_j] f\|_{L^2} \lesssim \mathcal{Q}_j, \quad j = 1, 2.$$

We now apply Theorem 3.1 to the function  $\psi_2 f$  with  $X = \Omega \setminus Y_{r_4}$  and  $U \subset X$  such that  $U \cap \text{supp } \psi_2 = \emptyset$ . Thus we obtain

$$\begin{aligned} & \|f\|_{H_h^1(\Omega \setminus Y_{r_3})} \leq \|\psi_2 f\|_{H_h^1(\Omega \setminus Y_{r_4})} \\ & \leq e^{\gamma h^{-4/(\alpha+3)}} \|(P(h) - E \pm i\varepsilon) \psi_2 f\|_{L^2(\Omega \setminus Y_{r_4})} \\ (4.7) \quad & \leq e^{\gamma h^{-4/(\alpha+3)}} \|(P(h) - E \pm i\varepsilon) f\|_{L^2(\Omega \setminus Y_{r_4})} + e^{\gamma h^{-4/(\alpha+3)}} \mathcal{Q}_2 \end{aligned}$$

with some constant  $\gamma > 0$ . In particular, (4.7) implies

$$(4.8) \quad \mathcal{Q}_1 \leq e^{\gamma h^{-4/(\alpha+3)}} \mathcal{Q}_0 + e^{\gamma h^{-4/(\alpha+3)}} \mathcal{Q}_2.$$

On the other hand, it is clear that if  $V$  satisfies (1.10) with  $\alpha = 1$  and  $\beta > 1$  (resp.  $0 < \alpha < 1$  and  $\beta = 4$ ), then  $V_\eta$  satisfies (1.3) (resp. (1.8)). Therefore, we can apply Theorem 4.2 to the function  $(1 - \psi_1) f$  to obtain

$$\begin{aligned} & \|(|x| + 1)^{-s} e^{\varphi/h} f\|_{L^2(Y_{r_2})} + \|(|x| + 1)^{-s} e^{\varphi/h} \mathcal{D}_r f\|_{L^2(Y_{r_2})} \\ & \leq \|(|x| + 1)^{-s} e^{\varphi/h} (1 - \psi_1) f\|_{L^2(Y_{r_1})} + \|(|x| + 1)^{-s} e^{\varphi/h} \mathcal{D}_r (1 - \psi_1) f\|_{L^2(Y_{r_1})} \\ & \leq C a^2 h^{-1} \|(|x| + 1)^s e^{\varphi/h} (P(h) - E \pm i\varepsilon) (1 - \psi_1) f\|_{L^2(Y_{r_1})} \\ & \quad + C \tau a (\varepsilon/h)^{1/2} \|e^{\varphi/h} f\|_{L^2(Y_{r_1})} \end{aligned}$$

$$\begin{aligned}
&\leq Ca^2h^{-1}\|(|x|+1)^s e^{\varphi/h}(P(h) - E \pm i\varepsilon)f\|_{L^2(Y_{r_1})} + Ca^2h^{-1}e^{\varphi(r_2)/h}\mathcal{Q}_1 \\
(4.9) \quad &+ C\tau a(\varepsilon/h)^{1/2}\|e^{\varphi/h}f\|_{L^2(Y_{r_1})}
\end{aligned}$$

for all  $0 < h \leq 1$ . In particular, (4.9) implies

$$\begin{aligned}
(4.10) \quad &e^{\varphi(r_3)/h}\mathcal{Q}_2 \leq Ca^2h^{-1}e^{\max \varphi/h}\mathcal{Q}_0 + C\tau a(\varepsilon/h)^{1/2}e^{\max \varphi/h}\|f\|_{L^2(\Omega)} \\
&+ Ca^2h^{-1}e^{\varphi(r_2)/h}\mathcal{Q}_1.
\end{aligned}$$

We have

$$\varphi(r_3) - \varphi(r_2) = \tau \int_{r_2}^{r_3} \left( (r+1)^{-k} - (a+1)^{-k} \right) dr \geq c\tau$$

with some constant  $c > 0$ . We deduce from (4.10)

$$\begin{aligned}
(4.11) \quad &\mathcal{Q}_2 \leq \exp\left(\beta h^{-4/(\alpha+3)} + \max \varphi/h\right)\mathcal{Q}_0 \\
&+ \varepsilon^{1/2} \exp\left(\beta h^{-4/(\alpha+3)} + \max \varphi/h\right)\|f\|_{L^2(\Omega)} \\
&+ \exp\left((\beta - c\tau_0)h^{-4/(\alpha+3)}\right)\mathcal{Q}_1
\end{aligned}$$

with some constant  $\beta > 0$ . Combining (4.8) and (4.11) we get

$$\begin{aligned}
(4.12) \quad &\mathcal{Q}_2 \leq \exp\left((\beta + \gamma)h^{-4/(\alpha+3)} + \max \varphi/h\right)\mathcal{Q}_0 \\
&+ \varepsilon^{1/2} \exp\left(\beta h^{-4/(\alpha+3)} + \max \varphi/h\right)\|f\|_{L^2(\Omega)} \\
&+ \exp\left((\beta + \gamma - c\tau_0)h^{-4/(\alpha+3)}\right)\mathcal{Q}_2.
\end{aligned}$$

