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SIGNATURE FOR PIECEWISE CONTINUOUS GROUPS

OCTAVE LACOURTE

ABSTRACT. Let $\widehat{\text{PC}}^{\text{pc}}$ be the group of bijections from $[0, 1[$ to itself which are continuous outside a finite set. Let PC^{pc} be its quotient by the subgroup of finitely supported permutations.

We show that the Kapoudjian class of PC^{pc} vanishes. That is, the quotient map $\widehat{\text{PC}}^{\text{pc}} \rightarrow \text{PC}^{\text{pc}}$ splits modulo the alternating subgroup of even permutations. This is shown by constructing a nonzero group homomorphism, called signature, from $\widehat{\text{PC}}^{\text{pc}}$ to $\mathbb{Z}/2\mathbb{Z}$. Then we use this signature to list normal subgroups of every subgroup \widehat{G} of $\widehat{\text{PC}}^{\text{pc}}$ which contains $\mathfrak{S}_{\text{fin}}$ and such that G , the projection of \widehat{G} in PC^{pc} , is simple.

1. INTRODUCTION

Let X be the right-open and left-closed interval $[0, 1[$. We denote by $\mathfrak{S}(X)$ the group of bijections of X to X . This group contains the subgroup composed of all finitely supported permutations is denoted by $\mathfrak{S}_{\text{fin}}$. The classical signature is well-defined on $\mathfrak{S}_{\text{fin}}$ and its kernel, denoted by $\mathfrak{A}_{\text{fin}}$, is the only subgroup of index 2 in $\mathfrak{S}_{\text{fin}}$. An observation, originally due to Vitali [10], is that the signature does not extend to $\mathfrak{S}(X)$.

For every subgroup G of $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$, we denote by \widehat{G} its inverse image in $\mathfrak{S}(X)$. The cohomology class of the central extension

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} = \mathfrak{S}_{\text{fin}}/\mathfrak{A}_{\text{fin}} \rightarrow \widehat{G}/\mathfrak{A}_{\text{fin}} \rightarrow G \rightarrow 1$$

is called the Kapoudjian class of G ; it belongs to $H^2(G, \mathbb{Z}/2\mathbb{Z})$. It appears in the work of Kapoudjian and Kapoudjian-Sergiescu [6, 7]. The vanishing of this class means that the above exact sequence splits; this means that there exists a group homomorphism from the preimage of G in $\mathfrak{S}(X)$ onto $\mathbb{Z}/2\mathbb{Z}$ which extends the signature on $\mathfrak{S}_{\text{fin}}$ (for more on the Kapoudjian class, see [3, §8.C]). This implies in particular that $\widehat{G}/\mathfrak{A}_{\text{fin}}$ is isomorphic to the direct product $G \times \mathbb{Z}/2\mathbb{Z}$. One can notice that for $G = \mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$ we have $\widehat{G} = \mathfrak{S}(X)$; in this case the Vitali's observation implies that the Kapoudjian class does not vanish.

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The set of all permutations of X continuous outside a finite set is a subgroup denoted by $\widehat{\text{PC}}^{\bowtie}$. The aim here is to show the following theorem:

Theorem 1.1. *There exists a group homomorphism $\varepsilon : \widehat{\text{PC}}^{\bowtie} \rightarrow \mathbb{Z}/2\mathbb{Z}$ that extends the classical signature on $\mathfrak{S}_{\text{fin}}$.*

Corollary 1.2. *Let G be a subgroup of PC^{\bowtie} . Then the Kapoudjian class of G is zero.*

This solves a question asked by Y. Cornulier [4, Question 1.15].

The subgroup of $\widehat{\text{PC}}^{\bowtie}$ consisting of all permutations of X that are piecewise isometric elements is denoted by $\widehat{\text{IET}}^{\bowtie}$ and the one consisting of all piecewise affine permutations of X is denoted by $\widehat{\text{PAff}}^{\bowtie}$. We also consider for each of these groups the subgroup composed of all piecewise orientation-preserving elements by replacing the symbol “ \bowtie ” by the symbol “ $+$ ”.

Let us observe that when $G \subset \text{PC}^+$ Corollary 1.2 is trivial. Indeed, in this case G can be lifted inside $\widehat{\text{PC}}^+$ itself. However, such a lift does not exist for PC^{\bowtie} or even IET^{\bowtie} , as was proved in [4].

The idea of proof of Theorem 1.1 is to associate for every $f \in \widehat{\text{PC}}^{\bowtie}$ and every finite partition \mathcal{P} of $[0, 1[$ into intervals associated with f , two numbers. The first is the number of interval of \mathcal{P} where f is order-reversing and the second is the signature of a particular finitely supported permutation. The next step is to prove that the sum modulo 2 of this two numbers is independent from the choice of partition. Then we show that it is enough to prove that $\varepsilon|_{\widehat{\text{IET}}^{\bowtie}}$ is a group homomorphism. For this we show that it is additive when we look at the composition of two elements of $\widehat{\text{IET}}^{\bowtie}$ by calculate the value of the signature with a particular partition.

