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► **To cite this version:**

Bruno Angles, Tuan Ngo Dac, Floric Tavares-Ribeiro. A class formula for admissible Anderson modules. *Inventiones Mathematicae*, 2022, 229, pp.563-606. 10.1007/s00222-022-01110-3. hal-02490566v2

**HAL Id: hal-02490566**

**<https://hal.science/hal-02490566v2>**

Submitted on 16 Sep 2021

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# A CLASS FORMULA FOR ADMISSIBLE ANDERSON MODULES

BRUNO ANGLÈS, TUAN NGO DAC, AND FLORIC TAVARES RIBEIRO

ABSTRACT. In 2012 Taelman proved a class formula for Drinfeld  $\mathbb{F}_q[t]$ -modules. For an arbitrary coefficient ring  $A$ , several deep but partial results in the direction of a class formula for Drinfeld  $A$ -modules have been obtained. In this paper, using a completely new approach based on the notion of Stark units, we establish the class formula for Drinfeld  $A$ -modules in full generality. Further we extend this class formula for  $A$ -finite or abelian Anderson modules and more generally for admissible Anderson modules.

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## INTRODUCTION

The celebrated class number formula relates many important invariants of a number field (e.g., the ideal class number, the regulator of units) to a special value of its Dedekind zeta function. By analogy between number fields and function fields, Carlitz suggested to transport the classical results to the function field setting in positive characteristic. In 1935, Carlitz [13] succeeded to prove the first instance of the class formula for special Carlitz zeta values. These zeta values are attached to the field of rational functions equipped with the infinity place (i.e., when  $A = \mathbb{F}_q[t]$ ) and intimately related to the so-called *Carlitz module* which is the first example of a Drinfeld module. In 2010, an important breakthrough was due to Taelman who gave an elegant and simple conjectural class formula for Drinfeld  $\mathbb{F}_q[t]$ -modules (see [28], Conjecture 1). It states that the special value of the Goss  $L$ -function attached to a Drinfeld module at  $s = 1$  is the product of a regulator term that is the covolume

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*Date:* June 22, 2021.

2010 *Mathematics Subject Classification.* Primary 11G09; Secondary 11M38, 11R58.

*Key words and phrases.* Drinfeld modules, Anderson modules,  $L$ -series in characteristic  $p$ , class formula.

of the module of units and an algebraic term arising from a certain class module. Shortly after, Taelman [29] used the trace formula of Anderson [4] and proved this conjecture.

Following the fundamental work of Taelman, in recent years, the class formula has been rapidly developed in different directions. Fang [18] and Demeslay [15, 16] proved the class formula for Anderson  $\mathbb{F}_q[t]$ -modules that are generalizations of Drinfeld  $\mathbb{F}_q[t]$ -modules in higher dimensions. For general  $A$ , the class formula for Anderson  $A$ -modules is shown in particular cases and we refer the reader to [7, 8, 14, 31, 33]. Recently, motivated by the works of Böckle-Pink [11] and V. Lafforgue [21], Mornev [24] developed a theory of shtuka cohomology and proved the class formula for Drinfeld  $A$ -modules having good reduction everywhere.

In the present paper, for an arbitrary coefficient ring  $A$ , we prove the class formula for Drinfeld  $A$ -modules in full generality, which generalizes the works of Taelman [29] and Mornev [24]. Our approach is completely different from the aforementioned works and based on the notion of Stark units developed in [7, 9, 10]. Further, we extend our proof to establish the class formula for admissible Anderson modules, in particular for  $A$ -finite or abelian Anderson modules.

Let us give now more precise statements of our results.

Throughout this paper, let  $K$  be a global function field over a finite field  $\mathbb{F}_q$  of characteristic  $p$ , having  $q$  elements ( $\mathbb{F}_q$  is algebraically closed in  $K$ ). We fix a place  $\infty$  of  $K$  of degree  $d_\infty \geq 1$  and denote by  $A$  the ring of elements of  $K$  which are regular outside  $\infty$ . The  $\infty$ -adic completion  $K_\infty$  of  $K$  is equipped with the normalized  $\infty$ -adic valuation  $v_\infty : K_\infty \rightarrow \mathbb{Z} \cup \{+\infty\}$  and has residue field  $\mathbb{F}_\infty$ . The completion  $\mathbb{C}_\infty$  of a fixed algebraic closure  $\overline{K}_\infty$  of  $K_\infty$  comes with a unique valuation extending  $v_\infty$ , which will be denoted by  $v_\infty$ . With the valuation  $v_\infty$ , we can define the following norm on  $\mathbb{C}_\infty$ :  $|x| = q^{-v_\infty(x)}$ . We define the Frobenius map  $\tau : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  as the  $\mathbb{F}_q$ -algebra homomorphism which sends  $x$  to  $x^q$ .

Let  $\iota : K \hookrightarrow L$  be a field homomorphism (necessarily injective) such that  $L/\iota(K)$  is a finite extension and let  $O_L$  be the integral closure of  $\iota(A)$  in  $L$ . We set  $L_\infty := L \otimes_K K_\infty$ . We recall the definition of Anderson modules of generic characteristic and refer the reader to Section 2 for the general definition. An Anderson  $A$ -module (or an Anderson module for short)  $E$  of dimension  $d$  defined over  $O_L$  is an  $\mathbb{F}_q$ -algebra homomorphism  $E : A \rightarrow M_{d \times d}(O_L)\{\tau\}$  such that  $E$  does not take all its values in  $M_{d \times d}(O_L)$  and for all  $a \in A$ , if we write  $E_a = \sum_{k \geq 0} E_{a,k} \tau^k$  with  $E_{a,k} \in M_{d \times d}(O_L)$ , then we require

$$(E_{a,0} - \iota(a)I_d)^d = 0_d.$$

We denote by  $\partial_E : A \rightarrow M_{d \times d}(L)$  the  $\mathbb{F}_q$ -algebra homomorphism given by  $\partial_E(a) = E_{a,0}$  for  $a \in A$ . By definition, a Drinfeld  $A$ -module (of generic characteristic) is just an Anderson module of dimension 1.

Let  $R$  be an  $O_L$ -algebra. We can define two  $A$ -module structures on  $R^d$ : the first one is denoted by  $E(R)$  where  $A$  acts on  $R^d$  via  $E$ , and the second one is denoted by  $\text{Lie}_E(R)$  where  $A$  acts on  $R^d$  via  $\partial_E$ .

There exists a unique power series called the exponential series  $\exp_E \in I_d + M_{d \times d}(L)\{\{\tau\}\}\tau$  such that

$$\exp_E \partial_E(a) = E_a \exp_E, \quad \text{for all } a \in A.$$

The exponential series  $\exp_E$  converges on  $\text{Lie}_E(L_\infty)$ . Taelman [28] introduced the unit module attached to  $E/O_L$  as follows:

$$U(E/O_L) := \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(O_L)\}.$$

By [28],  $U(E/O_L)$  is in fact an  $A$ -lattice in  $\text{Lie}_E(L_\infty)$ . In other words,  $U(E/O_L)$  is a finitely generated  $A$ -module which is discrete and cocompact in  $\text{Lie}_E(L_\infty)$ .

Following the pioneer work of Anderson [3] in which he introduced the analogue of cyclotomic units for the Carlitz module, the authors have introduced and developed the theory of Stark units for Anderson modules. Roughly speaking, they are units coming from the canonical deformation of Drinfeld modules in Tate algebras in the sense of Pellarin [26]. The germs of the concept of Stark units can be found in [5, 6]. The notion has been conceptualized in [10] for Drinfeld modules over  $\mathbb{F}_q[t]$  and then further developed in the general context in [7, 8, 9].

The notion of Stark units is a powerful tool for investigating log-algebraicity identities. A log-algebraicity result consists of the construction of a specific unit in connection with special  $L$ -values of a Drinfeld module. The notion of log-algebraicity was first introduced by Thakur (see [31, 32]). It has been notably developed by Anderson [2] and has become a very active topic in function field arithmetic. We note that the units obtained by log-algebraicity theorems turn out to be always Stark units.

More precisely, the Stark units attached to  $E/O_L$  are units coming from the canonical  $z$ -deformation  $\tilde{E}$  of  $E$ . By definition, letting  $z$  be an indeterminate with the rule  $\tau z = z\tau$ , we denote by  $\tilde{E}$  the homomorphism of  $\mathbb{F}_q(z)$ -algebras  $\tilde{E} : \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} A \rightarrow M_{d \times d}(\mathbb{F}_q(z) \otimes_{\mathbb{F}_q} O_L)\{\tau\}$  such that

$$\tilde{E}_a = \sum_{k \geq 0} z^k E_{a,k} \tau^k, \quad \text{for all } a \in A.$$

We write  $\exp_E = \sum_{i \geq 0} e_i \tau^i$  with  $e_i \in M_{d \times d}(L)$  and set  $\exp_{\tilde{E}} := \sum_{i \geq 0} z^i e_i \tau^i$ . Then one sees easily that  $\exp_{\tilde{E}} \partial_E(a) = \tilde{E}_a \exp_{\tilde{E}}$  for all  $a \in A$ . Further,  $\exp_{\tilde{E}}$  converges on  $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty))$  where  $\mathbb{T}_z(L_\infty)$  is the  $\infty$ -adic completion of  $L_\infty[z]$ . Thus we can define

$$U(\tilde{E}/O_L[z]) := \{x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(O_L[z])\}.$$

Let  $\text{ev} : \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \text{Lie}_E(L_\infty)$  be the evaluation at  $z = 1$ . We define the module of Stark units by

$$U_{St}(E/O_L) := \text{ev}(U(\tilde{E}/O_L[z])).$$

One can prove that  $U_{St}(E/O_L)$  is a sub- $A$ -lattice of  $U(E/O_L)$  in  $\text{Lie}_E(L_\infty)$ . Consequently, we get a regulator term  $[\text{Lie}_E(O_L) : U_{St}(E/O_L)]_A$  which is an invertible  $A$ -lattice in  $K_\infty$ .

