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SIGNATURE FOR PIECEWISE CONTINUOUS GROUPS

OCTAVE LACOURTE

ABSTRACT. Let $\widehat{\mathrm{PC}^{\bowtie}}$ be the group of bijections from [0, 1] to itself which are continuous outside a finite set. Let PC^{\bowtie} be its quotient by the subgroup of finitely supported permutations.

We show that the Kapoudjian class of PC^{\bowtie} vanishes. That is, the quotient map $\widehat{\mathrm{PC}}^{\bowtie} \to \mathrm{PC}^{\bowtie}$ splits modulo the alternating subgroup of even permutations. This is shown by constructing a nonzero group homomorphism, called signature, from $\widehat{\mathrm{PC}}^{\bowtie}$ to $\mathbb{Z}_{2\mathbb{Z}}^{\sim}$. Then we use this signature to list normal subgroups of every subgroup \widehat{G} of $\widehat{\mathrm{PC}}^{\bowtie}$ which contains $\mathfrak{S}_{\mathrm{fin}}$ and such that G, the projection of \widehat{G} in PC^{\bowtie} , is simple.

1. INTRODUCTION

Let X be the right-open and left-closed interval [0, 1[. We denote by $\mathfrak{S}(X)$ the group of bijections of X to X. This group contains the subgroup composed of all finitely supported permutations is denoted by $\mathfrak{S}_{\text{fin}}$. The classical signature is well-defined on $\mathfrak{S}_{\text{fin}}$ and its kernel, denoted by $\mathfrak{A}_{\text{fin}}$, is the only subgroup of index 2 in $\mathfrak{S}_{\text{fin}}$. An observation, originally due to Vitali [10], is that the signature does not extend to $\mathfrak{S}(X)$.

For every subgroup G of $\mathfrak{S}(X)/\mathfrak{S}_{fin}$, we denote by \widehat{G} its inverse image in $\mathfrak{S}(X)$. The cohomology class of the central extension

$$0 \to \mathbb{Z}_{2\mathbb{Z}} = \mathfrak{S}_{\operatorname{fin}} \xrightarrow{\widehat{G}}_{\mathfrak{A}_{\operatorname{fin}}} \to \widehat{G} \to 1$$

is called the Kapoudjian class of G; it belongs to $H^2(G, \mathbb{Z}_{2\mathbb{Z}})$. It appears in the work of Kapoudjian and Kapoudjian-Sergiescu [6, 7]. The vanishing of this class means that the above exact sequence splits; this means that there exists a group homomorphism from the preimage of G in $\mathfrak{S}(X)$ onto $\mathbb{Z}_{2\mathbb{Z}}$ which extends the signature on $\mathfrak{S}_{\mathrm{fn}}$ (for more on the Kapoudjian class, see [3, §8.C]). This implies in particular that $\widehat{G}_{\mathrm{A}_{\mathrm{fn}}}$ is isomorphic to the direct product $G \times \mathbb{Z}_{2\mathbb{Z}}$. One can notice that for $G = \mathfrak{S}(X)_{\mathfrak{S}_{\mathrm{fn}}}$ we have $\widehat{G} = \mathfrak{S}(X)$; in this case the Vitali's observation implies that the Kapoudjian class does not vanish.

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The set of all permutations of X continuous outside a finite set is a subgroup denoted by $\widehat{\text{PC}}^{\bowtie}$. The aim here is to show the following theorem:

Theorem 1.1. There exists a group homomorphism $\varepsilon : \widehat{PC}^{\bowtie} \to \mathbb{Z}_{2\mathbb{Z}}$ that extends the classical signature on \mathfrak{S}_{fin} .

Corollary 1.2. Let G be a subgroup of PC^{\bowtie} . Then the Kapoudjian class of G is zero.

This solves a question asked by Y. Cornulier [4, Question 1.15].

The subgroup of $\widehat{\operatorname{PC}}^{\bowtie}$ consisting of all permutations of X that are piecewise isometric elements is denoted by $\widehat{\operatorname{IET}}^{\bowtie}$ and the one consisting of all piecewise affine permutations of X is denoted by $\widehat{\operatorname{PAff}}^{\bowtie}$. We also consider for each of these groups the subgroup composed of all piecewise orientation-preserving elements by replacing the symbol " \bowtie " by the symbol "+". Let us observe that when $G \subset \operatorname{PC}^+$ Corollary 1.2 is trivial. Indeed, in this case

G can be lifted inside $\widehat{\mathrm{PC}^+}$ itself. However, such a lift does not exist for PC^{\bowtie} or even IET^{\bowtie}, as was proved in [4].