In Section 4, we apply these results to the study of normal subgroups of $\widehat{\text{PC}}^{\bowtie}$ and certain subgroups. More specifically we prove:

Theorem 1.3. *Let \widehat{G} be a subgroup of $\widehat{\text{PC}}^{\bowtie}$ containing $\mathfrak{S}_{\text{fin}}$ and such that its projection G in PC^{\bowtie} is simple nonabelian. Then \widehat{G} has exactly five normal subgroups given by the list: $\{\{1\}, \mathfrak{A}_{\text{fin}}, \mathfrak{S}_{\text{fin}}, \text{Ker}(\varepsilon), \widehat{G}\}$.*

We denote by $\widehat{\text{IET}}_{\text{rc}}^+$ the subgroup of $\widehat{\text{IET}}^+$ composed of all right-continuous elements. We know that it is naturally isomorphic to IET^+ . The same is true when we replace IET^+ by PAff^+ or PC^+ . This allows us to use the work of P. Arnoux [2] and the one of N. Guelman and I. Liousse [5] where they prove that IET^{\bowtie} , PC^+ and PAff^+ are simple. From this we deduce:

Theorem 1.4. *The groups PC^{\bowtie} and PAff^{\bowtie} are simple.*

This gives us some examples of groups that satisfy the conditions of the Theorem 1.3.

Finally Section 5 is independent and we study some normalizers and in particular we show that the behaviour when we look the group inside $\widehat{\text{PC}}^{\boxtimes}$ or PC^{\boxtimes} may not be the same. We denote by $\mathcal{R} \in \text{IET}^{\boxtimes}$ the map $x \mapsto 1 - x$. Then we define IET^- as the coset $\mathcal{R}.\text{IET}^+$ and PC^- as the coset $\mathcal{R}.\text{PC}^+$. Then the groups $\text{IET}^{\pm} := \text{IET}^+ \cup \text{IET}^-$ and $\text{PC}^{\pm} := \text{PC}^+ \cup \text{PC}^-$ are well-defined.

Proposition 1.5. *The subgroup $\widehat{\text{IET}}_{\text{rc}}^+$ (resp. $\widehat{\text{PC}}_{\text{rc}}^+$) is its own normalizer in $\widehat{\text{IET}}^{\boxtimes}$ (resp. $\widehat{\text{PC}}_{\text{rc}}^{\boxtimes}$). The normalizer of IET^+ (PC^+ respectively) in IET^{\boxtimes} (PC^{\boxtimes} respectively) is IET^{\pm} (PC^{\pm} respectively).*

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2. PRELIMINARIES

For every real interval I we denote by I° its interior in \mathbb{R} and if $I = [0, t[$ we agree that its interior is $]0, t[$.

2.1. Partitions associated.

An important tool to study elements in $\widehat{\text{PC}}^{\boxtimes}$ and PC^{\boxtimes} are partitions into intervals of $[0, 1[$. All partitions are assumed to be finite.

Definition 2.1. For every f in $\widehat{\text{PC}}^{\boxtimes}$, a finite partition \mathcal{P} into right-open and left-closed intervals of $[0, 1[$ is called *a partition into intervals associated with f* if and only if f is continuous on the interior of every interval of \mathcal{P} . We denote by Π_f the set of all partitions into intervals associated with f . We define also *the arrival partition of f associated with \mathcal{P}* , denoted $f(\mathcal{P})$, the partition of $[0, 1[$ composed of all right-open and left-closed intervals such that their interior is equal to the image by f of the interior of an interval of \mathcal{P} .

Remark 2.2. For every f in $\widehat{\text{PC}}^{\boxtimes}$ there exists a unique partition $\mathcal{P}_f^{\text{min}}$ associated with f which has a minimal number of intervals. It is actually minimal in the sense of refinement: Π_f consists precisely of the set of partitions refining $\mathcal{P}_f^{\text{min}}$.

2.2. Decompositions.

We define a family of elements which plays an important role inside our groups:

Definition 2.3. Let I be a non-empty right-open and left-closed subinterval of $[0, 1[$. The element $f \in \widehat{\text{PC}}^{\boxtimes}$ which sends the interior of I on itself with slope -1 while fixing the rest of $[0, 1[$ is called the *I -flip*. We define *a flip* as any I -flip for some I .

From the definition we deduce a decomposition inside $\widehat{\text{IET}}^{\boxtimes}$ and $\widehat{\text{PC}}^{\boxtimes}$.