Next, we introduce the special  $L$ -value attached to  $E/O_L$ . For any prime  $\mathfrak{P}$  in  $O_L$ , we define the local factor

$$Z_{\mathfrak{P}}(E/O_L) := \frac{\text{Fitt}_A(\text{Lie}_E(O_L/\mathfrak{P}O_L))}{\text{Fitt}_A(E(O_L/\mathfrak{P}O_L))}$$

which is an  $A$ -lattice in  $K_\infty$ . We would like to deal with the infinite product of  $A$ -lattices in  $K_\infty$  given by

$$L(E/O_L) = \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L)$$

where the product runs over the set of primes  $\mathfrak{P}$  in  $O_L$ . The rough idea is to reduce this infinite product of  $A$ -lattices to an infinite product of elements in  $K_\infty$ . The precise idea is given in Definition 4.1. In particular, if  $E$  is an *admissible Anderson module*, then for all except finitely many primes  $\mathfrak{P}$  in  $O_L$ , the local factor  $Z_{\mathfrak{P}}(E/O_L)$  is a principal lattice generated by an element  $z_{\mathfrak{P}} \in K_\infty$ , and we require that the infinite product  $\prod_{\mathfrak{P}} z_{\mathfrak{P}}$  converges in  $K_\infty$ .

One of our main results is the following theorem:

**Theorem A** (Theorem 4.7). Let  $E$  be an admissible Anderson  $A$ -module defined over  $O_L$ . Then we have the class formula

$$[\mathrm{Lie}_E(O_L) : U_{\mathrm{St}}(E/O_L)]_A = L(E/O_L).$$

As we mentioned above, the notion of Stark units is conceived to prove log-algebraicity identities. It is quite unexpected that it can be used to prove the previous class formula.

As a first application, we prove that all Anderson  $A$ -modules are admissible when  $A = \mathbb{F}_q[t]$  and recover (see Theorem 5.2) the class formula for Anderson  $\mathbb{F}_q[t]$ -modules already proved in [18] and [15, 16]. By reduction from a general coefficient ring  $A$  to  $\mathbb{F}_q[t]$ , we show that admissibility is equivalent to an *a priori* weaker condition than that given in Definition 4.1, which is, roughly speaking, that for almost all primes  $\mathfrak{P}$  in  $O_L$ , the local factor of the  $z$ -deformation of  $E$  at  $\mathfrak{P}$  is principal (see Theorem 5.6 for a precise statement).

We then show that any  $A$ -finite or abelian Anderson module is admissible (see Theorems 6.1 and 6.3). In particular, the class formula holds for  $A$ -finite or abelian Anderson modules (see Theorem 6.4). As an immediate consequence, the class formula holds for Drinfeld  $A$ -modules since Drinfeld  $A$ -modules are always  $A$ -finite and abelian. This theorem generalizes the works of Taelman [29] and Mornev [24].

**Theorem B** (Corollary 6.5). Let  $\phi$  be a Drinfeld  $A$ -module defined over  $O_L$ . Then  $L(\phi/O_L)$  is well-defined and is a principal  $A$ -lattice in  $K_\infty$ . Furthermore we have the following class formula

$$[O_L : U_{\mathrm{St}}(\phi/O_L)]_A = [O_L : U(\phi/O_L)]_A \mathrm{Fitt}_A(H(\phi/O_L)) = L(\phi/O_L).$$

**Acknowledgments.** We would like to express our gratitude to the referee for his valuable suggestions and comments which helped improve the paper.

The second author (T. ND.) was partially supported by ANR Grant COLOSS ANR-19-CE40-0015-02.

## 1. BACKGROUND

**1.1. Notation.** We keep the notation in the Introduction. Further, we fix a uniformizer  $\pi$  of  $K_\infty$  and identify  $K_\infty = \mathbb{F}_\infty((\pi))$ .

Letting  $z$  be an indeterminate, we will consider a field  $k$  which is either  $\mathbb{F}_q$  or  $\mathbb{F}_q(z)$ . We endow  $k \otimes_{\mathbb{F}_q} K$  with the topology coming from the discrete topology on  $k$  and the  $v_\infty$ -adic topology on  $K$ , and consider its completion

$$k_\infty := \widehat{k \otimes_{\mathbb{F}_q} K}_\infty.$$

Note that  $k_\infty = K_\infty = \mathbb{F}_\infty((\pi))$  if  $k = \mathbb{F}_q$  and  $k_\infty = \mathbb{F}_\infty(z)((\pi))$  if  $k = \mathbb{F}_q(z)$ .

We still denote by  $v_\infty$  the  $\infty$ -adic Gauss valuation on  $K_\infty(z)$ . We denote by  $\mathbb{T}_z(K_\infty)$  the completion of  $K_\infty[z]$  for  $v_\infty$  and  $\widetilde{K}_\infty$  the completion of  $K_\infty(z)$  for  $v_\infty$ . Then the  $\mathbb{F}_q(z)$ -vector space generated by  $\mathbb{T}_z(K_\infty)$  is dense in  $\widetilde{K}_\infty$ . Observe that  $\mathbb{T}_z(K_\infty) = \mathbb{F}_\infty[z](\langle\pi\rangle)$ .

**1.2. Fitting ideals.** We recall basic facts on the theory of Fitting ideals. The standard references are the appendix to [23] and [17, 22, 25]. We will call here *Fitting ideal* what is usually called in the theory the initial or zero-th Fitting ideal.

We consider a commutative ring  $R$  and a finitely presented  $R$ -module  $M$ . Note that if  $R$  is Noetherian, then this is equivalent to  $M$  being finitely generated. If

$$R^a \longrightarrow R^b \longrightarrow M \longrightarrow 0$$

is a presentation of  $M$ , and if  $X$  is the matrix of the map  $R^a \rightarrow R^b$ , then one defines  $\text{Fitt}_R(M)$  to be the ideal of  $R$  generated by all the  $b \times b$  minors of  $X$  if  $b \leq a$ , and  $\text{Fitt}_R(M) = 0$  if  $b > a$ . This is independent of the choice of presentation for  $M$ . Note that if  $M$  is torsion, one always has  $b \leq a$ .

We list below basic properties of Fitting ideals.

- (1) If  $M \simeq M_1 \times M_2$  is a direct product of finitely presented  $R$ -modules, then  $\text{Fitt}_R(M) = \text{Fitt}_R(M_1) \cdot \text{Fitt}_R(M_2)$ .
- (2) One always has  $\text{Fitt}_R(M) \subset \text{Ann}_R(M)$ , and if  $M$  is generated by  $b$  elements, then  $\text{Ann}_R(M)^b \subset \text{Fitt}_R(M)$ .
- (3) If  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  is exact, then  $\text{Fitt}_R(M_1) \cdot \text{Fitt}_R(M_2) \subset \text{Fitt}_R(M)$ .
- (4) If  $I \subset R$  is an ideal, then  $\text{Fitt}_{R/I}(M/IM)$  is the image of  $\text{Fitt}_R(M)$  in  $R/I$ .
- (5) More generally, if  $R \rightarrow R'$  is a ring homomorphism, then  $\text{Fitt}_{R'}(R' \otimes_R M) = R' \otimes_R \text{Fitt}_R(M)$ .

All the assertions above can be found in [23], except the last one which appears in [17], Corollary 20.5 as an immediate consequence of the right exactness of the tensor product.

In the case where  $R$  is a Dedekind ring, the structure theorem asserts that if  $M$  is a finitely generated and torsion  $R$ -module, then there exist ideals  $I_1, \dots, I_n$  of  $R$  such that  $M \simeq R/I_1 \times \dots \times R/I_n$ . We deduce that  $\text{Fitt}_R(M) = \prod_{i=1}^n I_i$ . Moreover, Fitting ideals are multiplicative in exact sequences. That is, if  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  is exact, then  $\text{Fitt}_R(M_1) \cdot \text{Fitt}_R(M_2) = \text{Fitt}_R(M)$ . This can be deduced from [12], Chapter VII, Section 4 n.5, Proposition 10.

**1.3. Ratio of co-volumes.** We define the ratio of co-volumes by following the ideas presented in [12], Chapter VII, Section 4 n.6. The reader can compare this section with the original construction of [29] or with [7], Section 2.3.

In this section,  $k$  is either  $\mathbb{F}_q$  or  $\mathbb{F}_q(z)$  and we recall that the field  $k_\infty := k \widehat{\otimes}_{\mathbb{F}_q} K_\infty$  can be identified with  $k(\langle\pi\rangle)$ . If  $V$  is a finite dimensional  $k_\infty$ -vector space, then it is endowed with the natural topology coming from  $k_\infty$ . We set  $kA := k \otimes_{\mathbb{F}_q} A$ .

