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# First and Second Order Moments of the Normalized Sample Covariance Matrix of Spherically Invariant Random Vectors

Sébastien Bausson, Frédéric Pascal, Philippe Forster, Jean-Philippe Ovarlez and Pascal Larzabal

## Abstract

Under Gaussian assumptions, the Sample Covariance Matrix (SCM) is encountered in many covariance based processing algorithms. In case of impulsive noise, this estimate is no more appropriate. This is the reason why when the noise is modeled by Spherically Invariant Random Vectors (SIRV), a natural extension of the SCM is extensively used in the literature: the well-known Normalized Sample Covariance Matrix (NSCM) which estimates the covariance of SIRV. Indeed, this estimate gets rid of a fluctuating noise power and is widely used in radar applications. The aim of this paper is to derive closed-form expressions of the first and second order moments of the NSCM.

## Index Terms

SIRV, NSCM, estimation, performance analysis.

## I. INTRODUCTION

Given independent identically distributed observations of a zero-mean complex Gaussian random vector, the Sample Covariance Matrix (SCM) is the Maximum Likelihood estimate of the data covariance matrix.

S. Bausson and P. Forster are with the Groupe d'Électromagnétisme Appliqué (GEA), Université Paris X, 92410 Ville d'Avray, France (email: sebastien.bausson, philippe.forster@u-paris10.fr).

F. Pascal and J-P. Ovarlez are with the Office National d'Études et de Recherches Aérospatiales (ONERA), DEMR/TSI, BP 72, 92322 Chatillon Cedex, France (email: frederic.pascal, jean-philippe.ovarlez@onera.fr).

P. Larzabal is with the IUT de Cachan, C.R.I.I.P., Université Paris Sud, 94234 Cachan Cedex, France, and also the SATIE, ENS Cachan, UMR CNRS 8029, 94235 Cachan Cedex, France (email: larzabal@satie.ens-cachan.fr).

It is well-known that the SCM is complex Wishart distributed, unbiased, and its second order moments have simple expressions [1]. The full statistical characterization of the SCM allowed performance analysis of numerous algorithms relying on this estimate. However, this widespread estimate is no more appropriate when observations are not Gaussian. For instance, this is the case for radar clutter returns [2], radio fading analysis [3], sonar interferences [4]... In these contexts, Spherically Invariant Random Vectors (SIRV) have been appropriately used in modeling non-Gaussian problems. A SIRV is a complex compound Gaussian process with random power. More precisely, a SIRV  $\mathbf{c}$  [5] is the product of the square root of a positive random variable  $\tau$ , called the *texture*, and a  $m$ -dimensional independent zero mean complex Gaussian vector  $\mathbf{x}$  with covariance matrix  $\Sigma = \mathbb{E}[\mathbf{x}\mathbf{x}^H]$  normalized according to  $\text{tr}(\Sigma) = m$ , where  $\mathbb{E}[\cdot]$  stands for the statistical mean,  $\text{tr}(\cdot)$  is the trace of a matrix, and  $^H$  denotes the transpose conjugate operator,

$$\mathbf{c} = \sqrt{\tau}\mathbf{x}.$$

The notation  $\mathbf{x} \sim \mathbb{CN}(\mathbf{0}, \Sigma)$  means that  $\mathbf{x}$  is a zero mean complex Gaussian vector with covariance matrix  $\Sigma$ . In this paper, we consider the estimation scheme of  $\Sigma$  from  $N$  independent SIRV observations,  $\mathbf{c}_k = \sqrt{\tau_k}\mathbf{x}_k$ , for  $k = 1, \dots, N$ . In this context, we analyze the statistical properties of the well-known Normalized Sample Covariance Matrix (NSCM), introduced in [6], and defined by

$$\hat{\Sigma} = \frac{m}{N} \sum_{k=1}^N \frac{\mathbf{c}_k \mathbf{c}_k^H}{\mathbf{c}_k^H \mathbf{c}_k} = \frac{m}{N} \sum_{k=1}^N \frac{\mathbf{c}_k \mathbf{c}_k^H}{\|\mathbf{c}_k\|^2} = \frac{m}{N} \sum_{k=1}^N \frac{\mathbf{x}_k \mathbf{x}_k^H}{\|\mathbf{x}_k\|^2}. \quad (1)$$

Notice that, in Eq. (1), the NSCM does not depend on the texture. Moreover, the Central Limit Theorem (CLT) ensures that the NSCM is asymptotically Gaussian, but, first and second order moments of this estimate never appeared in the literature. Thus, the goal of this paper is to fill these gaps when the  $\Sigma$ -eigenvalues are distinct, *i.e.*, the most common and realistic case.