Proposition 2.4. *Let h be an element of $\widehat{\text{IET}}^{\boxtimes}$. There exist $f, g \in \widehat{\text{IET}}_{\text{rc}}^+$ and r, s finite products of flips and σ, τ finitely supported permutations such that $h = r\sigma f = g\tau s$.*

Proof. Let h be an element of $\widehat{\text{IET}}^{\boxtimes}$, $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \dots, I_n\} \in \Pi_h$ (§ 2.1). We denote by $h(\mathcal{P}) := \{J_1, J_2, \dots, J_n\}$ the arrival partition of h associated with \mathcal{P} . Let g be the map that sends I_j° on J_j° by preserving the order and acts as h for every left endpoints of I_j for every $1 \leq j \leq n$. Note that g is bijective and then belongs to $\widehat{\text{IET}}^+$. For $1 \leq j \leq n$ let r_j be the J_j -flip if h is order-reversing on I_j otherwise let r_j be the identity. Let r be the product of all r_j , we can notice that r fixes all endpoints of J_j for every $1 \leq j \leq n$. Then it is just a verification to check that $h = rg$. Now as g belongs to $\widehat{\text{IET}}^+$ there exists σ in \mathfrak{S}_n such that $g = \sigma f$ with f in $\widehat{\text{IET}}_{\text{rc}}^+$.

The other decomposition follows by decomposing h^{-1} under the previous decomposition. \square

Proposition 2.5. *For every h in $\widehat{\text{PC}}^{\boxtimes}$ there exist ϕ and ψ two order-preserving homeomorphisms of $[0, 1[$ and f, g in $\widehat{\text{IET}}^{\boxtimes}$ such that $h = \psi \circ f = g \circ \phi$.*

Proof. Let λ be the Lebesgue measure on $[0, 1[$. Let $h \in \widehat{\text{PC}}^{\boxtimes}$ and $\mathcal{P} \in \Pi_h$. Then there exist $\phi, \psi \in \text{Homeo}^+([0, 1[)$ such that for every $I \in \mathcal{P}$, $\lambda(\phi(I)) = \lambda(h(I))$ and $\lambda(\psi(h(I))) = \lambda(I)$. Then $h \circ \phi$ and $\psi \circ h$ belongs to $\widehat{\text{IET}}^{\boxtimes}$. \square

3. CONSTRUCTION OF THE SIGNATURE HOMOMORPHISM

In our case we have $X = [0, 1[$ and $\widehat{\text{PC}}^{\boxtimes}$ is a subgroup of $\mathfrak{S}(X)$. We denote here $\mathfrak{S}_{\text{fin}} = \mathfrak{S}_{\text{fin}}(X)$ and ε_{fin} the classical signature on $\mathfrak{S}_{\text{fin}}$ taking values in $(\mathbb{Z}/2\mathbb{Z}, +)$.

3.1. Definitions.

Definition 3.1. Let h be an element of $\widehat{\text{PC}}^{\boxtimes}$, $n \in \mathbb{N}$ and $\mathcal{P} = \{I_1, I_2, \dots, I_n\} \in \Pi_h$. For every $1 \leq j \leq n$, let α_j be the left endpoint of I_j and β_j be the left endpoint of $h(I_j^\circ)$. We define the *default of pseudo right continuity for h about \mathcal{P}* denoted $\sigma_{(h, \mathcal{P})}$ as the finitely supported permutation which sends $h(\alpha_j)$ to β_j for every $1 \leq j \leq n$ (this is well-defined because the set of all $h(\alpha_j)$ is equal to the set of all β_j).

Definition 3.2. Let h be an element of $\widehat{\text{PC}}^{\boxtimes}$ and $\mathcal{P} \in \Pi_h$. Let k be the number of interval of \mathcal{P} on which h is order-reversing. We called the *flip number of h about \mathcal{P}* the number k . We denote it by $R(h, \mathcal{P})$.

Definition 3.3. For $h \in \widehat{\text{PC}}^{\boxtimes}$ and $\mathcal{P} \in \Pi_h$, define $\varepsilon(h, \mathcal{P}) \in \mathbb{Z}/2\mathbb{Z}$ as $R(h, \mathcal{P}) + \varepsilon_{\text{fin}}(\sigma_{(h, \mathcal{P})}) \pmod{2}$. We define also $\varepsilon(h) = \varepsilon(h, \mathcal{P}_h^{\text{fin}})$.

Proposition 3.4. *For every $\tau \in \mathfrak{S}_{\text{fin}}$ and every $\mathcal{P} \in \Pi_\tau$ we have $\varepsilon(\tau, \mathcal{P}) = \varepsilon_{\text{fin}}(\tau)$.*

Proof. It is clear that for every $\tau \in \mathfrak{S}_{\text{fin}}$ and every partition \mathcal{P} associated with τ we have $R(\tau, \mathcal{P}) = 0$ and $\sigma_{(\tau, \mathcal{P})} = \tau$. \square

We deduce that ε extends the classical signature ε_{fin} . Thus we will write ε instead of ε_{fin} .

