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| Octave Lacourte. Signature for piecewise continuous groups. 2020. hal-02491103

HAL Id: hal-02491103

<https://hal.science/hal-02491103>

Preprint submitted on 27 Feb 2020

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SIGNATURE FOR PIECEWISE CONTINUOUS GROUPS

OCTAVE LACOURTE

ABSTRACT. Let $\widehat{\text{PC}}^{\text{pc}}$ be the group of bijections from $[0, 1[$ to itself which are continuous outside a finite set. Let PC^{pc} be its quotient by the subgroup of finitely supported permutations.

We show that the Kapoudjian class of PC^{pc} vanishes. That is, the quotient map $\widehat{\text{PC}}^{\text{pc}} \rightarrow \text{PC}^{\text{pc}}$ splits modulo the alternating subgroup of even permutations. This is shown by constructing a nonzero group homomorphism, called signature, from $\widehat{\text{PC}}^{\text{pc}}$ to $\mathbb{Z}/2\mathbb{Z}$. Then we use this signature to list normal subgroups of every subgroup \widehat{G} of $\widehat{\text{PC}}^{\text{pc}}$ which contains $\mathfrak{S}_{\text{fin}}$ and such that G , the projection of \widehat{G} in PC^{pc} , is simple.

1. INTRODUCTION

Let X be the right-open and left-closed interval $[0, 1[$. We denote by $\mathfrak{S}(X)$ the group of bijections of X to X . This group contains the subgroup composed of all finitely supported permutations is denoted by $\mathfrak{S}_{\text{fin}}$. The classical signature is well-defined on $\mathfrak{S}_{\text{fin}}$ and its kernel, denoted by $\mathfrak{A}_{\text{fin}}$, is the only subgroup of index 2 in $\mathfrak{S}_{\text{fin}}$. An observation, originally due to Vitali [10], is that the signature does not extend to $\mathfrak{S}(X)$.

For every subgroup G of $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$, we denote by \widehat{G} its inverse image in $\mathfrak{S}(X)$. The cohomology class of the central extension

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} = \mathfrak{S}_{\text{fin}}/\mathfrak{A}_{\text{fin}} \rightarrow \widehat{G}/\mathfrak{A}_{\text{fin}} \rightarrow G \rightarrow 1$$

is called the Kapoudjian class of G ; it belongs to $H^2(G, \mathbb{Z}/2\mathbb{Z})$. It appears in the work of Kapoudjian and Kapoudjian-Sergiescu [6, 7]. The vanishing of this class means that the above exact sequence splits; this means that there exists a group homomorphism from the preimage of G in $\mathfrak{S}(X)$ onto $\mathbb{Z}/2\mathbb{Z}$ which extends the signature on $\mathfrak{S}_{\text{fin}}$ (for more on the Kapoudjian class, see [3, §8.C]). This implies in particular that $\widehat{G}/\mathfrak{A}_{\text{fin}}$ is isomorphic to the direct product $G \times \mathbb{Z}/2\mathbb{Z}$. One can notice that for $G = \mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$ we have $\widehat{G} = \mathfrak{S}(X)$; in this case the Vitali's observation implies that the Kapoudjian class does not vanish.

Date: February 25, 2020.

2010 Mathematics Subject Classification. 37E05, 20F65, 20J06.

The set of all permutations of X continuous outside a finite set is a subgroup denoted by $\widehat{\text{PC}}^{\bowtie}$. The aim here is to show the following theorem:

Theorem 1.1. *There exists a group homomorphism $\varepsilon : \widehat{\text{PC}}^{\bowtie} \rightarrow \mathbb{Z}/2\mathbb{Z}$ that extends the classical signature on $\mathfrak{S}_{\text{fin}}$.*

Corollary 1.2. *Let G be a subgroup of PC^{\bowtie} . Then the Kapoudjian class of G is zero.*

This solves a question asked by Y. Cornulier [4, Question 1.15].

The subgroup of $\widehat{\text{PC}}^{\bowtie}$ consisting of all permutations of X that are piecewise isometric elements is denoted by $\widehat{\text{IET}}^{\bowtie}$ and the one consisting of all piecewise affine permutations of X is denoted by $\widehat{\text{PAff}}^{\bowtie}$. We also consider for each of these groups the subgroup composed of all piecewise orientation-preserving elements by replacing the symbol “ \bowtie ” by the symbol “ $+$ ”.

Let us observe that when $G \subset \text{PC}^+$ Corollary 1.2 is trivial. Indeed, in this case G can be lifted inside $\widehat{\text{PC}}^+$ itself. However, such a lift does not exist for PC^{\bowtie} or even IET^{\bowtie} , as was proved in [4].