**Definition 1.1.** Let  $V$  be a finite dimensional  $k_\infty$ -vector space. A sub- $kA$ -module  $M$  of  $V$  is a  $kA$ -lattice in  $V$  if  $M$  is discrete in  $V$  and  $M$  generates  $V$  over  $k_\infty$ .

We recall the following standard fact.

**Lemma 1.2.** *Let  $V$  be a finite dimensional  $k_\infty$ -vector space, and let  $M$  be a sub- $kA$ -module of  $V$ . Then  $M$  is discrete in  $V$  if and only if there exist  $e_1, \dots, e_n \in M$  which are  $k_\infty$ -linearly independent and such that the quotient  $M / \bigoplus_{i=1}^n kAe_i$  is a finite dimensional  $k$ -vector space.*

As an immediate consequence, we obtain the following proposition.

**Proposition 1.3.** *Let  $V$  be a finite dimensional  $k_\infty$ -vector space, and let  $M$  be a sub- $kA$ -module of  $V$ . The following assertions are equivalent:*

- (i)  $M$  is a  $kA$ -lattice in  $V$ .
- (ii)  $M$  is discrete in  $V$ , finitely generated as a  $kA$ -module such that

$$\dim_{\text{Frac}(kA)} M \otimes_{kA} \text{Frac}(kA) = \dim_{k_\infty} V.$$

- (iii) There exists a  $k_\infty$ -basis  $(e_1, \dots, e_n)$  of  $V$  such that for all  $1 \leq i \leq n$ ,  $e_i \in M$ , and the quotient  $\frac{M}{\bigoplus_{i=1}^n kAe_i}$  is a finite dimensional  $k$ -vector space.

We now introduce the notion of ratio of co-volumes of lattices.

Let us start with the case where  $V$  is a  $k_\infty$ -vector space of dimension one. We write  $V = k_\infty e$ . Let  $M$  be a  $kA$ -lattice in  $V$ . By Proposition 1.3, there exist a non-zero fractional ideal  $I_M$  of  $kA$  and  $\alpha_M \in k_\infty^\times$  such that  $M = \alpha_M I_M e$ . Letting  $M, M'$  be two  $kA$ -lattices in  $V$ , we define

$$[M' : M]_{kA} := \alpha_M \alpha_{M'}^{-1} I_M I_{M'}^{-1}.$$

We see that  $[M' : M]_{kA}$  is a well-defined  $kA$ -lattice in  $k_\infty$  and does not depend on the choice of  $e$ . Furthermore, if  $M \subset M'$ , then

$$[M' : M]_{kA} = \text{Fitt}_{kA}(M'/M).$$

We also observe that, if  $M_1, M_2, M_3$  are three  $kA$ -lattices in  $V$ , then

$$[M_1 : M_3]_{kA} = [M_1 : M_2]_{kA} [M_2 : M_3]_{kA}.$$

Let us deal now with the general case. We put  $n = \dim_{k_\infty} V$ . Then  $\wedge^n V$  is a  $k_\infty$ -vector space of dimension one, and we denote by  $\wedge^n : V^n \rightarrow \wedge^n V$  the natural  $k_\infty$ -multilinear map. Let  $M, M'$  be two  $kA$ -lattices in  $V$ . We denote by  $N$  and  $N'$  the respective images of  $M^n$  and  $M'^n$  in  $\wedge^n V$  via the map  $\wedge^n$ . By Proposition 1.3, both  $N$  and  $N'$  are  $kA$ -lattices in  $\wedge^n V$ . We set

$$[M' : M]_{kA} := [N' : N]_{kA}.$$

**Remark 1.4.** Let us emphasize that  $[M' : M]_{kA}$  is a  $kA$ -lattice in  $k_\infty$  and not an element of  $k_\infty^\times$  as it is the case in many previous works, notably those of Taelman [27, 28, 29].

Since  $kA$  is a Dedekind domain, it follows that there exist two  $k_\infty$ -bases  $(e_1, \dots, e_n)$  and  $(f_1, \dots, f_n)$  of  $V$  such that

$$M = \bigoplus_{i=1}^{n-1} kAe_i \oplus Ie_n,$$

$$M' = \bigoplus_{i=1}^{n-1} kAf_i \oplus Jf_n,$$

where  $I, J$  are two non-zero fractional ideals of  $kA$ . Let  $\sigma : V \rightarrow V$  be the  $k_\infty$ -linear isomorphism of  $V$  such that  $\sigma(e_i) = f_i$ . Then

$$N = I e_1 \wedge \dots \wedge e_n,$$

$$N' = \det(\sigma) J e_1 \wedge \dots \wedge e_n.$$

Thus we get

$$[M' : M]_{kA} = \det(\sigma)^{-1} IJ^{-1}.$$

Note that the set of  $kA$ -lattices in  $k_\infty$  is naturally an abelian group, and

$$[M : M']_{kA} = [M' : M]_{kA}^{-1}.$$

Finally, if  $M_1, M_2, M_3$  are three  $kA$ -lattices in  $V$ , then

$$[M_1 : M_3]_{kA} = [M_1 : M_2]_{kA} [M_2 : M_3]_{kA}.$$

**Proposition 1.5.** *If  $M$  and  $M'$  are two  $kA$ -lattices of a finite dimensional  $k_\infty$ -vector space  $V$  with  $M' \subset M$ , then  $M/M'$  is torsion and*

$$[M : M']_{kA} = \text{Fitt}_{kA}(M/M').$$

*Proof.* It is clear that a quotient of lattices is torsion. By Proposition 1.3,  $M$  is contained in a lattice  $L$  which is free over  $kA$ . It implies that

$$0 \longrightarrow M \longrightarrow L \longrightarrow L/M \longrightarrow 0$$

is a presentation of  $L/M$ . If  $(e_1, \dots, e_n)$  is a  $kA$ -basis of  $L$ , and  $N$  is the image of  $M^n$  in  $\bigwedge^n V$ , it is equal to  $Ie_1 \wedge \dots \wedge e_n$  where  $I$  is the ideal of  $kA$  generated by the  $n \times n$  determinants of elements of  $M$  written in the basis  $(e_1, \dots, e_n)$ . Thus the definitions of  $[L : M]_{kA}$  and  $\text{Fitt}_{kA}(L/M)$  coincide. Moreover, we have an exact sequence of finitely generated and torsion  $kA$ -modules

$$0 \longrightarrow M/M' \longrightarrow L/M' \longrightarrow L/M \longrightarrow 0,$$

so that

$$\begin{aligned} \text{Fitt}_{kA}(M/M') &= \text{Fitt}_{kA}(L/M') \text{Fitt}_{kA}(L/M)^{-1} \\ &= [L : M']_{kA} [L : M]_{kA}^{-1} \\ &= [M : M']_{kA}. \end{aligned}$$

The proof is finished.  $\square$

We obtain immediately the following corollary.

**Corollary 1.6.** *Let  $V$  be a finite dimensional  $k_\infty$ -vector space and let  $M, M'$  be two  $kA$ -lattices in  $V$ . Let  $\sigma : V \rightarrow V$  be a  $k_\infty$ -linear isomorphism such that  $\sigma(M) \subset M'$ . Then*

$$[M' : M]_{kA} = \det \sigma^{-1} \cdot \text{Fitt}_{kA} \left( \frac{M'}{\sigma(M)} \right).$$

## 2. ANDERSON MODULES

Let  $d \geq 1$  be an integer. For an  $\mathbb{F}_q$ -algebra  $R$ , let  $M = (a_{i,j})_{1 \leq i,j \leq d} \in M_{d \times d}(R)$  be a  $d \times d$  matrix with coefficients in  $R$ . If  $k$  is an integer, we set  $M^{(k)}$  to be the matrix whose  $ij$ -entry is given by  $(a_{i,j})^{(k)} := (a_{i,j}^q)^k$ . We denote by  $M_{d \times d}(R)\{\tau\}$  the non-commutative ring of twisted polynomials in  $\tau$  with coefficients in  $M_{d \times d}(R)$  equipped with the usual addition and the commutation rule

$$\tau^k M = M^{(k)} \tau^k, \quad \text{for all } k \in \mathbb{Z}.$$

Now let  $R$  be a field extension of  $\mathbb{F}_q$  equipped with an  $\mathbb{F}_q$ -algebra homomorphism  $\iota : A \rightarrow R$ . An *Anderson  $A$ -module* (or an Anderson module for short) of dimension  $d$  over  $R$  is an  $\mathbb{F}_q$ -algebra homomorphism  $E : A \rightarrow M_{d \times d}(R)\{\tau\}$  such that

1) For all  $a \in A$ , if we write  $E_a = \sum_{k \geq 0} E_{a,k} \tau^k$  with  $E_{a,k} \in M_{d \times d}(R)$ , then we require

$$(E_{a,0} - \iota(a)I_d)^d = 0_d.$$

2) There exists  $a \in A$  such that  $E_a \neq E_{a,0}$ .

We denote by  $\partial_E : A \rightarrow M_{d \times d}(R)$  the  $\mathbb{F}_q$ -algebra homomorphism given by  $\partial_E(a) = E_{a,0}$  for all  $a \in A$ .

By definition, a *Drinfeld module* over  $R$  is an Anderson module of dimension 1 over  $R$ .