The idea of proof of Theorem 1.1 is to associate for every $f \in \widehat{\mathrm{PC}}^{\bowtie}$ and every finite partition \mathcal{P} of [0, 1] into intervals associated with f, two numbers. The first is the number of interval of \mathcal{P} where f is order-reversing and the second is the signature of a particular finitely supported permutation. The next step is to prove that the sum modulo 2 of this two numbers is independent from the choice of partition. Then we show that it is enough to prove that $\varepsilon|_{\mathrm{IET}^{\bowtie}}$ is a group homomorphism. For this we show that it is additive when we look at the composition of two elements of $\widehat{\mathrm{IET}^{\bowtie}}$ by calculate the value of the signature with a particular partition.

In Section 4, we apply these results to the study of normal subgroups of \widehat{PC}^{\bowtie} and certain subgroups. More specifically we prove:

Theorem 1.3. Let \widehat{G} be a subgroup of $\widehat{\mathrm{PC}}^{\bowtie}$ containing $\mathfrak{S}_{\mathrm{fin}}$ and such that its projection G in PC^{\bowtie} is simple nonabelian. Then \widehat{G} has exactly five normal subgroups given by the list: $\{\{1\}, \mathfrak{A}_{\mathrm{fin}}, \mathfrak{S}_{\mathrm{fin}}, \mathrm{Ker}(\varepsilon), \widehat{G}\}.$

We denote by $\widehat{\operatorname{IET}_{\operatorname{rc}}^+}$ the subgroup of $\widehat{\operatorname{IET}^+}$ composed of all right-continuous elements. We know that it is naturally isomorphic to IET^+ . The same is true when we replace IET^+ by PAff⁺ or PC⁺. This allows us to use the work of P. Arnoux [2] and the one of N. Guelman and I. Liousse [5] where they prove that $\operatorname{IET}^{\bowtie}$, PC⁺ and PAff⁺ are simple. From this we deduce:

Theorem 1.4. The groups PC^{\bowtie} and $PAff^{\bowtie}$ are simple.

This gives us some examples of groups that satisfy the conditions of the Theorem 1.3.

Finally Section 5 is independent and we study some normalizers and in particular we show that the behaviour when we look the group inside \widehat{PC}^{\bowtie} or PC^{\bowtie} may not be the same. We denote by $\mathcal{R} \in \operatorname{IET}^{\bowtie}$ the map $x \mapsto 1 - x$. Then we define IET^- as the coset \mathcal{R} . IET^+ and PC^- as the coset \mathcal{R} . PC^+ . Then the groups $\operatorname{IET}^{\pm} := \operatorname{IET}^+ \cup \operatorname{IET}^+$ and $PC^{\pm} := PC^+ \cup PC^-$ are well-defined.

Proposition 1.5. The subgroup $\widehat{\operatorname{IET}}_{\operatorname{rc}}^+$ (resp. $\widehat{\operatorname{PC}}_{\operatorname{rc}}^+$) is its own normalizer in $\widehat{\operatorname{IET}}^{\bowtie}$ (resp. $\widehat{\operatorname{PC}}_{\operatorname{rc}}^+$). The normalizer of IET^+ (PC⁺ respectively) in $\operatorname{IET}^{\bowtie}$ (PC^{\bowtie} respectively) is IET^{\pm} (PC^{\pm} respectively).

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2. Preliminaries

For every real interval I we denote by I° its interior in \mathbb{R} and if I = [0, t] we agree that its interior is [0, t].

2.1. Partitions associated.

An important tool to study elements in \widehat{PC}^{\bowtie} and PC^{\bowtie} are partitions into intervals of [0, 1]. All partitions are assumed to be finite.

Definition 2.1. For every f in $\widehat{\mathrm{PC}}^{\bowtie}$, a finite partition \mathcal{P} into right-open and left-closed intervals of [0, 1] is called a partition into intervals associated with f if and only if f is continuous on the interior of every interval of \mathcal{P} . We denote by Π_f the set of all partitions into intervals associated with f.

We define also the arrival partition of f associated with \mathcal{P} , denoted $f(\mathcal{P})$, the partition of [0, 1] composed of all right-open and left-closed intervals such that their interior is equal to the image by f of the interior of an interval of \mathcal{P} .

Remark 2.2. For every f in $\widehat{\mathrm{PC}}^{\bowtie}$ there exists a unique partition \mathcal{P}_{f}^{\min} associated with f which has a minimal number of intervals. It is actually minimal in the sense of refinement: Π_{f} consists precisely of the set of partitions refining \mathcal{P}_{f}^{\min} .

2.2. Decompositions.

We define a family of elements which plays an important role inside our groups:

Definition 2.3. Let I be a non-empty right-open and left-closed subinterval of [0, 1[. The element $f \in \widehat{PC}^{\bowtie}$ which sends the interior of I on itself with slope -1 while fixing the rest of [0, 1[is called the *I-flip*. We define a *flip* as any *I*-flip for some I.