## II. FIRST AND SECOND ORDER MOMENTS OF THE NSCM

This section is devoted to the presentation of the main results while computational details will be provided in Appendix.

First, let us introduce the eigenvalue decomposition of  $\Sigma$

$$\Sigma = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H = \sum_{k=1}^m \lambda_k \mathbf{u}_k \mathbf{u}_k^H, \quad (2)$$

where

- $\mathbf{\Lambda}$  is the diagonal matrix of the  $\Sigma$ -eigenvalues,  $\lambda_1 > \dots > \lambda_m > 0$ ,

- $\mathbf{U}$  is the unitary matrix of the  $\Sigma$ -eigenvectors.

Notice that we assume that all eigenvalues  $\lambda_1, \dots, \lambda_m$ , are strictly positive and different, *i.e.*, their multiplicity order is 1.

**Theorem 1** *The first order moment of the NSCM is given by*

$$\mathbb{E} [\widehat{\Sigma}] = m \mathbf{U} \Delta \mathbf{U}^H, \quad (3)$$

where

$$\delta_k = \sum_{\substack{n=1 \\ n \neq k}}^m d_n \lambda_k \left( \frac{\log \lambda_n - \log \lambda_k}{\lambda_n - \lambda_k} - \frac{1}{\lambda_n} \right), \quad (4)$$

$$d_n = \prod_{\substack{p=1 \\ p \neq n}}^m (1 - \lambda_p / \lambda_n)^{-1}, \quad (5)$$

and where  $\Delta$  is the diagonal matrix of the  $\delta_k$ 's, with  $\delta_1 > \dots > \delta_m > 0$ .

*Proof:* See Appendix I. ■

**Remark 1** *This theorem provides as a by-product the eigen-decomposition of  $\mathbb{E} [\widehat{\Sigma}]$ . It shows also that  $\mathbb{E} [\widehat{\Sigma}]$  and  $\Sigma$  share the same eigenvectors but have different eigenvalues. Consequently, the NSCM is a biased estimate of  $\Sigma$ .*

**Remark 2** *The NSCM preserves the ordering of the eigenvectors.*

Let us denote  $\text{vec}(\cdot)$  the operator which reshapes a  $m \times n$  matrix elements into a  $mn$  column vector and let us introduce the two matrices

$$\mathbf{V}_1 = \mathbb{E} \left[ \text{vec} (\widehat{\Sigma}) \text{vec} (\widehat{\Sigma})^H \right] \quad \text{and} \quad \mathbf{V}_2 = \mathbb{E} \left[ \text{vec} (\widehat{\Sigma}) \text{vec} (\widehat{\Sigma})^T \right], \quad (6)$$

from which the covariances of the real and imaginary parts of the NSCM are straightforwardly derived.

**Theorem 2** *The NSCM is asymptotically Gaussian and*

$$\begin{aligned} \mathbf{V}_1 &= \frac{m^2}{N} \sum_{p=1}^m \sum_{k=1}^m (w_{pk} + (N-1) \delta_p \delta_k) \text{vec} (\mathbf{u}_p \mathbf{u}_p^H) \text{vec} (\mathbf{u}_k \mathbf{u}_k^H)^H \\ &+ \frac{m^2}{N} \sum_{p=1}^m \sum_{\substack{k=1 \\ k \neq p}}^m w_{pk} \text{vec} (\mathbf{u}_p \mathbf{u}_k^H) \text{vec} (\mathbf{u}_p \mathbf{u}_k^H)^H, \end{aligned} \quad (7)$$