Proposition 3.5. *Every right-continuous element f of $\widehat{\text{PC}}^+$ satisfies $\varepsilon(f, \mathcal{P}) = 0$ for every $\mathcal{P} \in \Pi_f$.*

Proof. In this case, for every partition \mathcal{P} into intervals associated with f we always have $R(f, \mathcal{P}) = 0$ and $\sigma_{(f, \mathcal{P})} = \text{Id}$. \square

3.2. Proof of Theorem 1.1.

In order to prove that ε is a group homomorphism, it is useful to calculate $\varepsilon(h)$ thanks to $\varepsilon(h, \mathcal{P})$ for every $h \in \widehat{\text{PC}}^{\boxtimes}$ and $\mathcal{P} \in \Pi_h$.

Lemma 3.6. *For every $h \in \widehat{\text{PC}}^{\boxtimes}$ and every $\mathcal{P} \in \Pi_h$ we have $\varepsilon(h) = \varepsilon(h, \mathcal{P})$.*

Proof. Let h and \mathcal{P} be as in the statement. By minimality of $\mathcal{P}_h^{\text{min}}$, in term of refinement, we deduce that there exist $n \in \mathbb{N}$ and $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n \in \Pi_h$ such that:

- (1) $\mathcal{P}_1 = \mathcal{P}_h^{\text{min}}$;
- (2) $\mathcal{P}_n = \mathcal{P}$;
- (3) for every $2 \leq i \leq n$ the partition \mathcal{P}_i is a refinement of the partition \mathcal{P}_{i-1} where only one interval of \mathcal{P}_{i-1} is cut into two.

Hence it is enough to show $\varepsilon(h, \mathcal{Q}) = \varepsilon(h, \mathcal{Q}')$ where $\mathcal{Q}, \mathcal{Q}' \in \Pi_h$ such that there exist consecutive intervals $I, J \in \mathcal{Q}$ with $I \cup J \in \mathcal{Q}'$ and $\mathcal{Q}' \setminus \{I \cup J\} = \mathcal{Q} \setminus \{I, J\}$.

Let α be the left endpoint of I and let x be the right endpoint of I (x is also the left endpoint of J). There are only two cases but in both cases, we know that $\sigma_{(h, \mathcal{Q})} = \sigma_{(h, \mathcal{Q}')}$ except maybe on $h(\alpha)$ and $h(x)$:

- (1) The first case is when h is order-preserving on $I \cup J$. Then as $\mathcal{Q} \setminus \{I, J\} = \mathcal{Q}' \setminus \{I \cup J\}$ we get $R(h, \mathcal{Q}) = R(h, \mathcal{Q}')$. As h is order-preserving on the interior of $I \cup J$ we know that $\sigma_{(h, \mathcal{Q}')} (h(\alpha))$ is the left endpoint of $h(I \cup J)$ which is the left endpoint of $h(I)$ thus equals to $\sigma_{(h, \mathcal{Q})} (h(\alpha))$. With the same reasoning we deduce that $\sigma_{(h, \mathcal{Q}')} (h(x)) = \sigma_{(h, \mathcal{Q})} (h(x))$ hence $\sigma_{(h, \mathcal{Q})} = \sigma_{(h, \mathcal{Q}')}$. Thus in $\mathbb{Z}/2\mathbb{Z}$ we have $R(h, \mathcal{Q}') + \varepsilon(\sigma_{(h, \mathcal{Q}')}) = R(h, \mathcal{Q}) + \varepsilon(\sigma_{(h, \mathcal{Q})})$.
- (2) The second case is when h is order-reversing on $I \cup J$. Then we get $R(h, \mathcal{Q}) = R(h, \mathcal{Q}') + 1$. This time $\sigma_{(h, \mathcal{Q}')} (h(\alpha))$ is still the left endpoint of $h(I \cup J)$ which is the left endpoint of $h(J)$ thus equals to $\sigma_{(h, \mathcal{Q})} (h(x))$. With the same reasoning we deduce that $\sigma_{(h, \mathcal{Q}')} (h(x)) = \sigma_{(h, \mathcal{Q})} (h(\alpha))$. Then by denoting τ the transposition $(h(x) \sigma_{(h, \mathcal{Q}')} (h(\alpha)))$, we obtain $\sigma_{(h, \mathcal{Q})} = \tau \circ \sigma_{(h, \mathcal{Q}')}$. We must notice that the transposition is not the identity because $h^{-1}(\sigma_{(h, \mathcal{Q}')} (h(\alpha)))$ is an endpoint of one of the intervals of \mathcal{Q}' and

x is not.