Let  $E$  be an Anderson module of dimension  $d$  over  $R$  as above and let  $B$  be an  $R$ -algebra. We can define two  $A$ -module structures on  $B^d$ . The first one is denoted by  $E(B)$  where  $A$  acts on  $B^d$  via  $E$ :

$$a \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = \sum_{k \geq 0} E_{a,k} \begin{pmatrix} m_1^{q^k} \\ \vdots \\ m_d^{q^k} \end{pmatrix}, \quad \text{for all } a \in A, \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} \in B^d.$$

The second one is denoted by  $\text{Lie}_E(B)$  where  $A$  acts on  $B^d$  via  $\partial_E$ :

$$a \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = E_{a,0} \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix}, \quad \text{for all } a \in A, \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} \in B^d.$$

**2.1. Setup.** From now on,  $L$  will denote a field equipped with a field homomorphism  $\iota : K \hookrightarrow L$  such that  $L/\iota(K)$  is a finite extension. Let  $O_L$  be the integral closure of  $\iota(A)$  in  $L$ . We set  $L_\infty := L \otimes_K K_\infty$ . We view  $L$  as contained in  $L_\infty$  via the diagonal embedding  $x \mapsto x \otimes 1$ . We let the Frobenius map  $\tau$  act on  $L$  or  $L_\infty$  by  $x \mapsto x^q$ .

We now consider an Anderson module  $E$  defined over  $O_L$ , that means  $E_a \in M_{d \times d}(O_L)\{\tau\}$  for all  $a \in A$ . It follows that we have an  $\mathbb{F}_q$ -algebra homomorphism  $\partial_E : A \rightarrow M_{d \times d}(O_L)$  which extends uniquely to a continuous map  $\partial_E : K_\infty \rightarrow M_{d \times d}(L_\infty)$  (see for example [16], Section 2.1.2).

Let  $M_{d \times d}(L)\{\{\tau\}\}$  be the non-commutative ring of twisted power series in  $\tau$  with coefficient in  $M_{d \times d}(L)$ . One can show that there exist unique power series  $\exp_E, \log_E \in I_d + M_{d \times d}(L)\{\{\tau\}\}\tau$  (see [19], Theorem 5.9.6) satisfying the following equalities in  $M_{d \times d}(L)\{\{\tau\}\}$ :

$$\begin{aligned} \exp_E \partial_E(a) &= E_a \exp_E, \quad \text{for all } a \in A, \\ \log_E E_a &= \partial_E(a) \log_E, \quad \text{for all } a \in A, \\ \exp_E \log_E &= \log_E \exp_E = I_d. \end{aligned}$$

These power series  $\exp_E$  and  $\log_E$  are called *the exponential series* and *the logarithm series* attached to  $E$ . The exponential series  $\exp_E$  converges on  $\text{Lie}_E(L_\infty)$  (see for example [19], Section 5.9, and the original article of Anderson [1]). In particular,  $\exp_E$  induces a homomorphism of  $A$ -modules

$$\exp_E : \text{Lie}_E(L_\infty) \rightarrow E(L_\infty).$$

We should stress that the logarithm series  $\log_E$  does not converge everywhere but only on a neighborhood of 0.

Following Taelman [28], we introduce *the unit module* attached to  $E/O_L$  as follows:

$$U(E/O_L) := \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(O_L)\}.$$

By [28], one can easily deduce that  $U(E/O_L)$  is in fact an  $A$ -lattice in  $\text{Lie}_E(L_\infty)$ . Since  $U(E/O_L) = \exp_E^{-1}(E(O_L))$ , we deduce that the exponential series  $\exp_E$  induces an exact sequence of  $A$ -modules

$$0 \rightarrow U(E/O_L) \rightarrow \text{Lie}_E(L_\infty) \rightarrow \frac{E(L_\infty)}{E(O_L)} \rightarrow \frac{E(L_\infty)}{E(O_L) + \exp_E(\text{Lie}_E(L_\infty))} \rightarrow 0.$$

One can show (see [28], Theorem 1) that

$$H(E/O_L) := \frac{E(L_\infty)}{E(O_L) + \exp_E(\text{Lie}_E(L_\infty))}$$

is a finite  $A$ -module which is called *the class module* attached to  $E/O_L$ .

**2.2. The  $z$ -deformation of an Anderson module.** We keep the previous notation. We recall in particular that  $z$  is an indeterminate over  $K_\infty$  and  $\mathbb{T}_z(K_\infty)$  is the Tate algebra in the variable  $z$  with coefficients in  $K_\infty$ . We set

$$\mathbb{T}_z(L_\infty) := L_\infty \otimes_{K_\infty} \mathbb{T}_z(K_\infty).$$

The map  $\tau : L_\infty \rightarrow L_\infty, x \mapsto x^q$ , extends uniquely into a continuous homomorphism of  $\mathbb{F}_q[z]$ -algebras  $\tau : \mathbb{T}_z(L_\infty) \rightarrow \mathbb{T}_z(L_\infty)$ . Let  $\widetilde{L}_\infty := L \otimes_K \widetilde{K}_\infty$  and  $\widetilde{O}_L$  be the  $\mathbb{F}_q(z)$ -vector space generated by  $O_L$  in  $\widetilde{L}_\infty$ .

Similarly, we set  $\widetilde{A} := \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} A$ . Then the  $z$ -deformation of the Anderson module  $E$  denoted by  $\widetilde{E}$  is defined to be the homomorphism of  $\mathbb{F}_q(z)$ -algebras  $\widetilde{E} : \widetilde{A} \rightarrow M_{d \times d}(\widetilde{O}_L)\{\tau\}$  such that

$$\widetilde{E}_a = \sum_{k \geq 0} z^k E_{a,k} \tau^k, \quad \text{for all } a \in A.$$

If  $R$  is an  $O_L[z]$ -module endowed with a semilinear map  $\tau : R \rightarrow R$  with respect to  $\tau : O_L[z] \rightarrow O_L[z]$  (e.g.,  $R = \mathbb{T}_z(L_\infty)$  or  $R = \widetilde{L}_\infty$ ), then we denote by  $\text{Lie}_{\widetilde{E}}(R)$  the  $A[z]$ -module  $R^d$  where  $a$  acts via  $\partial_E(a)$  and by  $\widetilde{E}(R)$  the  $A[z]$ -module  $R^d$  where  $a$  acts via  $\widetilde{E}_a$ . In both cases, if  $R$  is an  $\mathbb{F}_q(z)$ -algebra, then we can extend the structures of  $A[z]$ -modules to structures of  $\widetilde{A}$ -modules by linearity.

**Lemma 2.1.** *There exist unique elements  $\exp_{\widetilde{E}}, \log_{\widetilde{E}} \in I_d + M_{d \times d}(L[z])\{\tau\}$  such that*

- (1)  $\exp_{\widetilde{E}} \partial_E(a) = \widetilde{E}_a \exp_{\widetilde{E}}$  for all  $a \in A$ ,
- (2)  $\partial_E(a) \log_{\widetilde{E}} = \log_{\widetilde{E}} \widetilde{E}_a$  for all  $a \in A$ .

Moreover, if  $\exp_E = \sum_{i \geq 0} e_i \tau^i$ , and  $\log_E = \sum_{i \geq 0} l_i \tau^i$  with  $e_i, l_i \in M_{d \times d}(L)$ , then

$$\exp_{\widetilde{E}} = \sum_{i \geq 0} z^i e_i \tau^i, \quad \text{and} \quad \log_{\widetilde{E}} = \sum_{i \geq 0} z^i l_i \tau^i.$$

In particular,  $\exp_{\widetilde{E}} \circ \log_{\widetilde{E}} = I_d = \log_{\widetilde{E}} \circ \exp_{\widetilde{E}}$  and  $\exp_{\widetilde{E}}$  converges on  $\text{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty))$ , is locally an isometry and induces a homomorphism of  $A[z]$ -modules

$$\exp_{\widetilde{E}} : \text{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \widetilde{E}(\mathbb{T}_z(L_\infty)).$$

*Proof.* Let us fix  $a \in A$ ,  $a \notin \mathbb{F}_q$ ; Then the functional equation  $\exp_{\tilde{E}} \partial_E(a) = \tilde{E}_a \exp_{\tilde{E}}$  induces a triangular system for the coefficients of  $\exp_{\tilde{E}}$ , from which we deduce their uniqueness. From the identity  $\exp_E \partial_E(a) = E_a \exp_E$ , it follows formally that  $\exp_{\tilde{E}} = \sum_{i \geq 0} z^i e_i \tau^i$  satisfies the desired equation. The same argument can be used for  $\log_{\tilde{E}}$ . The other statements are standard consequences of the properties of the coefficients  $e_i$  of  $\exp_E$  and are left to the reader.  $\square$

We define

$$U(\tilde{E}/O_L[z]) := \{x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(O_L[z])\},$$

$$U(\tilde{E}/\tilde{O}_L) := \{x \in \text{Lie}_{\tilde{E}}(\tilde{L}_\infty) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\tilde{O}_L)\}.$$

We can show (see [10], Proposition 1) that  $U(\tilde{E}/O_L[z])$  is a finitely generated  $A[z]$ -module in  $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty))$  and that  $U(\tilde{E}/\tilde{O}_L)$  is an  $A$ -lattice in  $\text{Lie}_{\tilde{E}}(\tilde{L}_\infty)$ . Further,  $U(\tilde{E}/\tilde{O}_L)$  is the  $\mathbb{F}_q(z)$ -vector space generated by  $U(\tilde{E}/O_L[z])$ .