$$\begin{aligned} \mathbf{V}_2 &= \frac{m^2}{N} \sum_{p=1}^m \sum_{k=1}^m (w_{pk} + (N-1) \delta_p \delta_k) \text{vec}(\mathbf{u}_p \mathbf{u}_p^H) \text{vec}(\mathbf{u}_k \mathbf{u}_k^H)^T \\ &\quad + \frac{m^2}{N} \sum_{p=1}^m \sum_{\substack{k=1 \\ k \neq p}}^m w_{pk} \text{vec}(\mathbf{u}_p \mathbf{u}_p^H) \text{vec}(\mathbf{u}_k \mathbf{u}_p^H)^T, \end{aligned} \quad (8)$$

where

$$w_{kk} = \sum_{\substack{n=1 \\ n \neq k}}^m d_n \lambda_k \left( \frac{2 \lambda_k \log(\lambda_k / \lambda_n)}{(\lambda_k - \lambda_n)^2} - \frac{1}{\lambda_n} \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n} \right), \quad (9)$$

$$w_{pk} = \sum_{\substack{n=1 \\ n \neq p \\ n \neq k}}^m d_n \tilde{w}_{pkn}, \text{ for } p \neq k, \quad (10)$$

with

$$\begin{aligned} \tilde{w}_{pkn} &= \left[ \frac{\lambda_n (\lambda_p + \lambda_k) - 2 \lambda_p \lambda_k}{\lambda_n^2 (\lambda_p - \lambda_k)^2} \right] \lambda_p \lambda_k - \left[ \frac{\lambda_p \lambda_k}{(\lambda_n - \lambda_p) (\lambda_n - \lambda_k)} \right] \log \frac{\lambda_n}{\lambda_p} \\ &\quad + \left[ \frac{\lambda_p \lambda_k^2 (\lambda_n - \lambda_p)}{\lambda_n^2 (\lambda_n - \lambda_k) (\lambda_p - \lambda_k)^3} \right] (2 \lambda_p \lambda_n - \lambda_p \lambda_k - \lambda_k^2) \log \frac{\lambda_k}{\lambda_p}, \end{aligned} \quad (11)$$

where  $\delta_n$  and  $d_n$  are respectively defined in equations (4) and (5).

*Proof:* See Appendix II, III, IV. ■

### III. CONCLUSION

The closed-form expressions of the first and second order moments of the NSCM for SIRV modeling have been provided in this paper with full detailed proofs.

These analytical equations are essential for analyzing performance of signal processing methods based on NSCM: detection schemes in radar applications, direction of arrivals (DOA), estimation in array processing, ...

#### APPENDIX I

##### PROOF OF THEOREM 1

Using the eigen-decomposition of Eq. (2), let us whiten  $\mathbf{x}$  according to  $\mathbf{y} = \mathbf{\Lambda}^{-1/2} \mathbf{U}^H \mathbf{x}$ . Hence,  $\mathbf{y} \sim \mathbb{CN}(\mathbf{0}, \mathbf{I})$  and consequently

$$\frac{\mathbf{x} \mathbf{x}^H}{\|\mathbf{x}\|^2} = \mathbf{U} \mathbf{\Lambda}^{-1/2} \frac{\mathbf{y} \mathbf{y}^H}{\mathbf{y}^H \mathbf{\Lambda}^{-1} \mathbf{y}} \mathbf{\Lambda}^{-1/2} \mathbf{U}^H.$$

The NSCM statistical mean can be rewritten as

$$\mathbb{E} \left[ \widehat{\Sigma} \right] = \mathbb{E} \left[ \frac{m}{N} \sum_{k=1}^N \frac{\mathbf{x}_k \mathbf{x}_k^H}{\|\mathbf{x}_k\|^2} \right] = m \mathbb{E} \left[ \frac{\mathbf{x} \mathbf{x}^H}{\|\mathbf{x}\|^2} \right] = m \mathbf{U} \mathbf{\Lambda}^{-1/2} \mathbb{E} \left[ \frac{\mathbf{y} \mathbf{y}^H}{\mathbf{y}^H \mathbf{\Lambda}^{-1} \mathbf{y}} \right] \mathbf{\Lambda}^{-1/2} \mathbf{U}^H. \quad (\text{I.12})$$

Each component  $y_k$  of the vector  $\mathbf{y}$  is a zero-mean unit variance circular complex Gaussian variable and can be expressed as:

$$y_k = \sqrt{\frac{1}{2} \chi_k^2} \exp(i\theta_k),$$

where  $\chi_k^2$  is Chi-squared-distributed with 2 degrees of freedom,  $\theta_k$  is uniformly distributed on  $[0, 2\pi]$ .