In conclusion in $\mathbb{Z}/2\mathbb{Z}$ we have:

$$R(h, \mathcal{Q}') + \varepsilon(\sigma_{(h, \mathcal{Q}')}) = R(h, \mathcal{Q}') + 1 + 1 + \varepsilon(\sigma_{(h, \mathcal{Q}')}) = R(h, \mathcal{Q}) + \varepsilon(\sigma_{(h, \mathcal{Q})})$$

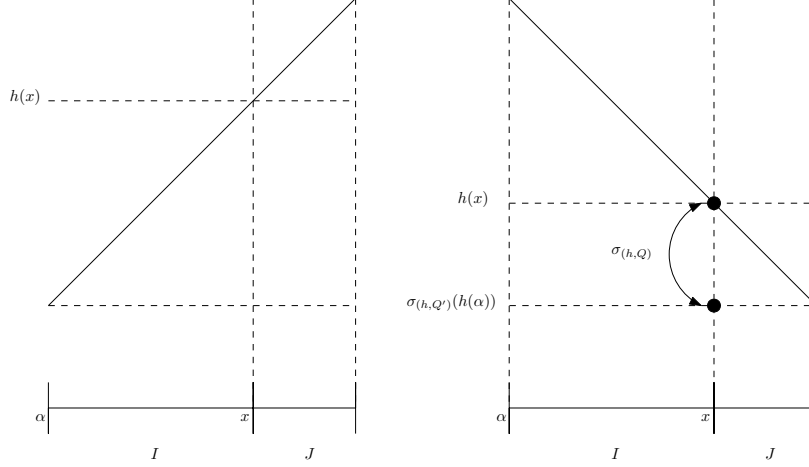


FIGURE 1. Illustrations of the two cases appearing in Lemma 3.6.

On the left we assume h order-preserving on $I \cup J$ and see that $\sigma_{(h, \mathcal{Q})}(h(x)) = \sigma_{(h, \mathcal{Q}')} (h(x))$. On the right we assume h order-reversing on $I \cup J$ and see that $\sigma_{(h, \mathcal{Q})}(h(x)) = (h(x) \sigma_{(h, \mathcal{Q}')} (h(\alpha))) \circ \sigma_{(h, \mathcal{Q}')} (h(x))$.

□

If $\phi \in \text{Homeo}^+([0, 1])$, then it follows from Proposition 3.5 that $\varepsilon(\phi) = 0$. We improve this, showing that ε is invariant by the action of $\text{Homeo}^+([0, 1])$ on $\widehat{\text{PC}}^{\boxtimes}$.

Lemma 3.7. *For every $h \in \widehat{\text{PC}}^{\boxtimes}$ and every $\phi \in \text{Homeo}^+([0, 1])$ we have $\varepsilon(h\phi) = \varepsilon(h) = \varepsilon(\phi h)$.*

Proof. Let $h \in \widehat{\text{PC}}^{\boxtimes}$ and $\phi \in \text{Homeo}^+([0, 1])$ be as in the statement. Let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \dots, I_n\} \in \Pi_h$. Then $\mathcal{Q} := \{\phi^{-1}(I_1), \phi^{-1}(I_2), \dots, \phi^{-1}(I_n)\}$ is in $\Pi_{h\phi}$. We know that ϕ is order preserving then for every $1, \leq i \leq n$, $h\phi$ preserves (reverses respectively) the order on $\phi^{-1}(I_i)$ if and only if h preserves (reverses respectively) the order on I_i , so $R(h, \mathcal{P}) = R(h\phi, \mathcal{Q})$. We can notice that the left endpoint of $\phi^{-1}(I_i)$ (denoted by α_i) is sent on the left endpoint of I_i (denoted by a_i) by ϕ hence $h(a_i) = h\phi(\alpha_i)$ has to be sent on $\sigma_{(h, \mathcal{P})}(h(a_i))$ so $\sigma_{(h\phi, \mathcal{Q})} = \sigma_{(h, \mathcal{P})}$. we deduce that $\varepsilon(h\phi) = \varepsilon(h)$.

The other equality has a similar proof. We denote $h(\mathcal{P})$ the arrival partition of h associated with \mathcal{P} . We know that ϕ is continuous thus $h(\mathcal{P})$ is in Π_ϕ and we deduce that $\mathcal{P} \in \Pi_{\phi h}$. Also ϕ is order-preserving then $R(h, \mathcal{P}) = R(\phi h, \mathcal{P})$. We know that $\sigma_{(\phi, h(\mathcal{P}))} = \text{Id}$ then we can notice that $\phi \circ \sigma_{(h, \mathcal{P})} \circ h$ send the left endpoint of I_i to the left endpoint of $\phi h(I_i^\circ)$. Then $\sigma_{(\phi h, \mathcal{P})} = \phi \sigma_{(h, \mathcal{P})} \phi^{-1}$ and we deduce that $\varepsilon(\sigma_{(\phi h, \mathcal{P})}) = \varepsilon(\sigma_{(h, \mathcal{P})})$. Hence $\varepsilon(\phi h) = \varepsilon(h)$. □

Thanks to Proposition 2.5 it is enough to prove that $\varepsilon|_{\widehat{\text{IET}}^\boxtimes}$ is a group homomorphism.