Taking  $\tau_0$  big enough, independent of  $h$ , we can arrange that

$$\exp\left((\beta + \gamma - c\tau_0)h^{-4/(\alpha+3)}\right) \leq \exp\left(-c\tau_0 h^{-4/(\alpha+3)}/2\right) \leq \exp(-c\tau_0/2) \leq 1/2.$$

Thus we can absorb the last term in the right-hand side of (4.12) to conclude that

$$\begin{aligned}
(4.13) \quad &\mathcal{Q}_1 + \mathcal{Q}_2 \leq \exp\left(\beta_1 h^{-4/(\alpha+3)} + \max \varphi/h\right)\mathcal{Q}_0 \\
&+ \varepsilon^{1/2} \exp\left(\beta_1 h^{-4/(\alpha+3)} + \max \varphi/h\right)\|f\|_{L^2(\Omega)}
\end{aligned}$$

with some constant  $\beta_1 > 0$ . By (4.7), (4.9) and (4.13) we obtain

$$(4.14) \quad \|\chi_s f\|_{L^2(\Omega)} \leq N\mathcal{Q}_0 + \varepsilon^{1/2}N\|f\|_{L^2(\Omega)}$$

where

$$N = \exp\left(\beta_2 h^{-4/(\alpha+3)} + \max \varphi/h\right)$$

with some constant  $\beta_2 > 0$ . In the same way as above, using the fact that the operator  $P(h)$  is symmetric, we get from (4.14) that the resolvent estimate

$$(4.15) \quad \|\chi_s(P(h) - E \pm i\varepsilon)^{-1}\chi_s\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq 4N^2$$

holds for all  $0 < h \leq 1$ ,  $0 < \varepsilon \leq 1$  and  $s$  satisfying (2.4), which together with Lemma 2.2 clearly imply the desired bound.

## 5. PROOF OF THEOREMS 4.1 AND 4.2

The proof of Theorem 4.1 is similar to the proof of Theorem 3.1 of [14], while the proof of Theorem 4.2 requires some modifications. We will first prove Theorem 4.1. The main point is to work with the polar coordinates  $(r, w) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}$ ,  $r = |x|$ ,  $w = x/|x|$  and to use that  $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^+ \times \mathbb{S}^{d-1}, r^{d-1} dr dw)$ . In what follows in this section we denote by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  the norm and the scalar product in  $L^2(\mathbb{S}^{d-1})$ . We will make use of the identity

$$(5.1) \quad r^{(d-1)/2} \Delta r^{-(d-1)/2} = \partial_r^2 + \frac{\tilde{\Delta}_w}{r^2}$$

where  $\tilde{\Delta}_w = \Delta_w - \frac{1}{4}(d-1)(d-3)$  and  $\Delta_w$  denotes the negative Laplace-Beltrami operator on  $\mathbb{S}^{d-1}$ . Set  $u = r^{(d-1)/2} e^{\varphi/h} f$  and

$$\begin{aligned} \mathcal{P}^\pm(h) &= r^{(d-1)/2} (P(h) - E \pm i\varepsilon) r^{-(d-1)/2}, \\ \mathcal{P}_\varphi^\pm(h) &= e^{\varphi/h} \mathcal{P}^\pm(h) e^{-\varphi/h}. \end{aligned}$$

Using (5.1) we can write the operator  $\mathcal{P}^\pm(h)$  in the coordinates  $(r, w)$  as follows

$$\mathcal{P}^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon + V$$

where we have put  $\mathcal{D}_r = -ih\partial_r$  and  $\Lambda_w = -h^2 \tilde{\Delta}_w$ . Since the function  $\varphi$  depends only on the variable  $r$ , we get

$$\mathcal{P}_\varphi^\pm(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - E \pm i\varepsilon - \varphi'^2 + h\varphi'' + 2i\varphi' \mathcal{D}_r + V.$$