All the  $\chi_k^2$ 's and  $\theta_k$ 's are two-by-two independent. It follows that Eq. (I.12) yields

$$\mathbb{E} \left[ \widehat{\Sigma} \right] = m \sum_{k=1}^m \lambda_k \mathbb{E} \left[ \chi_k^2 / \sum_{n=1}^m \lambda_n \chi_n^2 \right] \mathbf{u}_k \mathbf{u}_k^H.$$

Let us set

$$\delta_k = \mathbb{E} \left[ \lambda_k \chi_k^2 / \sum_{n=1}^m \lambda_n \chi_n^2 \right] = \mathbb{E} \left[ \frac{1}{1 + X_2/X_1} \right], \quad (\text{I.13})$$

where  $X_1 = \lambda_k \chi_k^2$  and  $X_2 = \sum_{\substack{n=1 \\ n \neq k}}^m \lambda_n \chi_n^2$ .

To complete the proof, the PDF of  $X_2$  has to be derived. Since all  $\chi_k^2$ 's are independent, the characteristic function of  $X_2$  is

$$\phi_{X_2}(u) = \prod_{\substack{n=1 \\ n \neq k}}^m (1 - 2i \lambda_n u)^{-1} = \sum_{\substack{n=1 \\ n \neq k}}^m \frac{c_n}{1 - 2i \lambda_n u},$$

where  $c_n = \prod_{\substack{p=1 \\ p \neq n \\ p \neq k}}^m \left( 1 - \frac{\lambda_p}{\lambda_n} \right)^{-1}$ . Thus, the PDF of  $X_2$  follows

$$p_{X_2}(x) = \frac{1}{2} \sum_{\substack{n=1 \\ n \neq k}}^m \frac{c_n}{\lambda_n} \exp \left( -\frac{x}{2\lambda_n} \right), \quad x \geq 0. \quad (\text{I.14})$$

So, the density of  $X_2$  is obtained by the weighted sum of the densities of  $\lambda_n \chi_n^2$  by the coefficient  $c_n$ .

Now, the PDF of the ratio  $X_2/X_1$  is a weighted sum of F laws (Fisher-Snedecor)

$$p_{X_2/X_1}(x) = \sum_{\substack{n=1 \\ n \neq k}}^m c_n \frac{\lambda_k}{\lambda_n} \left( 1 + \frac{\lambda_k}{\lambda_n} x \right)^{-2}, \quad x \geq 0, \quad (\text{I.15})$$

and after some manipulations, Eq. (I.13) yields

$$\delta_k = \sum_{\substack{n=1 \\ n \neq k}}^m c_n \left( \frac{\lambda_n/\lambda_k}{(1-\lambda_n/\lambda_k)^2} \log(\lambda_n/\lambda_k) + \frac{1}{1-\lambda_n/\lambda_k} \right).$$

It remains to show that  $\delta_1 > \dots > \delta_m > 0$ . First, the  $\delta_k$ 's, defined in Eq. (I.13), are strictly positive. Now, let us consider the function

$$f_w(x, y) = \frac{1}{4} \int_{\mathbb{R}_+^2} \frac{xu}{xu + yv + w} \exp\left(-\frac{u+v}{2}\right) du dv,$$

where it follows from (I.13) that we have  $\delta_k = \mathbb{E}_w[f_w(\lambda_k, \lambda_p)]$  and  $\delta_p = \mathbb{E}_w[f_w(\lambda_p, \lambda_k)]$ , for  $w = \sum_{\substack{n=1 \\ n \neq k \\ n \neq p}}^m \lambda_n \chi_n^2$ , and where  $\mathbb{E}_w[\cdot]$  stands for the statistical mean related to  $w$ .