From the definition we deduce a decomposition inside $\widehat{\operatorname{IET}^{\bowtie}}$ and $\widehat{\operatorname{PC}^{\bowtie}}$.

Proposition 2.4. Let h be an element of $\widehat{\operatorname{IET}}^{\bowtie}$. There exist $f, g \in \widehat{\operatorname{IET}}_{\operatorname{rc}}^+$ and r, s finite products of flips and σ, τ finitely supported permutations such that $h = r\sigma f = g\tau s$.

Proof. Let h be an element of $\widehat{\operatorname{IET}}^{\bowtie}$, $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \ldots, I_n\} \in \Pi_h$ (§ 2.1). We denote by $h(\mathcal{P}) := \{J_1, J_2, \ldots, J_n\}$ the arrival partition of h associated with \mathcal{P} . Let g be the map that sends I_j° on J_j° by preserving the order and acts as h for every left endpoints of I_j for every $1 \leq j \leq n$. Note that g is bijective and then belongs to $\widehat{\operatorname{IET}}^+$. For $1 \leq j \leq n$ let r_j be the J_j -flip if h is order-reversing on I_j otherwise let r_j be the identity. Let r be the product of all r_j , we can notice that r fixes all endpoints of J_j for every $1 \leq j \leq n$. Then it is just a verification to check that h = rg. Now as g belongs to $\widehat{\operatorname{IET}}^+$ there exists σ in \mathfrak{S}_n such that $g = \sigma f$ with f in $\widehat{\operatorname{IET}}^+_r$.

The other decomposition follows by decomposing h^{-1} under the previous decomposition.

Proposition 2.5. For every h in \widehat{PC}^{\bowtie} there exist ϕ and ψ two order-preserving homeomorphisms of [0, 1[and f, g in \widehat{IET}^{\bowtie} such that $h = \psi \circ f = g \circ \phi$.

Proof. Let λ be the Lebesgue measure on [0, 1[. Let $h \in \widehat{\mathrm{PC}}^{\bowtie}$ and $\mathcal{P} \in \Pi_h$. Then there exist $\phi, \psi \in \mathrm{Homeo}^+([0, 1[)$ such that for every $I \in \mathcal{P}, \lambda(\phi(I)) = \lambda(h(I))$ and $\lambda(\psi(h(I))) = \lambda(I)$. Then $h \circ \phi$ and $\psi \circ h$ belongs to $\widehat{\mathrm{IET}^{\bowtie}}$. \Box

3. Construction of the signature homomorphism

In our case we have X = [0, 1[and $\widehat{\mathrm{PC}}^{\bowtie}$ is a subgroup of $\mathfrak{S}(X)$. We denote here $\mathfrak{S}_{\mathrm{fin}} = \mathfrak{S}_{\mathrm{fin}}(X)$ and $\varepsilon_{\mathrm{fin}}$ the classical signature on $\mathfrak{S}_{\mathrm{fin}}$ taking values in $(\mathbb{Z}_{2\mathbb{Z}}, +)$.

3.1. Definitions.

Definition 3.1. Let *h* be an element of \widehat{PC}^{\bowtie} , $n \in \mathbb{N}$ and $\mathcal{P} = \{I_1, I_2, \ldots, I_n\} \in \Pi_h$. For every $1 \leq j \leq n$, let α_j be the left endpoint of I_j and β_j be the left endpoint of $h(I_j^{\circ})$. We define the *default of pseudo right continuity for h about* \mathcal{P} denoted $\sigma_{(h,\mathcal{P})}$ as the finitely supported permutation which sends $h(\alpha_j)$ to β_j for every $1 \leq j \leq n$ (this is well-defined because the set of all $h(\alpha_j)$ is equal to the set of all β_j).

Definition 3.2. Let h be an element of \widehat{PC}^{\bowtie} and $\mathcal{P} \in \Pi_h$. Let k be the number of interval of \mathcal{P} on which h is order-reversing. We called the *flip number of* h about \mathcal{P} the number k. We denote it by $R(h, \mathcal{P})$.

Definition 3.3. For $h \in \widehat{\mathrm{PC}}^{\bowtie}$ and $\mathcal{P} \in \Pi_h$, define $\varepsilon(h, \mathcal{P}) \in \mathbb{Z}_{2\mathbb{Z}}$ as $R(h, \mathcal{P}) + \varepsilon_{\mathrm{fin}}(\sigma_{(h,\mathcal{P})}) \pmod{2}$. We define also $\varepsilon(h) = \varepsilon(h, \mathcal{P}_h^{\mathrm{fin}})$.