We write  $V = V_L + V_S$  with  $V_L := V_\theta$  and  $V_S := V - V_\theta$  if  $V$  satisfies (1.8), and  $V_L := V$  and  $V_S := 0$  if  $V$  satisfies (1.3). For  $r > 0$ ,  $r \neq a$ , introduce the function

$$F(r) = -\langle (r^{-2} \Lambda_w - E - \varphi'(r)^2 + V_L(r, \cdot)) u(r, \cdot), u(r, \cdot) \rangle + \|\mathcal{D}_r u(r, \cdot)\|^2$$

where  $V_L(r, w) := V_L(rw)$ . Then its first derivative is given by

$$\begin{aligned} F'(r) &= \frac{2}{r} \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle + ((\varphi')^2 - V_L)' \|u(r, \cdot)\|^2 \\ &\quad - 2h^{-1} \operatorname{Im} \langle \mathcal{P}_\varphi^\pm(h) u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \\ &\quad \pm 2\varepsilon h^{-1} \operatorname{Re} \langle u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle + 4h^{-1} \varphi' \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad + 2h^{-1} \operatorname{Im} \langle (V_S + h\varphi'') u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle. \end{aligned}$$

Thus we obtain the identity

$$\begin{aligned} (\mu F)' &= \mu' F + \mu F' \\ &= (2r^{-1} \mu - \mu') \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle \\ &\quad + (E\mu' + (\mu(\varphi')^2 - \mu V_L)') \|u(r, \cdot)\|^2 \\ &\quad - 2h^{-1} \mu \operatorname{Im} \langle \mathcal{P}_\varphi^\pm(h) u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle \\ &\quad \pm 2\varepsilon h^{-1} \mu \operatorname{Re} \langle u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle + (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad + 2h^{-1} \mu \operatorname{Im} \langle (V_S + h\varphi'') u(r, \cdot), \mathcal{D}_r u(r, \cdot) \rangle. \end{aligned}$$

Using that  $\Lambda_w \geq 0$  as long as  $d \geq 3$  together with (2.5) we get the inequality

$$\begin{aligned} \mu' F + \mu F' &\geq (E\mu' + (\mu(\varphi')^2 - \mu V_L)') \|u(r, \cdot)\|^2 \\ &\quad + (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\ &\quad - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h) u(r, \cdot)\|^2 - \frac{\mu'}{3} \|\mathcal{D}_r u(r, \cdot)\|^2 \end{aligned}$$



$$\begin{aligned}
& -\varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) \\
& -3h^{-2} \mu^2 (\mu' + 4h^{-1} \varphi' \mu)^{-1} \|(V_S + h\varphi'')u(r, \cdot)\|^2 \\
& \quad - \frac{1}{3} (\mu' + 4h^{-1} \varphi' \mu) \|\mathcal{D}_r u(r, \cdot)\|^2 \\
& \geq (E\mu' + (\mu(\varphi')^2)' - T_L \mu - Z_L \mu') \|u(r, \cdot)\|^2 \\
& \quad + \frac{\mu'}{3} \|\mathcal{D}_r u(r, \cdot)\|^2 - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 \\
& \quad - \varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) \\
& -3h^{-2} \mu^2 (\mu' + 4h^{-1} \varphi' \mu)^{-1} (Q_S + h|\varphi''|)^2 \|u(r, \cdot)\|^2
\end{aligned}$$

where

$$T_L = \mathcal{O}\left((r+1)^{-\beta}\right), \quad Z_L = p(r), \quad Q_S = 0,$$

if  $V$  satisfies (1.3),

$$T_L = \mathcal{O}(\theta^{-1+\alpha}(r+1)^{-4}), \quad Z_L = p(r) + \mathcal{O}((r+1)^{-4}), \quad Q_S = \mathcal{O}(\theta^\alpha(r+1)^{-4}),$$

if  $V$  satisfies (1.8), and we have used the bounds (2.1), (2.2) and (2.3) in the second case. Hence we can rewrite the above inequality in the form

$$\begin{aligned}
\mu' F + \mu F' & \geq (E\mu' + A(r) - CB(r)) \|u(r, \cdot)\|^2 + \frac{\mu'}{3} \|\mathcal{D}_r u(r, \cdot)\|^2 \\
& \quad - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 - \varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2)
\end{aligned}$$

with a suitable constant  $C > 0$ . Now we use Lemma 2.3 to conclude that

$$\begin{aligned}
(5.2) \quad \mu' F + \mu F' & \geq \frac{E}{2} \mu' \|u(r, \cdot)\|^2 + \frac{\mu'}{3} \|\mathcal{D}_r u(r, \cdot)\|^2 - \frac{3h^{-2} \mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 \\
& \quad - \varepsilon h^{-1} \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2)
\end{aligned}$$

provided  $h$  is taken small enough. We integrate this inequality with respect to  $r$  and use that  $\mu(0) = 0$ . We have

$$\int_0^\infty (\mu' F + \mu F') dr = 0.$$

Thus we obtain the estimate

$$\begin{aligned}
(5.3) \quad \frac{E}{2} \int_0^\infty \mu' \|u(r, \cdot)\|^2 dr + \int_0^\infty \frac{\mu'}{3} \|\mathcal{D}_r u(r, \cdot)\|^2 dr & \leq 3h^{-2} \int_0^\infty \frac{\mu^2}{\mu'} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \\
& \quad + \varepsilon h^{-1} \int_0^\infty \mu (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr.
\end{aligned}$$