Lemma 3.8. *The map $\varepsilon|_{\widehat{\text{IET}}^\boxtimes}$ is a group homomorphism.*

Proof. Let $f, g \in \widehat{\text{IET}}^\boxtimes$. Let $\mathcal{P} \in \Pi_f$ and $\mathcal{Q} \in \Pi_g$. For every $I \in \mathcal{Q}$ (resp. $J \in \mathcal{P}$) we denote by α_I (resp. β_J) the left endpoint of I (resp. J). Up to refine \mathcal{P} and \mathcal{Q} we can assume that $\mathcal{P} = g(\mathcal{Q})$ thus $g(\{\alpha_I\}_{I \in \mathcal{Q}}) = \{\beta_J\}_{J \in \mathcal{P}}$. Then $Q \in \Pi_{f \circ g}$ and for every $K \in f \circ g(Q)$ we denote by γ_K the left endpoint of K .

In $\mathbb{Z}/2\mathbb{Z}$, we get immediately that $R(f \circ g, Q) = R(g, Q) + R(f, g(Q))$. Now we want to describe the default of pseudo right continuity for $f \circ g$ about Q . We recall that $\sigma_{(f \circ g, Q)}$ is the permutation that sends $f \circ g(\alpha_I)$ on $\gamma_{f \circ g(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of $[0, 1[$. Furthermore $\sigma_{(g, Q)}(g(\alpha_I)) = \beta_{g(I)}$ and $\sigma_{(f, g(Q))}(f(\beta_{g(I)})) = \gamma_{f \circ g(I)}$. Then $\sigma_{(f, g(Q))} \circ f \circ \sigma_{(g, Q)} \circ g(\alpha_I) = \gamma_{f \circ g(I)}$ and we deduce that the permutation $\sigma_{(f, g(Q))} \circ f \circ \sigma_{(g, Q)} \circ f^{-1}$ sends $f \circ g(\alpha_I)$ on $\gamma_{f \circ g(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of $[0, 1[$. Thus $\sigma_{(f \circ g, Q)} = \sigma_{f, g(Q)} \circ f \circ \sigma_{(g, Q)} \circ f^{-1}$. Then $\varepsilon(\sigma_{(f \circ g, Q)}) = \varepsilon(\sigma_{f, g(Q)}) + \varepsilon(\sigma_{(g, Q)})$ and we conclude that $\varepsilon(f \circ g) = \varepsilon(f) + \varepsilon(g)$. \square

Corollary 3.9. *The map ε is a group homomorphism.* \square

4. NORMAL SUBGROUPS OF $\widehat{\text{PC}}^\boxtimes$ AND SOME SUBGROUPS

Here we present some corollaries of Theorem 1.1. For every group G we denote by $D(G)$ its derived subgroup.

Definition 4.1. For every group H , we define $J_3(H)$ as the subgroup generated by elements of order 3.

Let \widehat{G} be a subgroup of $\widehat{\text{PC}}^\boxtimes$ containing $\mathfrak{S}_{\text{fin}}$. We denote by G its projection on PC^\boxtimes . We recall that $\mathfrak{A}_{\text{fin}}$ is a normal subgroup of \widehat{G} , and has a trivial centraliser. We deduce that for every nontrivial normal subgroup H of \widehat{G} contains $\mathfrak{A}_{\text{fin}}$.

From the short exact sequence:

$$1 \longrightarrow \mathfrak{S}_{\text{fin}} \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

we deduce the next short exact sequence which is a central extension:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widehat{G}/\mathfrak{A}_{\text{fin}} \longrightarrow G \longrightarrow 1.$$

This short exact sequence splits because the signature $\varepsilon|_{\widehat{G}} : \widehat{G} \rightarrow \mathbb{Z}/2\mathbb{Z}$ constructed in § 3 is a retraction. Then we deduce that $\widehat{G}/\mathfrak{A}_{\text{fin}}$ is isomorphic to the direct product $\mathbb{Z}/2\mathbb{Z} \times G$.

Corollary 4.2. *The projection $\widehat{G}_{\text{ab}} \rightarrow G_{\text{ab}}$ extends in an isomorphism $\widehat{G}_{\text{ab}} \sim G_{\text{ab}} \times \mathbb{Z}/2\mathbb{Z}$. Furthermore $D(\widehat{G}) = \text{Ker}(\varepsilon) \cap \widehat{D(G)}$ is a subgroup of index 2 in $\widehat{D(G)}$. In particular, if G is a perfect group then $\widehat{G}_{\text{ab}} = \mathbb{Z}/2\mathbb{Z}$.*

Corollary 4.3. *Let \widehat{G} be a subgroup of $\widehat{\text{PC}}^{\boxtimes}$ containing $\mathfrak{S}_{\text{fin}}$ and such that its projection G in PC^{\boxtimes} is simple nonabelian. Then \widehat{G} has exactly 5 normal subgroups given by the list: $\{\{1\}, \mathfrak{A}_{\text{fin}}, \mathfrak{S}_{\text{fin}}, \text{Ker}(\varepsilon), \widehat{G}\}$.*

Proof. Let \widehat{G} as in the statement. First we immediately check that the subgroups in the list are distinct normal subgroups of \widehat{G} . In the case of $\text{Ker}(\varepsilon)$, there exists $g \in \widehat{G} \setminus \mathfrak{S}_{\text{fin}}$ thus either $g \in \text{Ker}(\varepsilon) \setminus \mathfrak{S}_{\text{fin}}$ or $\sigma g \in \text{Ker}(\varepsilon) \setminus \mathfrak{S}_{\text{fin}}$ for any transposition σ .