Proposition 3.4. For every $\tau \in \mathfrak{S}_{\text{fin}}$ and every $\mathcal{P} \in \Pi_{\tau}$ we have $\varepsilon(\tau, \mathcal{P}) = \varepsilon_{\text{fin}}(\tau)$.

Proof. It is clear that for every $\tau \in \mathfrak{S}_{\text{fin}}$ and every partition \mathcal{P} associated with τ we have $R(\tau, \mathcal{P}) = 0$ and $\sigma_{(\tau, \mathcal{P})} = \tau$.

We deduce that ε extends the classical signature ε_{fin} . Thus we will write ε instead of ε_{fin} .

Proposition 3.5. Every right-continuous element f of $\widehat{PC^+}$ satisfies $\varepsilon(f, \mathcal{P}) = 0$ for every $\mathcal{P} \in \Pi_f$.

Proof. In this case, for every partition \mathcal{P} into intervals associated with f we always have $R(f, \mathcal{P}) = 0$ and $\sigma_{(f, \mathcal{P})} = \text{Id.}$

3.2. Proof of Theorem 1.1.

In order to prove that ε is a group homomorphism, it is useful to calculate $\varepsilon(h)$ thanks to $\varepsilon(h, \mathcal{P})$ for every $h \in \widehat{\mathrm{PC}}^{\bowtie}$ and $\mathcal{P} \in \Pi_h$.

Lemma 3.6. For every $h \in \widehat{PC}^{\bowtie}$ and every $\mathcal{P} \in \Pi_h$ we have $\varepsilon(h) = \varepsilon(h, \mathcal{P})$.

Proof. Let h and \mathcal{P} be as in the statement. By minimality of \mathcal{P}_h^{\min} , in term of refinement, we deduce that there exist $n \in \mathbb{N}$ and $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \in \Pi_h$ such that:

- (1) $\mathcal{P}_1 = \mathcal{P}_h^{\min};$
- (2) $\mathcal{P}_n = \mathcal{P};$
- (3) for every $2 \le i \le n$ the partition \mathcal{P}_i is a refinement of the partition \mathcal{P}_{i-1} where only one interval of \mathcal{P}_{i-1} is cut into two.

Hence it is enough to show $\varepsilon(h, \mathcal{Q}) = \varepsilon(h, \mathcal{Q}')$ where $\mathcal{Q}, \mathcal{Q}' \in \Pi_h$ such that there exist consecutive intervals $I, J \in \mathcal{Q}$ with $I \cup J \in \mathcal{Q}'$ and $\mathcal{Q}' \setminus \{I \cup J\} = \mathcal{Q} \setminus \{I, J\}$.

Let α be the left endpoint of I and let x be the right endpoint of I (x is also the left endpoint of J). There are only two cases but in both cases, we know that $\sigma_{(h,Q)} = \sigma_{(h,Q')}$ except maybe on $h(\alpha)$ and h(x):

- (1) The first case is when h is order-preserving on $I \cup J$. Then as $\mathcal{Q} \setminus \{I, J\} = \mathcal{Q}' \setminus \{I \cup J\}$ we get $R(h, \mathcal{Q}) = R(h, \mathcal{Q}')$. As h is order-preserving on the interior of $I \cup J$ we know that $\sigma_{(h,\mathcal{Q}')}(h(\alpha))$ is the left endpoint of $h(I \cup J)$ which is the left endpoint of h(I) thus equals to $\sigma_{(h,\mathcal{Q})}(h(\alpha))$. With the same reasoning we deduce that $\sigma_{(h,\mathcal{Q}')}(h(x)) = \sigma_{(h,\mathcal{Q})}(h(x))$ hence $\sigma_{(h,\mathcal{Q})} = \sigma_{(h,\mathcal{Q}')}$. Thus in $\mathbb{Z}_{2\mathbb{Z}}$ we have $R(h,\mathcal{Q}') + \varepsilon(\sigma_{(h,\mathcal{Q}')}) = R(h,\mathcal{Q}) + \varepsilon(\sigma_{(h,\mathcal{Q})})$.
- (2) The second case is when h is order-reversing on $I \cup J$. Then we get R(h, Q) = R(h, Q') + 1. This time $\sigma_{(h,Q')}(h(\alpha))$ is still the left endpoint of $h(I \cup J)$ which is the left endpoint of h(J) thus equals to $\sigma_{(h,Q)}(h(\alpha))$. With the same reasoning we deduce that $\sigma_{(h,Q')}(h(\alpha)) = \sigma_{(h,Q)}(h(\alpha))$. Then by denoting τ the transposition $(h(x) \ \sigma_{(h,Q')}(h(\alpha)))$, we obtain $\sigma_{(h,Q)} = \tau \circ \sigma_{(h,Q')}$. We must notice that the transposition is not the identity because $h^{-1}(\sigma_{(h,Q')}(h(\alpha)))$ is an endpoint of one of the intervals of Q' and



FIGURE 1. Illustrations of the two cases appearing in Lemma 3.6. On the left we assume *h* order-preserving on $I \cup J$ and see that $\sigma_{(h,\mathcal{Q})}(h(x)) = \sigma_{(h,\mathcal{Q}')}(h(x))$. On the right we assume *h* order-reversing on $I \cup J$ and see that $\sigma_{(h,\mathcal{Q})}(h(x)) = (h(x) \sigma_{(h,\mathcal{Q}')}(h(\alpha))) \circ \sigma_{(h,\mathcal{Q}')}(h(x))$.