The idea of proof of Theorem 1.1 is to associate for every $f \in \widehat{\text{PC}}^{\bowtie}$ and every finite partition \mathcal{P} of $[0, 1[$ into intervals associated with f , two numbers. The first is the number of interval of \mathcal{P} where f is order-reversing and the second is the signature of a particular finitely supported permutation. The next step is to prove that the sum modulo 2 of this two numbers is independent from the choice of partition. Then we show that it is enough to prove that $\varepsilon|_{\widehat{\text{IET}}^{\bowtie}}$ is a group homomorphism. For this we show that it is additive when we look at the composition of two elements of $\widehat{\text{IET}}^{\bowtie}$ by calculate the value of the signature with a particular partition.

In Section 4, we apply these results to the study of normal subgroups of $\widehat{\text{PC}}^{\bowtie}$ and certain subgroups. More specifically we prove:

Theorem 1.3. *Let \widehat{G} be a subgroup of $\widehat{\text{PC}}^{\bowtie}$ containing $\mathfrak{S}_{\text{fin}}$ and such that its projection G in PC^{\bowtie} is simple nonabelian. Then \widehat{G} has exactly five normal subgroups given by the list: $\{\{1\}, \mathfrak{A}_{\text{fin}}, \mathfrak{S}_{\text{fin}}, \text{Ker}(\varepsilon), \widehat{G}\}$.*

We denote by $\widehat{\text{IET}}_{\text{rc}}^+$ the subgroup of $\widehat{\text{IET}}^+$ composed of all right-continuous elements. We know that it is naturally isomorphic to IET^+ . The same is true when we replace IET^+ by PAff^+ or PC^+ . This allows us to use the work of P. Arnoux [2] and the one of N. Guelman and I. Liousse [5] where they prove that IET^{\bowtie} , PC^+ and PAff^+ are simple. From this we deduce:

Theorem 1.4. *The groups PC^{\bowtie} and PAff^{\bowtie} are simple.*

This gives us some examples of groups that satisfy the conditions of the Theorem 1.3.

Finally Section 5 is independent and we study some normalizers and in particular we show that the behaviour when we look the group inside $\widehat{\text{PC}}^{\boxtimes}$ or PC^{\boxtimes} may not be the same. We denote by $\mathcal{R} \in \text{IET}^{\boxtimes}$ the map $x \mapsto 1 - x$. Then we define IET^- as the coset $\mathcal{R}.\text{IET}^+$ and PC^- as the coset $\mathcal{R}.\text{PC}^+$. Then the groups $\text{IET}^{\pm} := \text{IET}^+ \cup \text{IET}^-$ and $\text{PC}^{\pm} := \text{PC}^+ \cup \text{PC}^-$ are well-defined.

Proposition 1.5. *The subgroup $\widehat{\text{IET}}_{\text{rc}}^+$ (resp. $\widehat{\text{PC}}_{\text{rc}}^+$) is its own normalizer in $\widehat{\text{IET}}^{\boxtimes}$ (resp. $\widehat{\text{PC}}_{\text{rc}}^{\boxtimes}$). The normalizer of IET^+ (PC^+ respectively) in IET^{\boxtimes} (PC^{\boxtimes} respectively) is IET^{\pm} (PC^{\pm} respectively).*

Acknowledgments. I would like to thank Y. Cornulier, P. de la Harpe and N. Matte Bon for corrections, remarks and discussions on preliminary versions of this paper.

2. PRELIMINARIES

For every real interval I we denote by I° its interior in \mathbb{R} and if $I = [0, t[$ we agree that its interior is $]0, t[$.

2.1. Partitions associated.

An important tool to study elements in $\widehat{\text{PC}}^{\boxtimes}$ and PC^{\boxtimes} are partitions into intervals of $[0, 1[$. All partitions are assumed to be finite.

Definition 2.1. For every f in $\widehat{\text{PC}}^{\boxtimes}$, a finite partition \mathcal{P} into right-open and left-closed intervals of $[0, 1[$ is called *a partition into intervals associated with f* if and only if f is continuous on the interior of every interval of \mathcal{P} . We denote by Π_f the set of all partitions into intervals associated with f . We define also *the arrival partition of f associated with \mathcal{P}* , denoted $f(\mathcal{P})$, the partition of $[0, 1[$ composed of all right-open and left-closed intervals such that their interior is equal to the image by f of the interior of an interval of \mathcal{P} .

Remark 2.2. For every f in $\widehat{\text{PC}}^{\boxtimes}$ there exists a unique partition $\mathcal{P}_f^{\text{min}}$ associated with f which has a minimal number of intervals. It is actually minimal in the sense of refinement: Π_f consists precisely of the set of partitions refining $\mathcal{P}_f^{\text{min}}$.

2.2. Decompositions.

We define a family of elements which plays an important role inside our groups:

Definition 2.3. Let I be a non-empty right-open and left-closed subinterval of $[0, 1[$. The element $f \in \widehat{\text{PC}}^{\boxtimes}$ which sends the interior of I on itself with slope -1 while fixing the rest of $[0, 1[$ is called the *I -flip*. We define *a flip* as any I -flip for some I .

From the definition we deduce a decomposition inside $\widehat{\text{IET}}^{\boxtimes}$ and $\widehat{\text{PC}}^{\boxtimes}$.