Second let H be a normal subgroup of \widehat{G} distinct from $\{1\}$. Then it contains $\mathfrak{A}_{\text{fin}}$. Also $H/\mathfrak{A}_{\text{fin}}$ is a normal subgroup of $\widehat{G}/\mathfrak{A}_{\text{fin}} \simeq \mathbb{Z}/2\mathbb{Z} \times G$. Furthermore G is simple then there are only four possibilities for $H/\mathfrak{A}_{\text{fin}}$. As two normal subgroups H, K of \widehat{G} containing $\mathfrak{A}_{\text{fin}}$ such that $H/\mathfrak{A}_{\text{fin}} = K/\mathfrak{A}_{\text{fin}}$ are equal, we deduce that \widehat{G} has at most 5 normal subgroups. \square

Corollary 4.4. *Let \widehat{G} be a subgroup of $\widehat{\text{PC}}^{\boxtimes}$ containing $\mathfrak{S}_{\text{fin}}$ and such that its projection G in PC^{\boxtimes} is simple nonabelian. If there exists an element of order 3 in $G \setminus \mathfrak{A}_{\text{fin}}$ then $J_3(\widehat{G}) = \text{Ker}(\varepsilon) = D(\widehat{G})$. \square*

Remark 4.5. In the context of topological-full groups, the group $J_3(G)$ appears naturally (with some mild assumptions) and is denoted by $A(G)$ by Nekrashevych in [9]. In some case of topological-full groups of minimal groupoids (see [8]) we have the equality $A(G) = D(G)$ thanks to the simplicity of $D(G)$. In spite of the analogy, it is not clear that the corollary can be obtained as particular case of this result.

Remark 4.6. A lot of groups satisfy the conditions of Corollary 4.4. When \widehat{G} contains $\widehat{\text{IET}}^+$ there is an element of order 3 in $G \setminus \mathfrak{A}_{\text{fin}}$. We recall that $\widehat{\text{IET}}^{\boxtimes}$, PC^+ and PAff^+ are simple (see [2, 5]). Thus these groups satisfy the conditions of Corollary 4.4. The next theorem add PC^{\boxtimes} and PAff^{\boxtimes} to the list of examples.

Theorem 4.7. *The groups PC^{\boxtimes} and PAff^{\boxtimes} are simple.*

Lemma 4.8. *The group $\widehat{\text{IET}}^{\boxtimes}$ is generated by flips (=images of flips from $\widehat{\text{IET}}^{\boxtimes}$).*

Proof. By Proposition 2.4 it is enough to show that $\widehat{\text{IET}}^+$ is generated by flips. For every consecutive, right-open and left-closed subintervals I and J of $[0, 1[$, we define $R_{I,J}$ the map that exchanges I and J . They are elements of $\widehat{\text{IET}}_{\text{rc}}^+$ and they formed a generating set. Then their image $r_{I,J}$ in $\widehat{\text{IET}}^{\boxtimes}$ is a generating set of $\widehat{\text{IET}}^+$. For every right-open and left-closed subinterval I of $[0, 1[$, we define s_I the I -flip. Take I and J be two consecutive, right-open and left-closed subintervals of $[0, 1[$. Then $r_{I,J} = s_I s_J s_{I \cup J}$. \square

Proof of Theorem 4.7 (sketched).

Since the argument in [2] could also be adapted, we only provide a sketch.

We work with elements of PC^{\boxtimes} ; all intervals below are meant modulo finite subsets. Let N be a nontrivial normal subgroup of PC^{\boxtimes} (resp. PAff^{\boxtimes}). Let g be a nontrivial element of N . There exists a subinterval I of $[0, 1[$ such that:

- (1) g is continuous (resp. affine) on I ,
- (2) $g(I) \cap I = \emptyset$ (modulo finite subsets),
- (3) $I \cup g(I) \neq [0, 1[$ (modulo finite subsets).

Let f be the I -flip. If g is affine on I then $h = gfg^{-1}f^{-1}$ is the product of the I -flip with the $g(I)$ -flip. Observe that h is conjugate to a single flip by a suitable element of IET^+ . If g is only continuous then h is still of order 2 and it is conjugate in PC^{\boxtimes} to a single flip. Conjugating by elements of PAff^+ , one obtains that N contains flips of intervals of all possible lengths, and hence contains all flips. Thanks to Lemma 4.8 we know that IET^{\boxtimes} is generated by the set of flips thus N contains IET^{\boxtimes} , in particular N intersects PC^+ (resp. PAff^+) nontrivially. By simplicity of PC^+ (resp. PAff^+) we deduce that N contains $\text{PC}^{\boxtimes} = \langle \text{PC}^+, \text{IET}^{\boxtimes} \rangle$ (resp. $\text{PAff}^{\boxtimes} = \langle \text{PAff}^+, \text{IET}^{\boxtimes} \rangle$). \square

5. ABOUT SOME NORMALIZERS

Here we show that computing normalizers inside $\widehat{\text{PC}^{\boxtimes}}$ and PC^{\boxtimes} may leads to different behaviour. We look the case of PC^+ , IET^+ and $\widehat{\text{PC}}_{\text{rc}}^+$ and $\widehat{\text{IET}}_{\text{rc}}^+$.