To show that  $\delta_k < \delta_p$ , we prove that  $f_w(\lambda_k, \lambda_p) < f_w(\lambda_p, \lambda_k)$  for all  $w$ , assuming  $\lambda_k < \lambda_p$ .

Let us define the functions

$$\begin{aligned} f_1(t) &= f_w((1-t)\lambda_p + t\lambda_k, \lambda_p), \\ f_2(t) &= f_w(\lambda_p, (1-t)\lambda_p + t\lambda_k), \end{aligned}$$

which verify  $f_1(0) = f_2(0)$ ,  $f_1(1) = \delta_k$  and  $f_2(1) = \delta_p$ . To demonstrate that  $\delta_k < \delta_p$ , we show hereafter that  $f_1$  and  $f_2$  are respectively strictly decreasing and strictly increasing functions of  $t$  on the interval  $[0, 1]$ . To proceed, we compute and study the signs of the partial derivatives of  $f_w$ ,

$$\begin{aligned} \frac{\partial f_w}{\partial x}(x, y) &= \frac{1}{4} \int_{\mathbb{R}_+^2} \frac{u(yv + w)}{(xu + yv + w)^2} \exp\left(-\frac{u+v}{2}\right) du dv > 0, \\ \frac{\partial f_w}{\partial y}(x, y) &= -\frac{1}{4} \int_{\mathbb{R}_+^2} \frac{vxu}{(xu + yv + w)^2} \exp\left(-\frac{u+v}{2}\right) du dv < 0, \end{aligned}$$

which allow to find the signs of the derivatives of  $f_1$  and  $f_2$ ,

$$\begin{aligned} \frac{df_1}{dt} &= \left( \frac{\partial f_w}{\partial x} \right)_{((1-t)\lambda_p + t\lambda_k, \lambda_p)} (\lambda_k - \lambda_p) < 0, \\ \frac{df_2}{dt} &= \left( \frac{\partial f_w}{\partial y} \right)_{(\lambda_p, (1-t)\lambda_p + t\lambda_k)} (\lambda_k - \lambda_p) > 0. \end{aligned}$$

In summary,  $\delta_k < \delta_p$  for any  $k, p$  such that  $\lambda_k < \lambda_p$ . This completes the proof of theorem 1.

## APPENDIX II

## PROOF OF THE EQS. (7), (8) AND (9) OF THEOREM 2

By expressing the variance of the NSCM as a linear combination of functions of the  $\Sigma$ -eigenvectors, we compute the statistical means of the coefficients. Eqs. (1), (3), (6) and (I.12) leads to

$$\begin{aligned} \mathbf{V}_1 &= \frac{m^2}{N^2} \sum_{k=1}^m \sum_{p=1}^m \mathbb{E} \left[ \text{vec} \left( \frac{\mathbf{x}_k \mathbf{x}_k^H}{\|\mathbf{x}_k\|^2} \right) \text{vec} \left( \frac{\mathbf{x}_p \mathbf{x}_p^H}{\|\mathbf{x}_p\|^2} \right)^H \right], \\ &= \frac{m^2}{N} \mathbb{E} \left[ \text{vec} \left( \frac{\mathbf{x} \mathbf{x}^H}{\|\mathbf{x}\|^2} \right) \text{vec} \left( \frac{\mathbf{x} \mathbf{x}^H}{\|\mathbf{x}\|^2} \right)^H \right] + m^2 \frac{N-1}{N} \text{vec} \left( \mathbb{E} \left[ \frac{\mathbf{x} \mathbf{x}^H}{\|\mathbf{x}\|^2} \right] \right) \text{vec} \left( \mathbb{E} \left[ \frac{\mathbf{x} \mathbf{x}^H}{\|\mathbf{x}\|^2} \right] \right)^H, \\ &= \frac{m^2}{N} \sum_{p=1}^m \sum_{j=1}^m \sum_{n=1}^m \sum_{k=1}^m (\omega_{pjnk} + (N-1) \delta_p \delta_n \delta(p-j) \delta(n-k)) \text{vec}(\mathbf{u}_p \mathbf{u}_j^H) \text{vec}(\mathbf{u}_n \mathbf{u}_k^H)^H, \end{aligned}$$