Proposition 2.4. *Let h be an element of $\widehat{\text{IET}}^{\boxtimes}$. There exist $f, g \in \widehat{\text{IET}}_{\text{rc}}^+$ and r, s finite products of flips and σ, τ finitely supported permutations such that $h = r\sigma f = g\tau s$.*

Proof. Let h be an element of $\widehat{\text{IET}}^{\boxtimes}$, $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \dots, I_n\} \in \Pi_h$ (§ 2.1). We denote by $h(\mathcal{P}) := \{J_1, J_2, \dots, J_n\}$ the arrival partition of h associated with \mathcal{P} . Let g be the map that sends I_j° on J_j° by preserving the order and acts as h for every left endpoints of I_j for every $1 \leq j \leq n$. Note that g is bijective and then belongs to $\widehat{\text{IET}}^+$. For $1 \leq j \leq n$ let r_j be the J_j -flip if h is order-reversing on I_j otherwise let r_j be the identity. Let r be the product of all r_j , we can notice that r fixes all endpoints of J_j for every $1 \leq j \leq n$. Then it is just a verification to check that $h = rg$. Now as g belongs to $\widehat{\text{IET}}^+$ there exists σ in \mathfrak{S}_n such that $g = \sigma f$ with f in $\widehat{\text{IET}}_{\text{rc}}^+$.

The other decomposition follows by decomposing h^{-1} under the previous decomposition. \square

Proposition 2.5. *For every h in $\widehat{\text{PC}}^{\boxtimes}$ there exist ϕ and ψ two order-preserving homeomorphisms of $[0, 1[$ and f, g in $\widehat{\text{IET}}^{\boxtimes}$ such that $h = \psi \circ f = g \circ \phi$.*

Proof. Let λ be the Lebesgue measure on $[0, 1[$. Let $h \in \widehat{\text{PC}}^{\boxtimes}$ and $\mathcal{P} \in \Pi_h$. Then there exist $\phi, \psi \in \text{Homeo}^+([0, 1[)$ such that for every $I \in \mathcal{P}$, $\lambda(\phi(I)) = \lambda(h(I))$ and $\lambda(\psi(h(I))) = \lambda(I)$. Then $h \circ \phi$ and $\psi \circ h$ belongs to $\widehat{\text{IET}}^{\boxtimes}$. \square

3. CONSTRUCTION OF THE SIGNATURE HOMOMORPHISM

In our case we have $X = [0, 1[$ and $\widehat{\text{PC}}^{\boxtimes}$ is a subgroup of $\mathfrak{S}(X)$. We denote here $\mathfrak{S}_{\text{fin}} = \mathfrak{S}_{\text{fin}}(X)$ and ε_{fin} the classical signature on $\mathfrak{S}_{\text{fin}}$ taking values in $(\mathbb{Z}/2\mathbb{Z}, +)$.

3.1. Definitions.

Definition 3.1. Let h be an element of $\widehat{\text{PC}}^{\boxtimes}$, $n \in \mathbb{N}$ and $\mathcal{P} = \{I_1, I_2, \dots, I_n\} \in \Pi_h$. For every $1 \leq j \leq n$, let α_j be the left endpoint of I_j and β_j be the left endpoint of $h(I_j^\circ)$. We define the *default of pseudo right continuity for h about \mathcal{P}* denoted $\sigma_{(h, \mathcal{P})}$ as the finitely supported permutation which sends $h(\alpha_j)$ to β_j for every $1 \leq j \leq n$ (this is well-defined because the set of all $h(\alpha_j)$ is equal to the set of all β_j).

Definition 3.2. Let h be an element of $\widehat{\text{PC}}^{\boxtimes}$ and $\mathcal{P} \in \Pi_h$. Let k be the number of interval of \mathcal{P} on which h is order-reversing. We called the *flip number of h about \mathcal{P}* the number k . We denote it by $R(h, \mathcal{P})$.

Definition 3.3. For $h \in \widehat{\text{PC}}^{\boxtimes}$ and $\mathcal{P} \in \Pi_h$, define $\varepsilon(h, \mathcal{P}) \in \mathbb{Z}/2\mathbb{Z}$ as $R(h, \mathcal{P}) + \varepsilon_{\text{fin}}(\sigma_{(h, \mathcal{P})}) \pmod{2}$. We define also $\varepsilon(h) = \varepsilon(h, \mathcal{P}_h^{\text{fin}})$.

Proposition 3.4. *For every $\tau \in \mathfrak{S}_{\text{fin}}$ and every $\mathcal{P} \in \Pi_\tau$ we have $\varepsilon(\tau, \mathcal{P}) = \varepsilon_{\text{fin}}(\tau)$.*

Proof. It is clear that for every $\tau \in \mathfrak{S}_{\text{fin}}$ and every partition \mathcal{P} associated with τ we have $R(\tau, \mathcal{P}) = 0$ and $\sigma_{(\tau, \mathcal{P})} = \tau$. \square

We deduce that ε extends the classical signature ε_{fin} . Thus we will write ε instead of ε_{fin} .