If $\phi \in \text{Homeo}^+([0, 1[), \text{ then it follows from Proposition 3.5 that } \varepsilon(\phi) = 0$. We improve this, showing that ε is invariant by the action of $\text{Homeo}^+([0, 1[) \text{ on } \widehat{\text{PC}}^{\bowtie}$.

Lemma 3.7. For every $h \in \widehat{PC}^{\bowtie}$ and every $\phi \in \text{Homeo}^+([0,1[) \text{ we have } \varepsilon(h\phi) = \varepsilon(h) = \varepsilon(\phi h).$

Proof. Let $h \in \widehat{\mathrm{PC}}^{\bowtie}$ and $\phi \in \mathrm{Homeo}^+([0,1[))$ be as in the statement. Let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \ldots, I_n\} \in \Pi_h$. Then $\mathcal{Q} := \{\phi^{-1}(I_1), \phi^{-1}(I_2), \ldots, \phi^{-1}(I_n)\}$ is in $\Pi_{h\phi}$. We know that ϕ is order preserving then for every $1, \leq i \leq n, h\phi$ preserves (reverses respectively) the order on $\phi^{-1}(I_i)$ if and only if h preserves (reverses respectively) the order on I_i , so $R(h, \mathcal{P}) = R(h\phi, \mathcal{Q})$. We can notice that the left endpoint of $\phi^{-1}(I_i)$ (denoted by α_i) is send on the left endpoint of I_i (denoted by a_i) by ϕ hence $h(a_i) = h\phi(\alpha_i)$ has to be send on $\sigma_{(h,\mathcal{P})}(h(a_i))$ so $\sigma_{(h\phi,\mathcal{Q})} = \sigma_{(h,\mathcal{P})}$. we deduce that $\varepsilon(h\phi) = \varepsilon(h)$.

The other equality has a similar proof. We denote $h(\mathcal{P})$ the arrival partition of h associated with \mathcal{P} . We know that ϕ is continuous thus $h(\mathcal{P})$ is in Π_{ϕ} and we deduce that $\mathcal{P} \in \Pi_{\phi h}$. Also ϕ is order-preserving then $R(h, \mathcal{P}) = R(\phi h, \mathcal{P}))$. We know that $\sigma_{(\phi,h(\mathcal{P}))} = \text{Id}$ then we can notice that $\phi \circ \sigma_{(h,\mathcal{P})} \circ h$ send the left endpoint of I_i to the left endpoint of $\phi h(I_i^\circ)$. Then $\sigma_{(\phi h,\mathcal{P})} = \phi \sigma_{(h,\mathcal{P})} \phi^{-1}$ and we deduce that $\varepsilon(\sigma_{(\phi h,\mathcal{P})}) = \varepsilon(\sigma_{(h,\mathcal{P})})$. Hence $\varepsilon(\phi h) = \varepsilon(h)$. Thanks to Proposition 2.5 it is enough to prove that $\varepsilon|_{\widehat{IET}^{\bowtie}}$ is a group homomorphism.

Lemma 3.8. The map $\varepsilon|_{\widehat{I} \in \Gamma^{\boxtimes}}$ is a group homomorphism.