where  $\omega_{pjnk} = (\lambda_p \lambda_j \lambda_n \lambda_k)^{1/2} \mathbb{E} \left[ \frac{(\chi_p^2 \chi_j^2 \chi_n^2 \chi_k^2)^{1/2}}{\left( \sum_{t=1}^m \lambda_t \chi_t^2 \right)^2} \right] \mathbb{E} [\exp(i(\theta_p - \theta_j + \theta_k - \theta_n))]$  and  $\delta(\cdot)$  is the Kronecker delta. The  $\theta$ 's being independent uniform variables, the last term of previous equations is zero if  $p \neq j$  and  $n \neq k$ , or if  $p \neq n$  and  $k \neq j$ , which leads to

$$\begin{aligned} \mathbf{V}_1 &= \frac{m^2}{N} \sum_{p=1}^m \sum_{k=1}^m (w_{pk} + (N-1) \delta_p \delta_k) \text{vec}(\mathbf{u}_p \mathbf{u}_p^H) \text{vec}(\mathbf{u}_k \mathbf{u}_k^H)^H \\ &\quad + \frac{m^2}{N} \sum_{p=1}^m \sum_{\substack{k=1 \\ k \neq p}}^m w_{pk} \text{vec}(\mathbf{u}_p \mathbf{u}_k^H) \text{vec}(\mathbf{u}_p \mathbf{u}_k^H)^H \end{aligned}$$

where

$$w_{pk} = \lambda_p \lambda_k \mathbb{E} \left[ \chi_p^2 \chi_k^2 / \left( \sum_{n=1}^m \lambda_n \chi_n^2 \right)^2 \right], \quad (\text{II.16})$$

This is Eq. (7) of theorem 2 and Eq. (8) is derived from the same reasoning. Concerning Eq. (9), one has, from Eq. (II.16), for  $p = k$ ,  $w_{kk} = \mathbb{E} \left[ (1 + X_2/X_1)^{-2} \right]$ , where  $X_1$  and  $X_2$  are defined in (I.13). Thus,  $w_{kk} = \int_0^{+\infty} (1+x)^{-2} p_{X_2/X_1}(x) dx$ . Eq. (10) will be derived in Appendix IV. Its proof needs some results on exponential integrals, recaped in Appendix III.



## APPENDIX III

## EXPONENTIAL INTEGRALS AND RELATED FUNCTIONS

This section contains some mathematical tools used in Appendix IV. From pp. 228 of [7], let us recall the definition of the exponential integral

$$\begin{aligned} E_n(z) &= \int_1^{+\infty} \frac{e^{-zt}}{t^n} dt, \quad n \in \mathbb{N}, \Re(z) > 0, \\ E_1(z) &= \int_1^{+\infty} \frac{e^{-zt}}{t} dt = \int_z^{+\infty} \frac{e^{-t}}{t} dt = -\gamma - \ln z - \sum_{n=1}^{+\infty} \frac{(-1)^n z^n}{n n!}, \end{aligned} \quad (\text{III.17})$$

where  $\Re(z)$  denotes the real part of  $z$  and  $\gamma \simeq 0.57721$  is Euler's constant. Let us introduce the function

$$F_n(a, z) = \int_z^{+\infty} t^n e^{-at} E_1(t) dt, \quad n \in \mathbb{N}, \Re(z) > 0, a > -1, \quad (\text{III.18})$$

which gives for  $n = 0$  and  $n = 1$

$$\begin{aligned} F_0(a, z) &= \frac{1}{a} e^{-az} E_1(z) - \frac{1}{a} E_1((1+a)z), \\ F_1(a, z) &= \frac{1+az}{a^2} e^{-az} E_1(z) - \frac{1}{a^2} E_1((1+a)z) - \frac{1}{a(1+a)} e^{-(1+a)z}. \end{aligned}$$