Proposition 3.5. *Every right-continuous element f of $\widehat{\text{PC}}^+$ satisfies $\varepsilon(f, \mathcal{P}) = 0$ for every $\mathcal{P} \in \Pi_f$.*

Proof. In this case, for every partition \mathcal{P} into intervals associated with f we always have $R(f, \mathcal{P}) = 0$ and $\sigma_{(f, \mathcal{P})} = \text{Id}$. \square

3.2. Proof of Theorem 1.1.

In order to prove that ε is a group homomorphism, it is useful to calculate $\varepsilon(h)$ thanks to $\varepsilon(h, \mathcal{P})$ for every $h \in \widehat{\text{PC}}^{\boxtimes}$ and $\mathcal{P} \in \Pi_h$.

Lemma 3.6. *For every $h \in \widehat{\text{PC}}^{\boxtimes}$ and every $\mathcal{P} \in \Pi_h$ we have $\varepsilon(h) = \varepsilon(h, \mathcal{P})$.*

Proof. Let h and \mathcal{P} be as in the statement. By minimality of $\mathcal{P}_h^{\text{min}}$, in term of refinement, we deduce that there exist $n \in \mathbb{N}$ and $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n \in \Pi_h$ such that:

- (1) $\mathcal{P}_1 = \mathcal{P}_h^{\text{min}}$;
- (2) $\mathcal{P}_n = \mathcal{P}$;
- (3) for every $2 \leq i \leq n$ the partition \mathcal{P}_i is a refinement of the partition \mathcal{P}_{i-1} where only one interval of \mathcal{P}_{i-1} is cut into two.

Hence it is enough to show $\varepsilon(h, \mathcal{Q}) = \varepsilon(h, \mathcal{Q}')$ where $\mathcal{Q}, \mathcal{Q}' \in \Pi_h$ such that there exist consecutive intervals $I, J \in \mathcal{Q}$ with $I \cup J \in \mathcal{Q}'$ and $\mathcal{Q}' \setminus \{I \cup J\} = \mathcal{Q} \setminus \{I, J\}$.

Let α be the left endpoint of I and let x be the right endpoint of I (x is also the left endpoint of J). There are only two cases but in both cases, we know that $\sigma_{(h, \mathcal{Q})} = \sigma_{(h, \mathcal{Q}')}$ except maybe on $h(\alpha)$ and $h(x)$:

- (1) The first case is when h is order-preserving on $I \cup J$. Then as $\mathcal{Q} \setminus \{I, J\} = \mathcal{Q}' \setminus \{I \cup J\}$ we get $R(h, \mathcal{Q}) = R(h, \mathcal{Q}')$. As h is order-preserving on the interior of $I \cup J$ we know that $\sigma_{(h, \mathcal{Q}')} (h(\alpha))$ is the left endpoint of $h(I \cup J)$ which is the left endpoint of $h(I)$ thus equals to $\sigma_{(h, \mathcal{Q})} (h(\alpha))$. With the same reasoning we deduce that $\sigma_{(h, \mathcal{Q}')} (h(x)) = \sigma_{(h, \mathcal{Q})} (h(x))$ hence $\sigma_{(h, \mathcal{Q})} = \sigma_{(h, \mathcal{Q}')}$. Thus in $\mathbb{Z}/2\mathbb{Z}$ we have $R(h, \mathcal{Q}') + \varepsilon(\sigma_{(h, \mathcal{Q}')}) = R(h, \mathcal{Q}) + \varepsilon(\sigma_{(h, \mathcal{Q})})$.
- (2) The second case is when h is order-reversing on $I \cup J$. Then we get $R(h, \mathcal{Q}) = R(h, \mathcal{Q}') + 1$. This time $\sigma_{(h, \mathcal{Q}')} (h(\alpha))$ is still the left endpoint of $h(I \cup J)$ which is the left endpoint of $h(J)$ thus equals to $\sigma_{(h, \mathcal{Q})} (h(x))$. With the same reasoning we deduce that $\sigma_{(h, \mathcal{Q}')} (h(x)) = \sigma_{(h, \mathcal{Q})} (h(\alpha))$. Then by denoting τ the transposition $(h(x) \sigma_{(h, \mathcal{Q}')} (h(\alpha)))$, we obtain $\sigma_{(h, \mathcal{Q})} = \tau \circ \sigma_{(h, \mathcal{Q}')}$. We must notice that the transposition is not the identity because $h^{-1}(\sigma_{(h, \mathcal{Q}')} (h(\alpha)))$ is an endpoint of one of the intervals of \mathcal{Q}' and

x is not.