Proposition 5.1. *The normalizer of IET^+ in IET^{\boxtimes} is reduced to IET^{\pm} .*

Proof. Let $f \in \text{IET}^+$ and $g \in \text{IET}^{\pm}$. If $g \in \text{IET}^+$ then $gfg^{-1} \in \text{IET}^+$. We assume $g \in \text{IET}^-$ then $gfg^{-1} = (g \circ \mathcal{R}) \circ (\mathcal{R} \circ f \circ \mathcal{R}) \circ (\mathcal{R} \circ g) \in \text{IET}^+$. For the inclusion from left to right, let $g \in \text{IET}^{\boxtimes} \setminus \text{IET}^{\pm}$ and let \widehat{g} be a representative of g in $\widehat{\text{IET}^{\boxtimes}}$. Hence we can find I, J, K, L four right-open and left-closed intervals of the same length such that their image by \widehat{g} are intervals and such that \widehat{g} is order-reversing on I and order-preserving on J, K and L . We define $\widehat{f} \in \widehat{\text{IET}}^+$ as the element which exchanges $\widehat{g}(I)$ with $\widehat{g}(J)$ and $\widehat{g}(K)$ with $\widehat{g}(L)$ while fixing the rest of $[0, 1[$. Then the image f of \widehat{f} in IET^+ is not trivial and $\widehat{g}\widehat{f}\widehat{g}^{-1} \notin \widehat{\text{IET}}^+$ implies $gfg^{-1} \notin \text{IET}^+$. \square

A similar argument stands for the case of PC thus we obtain:

Proposition 5.2. *The normalizer of PC^+ in PC^{\boxtimes} is reduced to PC^{\pm} .* \square

We now take a look to inside $\widehat{\text{PC}^{\boxtimes}}$:

Proposition 5.3. *The normalizer of $\widehat{\text{IET}}_{\text{rc}}^+$ in $\widehat{\text{IET}}^{\boxtimes}$ is $\widehat{\text{IET}}_{\text{rc}}^+$.*

Proof. Let g be an element of $\widehat{\text{IET}}^{\boxtimes}$ which is not the identity. There are two cases:

- (1) If $g \in \widehat{\text{IET}}^+ \setminus \widehat{\text{IET}}_{\text{rc}}^+$ then $g = \sigma g'$ with $\sigma \in \mathfrak{S}_{\text{fin}} \setminus \{\text{Id}\}$ and $g' \in \widehat{\text{IET}}_{\text{rc}}^+$. Then for every $f \in \widehat{\text{IET}}_{\text{rc}}^+$ we have $gfg^{-1} = \sigma g'fg'^{-1}\sigma^{-1}$. Thus it is enough to treat the case of $\mathfrak{S}_{\text{fin}}$. Let us assume $g \in \mathfrak{S}_{\text{fin}}$ then let x in the support of g . There exist two consecutive right-open and left-closed intervals I and J of the same length such that x is the right endpoint of I (and the left endpoint of J). Up to reduce I and J we can assume that I does not intersect the support of g . Then let $f \in \widehat{\text{IET}}_{\text{rc}}^+$ which exchanges I and J while fixing the rest of $[0, 1[$. Then gfg^{-1} exchanges the interior of I with the interior of J but $gfg^{-1}(x)$ is not equal to $f(x)$ because $f(x)$ is the left endpoint of I and I does not intersect the support of g . Then we deduce that gfg^{-1} is not right-continuous on J .
- (2) If $g \in \widehat{\text{IET}}^{\boxtimes} \setminus \widehat{\text{IET}}^+$. Then we can find two consecutive subinterval I and J where g is continuous and order-reversing on $I \cup J$. Let a be the right endpoint of J . Let f be the element in $\widehat{\text{IET}}_{\text{rc}}^+$ which exchanges I and J . Then gfg^{-1} exchanges the interior of $g(J)$ with the interior of $g(I)$. However the left endpoint of $g(J)$ is send by g^{-1} on a which is fixed by f . Then gfg^{-1} fixes the left endpoint of $g(J)$, thus gfg^{-1} is not right-continuous on $g(J)$.

□

A similar argument stands for the case of PC thus we obtain:

Proposition 5.4. *The normalizer of $\widehat{\text{PC}}_{\text{rc}}^+$ in $\widehat{\text{PC}}^{\boxtimes}$ is $\widehat{\text{PC}}_{\text{rc}}^+$.*

□

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