In particular, for  $a > -1$ , we will use the values

$$\begin{aligned} F_1(a, 0) &= \frac{1}{a} \left[ \frac{1}{a} \ln(1+a) - \frac{1}{1+a} \right], \\ F_2(a, 0) &= \frac{2}{a^3} \ln(1+a) - \frac{3a+2}{a^2(1+a)^2}. \end{aligned} \quad (\text{III.19})$$

## APPENDIX IV

END OF PROOF OF THEOREM 2 ( $w_{pk}$  FOR  $p \neq k$ , SEE EQ. (10))

At the end of Appendix II, it remained to compute Eq. (II.16) to complete the proof of theorem 2. A PDF decomposition similar to Eq. (I.14) provides

$$w_{pk} = \sum_{\substack{n=1 \\ n \neq p \\ n \neq k}}^m \delta_{pkn} \prod_{\substack{j=1 \\ j \neq n \\ j \neq p \\ j \neq k}}^m \left( 1 - \frac{\lambda_j}{\lambda_n} \right)^{-1}, \quad (\text{IV.20})$$

where  $\delta_{pkn} = \lambda_p \lambda_k \mathbb{E} \left[ \frac{\chi_p^2 \chi_k^2}{(\lambda_p \chi_p^2 + \lambda_k \chi_k^2 + \lambda_n \chi_n^2)^2} \right]$  can be rewritten as

$$\delta_{pkn} = \frac{\lambda_p \lambda_k}{8} \int_{\mathbb{R}_+^3} \frac{x_p x_k}{(\lambda_p x_p + \lambda_k x_k + \lambda_n x_n)^2} \exp \left( -\frac{1}{2}(x_p + x_k + x_n) \right) dx_p dx_k dx_n. \quad (\text{IV.21})$$

An analytic expression of  $\delta_{pkn}$  is obtained by computing the above integral. Eq. (IV.21) is rewritten as

$$\delta_{pkn} = \frac{\lambda_p \lambda_k}{8} \int_0^{+\infty} \int_0^{+\infty} x_p x_k \exp\left(-\frac{1}{2}(x_p + x_k)\right) t_1 dx_p dx_k, \quad (\text{IV.22})$$

where  $t_1 = \int_0^{+\infty} \frac{\exp(-x_n/2)}{(\lambda_p x_p + \lambda_k x_k + \lambda_n x_n)^2} dx_n$ . Then, by setting  $C = \lambda_p x_p + \lambda_k x_k$ ,  $t_1$  is rewritten as

$$t_1 = \frac{1}{\lambda_n C} \exp\left(\frac{C}{2\lambda_n}\right) E_2\left(\frac{C}{2\lambda_n}\right) = \frac{1}{\lambda_n C} \left[1 - \frac{C}{2\lambda_n} \exp\left(\frac{C}{2\lambda_n}\right) E_1\left(\frac{C}{2\lambda_n}\right)\right].$$

where  $E_1$  and  $E_2$  are defined in (III.17). Now, by replacing  $t_1$  in Eq. (IV.22), we obtain

$$\delta_{pkn} = \frac{\lambda_p \lambda_k}{8 \lambda_n} \left(t_2 - \frac{1}{2\lambda_n} t_3\right), \quad (\text{IV.23})$$

where

$$\begin{aligned} t_2 &= \int_0^{+\infty} \int_0^{+\infty} \frac{x_p x_k}{(\lambda_p x_p + \lambda_k x_k)} \exp((x_p + x_k)/2) dx_p dx_k, \\ t_3 &= \int_0^{+\infty} \int_0^{+\infty} x_p x_k \exp\left(-\frac{1}{2}\left(x_p + x_k - \frac{\lambda_p x_p + \lambda_k x_k}{\lambda_n}\right)\right) E_1\left(\frac{\lambda_p x_p + \lambda_k x_k}{2\lambda_n}\right) dx_p dx_k. \end{aligned}$$