In conclusion in $\mathbb{Z}/2\mathbb{Z}$ we have:

$$R(h, \mathcal{Q}') + \varepsilon(\sigma_{(h, \mathcal{Q}')}) = R(h, \mathcal{Q}') + 1 + 1 + \varepsilon(\sigma_{(h, \mathcal{Q}')}) = R(h, \mathcal{Q}) + \varepsilon(\sigma_{(h, \mathcal{Q})})$$

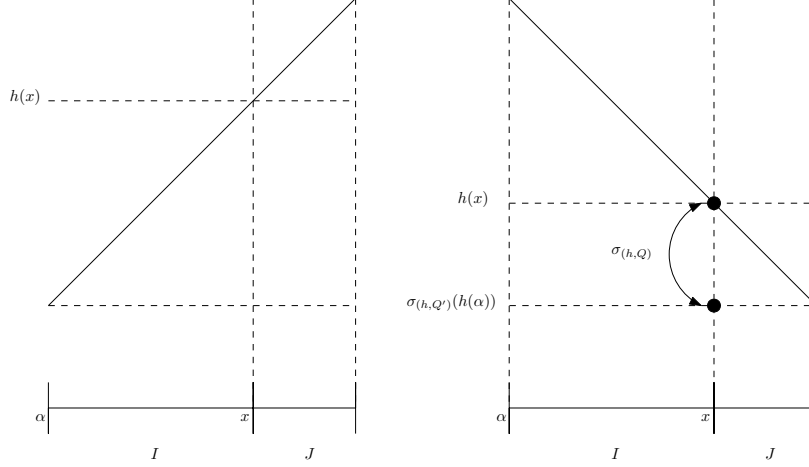


FIGURE 1. Illustrations of the two cases appearing in Lemma 3.6.

On the left we assume h order-preserving on $I \cup J$ and see that $\sigma_{(h, \mathcal{Q})}(h(x)) = \sigma_{(h, \mathcal{Q}')} (h(x))$. On the right we assume h order-reversing on $I \cup J$ and see that $\sigma_{(h, \mathcal{Q})}(h(x)) = (h(x) \sigma_{(h, \mathcal{Q}')} (h(\alpha))) \circ \sigma_{(h, \mathcal{Q}')} (h(x))$.

□

If $\phi \in \text{Homeo}^+([0, 1])$, then it follows from Proposition 3.5 that $\varepsilon(\phi) = 0$. We improve this, showing that ε is invariant by the action of $\text{Homeo}^+([0, 1])$ on $\widehat{\text{PC}}^{\boxtimes}$.

Lemma 3.7. *For every $h \in \widehat{\text{PC}}^{\boxtimes}$ and every $\phi \in \text{Homeo}^+([0, 1])$ we have $\varepsilon(h\phi) = \varepsilon(h) = \varepsilon(\phi h)$.*

Proof. Let $h \in \widehat{\text{PC}}^{\boxtimes}$ and $\phi \in \text{Homeo}^+([0, 1])$ be as in the statement. Let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \dots, I_n\} \in \Pi_h$. Then $\mathcal{Q} := \{\phi^{-1}(I_1), \phi^{-1}(I_2), \dots, \phi^{-1}(I_n)\}$ is in $\Pi_{h\phi}$. We know that ϕ is order preserving then for every $1, \leq i \leq n$, $h\phi$ preserves (reverses respectively) the order on $\phi^{-1}(I_i)$ if and only if h preserves (reverses respectively) the order on I_i , so $R(h, \mathcal{P}) = R(h\phi, \mathcal{Q})$. We can notice that the left endpoint of $\phi^{-1}(I_i)$ (denoted by α_i) is sent on the left endpoint of I_i (denoted by a_i) by ϕ hence $h(a_i) = h\phi(\alpha_i)$ has to be sent on $\sigma_{(h, \mathcal{P})}(h(a_i))$ so $\sigma_{(h\phi, \mathcal{Q})} = \sigma_{(h, \mathcal{P})}$. we deduce that $\varepsilon(h\phi) = \varepsilon(h)$.

The other equality has a similar proof. We denote $h(\mathcal{P})$ the arrival partition of h associated with \mathcal{P} . We know that ϕ is continuous thus $h(\mathcal{P})$ is in Π_ϕ and we deduce that $\mathcal{P} \in \Pi_{\phi h}$. Also ϕ is order-preserving then $R(h, \mathcal{P}) = R(\phi h, \mathcal{P})$. We know that $\sigma_{(\phi, h(\mathcal{P}))} = \text{Id}$ then we can notice that $\phi \circ \sigma_{(h, \mathcal{P})} \circ h$ send the left endpoint of I_i to the left endpoint of $\phi h(I_i^\circ)$. Then $\sigma_{(\phi h, \mathcal{P})} = \phi \sigma_{(h, \mathcal{P})} \phi^{-1}$ and we deduce that $\varepsilon(\sigma_{(\phi h, \mathcal{P})}) = \varepsilon(\sigma_{(h, \mathcal{P})})$. Hence $\varepsilon(\phi h) = \varepsilon(h)$. □

Thanks to Proposition 2.5 it is enough to prove that $\varepsilon|_{\widehat{\text{IET}}^\boxtimes}$ is a group homomorphism.