We set

$$H(\tilde{E}/O_L[z]) := \frac{\tilde{E}(\mathbb{T}_z(L_\infty))}{\tilde{E}(O_L[z]) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)))},$$

$$H(\tilde{E}/\tilde{O}_L) := \frac{\tilde{E}(\tilde{L}_\infty)}{\tilde{E}(\tilde{O}_L) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\tilde{L}_\infty))}.$$

**Proposition 2.2.** *The  $A[z]$ -module  $H(\tilde{E}/O_L[z])$  is a finitely generated and torsion  $\mathbb{F}_q[z]$ -module, with no  $z$ -torsion. In particular,*

$$H(\tilde{E}/\tilde{O}_L) = 0.$$

*Proof.* The proof is similar to that of [9], Theorem 3.3 for Anderson  $\mathbb{F}_q[t]$ -modules (see also [10], Proposition 2 and [7], Section 2.2 for Drinfeld modules). We quickly recall the proof for the convenience of the reader.

Since  $\exp_{\tilde{E}}$  is a local isometry,  $\exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)))$  contains a neighborhood  $V$  of 0 in  $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty))$ . Since  $\frac{\tilde{E}(\mathbb{T}_z(L_\infty))}{\tilde{E}(O_L[z]) + V}$  is a finitely generated  $\mathbb{F}_q[z]$ -module, so is  $H(\tilde{E}/O_L[z])$ .

Since  $\exp_{\tilde{E}} \equiv I_d \pmod{z}$ , we see that  $H(\tilde{E}/O_L[z])/zH(\tilde{E}/O_L[z])$  vanishes. The result is then a consequence of the structure theorem for finitely generated  $\mathbb{F}_q[z]$ -modules.  $\square$

Let  $\text{ev} : \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \text{Lie}_E(L_\infty)$  be the evaluation at  $z = 1$ . Observe that  $\text{ev}$  induces a short exact sequence of  $A$ -modules

$$0 \rightarrow (z-1)\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \xrightarrow{\text{ev}} \text{Lie}_E(L_\infty) \rightarrow 0.$$

We define

$$U_{St}(E/O_L) := \text{ev}(U(\tilde{E}/O_L[z])).$$

We observe that  $U_{St}(E/O_L) \subset U(E/O_L)$ . The  $A$ -module  $U_{St}(E/O_L)$  is called *the module of Stark units* attached to  $E/O_L$ . One can prove that  $U_{St}(E/O_L)$  is an  $A$ -lattice in  $\text{Lie}_E(L_\infty)$  and that we have (see [7], Theorem 2.7, [9], proof of Theorem 3.3 or [10], Theorem 1)

$$(2.1) \quad \text{Fitt}_A \left( \frac{U(E/O_L)}{U_{St}(E/O_L)} \right) = \text{Fitt}_A(H(E/O_L)).$$

## 3. LOCAL FACTORS

**3.1. Local factors.** For all non-zero ideals  $\mathcal{I}$  of  $O_L$ , we set

$$Z_{\mathcal{I}}(E/O_L) := \frac{\text{Fitt}_A(\text{Lie}_E(O_L/\mathcal{I}O_L))}{\text{Fitt}_A(E(O_L/\mathcal{I}O_L))} \subset K_{\infty},$$

$$Z_{\mathcal{I}}(\tilde{E}/\tilde{O}_L) := \frac{\text{Fitt}_{\tilde{A}}(\text{Lie}_{\tilde{E}}(\tilde{O}_L/\mathcal{I}\tilde{O}_L))}{\text{Fitt}_{\tilde{A}}(\tilde{E}(\tilde{O}_L/\mathcal{I}\tilde{O}_L))} \subset \tilde{K}_{\infty}.$$

If  $\mathfrak{P}$  is a non-zero prime ideal of  $O_L$ , then  $Z_{\mathfrak{P}}$  will be called the local factor at  $\mathfrak{P}$ . For an ideal  $I$  of  $A$ , we will write  $Z_I$  instead of  $Z_{IO_L}$ .

**Lemma 3.1.** *Let  $I$  and  $J$  be two non-zero coprime ideals of  $O_L$ . Then we have*

$$Z_{IJ}(E/O_L) = Z_I(E/O_L) Z_J(E/O_L),$$

$$Z_{IJ}(\tilde{E}/\tilde{O}_L) = Z_I(\tilde{E}/\tilde{O}_L) Z_J(\tilde{E}/\tilde{O}_L).$$

*Proof.* This is an easy consequence of the Chinese Remainder Theorem.  $\square$

**Lemma 3.2.** *Let  $I$  be a non-zero ideal of  $O_L$  and  $n \geq 1$ . Then we have*

$$Z_I(E/O_L) = Z_{I^n}(E/O_L),$$

$$Z_I(\tilde{E}/\tilde{O}_L) = Z_{I^n}(\tilde{E}/\tilde{O}_L).$$

*Proof.* We give a proof by induction for the first equality since the second one can be proved similarly. The case  $n = 1$  is clear. Next, the short exact sequence

$$0 \rightarrow I^n/I^{n+1} \rightarrow O_L/I^{n+1} \rightarrow O_L/I^n \rightarrow 0$$

gives

$$\text{Fitt}_A(\text{Lie}_E(O_L/I^{n+1})) = \text{Fitt}_A(\text{Lie}_E(I^n/I^{n+1})) \text{Fitt}_A(\text{Lie}_E(O_L/I^n)),$$

$$\text{Fitt}_A(E(O_L/I^{n+1})) = \text{Fitt}_A(E(I^n/I^{n+1})) \text{Fitt}_A(E(O_L/I^n)).$$

For any  $x \in I^n$  and  $a \in A$ , we know that  $E_a(x) \equiv \partial_E(a)x \pmod{I^{qn}}$ . Thus  $E(I^n/I^{n+1}) \simeq \text{Lie}_E(I^n/I^{n+1})$  as  $A$ -modules. It follows that  $\text{Fitt}_A(E(I^n/I^{n+1})) = \text{Fitt}_A(\text{Lie}_E(I^n/I^{n+1}))$ . Thus we get

$$Z_{I^n}(E/O_L) = Z_{I^{n+1}}(E/O_L)$$

as desired.  $\square$

The following proposition is an immediate consequence of Lemmas 3.1 and 3.2.

**Proposition 3.3.** *Let  $I$  be a non-zero ideal in  $O_L$ . Then*

$$Z_I(E/O_L) = \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L),$$

$$Z_I(\tilde{E}/\tilde{O}_L) = \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L),$$

where the product runs over all the primes  $\mathfrak{P}$  in  $O_L$  containing  $I$ .

**3.2. Elimination of local factors.** Let  $I = (\xi)$  be a principal ideal of  $O_L$ . Then there exists a unique Anderson  $A$ -module  $E'$  over  $O_L$  satisfying for all  $a \in A$ ,  $\xi E'_a = E_a \xi$ . For  $a \in A$ , if  $E_a = \sum_{i=0}^r E_{a,i} \tau^i$ , then  $E'_a = \sum_{i=0}^r \xi^{q^i-1} E_{a,i} \tau^i$ . Note that for all  $a \in A$ ,  $\partial_E(a) = \partial_{E'}(a)$ .

**Lemma 3.4.** *Let  $I = (\xi)$  be a principal ideal in  $O_L$  and  $E'$  be the Anderson  $A$ -module over  $O_L$  satisfying  $\xi E'_a = E_a \xi$  for all  $a \in A$ . Then  $IU(E'/O_L)$  is a sub- $A$ -module of  $U(E/O_L)$ . Further we have*

$$Z_I(E/O_L) = [U(E'/O_L) : U(E/O_L)]_A \frac{\text{Fitt}_A(H(E/O_L))}{\text{Fitt}_A(H(E'/O_L))}.$$

*Proof.* We denote by  $d$  the dimension of  $E$ . By the uniqueness of exponential series, we see that

$$\exp_{E'} = \xi^{-1} \exp_E \xi.$$

We deduce that

$$IU(E'/O_L) = \xi U(E'/O_L) = \{x \in \text{Lie}_E(L_\infty), \exp_E(x) \in \xi O_L^d\}.$$

Thus  $IU(E'/O_L)$  is a sub- $A$ -module of  $U(E/O_L)$ . We deduce also that  $\exp_E$  induces an injection  $\exp_E : U(E/O_L)/IU(E'/O_L) \hookrightarrow O_L^d/\xi O_L^d$ . We therefore get a short exact sequence of torsion  $A$ -modules

$$0 \longrightarrow \frac{U(E/O_L)}{IU(E'/O_L)} \longrightarrow E \left( \frac{O_L}{\xi O_L} \right) \longrightarrow \frac{O_L^d + \exp_E(\text{Lie}_E(L_\infty))}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} \longrightarrow 0$$

where  $A$  acts on the third term via  $E$ .