Integrating firstly along  $x_k$  in  $t_2$  allow to rewrite  $t_2$  as

$$t_2 = \frac{8}{\lambda_k} - \frac{8\lambda_p}{\lambda_k^2} F_2\left(\frac{\lambda_k - \lambda_p}{\lambda_p}, 0\right), \quad (\text{IV.24})$$

where the function  $F_2(\cdot)$  is defined in Appendix III. Now, let us compute  $t_3$  as

$$t_3 = \int_0^{+\infty} x_p \exp\left(-\frac{x_p}{2}\left(1 - \frac{\lambda_p}{\lambda_n}\right)\right) t_4 dx_p,$$

with

$$t_4 = \int_0^{+\infty} x_k \exp\left(-\frac{x_k}{2}\left(1 - \frac{\lambda_k}{\lambda_n}\right)\right) E_1\left(\frac{\lambda_p x_p + \lambda_k x_k}{2\lambda_n}\right) dx_k.$$

By a change of variable,  $t_4$  is rewritten as

$$\begin{aligned} t_4 &= \frac{2\lambda_n}{\lambda_k^2} \int_{\frac{\lambda_p x_p}{2\lambda_n}}^{+\infty} (2\lambda_n t - \lambda_p x_p) \exp\left(-\frac{1}{2\lambda_k}(2\lambda_n t - \lambda_p x_p)\left(1 - \frac{\lambda_k}{\lambda_n}\right)\right) E_1(t) dt, \\ &= \frac{2\lambda_n}{\lambda_k^2} \exp\left(\frac{\lambda_p}{2\lambda_k}\left(1 - \frac{\lambda_k}{\lambda_n}\right)x_p\right) \left[2\lambda_n F_1\left(\frac{\lambda_n}{\lambda_k} - 1, \frac{\lambda_p x_p}{2\lambda_n}\right) - \lambda_p x_p F_0\left(\frac{\lambda_n}{\lambda_k} - 1, \frac{\lambda_p x_p}{2\lambda_n}\right)\right]. \end{aligned}$$

and can be simplified with the following relation

$$F_1(a, z) - z F_0(a, z) = \frac{1}{a^2} e^{-az} E_1(z) + \left( \frac{z}{a} - \frac{1}{a^2} \right) E_1((1+a)z) - \frac{1}{a(1+a)} e^{-(1+a)z},$$

where  $a > -1$ ,  $\Re(z) > 0$ . The simplified expression of  $t_4$  allow to rewrite  $t_3$  as

$$t_3 = \frac{16 \lambda_n^2}{\lambda_n - \lambda_k} \left( \frac{\lambda_k}{\lambda_p^2 (\lambda_n - \lambda_k)} [\lambda_n^2 F_1(b_n, 0) - \lambda_k^2 F_1(b_k, 0)] - \frac{1}{\lambda_n} + \frac{\lambda_k^2}{\lambda_n \lambda_p^2} F_2(b_k, 0) \right).$$

where  $b_j = \frac{\lambda_j - \lambda_p}{\lambda_p}$ , for  $j = k, n$ .

Finally, combining the previous result with Eqs. (III.19), (IV.23) and (IV.24), one has

$$\begin{aligned} \delta_{pkn} = & \lambda_p \lambda_k \left[ \frac{\lambda_n (\lambda_p + \lambda_k) - 2 \lambda_p \lambda_k}{(\lambda_n - \lambda_p) (\lambda_n - \lambda_k) (\lambda_p - \lambda_k)^2} - \frac{\lambda_n^2 \log \lambda_n}{(\lambda_n - \lambda_p)^2 (\lambda_n - \lambda_k)^2} \right] \\ & + \frac{\lambda_p^2 \lambda_k^2}{(\lambda_p - \lambda_k)^3} \left[ \frac{2 \lambda_p \lambda_n - \lambda_p \lambda_k - \lambda_k^2}{\lambda_p (\lambda_n - \lambda_k)^2} \log \lambda_k - \frac{2 \lambda_k \lambda_n - \lambda_p \lambda_k - \lambda_p^2}{\lambda_k (\lambda_n - \lambda_p)^2} \log \lambda_p \right]. \end{aligned}$$

Thanks to Eq. (IV.20), the previous equation provides Eqs. (10) and (11). This concludes the proof of theorem 2.

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