Lemma 3.8. *The map $\varepsilon|_{\widehat{\text{IET}}^\boxtimes}$ is a group homomorphism.*

Proof. Let $f, g \in \widehat{\text{IET}}^\boxtimes$. Let $\mathcal{P} \in \Pi_f$ and $\mathcal{Q} \in \Pi_g$. For every $I \in \mathcal{Q}$ (resp. $J \in \mathcal{P}$) we denote by α_I (resp. β_J) the left endpoint of I (resp. J). Up to refine \mathcal{P} and \mathcal{Q} we can assume that $\mathcal{P} = g(\mathcal{Q})$ thus $g(\{\alpha_I\}_{I \in \mathcal{Q}}) = \{\beta_J\}_{J \in \mathcal{P}}$. Then $Q \in \Pi_{f \circ g}$ and for every $K \in f \circ g(Q)$ we denote by γ_K the left endpoint of K .

In $\mathbb{Z}/2\mathbb{Z}$, we get immediately that $R(f \circ g, Q) = R(g, Q) + R(f, g(Q))$. Now we want to describe the default of pseudo right continuity for $f \circ g$ about Q . We recall that $\sigma_{(f \circ g, Q)}$ is the permutation that sends $f \circ g(\alpha_I)$ on $\gamma_{f \circ g(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of $[0, 1[$. Furthermore $\sigma_{(g, Q)}(g(\alpha_I)) = \beta_{g(I)}$ and $\sigma_{(f, g(Q))}(f(\beta_{g(I)})) = \gamma_{f \circ g(I)}$. Then $\sigma_{(f, g(Q))} \circ f \circ \sigma_{(g, Q)} \circ g(\alpha_I) = \gamma_{f \circ g(I)}$ and we deduce that the permutation $\sigma_{(f, g(Q))} \circ f \circ \sigma_{(g, Q)} \circ f^{-1}$ sends $f \circ g(\alpha_I)$ on $\gamma_{f \circ g(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of $[0, 1[$. Thus $\sigma_{(f \circ g, Q)} = \sigma_{f, g(Q)} \circ f \circ \sigma_{(g, Q)} \circ f^{-1}$. Then $\varepsilon(\sigma_{(f \circ g, Q)}) = \varepsilon(\sigma_{f, g(Q)}) + \varepsilon(\sigma_{(g, Q)})$ and we conclude that $\varepsilon(f \circ g) = \varepsilon(f) + \varepsilon(g)$. \square

Corollary 3.9. *The map ε is a group homomorphism.* \square

4. NORMAL SUBGROUPS OF $\widehat{\text{PC}}^\boxtimes$ AND SOME SUBGROUPS

Here we present some corollaries of Theorem 1.1. For every group G we denote by $D(G)$ its derived subgroup.

Definition 4.1. For every group H , we define $J_3(H)$ as the subgroup generated by elements of order 3.

Let \widehat{G} be a subgroup of $\widehat{\text{PC}}^\boxtimes$ containing $\mathfrak{S}_{\text{fin}}$. We denote by G its projection on PC^\boxtimes . We recall that $\mathfrak{A}_{\text{fin}}$ is a normal subgroup of \widehat{G} , and has a trivial centraliser. We deduce that for every nontrivial normal subgroup H of \widehat{G} contains $\mathfrak{A}_{\text{fin}}$.

From the short exact sequence:

$$1 \longrightarrow \mathfrak{S}_{\text{fin}} \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

we deduce the next short exact sequence which is a central extension:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widehat{G}/\mathfrak{A}_{\text{fin}} \longrightarrow G \longrightarrow 1.$$

This short exact sequence splits because the signature $\varepsilon|_{\widehat{G}} : \widehat{G} \rightarrow \mathbb{Z}/2\mathbb{Z}$ constructed in § 3 is a retraction. Then we deduce that $\widehat{G}/\mathfrak{A}_{\text{fin}}$ is isomorphic to the direct product $\mathbb{Z}/2\mathbb{Z} \times G$.

Corollary 4.2. *The projection $\widehat{G}_{\text{ab}} \rightarrow G_{\text{ab}}$ extends in an isomorphism $\widehat{G}_{\text{ab}} \sim G_{\text{ab}} \times \mathbb{Z}/2\mathbb{Z}$. Furthermore $D(\widehat{G}) = \text{Ker}(\varepsilon) \cap \widehat{D(G)}$ is a subgroup of index 2 in $\widehat{D(G)}$. In particular, if G is a perfect group then $\widehat{G}_{\text{ab}} = \mathbb{Z}/2\mathbb{Z}$.*

Corollary 4.3. *Let \widehat{G} be a subgroup of $\widehat{\text{PC}}^{\boxtimes}$ containing $\mathfrak{S}_{\text{fin}}$ and such that its projection G in PC^{\boxtimes} is simple nonabelian. Then \widehat{G} has exactly 5 normal subgroups given by the list: $\{\{1\}, \mathfrak{A}_{\text{fin}}, \mathfrak{S}_{\text{fin}}, \text{Ker}(\varepsilon), \widehat{G}\}$.*

Proof. Let \widehat{G} as in the statement. First we immediately check that the subgroups in the list are distinct normal subgroups of \widehat{G} . In the case of $\text{Ker}(\varepsilon)$, there exists $g \in \widehat{G} \setminus \mathfrak{S}_{\text{fin}}$ thus either $g \in \text{Ker}(\varepsilon) \setminus \mathfrak{S}_{\text{fin}}$ or $\sigma g \in \text{Ker}(\varepsilon) \setminus \mathfrak{S}_{\text{fin}}$ for any transposition σ .