We then get

$$\text{Fitt}_A(E(O_L/IO_L)) = \text{Fitt}_A \left( \frac{U(E/O_L)}{IU(E'/O_L)} \right) \text{Fitt}_A \left( \frac{O_L^d + \exp_E(\text{Lie}_E(L_\infty))}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} \right).$$

Since  $U(E'/O_L)$  et  $\text{Lie}_E(O_L)$  have the same  $A$ -ranks, we obtain

$$\begin{aligned} \text{Fitt}_A \left( \frac{U(E/O_L)}{IU(E'/O_L)} \right) &= [U(E/O_L) : U(E'/O_L)]_A \text{Fitt}_A \left( \frac{U(E'/O_L)}{IU(E'/O_L)} \right) \\ &= [U(E/O_L) : U(E'/O_L)]_A \text{Fitt}_A(\text{Lie}_E(O_L/IO_L)). \end{aligned}$$

Further, we have

$$\begin{aligned} &\text{Fitt}_A \left( \frac{O_L^d + \exp_E(\text{Lie}_E(L_\infty))}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} \right) \\ &= \text{Fitt}_A \left( \frac{L_\infty^d}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} \right) \text{Fitt}_A(H(E/O_L))^{-1}. \end{aligned}$$

Since  $\exp_{E'} = \xi^{-1} \exp_E \xi$  and  $\xi L_\infty^d = L_\infty^d$ , we get isomorphisms of  $A$ -modules

$$\frac{L_\infty^d}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} = \frac{L_\infty^d}{\xi O_L^d + \xi \exp_{E'}(\text{Lie}_E(L_\infty))} \simeq H(E'/O_L).$$

The Lemma follows immediately.  $\square$

**Corollary 3.5.** *Let  $\xi \in O_L$  be a non-zero element. Denote by  $\mathfrak{P}_1, \dots, \mathfrak{P}_n$  the primes in  $O_L$  containing  $\xi$ . Let  $E'$  be the Anderson  $A$ -module over  $O_L$  satisfying for all  $a \in A$ ,  $\xi E'_a = E_a \xi$ . Then*

$$\prod_{i=1}^n Z_{\mathfrak{P}_i}(E/O_L) = [U(E'/O_L) : U(E/O_L)]_A \frac{\text{Fitt}_A(H(E/O_L))}{\text{Fitt}_A(H(E'/O_L))}.$$

*Proof.* This is an immediate consequence of Proposition 3.3 and Lemma 3.4.  $\square$

By similar arguments and Proposition 2.2, we obtain the  $z$ -version of Corollary 3.5:

**Proposition 3.6.** *Let  $\xi \in O_L$  be a non-zero element. Denote by  $\mathfrak{P}_1, \dots, \mathfrak{P}_n$  the primes in  $O_L$  containing  $\xi$ . Let  $E'$  be the Anderson  $A$ -module over  $O_L$  satisfying for all  $a \in A$ ,  $\xi E'_a = E_a \xi$ . Then we have*

$$\prod_{i=1}^n Z_{\mathfrak{P}_i}(\tilde{E}/\tilde{O}_L) = \left[ U(\tilde{E}'/\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}.$$

We also need the following lemma.

**Lemma 3.7.** *Let  $\xi \in O_L$  be a non-zero element and  $E'$  be the Anderson  $A$ -module over  $O_L$  satisfying for all  $a \in A$ ,  $\xi E'_a = E_a \xi$ . Then for all primes  $\mathfrak{P}$  in  $O_L$  containing  $\xi$ , we have  $Z_{\mathfrak{P}}(E'/O_L) = A$  and  $Z_{\mathfrak{P}}(\tilde{E}'/\tilde{O}_L) = \tilde{A}$ . Otherwise, we have  $Z_{\mathfrak{P}}(E'/O_L) = Z_{\mathfrak{P}}(E/O_L)$  and  $Z_{\mathfrak{P}}(\tilde{E}'/\tilde{O}_L) = Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$ .*

*Proof.* Suppose first that  $\mathfrak{P}$  is a prime in  $O_L$  containing  $\xi$ . By construction  $E'_a \in a + \xi O_L[\tau]$  for all  $a \in A$ . Thus for all  $x \in O_L$  and  $a \in A$ , we have  $E'_a(x) \equiv \iota(a)x \pmod{\mathfrak{P}O_L}$ . We deduce that  $Z_{\mathfrak{P}}(E'/O_L) = A$  and  $Z_{\mathfrak{P}}(\tilde{E}'/\tilde{O}_L) = \tilde{A}$ .

We now consider the case where  $\mathfrak{P}$  does not contain  $\xi$ . The desired equality is an immediate consequence of the fact that the multiplication by  $\xi$  is an isomorphism of  $O_L/\mathfrak{P}O_L$ .  $\square$

#### 4. CLASS FORMULA À LA TAE LMAN

We keep the notation of Section 3. Recall that  $L/\iota(K)$  is a finite extension,  $O_L$  denotes the integral closure of  $\iota(A)$  in  $L$  and  $E$  is an Anderson module defined over  $O_L$ . Then  $\tilde{E}$  denotes the  $z$ -deformation of  $E$  and we have defined local factors attached to  $E$  and  $\tilde{E}$  in Section 3.

##### 4.1. Admissible Anderson modules.

**Definition 4.1.** We say that an Anderson module  $E$  defined over  $O_L$  is *admissible* if the following conditions are satisfied:

**Condition (P):** There exists a finite set  $S$  of primes in  $O_L$  such that for all primes  $\mathfrak{P}$  in  $O_L$  with  $\mathfrak{P} \notin S$ , there exists  $x_{\mathfrak{P}} \in A[z]$  such that

$$\text{Fitt}_{\tilde{A}}\left(\tilde{E}(\tilde{O}_L/\mathfrak{P}\tilde{O}_L)\right) = x_{\mathfrak{P}}\tilde{A},$$

and

$$\begin{aligned} \text{Fitt}_A(\text{Lie}_E(O_L/\mathfrak{P})) &= x_{\mathfrak{P}}(0)A, \\ \text{Fitt}_A(E(O_L/\mathfrak{P})) &= x_{\mathfrak{P}}(1)A. \end{aligned}$$

**Condition (C):** The infinite product

$$\prod_{\mathfrak{p} \notin S} \frac{x_{\mathfrak{p}}(0)}{x_{\mathfrak{p}}}$$

converges in  $\widetilde{K}_{\infty}$ .

Let  $E$  be an admissible Anderson module defined over  $O_L$  as above. We set

$$x := \prod_{\mathfrak{p} \notin S} \frac{x_{\mathfrak{p}}(0)}{x_{\mathfrak{p}}} \in \widetilde{K}_{\infty},$$

$$y := \prod_{\mathfrak{p} \notin S} \frac{x_{\mathfrak{p}}(0)}{x_{\mathfrak{p}}(1)} \in K_{\infty}.$$

Then we define

$$L(\widetilde{E}/\widetilde{O}_L) := (x) \prod_{\mathfrak{p} \in S} Z_{\mathfrak{p}}(\widetilde{E}/\widetilde{O}_L) \subset \widetilde{K}_{\infty},$$

$$L(E/O_L) := (y) \prod_{\mathfrak{p} \in S} Z_{\mathfrak{p}}(E/O_L) \subset K_{\infty}.$$

Further, there exists a finite set  $S' \supset S$  of primes in  $O_L$  such that  $\prod_{\mathfrak{p} \notin S'} \frac{x_{\mathfrak{p}}(0)}{x_{\mathfrak{p}}}$  converges in  $\mathbb{T}_z(K_{\infty})$ .

**Remark 4.2.** We shall prove in Theorem 5.6 that the admissibility of an Anderson module is equivalent to a new condition called Condition ( $P'$ ) which is *a priori* weaker than Condition ( $P$ ). Further we show that Condition ( $P'$ ) implies Condition ( $C$ ). We would like to thank the referee for suggesting this improvement.

**4.2. A partial class formula over  $\mathbb{F}_q(z)$  for admissible Anderson modules.** This section is devoted to the proof of the class formula over  $\mathbb{F}_q(z)$  for admissible Anderson modules.

**Theorem 4.3.** *Let  $E$  be an admissible Anderson module defined over  $O_L$ . Then we have the partial class formula*

$$(4.1) \quad \left[ \text{Lie}_{\widetilde{E}}(\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L) \right]_{\widetilde{A}} \subset L(\widetilde{E}/\widetilde{O}_L).$$

The lattice  $\left[ \text{Lie}_{\widetilde{E}}(\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L) \right]_{\widetilde{A}}$  is also called the *regulator* of  $U(\widetilde{E}/\widetilde{O}_L)$ . This partial class formula thus states, roughly speaking, that the regulator of the  $z$ -units is a sub- $\widetilde{A}$ -lattice of the  $L$ -series lattice.

Before giving a proof of the above Theorem, we need some preparatory results. We denote by  $\mathbb{F}_q[z]_{(z)}$  the localization of  $\mathbb{F}_q[z]$  at the ideal  $(z)$ .