Using that  $\mu = \mathcal{O}(a^2)$  together with (2.6) we get from (5.3)

$$\begin{aligned}
(5.4) \quad & \int_0^\infty (r+1)^{-2s} (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr \\
& \leq C a^4 h^{-2} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \\
& \quad + C \varepsilon h^{-1} a^2 \int_0^\infty (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr
\end{aligned}$$

with some constant  $C > 0$  independent of  $h$  and  $\varepsilon$ . On the other hand, we have the identity

$$\operatorname{Re} \int_0^\infty \langle 2i\varphi' \mathcal{D}_r u(r, \cdot), u(r, \cdot) \rangle dr = \int_0^\infty h\varphi'' \|u(r, \cdot)\|^2 dr$$

and hence

$$\begin{aligned} \operatorname{Re} \int_0^\infty \langle \mathcal{P}_\varphi^\pm(h)u(r, \cdot), u(r, \cdot) \rangle dr &= \int_0^\infty \|\mathcal{D}_r u(r, \cdot)\|^2 dr + \int_0^\infty \langle r^{-2} \Lambda_w u(r, \cdot), u(r, \cdot) \rangle dr \\ &\quad - \int_0^\infty (E + \varphi'^2) \|u(r, \cdot)\|^2 dr + \int_0^\infty \langle V u(r, \cdot), u(r, \cdot) \rangle dr \\ &\geq \int_0^\infty \|\mathcal{D}_r u(r, \cdot)\|^2 dr - \mathcal{O}(\tau^2) \int_0^\infty \|u(r, \cdot)\|^2 dr. \end{aligned}$$

This implies

$$\begin{aligned} \varepsilon h^{-1} a^2 \int_0^\infty \|\mathcal{D}_r u(r, \cdot)\|^2 dr &\leq \mathcal{O}(\tau^2) \varepsilon h^{-1} a^2 \int_0^\infty \|u(r, \cdot)\|^2 dr \\ (5.5) \quad + \gamma \int_0^\infty (r+1)^{-2s} \|u(r, \cdot)\|^2 dr &+ \gamma^{-1} h^{-2} a^4 \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \end{aligned}$$

for every  $\gamma > 0$ . Taking  $\gamma$  small enough, independent of  $h$ ,  $\tau$  and  $a$ , and combining the estimates (5.4) and (5.5), we get

$$\begin{aligned} &\int_0^\infty (r+1)^{-2s} (\|u(r, \cdot)\|^2 + \|\mathcal{D}_r u(r, \cdot)\|^2) dr \\ &\leq C a^4 h^{-2} \int_0^\infty (r+1)^{2s} \|\mathcal{P}_\varphi^\pm(h)u(r, \cdot)\|^2 dr \\ (5.6) \quad &+ C \varepsilon h^{-1} a^2 \tau^2 \int_0^\infty \|u(r, \cdot)\|^2 dr \end{aligned}$$

with a new constant  $C > 0$  independent of  $h$  and  $\varepsilon$ . Clearly, the estimate (5.6) implies (4.1).

The proof of Theorem 4.2 in the case when  $d \geq 3$  goes very much like the proof of Theorem 4.1 above. The only difference in this case is that we have to integrate the function  $F(r)$  from  $r_0$  to  $\infty$  and use that  $F(r_0) = 0$  by assumption. Thus, by Lemma 2.3 we conclude that the inequality (5.2) holds for all  $r \geq r_0$  and small  $h$ , or for  $r \geq r_0$ ,  $r_0 > 0$  being big enough, and all  $0 < h \leq 1$ .

In the case  $d = 2$  the operator  $\Lambda_w$  is no longer non-negative. Instead, we will use that so is the operator  $-\Delta_w$ . Thus, it is easy to see that the above inequalities still hold with  $V_L$  replaced by  $V_L - h^2(2r)^{-2}$ . Since

$$(\mu(r)(2r)^{-2})' = \mu'(r)(2r)^{-2} - 2^{-1}r^{-3}\mu(r) > -r^{-3}\mu(r),$$

we can use Lemma 2.4 instead of Lemma 2.3 to conclude that the inequality (5.2) remains valid for  $r \geq r_0$ ,  $r_0 > 0$  being arbitrary, and  $0 < h \leq h_0 \ll 1$ , or for  $r \geq r_0$ ,  $r_0 > 0$  being big enough, and all  $h_0 \leq h \leq 1$ .  $\square$

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