Second let H be a normal subgroup of \widehat{G} distinct from $\{1\}$. Then it contains $\mathfrak{A}_{\text{fin}}$. Also $H/\mathfrak{A}_{\text{fin}}$ is a normal subgroup of $\widehat{G}/\mathfrak{A}_{\text{fin}} \simeq \mathbb{Z}/2\mathbb{Z} \times G$. Furthermore G is simple then there are only four possibilities for $H/\mathfrak{A}_{\text{fin}}$. As two normal subgroups H, K of \widehat{G} containing $\mathfrak{A}_{\text{fin}}$ such that $H/\mathfrak{A}_{\text{fin}} = K/\mathfrak{A}_{\text{fin}}$ are equal, we deduce that \widehat{G} has at most 5 normal subgroups. \square

Corollary 4.4. *Let \widehat{G} be a subgroup of $\widehat{\text{PC}}^{\boxtimes}$ containing $\mathfrak{S}_{\text{fin}}$ and such that its projection G in PC^{\boxtimes} is simple nonabelian. If there exists an element of order 3 in $G \setminus \mathfrak{A}_{\text{fin}}$ then $J_3(\widehat{G}) = \text{Ker}(\varepsilon) = D(\widehat{G})$. \square*

Remark 4.5. In the context of topological-full groups, the group $J_3(G)$ appears naturally (with some mild assumptions) and is denoted by $A(G)$ by Nekrashevych in [9]. In some case of topological-full groups of minimal groupoids (see [8]) we have the equality $A(G) = D(G)$ thanks to the simplicity of $D(G)$. In spite of the analogy, it is not clear that the corollary can be obtained as particular case of this result.

Remark 4.6. A lot of groups satisfy the conditions of Corollary 4.4. When \widehat{G} contains $\widehat{\text{IET}}^+$ there is an element of order 3 in $G \setminus \mathfrak{A}_{\text{fin}}$. We recall that $\widehat{\text{IET}}^{\boxtimes}$, PC^+ and PAff^+ are simple (see [2, 5]). Thus these groups satisfy the conditions of Corollary 4.4. The next theorem add PC^{\boxtimes} and PAff^{\boxtimes} to the list of examples.

Theorem 4.7. *The groups PC^{\boxtimes} and PAff^{\boxtimes} are simple.*

Lemma 4.8. *The group $\widehat{\text{IET}}^{\boxtimes}$ is generated by flips (=images of flips from $\widehat{\text{IET}}^{\boxtimes}$).*

Proof. By Proposition 2.4 it is enough to show that $\widehat{\text{IET}}^+$ is generated by flips. For every consecutive, right-open and left-closed subintervals I and J of $[0, 1[$, we define $R_{I,J}$ the map that exchanges I and J . They are elements of $\widehat{\text{IET}}_{\text{rc}}^+$ and they formed a generating set. Then their image $r_{I,J}$ in $\widehat{\text{IET}}^{\boxtimes}$ is a generating set of $\widehat{\text{IET}}^+$. For every right-open and left-closed subinterval I of $[0, 1[$, we define s_I the I -flip. Take I and J be two consecutive, right-open and left-closed subintervals of $[0, 1[$. Then $r_{I,J} = s_I s_J s_{I \cup J}$. \square

Proof of Theorem 4.7 (sketched).

Since the argument in [2] could also be adapted, we only provide a sketch.

We work with elements of PC^{\boxtimes} ; all intervals below are meant modulo finite subsets. Let N be a nontrivial normal subgroup of PC^{\boxtimes} (resp. PAff^{\boxtimes}). Let g be a nontrivial element of N . There exists a subinterval I of $[0, 1[$ such that:

- (1) g is continuous (resp. affine) on I ,
- (2) $g(I) \cap I = \emptyset$ (modulo finite subsets),
- (3) $I \cup g(I) \neq [0, 1[$ (modulo finite subsets).

Let f be the I -flip. If g is affine on I then $h = gfg^{-1}f^{-1}$ is the product of the I -flip with the $g(I)$ -flip. Observe that h is conjugate to a single flip by a suitable element of IET^+ . If g is only continuous then h is still of order 2 and it is conjugate in PC^{\boxtimes} to a single flip. Conjugating by elements of PAff^+ , one obtains that N contains flips of intervals of all possible lengths, and hence contains all flips. Thanks to Lemma 4.8 we know that IET^{\boxtimes} is generated by the set of flips thus N contains IET^{\boxtimes} , in particular N intersects PC^+ (resp. PAff^+) nontrivially. By simplicity of PC^+ (resp. PAff^+) we deduce that N contains $\text{PC}^{\boxtimes} = \langle \text{PC}^+, \text{IET}^{\boxtimes} \rangle$ (resp. $\text{PAff}^{\boxtimes} = \langle \text{PAff}^+, \text{IET}^{\boxtimes} \rangle$). \square

5. ABOUT SOME NORMALIZERS

Here we show that computing normalizers inside $\widehat{\text{PC}^{\boxtimes}}$ and PC^{\boxtimes} may leads to different behaviour. We look the case of PC^+ , IET^+ and $\widehat{\text{PC}}_{\text{rc}}^+$ and $\widehat{\text{IET}}_{\text{rc}}^+$.