Proof. Let $f, g \in \widehat{\operatorname{IET}}^{\bowtie}$. Let $\mathcal{P} \in \Pi_f$ and $\mathcal{Q} \in \Pi_g$. For every $I \in \mathcal{Q}$ (resp. $J \in \mathcal{P}$) we denote by α_I (resp. β_J) the left endpoint of I (resp. J). Up to refine \mathcal{P} and \mathcal{Q} we can assume that $\mathcal{P} = g(\mathcal{Q})$ thus $g(\{\alpha_I\}_{I \in \mathcal{Q}}) = \{\beta_J\}_{J \in \mathcal{P}}$. Then $Q \in \Pi_{f \circ g}$ and for every $K \in f \circ g(Q)$ we denote by γ_K the left endpoint of K. In $\mathbb{Z}_{2\mathbb{Z}}$, we get immediately that $R(f \circ g, Q) = R(g, Q) + R(f, g(Q))$. Now we want to describe the default of pseudo right continuity for $f \circ g$ about \mathcal{Q} . We recall that $\sigma_{(f \circ g, Q)}$ is the permutation that sends $f \circ g(\alpha_I)$ on $\gamma_{f \circ g(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of [0, 1[. Furthermore $\sigma_{(g,Q)}(g(\alpha_I)) = \beta_{g(I)}$ and $\sigma_{(f,g(Q))}(f(\beta_{g(I)})) = \gamma_{f \circ g(I)}$. Then $\sigma_{(f,g(Q))} \circ f \circ \sigma_{(g,Q)} \circ g(\alpha_I) = \gamma_{f \circ g(I)}$ and we deduce that the permutation $\sigma_{(f,g(Q))} \circ f \circ \sigma_{(g,Q)} \circ f^{-1}$ sends $f \circ g(\alpha_I)$ on $\gamma_{f \circ g(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of [0, 1[. Thus $\sigma_{(f \circ g,Q)} = \sigma_{f,g(Q)} \circ f \circ \sigma_{(g,Q)} \circ f^{-1}$. Then $\varepsilon(\sigma_{(f \circ g,Q)}) = \varepsilon(\sigma_{f,g(Q)}) + \varepsilon(\sigma_{(g,Q)})$ and we conclude that $\varepsilon(f \circ g) = \varepsilon(f) + \varepsilon(g)$. \Box

Corollary 3.9. The map ε is a group homomorphism.

4. Normal subgroups of \widehat{PC}^{\bowtie} and some subgroups

Here we present some corollaries of Theorem 1.1. For every group G we denote by D(G) its derived subgroup.

Definition 4.1. For every group H, we define $J_3(H)$ as the subgroup generated by elements of order 3.

Let \widehat{G} be a subgroup of $\widehat{\mathrm{PC}}^{\bowtie}$ containing $\mathfrak{S}_{\mathrm{fin}}$. We denote by G its projection on PC^{\bowtie} . We recall that $\mathfrak{A}_{\mathrm{fin}}$ is a normal subgroup of \widehat{G} , and has a trivial centraliser. We deduce that for every nontrivial normal subgroup H of \widehat{G} contains $\mathfrak{A}_{\mathrm{fin}}$.

From the short exact sequence:

$$1 \longrightarrow \mathfrak{S}_{\mathrm{fin}} \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

we deduce the next short exact sequence which is a central extension:

$$1 \longrightarrow \mathbb{Z}_{2\mathbb{Z}} \longrightarrow \widehat{G}_{\mathfrak{A}_{\mathrm{fin}}} \longrightarrow G \longrightarrow 1.$$

This short exact sequence splits because the signature $\varepsilon_{|\widehat{G}} : \widehat{G} \to \mathbb{Z}_{2\mathbb{Z}}$ constructed in § 3 is a retraction. Then we deduce that $\widehat{G}_{\mathfrak{A}_{\mathrm{fin}}}$ is isomorphic to the direct product $\mathbb{Z}_{2\mathbb{Z}} \times G$.

Corollary 4.2. The projection $\widehat{G}_{ab} \to G_{ab}$ extends in an isomorphism $\widehat{G}_{ab} \sim G_{ab} \times \mathbb{Z}_{2\mathbb{Z}}$. Furthermore $D(\widehat{G}) = \operatorname{Ker}(\varepsilon) \cap \widehat{D(G)}$ is a subgroup of index 2 in $\widehat{D(G)}$. In particular, if G is a perfect group then $\widehat{G}_{ab} = \mathbb{Z}_{2\mathbb{Z}}$.

Corollary 4.3. Let \hat{G} be a subgroup of $\widehat{\mathrm{PC}}^{\bowtie}$ containing $\mathfrak{S}_{\mathrm{fin}}$ and such that its projection G in PC^{\bowtie} is simple nonabelian. Then \hat{G} has exactly 5 normal subgroups given by the list: $\{\{1\}, \mathfrak{A}_{\mathrm{fin}}, \mathfrak{S}_{\mathrm{fin}}, \mathrm{Ker}(\varepsilon), \hat{G}\}.$

Proof. Let \hat{G} as in the statement. First we immediately check that the subgroups in the list are distinct normal subgroups of \hat{G} . In the case of $\text{Ker}(\varepsilon)$, there exists $g \in \hat{G} \setminus \mathfrak{S}_{\text{fin}}$ thus either $g \in \text{Ker}(\varepsilon) \setminus \mathfrak{S}_{\text{fin}}$ or $\sigma g \in \text{Ker}(\varepsilon) \setminus \mathfrak{S}_{\text{fin}}$ for any transposition σ . Second let H be a normal subgroup of \hat{G} distinct from {1}. Then it contains

Second let H be a normal subgroup of G distinct from {1}. Then it contains $\mathfrak{A}_{\text{fin}}$. Also $H_{\mathfrak{A}_{\text{fin}}}$ is a normal subgroup of $\widehat{G}_{\mathfrak{A}_{\text{fin}}} \simeq \mathbb{Z}_{2\mathbb{Z}} \times G$. Furthermore G is simple then there are only four possibilities for $H_{\mathfrak{A}_{\text{fin}}}$. As two normal subgroups H, K of \widehat{G} containing $\mathfrak{A}_{\text{fin}}$ such that $H_{\mathfrak{A}_{\text{fin}}} = K_{\mathfrak{A}_{\text{fin}}}$ are equal, we deduce that \widehat{G} has at most 5 normal subgroups.