**Lemma 4.4.** *Let  $E$  be an Anderson module of dimension  $d$  defined over  $O_L$ . We write  $\exp_E = \sum_{k \geq 0} e_k \tau^k$ . Let  $n \geq 1$  be an integer such that for all  $0 \leq k \leq n-1$ , the coefficient  $e_k$  is integral, i.e.,  $e_k \in M_{d \times d}(O_L)$ . Then any non-zero element in  $U(\widetilde{E}/\widetilde{O}_L)$  can be written in the form  $z^a(\alpha + z^n \beta)$  with  $a \in \mathbb{Z}$ ,  $\alpha \in (\mathbb{F}_q[z]_{(z)} O_L[z])^d$  not divisible by  $z$ , and  $\beta \in (\mathbb{F}_q[z]_{(z)} \mathbb{T}_z(L_{\infty}))^d$ .*

*Conversely, for any  $\alpha \in (\mathbb{F}_q[z]_{(z)} O_L[z])^d$ , there exists  $\beta \in (\mathbb{F}_q[z]_{(z)} \mathbb{T}_z(L_{\infty}))^d$  such that  $(\alpha + z^n \beta) \in \mathbb{F}_q[z]_{(z)} U(\widetilde{E}/O_L[z])$ .*

*Proof.* We write  $\log_E = \sum_{k \geq 0} l_k \tau^k$  with  $l_k \in M_{d \times d}(L)$ . Since  $e_k \in M_{d \times d}(O_L)$  for all  $0 \leq k \leq n-1$ , and  $\log_E \circ \exp_E = I_d$ , it follows by induction on  $k$  that for  $0 \leq k \leq n-1$ , we also have  $l_k \in M_{d \times d}(O_L)$ . Therefore, if  $u \in U(\tilde{E}/O_L[z])$ , since  $\log_{\tilde{E}} \circ \exp_{\tilde{E}} = I_d$ , then we have

$$u \equiv \sum_{k=0}^{n-1} l_k z^k \tau^k(\exp_{\tilde{E}}(u)) \pmod{z^n L_\infty[[z]]^d},$$

and thus

$$u \in O_L[z]^d \pmod{z^n \mathbb{T}_z(L_\infty)^d}.$$

Moreover, if  $u \in (\mathbb{F}_q[z]_{(z)} \mathbb{T}_z(L_\infty))^d$  such that  $zu \in (\mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z]))^d$ , then  $u \in (\mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z]))^d$ . Thus, if  $u \in U(\tilde{E}/O_L[z])$  is non-zero, then there exists  $a \in \mathbb{N}$ ,  $\alpha \in (O_L[z])^d$  not divisible by  $z$ , and  $\beta \in (\mathbb{T}_z(L_\infty))^d$  such that  $u = z^a(\alpha + z^n \beta)$ .

Since  $U(\tilde{E}/\tilde{O}_L)$  is the  $\mathbb{F}_q(z)$ -vector space generated by  $U(\tilde{E}/O_L[z])$ , we deduce that any element  $u \in U(\tilde{E}/\tilde{O}_L)$  can be written in the form  $u = z^a(\alpha + z^n \beta)$  with  $a \in \mathbb{Z}$ ,  $\alpha \in (\mathbb{F}_q[z]_{(z)} O_L[z])^d$  not divisible by  $z$ , and  $\beta \in (\mathbb{F}_q[z]_{(z)} \mathbb{T}_z(L_\infty))^d$ .

Conversely, let  $\alpha \in O_L[z]^d$ . By the assumption on the coefficients  $e_k$  ( $0 \leq k \leq n-1$ ) of  $\exp_E$ , there exist  $x \in O_L[z]^d$  and  $y \in \mathbb{T}_z(L_\infty)^d$  such that  $\exp_{\tilde{E}}(\alpha) = x + z^n y$ . By Proposition 2.2, there is  $f \in \mathbb{F}_q[z]$  not divisible by  $z$ ,  $y_1 \in O_L[z]^d$  and  $y_2 \in \mathbb{T}_z(L_\infty)^d$  such that  $fy = y_1 + \exp_{\tilde{E}}(y_2)$ . Therefore,  $f\alpha - z^n y_2 \in U(\tilde{E}/O_L[z])$ . This concludes the proof.  $\square$

**Proposition 4.5.** *We keep the hypothesis of Lemma 4.4. Then any non-zero element in  $\left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}$  can be written in the form  $z^a(\alpha + z^n \beta)$  with  $a \in \mathbb{Z}$ ,  $\alpha \in \mathbb{F}_q[z]_{(z)} A$  not divisible by  $z$ , and  $\beta \in \mathbb{F}_q[z]_{(z)} \mathbb{T}_z(K_\infty)$ .*

*Proof.* The lattice  $\left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}$  can be computed in the following way. We fix a  $K$ -basis  $\tilde{\mathcal{B}}$  of  $\text{Lie}_E(L)$  in  $\text{Lie}_E(\tilde{O}_L)$ . If we set  $M := A\tilde{\mathcal{B}}$ , then  $M$  is a sub- $A$ -lattice of  $\text{Lie}_E(O_L)$ . It follows that

$$\left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}} = \left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : \tilde{M} \right]_{\tilde{A}} \left[ \tilde{M} : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}.$$

Since

$$\left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : \tilde{M} \right]_{\tilde{A}} = \mathbb{F}_q(z) \left[ \text{Lie}_E(O_L) : M \right]_A,$$

we are reduced to computing  $\left[ \tilde{M} : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}$ , that is, determinants of vectors of elements of  $U(\tilde{E}/\tilde{O}_L)$  in the basis  $\tilde{\mathcal{B}}$ .

First we remark that by Lemma 4.4, the reduction map modulo  $z^n$  induces a surjective homomorphism of  $A$ -modules

$$\mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z]) \rightarrow \text{Lie}_E(O_L) \otimes_{\mathbb{F}_q} \mathbb{F}_q[z]/(z^n).$$

Since  $\text{Lie}_E(O_L) \otimes_{\mathbb{F}_q} \mathbb{F}_q[z]/(z^n)$  is finitely generated and torsion free over the Dedekind domain  $A$ , it is projective. This implies that there exists a section to the above reduction map. By restriction to  $\text{Lie}_E(O_L)$ , we obtain a homomorphism of  $A$ -modules  $\mu : \text{Lie}_E(O_L) \rightarrow \mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z])$  such that for all  $m \in \text{Lie}_E(O_L)$ ,

$$\mu(m) \equiv m \pmod{z^n \mathbb{F}_q[z]_{(z)} \mathbb{T}_z(L_\infty)}.$$

We see that  $\mu$  extends to a  $\widetilde{K_\infty}$ -linear map  $\text{Lie}_{\widetilde{E}}(\widetilde{L_\infty}) \rightarrow \text{Lie}_{\widetilde{E}}(\widetilde{L_\infty})$  so that its determinant  $\det \mu$  is well-defined. For all  $\mathbf{m} = (m_1, \dots, m_s) \in \text{Lie}_E(O_L)^s$  where  $s$  is the rank of  $O_L^d$  over  $A$ , we have  $\det_{\mathcal{B}}(\mu(\mathbf{m})) = \det \mu \det_{\mathcal{B}}(\mathbf{m})$ . We also note that

$$(4.2) \quad \det \mu \in 1 + z^n \mathbb{F}_q[z]_{(z)} \mathbb{T}_z(K_\infty).$$

By induction, we can prove that for all  $b \in \mathbb{N}$ ,  $u \in \mathbb{F}_q[z]_{(z)} U(\widetilde{E}/O_L[z])$ , there exist  $m_0, \dots, m_{b-1} \in \text{Lie}_E(O_L)$  such that

$$u \equiv \sum_{j=0}^{b-1} z^j \mu(m_j) \pmod{z^b}.$$

We now compute sums of determinants. We consider  $r$  vectors  $\mathbf{u}_i = (u_{i,1}, \dots, u_{i,s}) \in U(\widetilde{E}/\widetilde{O}_L)^s$ , and compute  $\sum_{i=1}^r \det_{\mathcal{B}}(\mathbf{u}_i)$ . Since  $U(\widetilde{E}/\widetilde{O}_L)$  is the  $\mathbb{F}_q(z)$ -vector space generated by  $U(\widetilde{E}/O_L[z])$ , we can suppose that all the coefficients  $u_{i,j}$  belong to  $\mathbb{F}_q[z]_{(z)} U(\widetilde{E}/O_L[z])$ . We also suppose that  $\sum_{i=1}^r \det_{\mathcal{B}}(\mathbf{u}_i)$  is not zero and let  $a$  be the biggest integer such that  $z^a$  divides  $\sum_{i=1}^r \det_{\mathcal{B}}(\mathbf{u}_i)$ .

By the previous discussion, for  $1 \leq i \leq r$  and  $1 \leq j \leq s$ , there exist  $m_{i,j,0}, \dots, m_{i,j,a+n-1} \in \text{Lie}_E(O_L)$  such that

$$u_{i,j} \equiv \sum_{k=0}^{a+n-1} z^k \mu(m_{i,j,k}) \pmod{z^{a+n}}.$$

We set  $u'_{i,j} := \sum_{k=0}^{a+n-1} z^k \mu(m_{i,j,k}) \in \mathbb{F}_q[z]_{(z)} U(\widetilde{E}/O_L[z])$  and  $\mathbf{u}'_i := (u'_{i,1}, \dots, u'_{i,s})$ . It follows that

$$(4.3) \quad \sum_{i=1}^r \det_{\mathcal{B}}(\mathbf{u}_i) \equiv \sum_{i=1}^r \det_{\mathcal{B}}(\mathbf{u}'_i) \pmod{z^{a+n}},$$

and

$$\sum_{i=1}^r \det_{\mathcal{B}}(\mathbf{u}'_i) = \sum_{k=0}^{a+n-1} z^k \sum_{i=1}^r \sum_{k_1+\dots+k_s=k} \det_{\mathcal{B}}(\mu(m_{i,1,k_1}, \dots, \mu(m_{i,s,k_s}))) \pmod{z^{a+n}}.$$

For  $0 \leq k < a$ , we put

$$\begin{aligned} \Sigma_k &:= \sum_{i=1}^r \sum_{k_1+\dots+k_s=k} \det_{\mathcal{B}}(\mu(m_{i,1,k_1}, \dots, \mu(m_{i,s,k_s}))) \\ &= (\det \mu) \sum_{i=1}^r \sum_{k_1+\dots+k_s=k} \det_{\mathcal{B}}(m_{i,1,k_1}, \dots, m_{i,s,k_s}). \end{aligned}$$

By (4.2), (4.3) and the assumption that  $z^a$  divides  $\sum_{i=1}^r \det_{\mathcal{B}}(\mathbf{u}_i)$ , we get by induction on  $k$  that for all  $0 \leq k < a$ ,  $\Sigma_k = 0$ .