Proposition 5.1. *The normalizer of IET^+ in IET^{\boxtimes} is reduced to IET^{\pm} .*

Proof. Let $f \in \text{IET}^+$ and $g \in \text{IET}^{\pm}$. If $g \in \text{IET}^+$ then $gfg^{-1} \in \text{IET}^+$. We assume $g \in \text{IET}^-$ then $gfg^{-1} = (g \circ \mathcal{R}) \circ (\mathcal{R} \circ f \circ \mathcal{R}) \circ (\mathcal{R} \circ g) \in \text{IET}^+$. For the inclusion from left to right, let $g \in \text{IET}^{\boxtimes} \setminus \text{IET}^{\pm}$ and let \widehat{g} be a representative of g in $\widehat{\text{IET}^{\boxtimes}}$. Hence we can find I, J, K, L four right-open and left-closed intervals of the same length such that their image by \widehat{g} are intervals and such that \widehat{g} is order-reversing on I and order-preserving on J, K and L . We define $\widehat{f} \in \widehat{\text{IET}}^+$ as the element which exchanges $\widehat{g}(I)$ with $\widehat{g}(J)$ and $\widehat{g}(K)$ with $\widehat{g}(L)$ while fixing the rest of $[0, 1[$. Then the image f of \widehat{f} in IET^+ is not trivial and $\widehat{g}\widehat{f}\widehat{g}^{-1} \notin \widehat{\text{IET}}^+$ implies $gfg^{-1} \notin \text{IET}^+$. \square

A similar argument stands for the case of PC thus we obtain:

Proposition 5.2. *The normalizer of PC^+ in PC^{\boxtimes} is reduced to PC^{\pm} .* \square

We now take a look to inside $\widehat{\text{PC}^{\boxtimes}}$:

Proposition 5.3. *The normalizer of $\widehat{\text{IET}}_{\text{rc}}^+$ in $\widehat{\text{IET}}^{\boxtimes}$ is $\widehat{\text{IET}}_{\text{rc}}^+$.*

Proof. Let g be an element of $\widehat{\text{IET}}^{\boxtimes}$ which is not the identity. There are two cases:

- (1) If $g \in \widehat{\text{IET}}^+ \setminus \widehat{\text{IET}}_{\text{rc}}^+$ then $g = \sigma g'$ with $\sigma \in \mathfrak{S}_{\text{fin}} \setminus \{\text{Id}\}$ and $g' \in \widehat{\text{IET}}_{\text{rc}}^+$. Then for every $f \in \widehat{\text{IET}}_{\text{rc}}^+$ we have $gfg^{-1} = \sigma g'fg'^{-1}\sigma^{-1}$. Thus it is enough to treat the case of $\mathfrak{S}_{\text{fin}}$. Let us assume $g \in \mathfrak{S}_{\text{fin}}$ then let x in the support of g . There exist two consecutive right-open and left-closed intervals I and J of the same length such that x is the right endpoint of I (and the left endpoint of J). Up to reduce I and J we can assume that I does not intersect the support of g . Then let $f \in \widehat{\text{IET}}_{\text{rc}}^+$ which exchanges I and J while fixing the rest of $[0, 1[$. Then gfg^{-1} exchanges the interior of I with the interior of J but $gfg^{-1}(x)$ is not equal to $f(x)$ because $f(x)$ is the left endpoint of I and I does not intersect the support of g . Then we deduce that gfg^{-1} is not right-continuous on J .
- (2) If $g \in \widehat{\text{IET}}^{\boxtimes} \setminus \widehat{\text{IET}}^+$. Then we can find two consecutive subinterval I and J where g is continuous and order-reversing on $I \cup J$. Let a be the right endpoint of J . Let f be the element in $\widehat{\text{IET}}_{\text{rc}}^+$ which exchanges I and J . Then gfg^{-1} exchanges the interior of $g(J)$ with the interior of $g(I)$. However the left endpoint of $g(J)$ is send by g^{-1} on a which is fixed by f . Then gfg^{-1} fixes the left endpoint of $g(J)$, thus gfg^{-1} is not right-continuous on $g(J)$.

□

A similar argument stands for the case of PC thus we obtain:

Proposition 5.4. *The normalizer of $\widehat{\text{PC}}_{\text{rc}}^+$ in $\widehat{\text{PC}}^{\boxtimes}$ is $\widehat{\text{PC}}_{\text{rc}}^+$.*

□

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