Corollary 4.4. Let \hat{G} be a subgroup of $\widehat{\mathrm{PC}}^{\bowtie}$ containing $\mathfrak{S}_{\mathrm{fin}}$ and such that its projection G in PC^{\bowtie} is simple nonabelian. If there exists an element of order 3 in $G \smallsetminus \mathfrak{A}_{\mathrm{fin}}$ then $J_3(\hat{G}) = \mathrm{Ker}(\varepsilon) = D(\hat{G})$.

Remark 4.5. In the context of topological-full groups, the group $J_3(G)$ appears naturally (with some mild assumptions) and is denoted by A(G) by Nekrashevych in [9]. In some case of topological-full groups of minimal groupoids (see [8]) we have the equality A(G) = D(G) thanks to the simplicity of D(G). In spite of the analogy, it is not clear that the corollary can be obtained as particular case of this result.

Remark 4.6. A lot of groups satisfy the conditions of Corollary 4.4. When \widehat{G} contains $\widehat{\operatorname{IET}^+}$ there is an element of order 3 in $G \smallsetminus \mathfrak{A}_{\operatorname{fn}}$. We recall that $\operatorname{IET}^{\bowtie}$, PC^+ and PAff^+ are simple (see [2, 5]). Thus these groups satisfy the conditions of Corollary 4.4. The next theorem add $\operatorname{PC}^{\bowtie}$ and $\operatorname{PAff}^{\bowtie}$ to the list of examples.

Theorem 4.7. The groups PC^{\bowtie} and $PAff^{\bowtie}$ are simple.

Lemma 4.8. The group $\operatorname{IET}^{\bowtie}$ is generated by flips (=images of flips from $\widehat{\operatorname{IET}^{\bowtie}}$).

Proof. By Proposition 2.4 it is enough to show that IET⁺ is generated by flips. For every consecutive, right-open and left-closed subintervals I and J of [0, 1[, we define $R_{I,J}$ the map that exchanges I and J. They are elements of $\widehat{\text{IET}}_{\text{rc}}^+$ and they formed a generating set. Then their image $r_{I,J}$ in $\operatorname{IET}^{\bowtie}$ is a generating set of IET⁺. For every right-open and left-closed subinterval I of [0, 1[, we define s_I the I-flip. Take I and J be two consecutive, right-open and left-closed subintervals of [0, 1[. Then $r_{I,J} = s_I s_J s_{I \cup J}$.

Proof of Theorem 4.7 (sketched). Since the argument in [2] could also be adapted, we only provide a sketch. We work with elements of PC^{\bowtie} ; all intervals below are meant modulo finite subsets. Let N be a nontrivial normal subgroup of PC^{\bowtie} (resp. PAff^{\bowtie}). Let g be a nontrivial element of N. There exists a subinterval I of [0, 1] such that:

- (1) g is continuous (resp. affine) on I,
- (2) $g(I) \cap I = \emptyset$ (modulo finite subsets),
- (3) $I \cup g(I) \neq [0, 1]$ (modulo finite subsets).

Let f be the *I*-flip. If g is affine on I then $h = gfg^{-1}f^{-1}$ is the product of the I-flip with the g(I)-flip. Observe that h is conjugate to a single flip by a suitable element of IET⁺. If g is only continuous then h is still of order 2 and it is conjugate in PC^{\bowtie} to a single flip. Conjugating by elements of PAff⁺, one obtains that N contains flips of intervals of all possible lengths, and hence contains all flips. Thanks to Lemma 4.8 we know that IET^{\bowtie} is generated by the set of flips thus N contains IET^{\bowtie}, in particular N intersects PC^+ (resp. PAff⁺) nontrivially. By simplicity of PC^+ (resp. PAff⁺) we deduce that N contains $PC^{\bowtie} = \langle PC^+, IET^{\bowtie} \rangle$ (resp. PAff^{\bowtie} = $\langle PAff^+, IET^{\bowtie} \rangle$).

5. About some Normalizers

Here we show that computing normalizers inside \widehat{PC}^{\bowtie} and PC^{\bowtie} may leads to different behaviour. We look the case of PC^+ , IET^+ and $\widehat{PC_{rc}^+}$ and $\widehat{IET_{rc}^+}$.

Proposition 5.1. The normalizer of IET^+ in IET^{\bowtie} is reduced to IET^{\pm} .

Proof. Let $f \in \operatorname{IET}^+$ and $g \in \operatorname{IET}^\pm$. If $g \in \operatorname{IET}^+$ then $gfg^{-1} \in \operatorname{IET}^+$. We assume $g \in \operatorname{IET}^-$ then $gfg^{-1} = (g \circ \mathcal{R}) \circ (\mathcal{R} \circ f \circ \mathcal{R}) \circ (\mathcal{R} \circ g) \in \operatorname{IET}^+$. For the inclusion from left to right, let $g \in \operatorname{IET}^{\bowtie} \smallsetminus \operatorname{IET}^\pm$ and let \hat{g} be a representative of g in $\widehat{\operatorname{IET}^{\bowtie}}$. Hence we can find I, J, K, L four right-open and left-closed intervals of the same length such that their image by \hat{g} are intervals and such that \hat{g} is order-reversing on I and order-preserving on J, K and L. We define $\hat{f} \in \widehat{\operatorname{IET}^+}$ as the element which exchanges $\hat{g}(I)$ with $\hat{g}(J)$ and $\hat{g}(K)$ with $\hat{g}(L)$ while fixing the rest of [0, 1[. Then the image f of \hat{f} in IET^+ is not trivial and $\hat{g}\hat{f}\hat{g}^{-1} \notin \widehat{\operatorname{IET}^+}$ implies $gfg^{-1} \notin \operatorname{IET}^+$. \Box

A similar argument stands for the case of PC thus we obtain:

Proposition 5.2. The normalizer of PC^+ in PC^{\bowtie} is reduced to PC^{\pm} .

We now take a look to inside $\widehat{\mathrm{PC}}^{\bowtie}$:

Proposition 5.3. The normalizer of $\widehat{\operatorname{IET}}_{\operatorname{rc}}^+$ in $\widehat{\operatorname{IET}}^{\bowtie}$ is $\widehat{\operatorname{IET}}_{\operatorname{rc}}^+$.

Proof. Let g be an element of $\widehat{\operatorname{IET}}^{\bowtie}$ which is not the identity. There are two cases:

- (1) If $g \in \widehat{\operatorname{IET}^+} \smallsetminus \widehat{\operatorname{IET}^+_{\operatorname{rc}}}$ then $g = \sigma g'$ with $\sigma \in \mathfrak{S}_{\operatorname{fin}} \smallsetminus \{\operatorname{Id}\}$ and $g' \in \widehat{\operatorname{IET}^+_{\operatorname{rc}}}$. Then for every $f \in \widehat{\operatorname{IET}^+_{\operatorname{rc}}}$ we have $gfg^{-1} = \sigma g'fg'^{-1}\sigma^{-1}$. Thus it is enough to treat the case of $\mathfrak{S}_{\operatorname{fin}}$. Let us assume $g \in \mathfrak{S}_{\operatorname{fin}}$ then let x in the support of g. There exist two consecutive right-open and left-closed intervals Iand J of the same length such that x is the right endpoint of I (and the left endpoint of J). Up to reduce I and J we can assume that I does not intersect the support of g. Then let $f \in \widehat{\operatorname{IET}^+_{\operatorname{rc}}}$ which exchanges I and Jwhile fixing the rest of [0, 1[. Then gfg^{-1} exchanges the interior of I with the interior of J but $gfg^{-1}(x)$ is not equal to f(x) because f(x) is the left endpoint of I and I does not intersect the support of g. Then we deduce that gfg^{-1} is not right-continuous on J.
- (2) If $g \in \widehat{\operatorname{IET}} \smallsetminus \widehat{\operatorname{IET}}^+$. Then we can find two consecutive subinterval I and J where g is continuous and order-reversing on $I \cup J$. Let a be the right endpoint of J. Let f be the element in $\widehat{\operatorname{IET}}^+_{\operatorname{rc}}$ which exchanges I and J. Then gfg^{-1} exchanges the interior of g(J) with the interior of g(I). However the left endpoint of g(J) is send by g^{-1} on a which is fixed by f. Then gfg^{-1} fixes the left endpoint of g(J), thus gfg^{-1} is not right-continuous on g(J).

A similar argument stands for the case of PC thus we obtain:

Proposition 5.4. The normalizer of $\widehat{\mathrm{PC}}^+_{\mathrm{rc}}$ in $\widehat{\mathrm{PC}}^{\bowtie}$ is $\widehat{\mathrm{PC}}^+_{\mathrm{rc}}$.

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