We obtain

$$\sum_{i=1}^r \det_{\mathcal{B}}(\mathbf{u}'_i) = (\det \mu) \sum_{k=a}^{a+n-1} z^k \sum_{i=1}^r \sum_{k_1+\dots+k_s=k} \det_{\mathcal{B}}(m_{i,1,k_1}, \dots, m_{i,s,k_s}) \pmod{z^{a+n}}.$$

Thus, by (4.2) again, if  $a \leq j < a+n$ , the coefficient of  $z^j$  of this determinant belongs to  $[M : \text{Lie}_E(O_L)]_A$ , as desired.  $\square$

We are now ready to prove the main theorem of this Section.

*Proof of Theorem 4.3.* We divide the proof into two steps.

**Step 1.** We first show that Theorem 4.3 holds for any admissible Anderson module  $E$  defined over  $O_L$  satisfying the following condition: for all primes  $\mathfrak{P}$  in  $O_L$ ,  $Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$  is principal.

In fact, we claim that there exists a sequence  $\xi_n \in O_L \setminus \{0\}$ ,  $n \geq 0$  such that:

- (i) for all  $n \in \mathbb{N}$ ,  $\xi_{n+1}$  divides  $\xi_n$ ,
- (ii) for all primes  $\mathfrak{P}$  in  $O_L$ , there exists  $n \in \mathbb{N}$  such that  $\xi_n \in \mathfrak{P}$ ,
- (iii) for all  $n \in \mathbb{N}$ ,  $E^{[n]}$  denotes the Anderson module such that  $\xi_n E_a^{[n]} = E_a \xi_n$  for all  $a \in A$ , we denote  $\exp_{E^{[n]}} = \sum_{k \geq 0} e_k^{[n]} \tau^k$ , then for all  $0 \leq k \leq n-1$ , the coefficient  $e_k^{[n]}$  is integral, that means  $e_k^{[n]} \in M_{d \times d}(O_L)$ .

Recall that  $\exp_E = \sum_{k \geq 0} e_k \tau^k$ . Then we have

$$\exp_{E^{[n]}} = \xi_n^{-1} \exp_E \xi_n = \sum_{k \geq 0} \xi_n^{q^k - 1} e_k \tau^k,$$

that is,  $e_k^{[n]} = \xi_n^{q^k - 1} e_k$ . Thus we can choose  $\xi_n \in O_L \setminus \{0\}$ ,  $n \geq 0$  such that Conditions (i) and (iii) are satisfied. Condition (ii) is verified by requiring  $\xi_n \in \mathfrak{P}$  for all primes  $\mathfrak{P}$  in  $O_L$  of degree smaller than  $n$ .

We now write

$$\begin{aligned} & \left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}} \\ &= \left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}^{[n]}/\tilde{O}_L) \right]_{\tilde{A}} \left[ U(\tilde{E}^{[n]}/\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}. \end{aligned}$$

By Proposition 3.6, we have

$$\left[ U(\tilde{E}^{[n]}/\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}} = \prod_{\xi_n \in \mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$$

so that

$$\left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}} = \left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}^{[n]}/\tilde{O}_L) \right]_{\tilde{A}} \prod_{\xi_n \in \mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L).$$

Let  $y_n \in \mathbb{T}_z(K_\infty)$  denote the generator of  $\prod_{\xi_n \in \mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$  given by the admissibility condition and  $y$  its limit, which generates  $\prod_{\mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$ . For all non-zero  $u \in \left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}$ , there exists  $x_n \in \left[ \text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}^{[n]}/\tilde{O}_L) \right]_{\tilde{A}}$  such that  $u = x_n y_n$ . Since  $y_n$  converges to  $y$ ,  $x_n$  converges to  $x = \frac{u}{y}$ . By Proposition 4.5,  $x_n$  is of the form  $z^{a_n}(\alpha_n + z^n \beta_n)$  with  $a_n \in \mathbb{Z}$ ,  $\alpha_n \in \mathbb{F}_q[z]_{(z)} A$  not divisible by  $z$ , and  $\beta_n \in \mathbb{F}_q[z]_{(z)} \mathbb{T}_z(K_\infty)$ . It implies that the sequence  $(a_n)_{n \in \mathbb{N}}$  becomes stationary. Moreover, for all  $n_0$ , the sequence  $(x_n \pmod{z^{n_0}})_{n \geq n_0}$  converges to  $x \pmod{z^{n_0}}$  in  $\mathbb{F}_q[z]_{(z)} A / (z^{n_0})$ , which is discrete. Thus, the coefficients of  $x$  must belong to  $A$ , and this implies that the limit  $x$  belongs to  $\tilde{A}$ . Thus we obtain the proof of Theorem 4.3 for  $E$  verifying  $Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$  is principal for all primes  $\mathfrak{P}$  in  $O_L$ .

**Step 2.** We now deal with the general case. Let  $E$  be an admissible Anderson module defined over  $O_L$ . Then there exists a finite set  $S$  of primes in  $O_L$  such that  $Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$  is principal for all primes  $\mathfrak{P}$  in  $O_L$  with  $\mathfrak{P} \notin S$ . We choose a non-zero element  $\xi$  of  $O_L$  such that  $\xi$  belongs to all primes  $\mathfrak{P}$  in  $O_L$  with  $\mathfrak{P} \in S$ .

Let  $E'$  be the Anderson  $A$ -module over  $O_L$  satisfying  $\xi E'_a = E_a \xi$  for all  $a \in A$ . By Lemma 3.7, for any prime  $\mathfrak{P}$  in  $O_L$  containing  $\xi$ , we have  $Z_{\mathfrak{P}}(\widetilde{E}'/\widetilde{O}_L) = \widetilde{A}$ . Otherwise, we have  $Z_{\mathfrak{P}}(\widetilde{E}'/\widetilde{O}_L) = Z_{\mathfrak{P}}(\widetilde{E}/\widetilde{O}_L)$ . In particular,

$$L(\widetilde{E}/\widetilde{O}_L) = L(\widetilde{E}'/\widetilde{O}_L) \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L),$$

where the product runs over the primes  $\mathfrak{P}$  in  $O_L$  containing  $\xi$ .

Further,  $E'$  is admissible and for all primes  $\mathfrak{P}$  in  $O_L$ ,  $Z_{\mathfrak{P}}(\widetilde{E}'/\widetilde{O}_L)$  is principal. By Step 1, Theorem 4.3 holds for  $E'$  and we get

$$\left[ \text{Lie}_{\widetilde{E}'}(\widetilde{O}_L) : U(\widetilde{E}'/\widetilde{O}_L) \right]_{\widetilde{A}} \subset L(\widetilde{E}'/\widetilde{O}_L).$$

Therefore, by Proposition 3.6, we deduce

$$\begin{aligned} \left[ \text{Lie}_{\widetilde{E}}(\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L) \right]_{\widetilde{A}} &= \left[ \text{Lie}_{\widetilde{E}'}(\widetilde{O}_L) : U(\widetilde{E}'/\widetilde{O}_L) \right]_{\widetilde{A}} \left[ U(\widetilde{E}'/\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L) \right]_{\widetilde{A}} \\ &= \left[ \text{Lie}_{\widetilde{E}'}(\widetilde{O}_L) : U(\widetilde{E}'/\widetilde{O}_L) \right]_{\widetilde{A}} \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L) \\ &\subset L(\widetilde{E}'/\widetilde{O}_L) \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L) = L(\widetilde{E}/\widetilde{O}_L) \end{aligned}$$

as required.  $\square$

**4.3. A class formula for admissible Anderson modules.** We now apply the class formula over  $\mathbb{F}_q(z)$  to deduce a class formula for admissible Anderson modules.

We will need the following lemma due to Taelman:

**Lemma 4.6.** *Let  $t \in A \setminus \mathbb{F}_q$ , then there exists  $\alpha_t \in \mathbb{F}_q[t]$  with  $v_\infty(\alpha_t) = 0$  such that*

$$\left[ \text{Lie}_E(O_L) : U(E/O_L) \right]_{\mathbb{F}_q[t]} \text{Fitt}_{\mathbb{F}_q[t]}(H(E/O_L)) = \alpha_t \mathbb{F}_q[t].$$

*In particular, we have*

$$\left[ \text{Lie}_E(O_L) : U_{\text{St}}(E/O_L) \right]_{\mathbb{F}_q[t]} = \alpha_t \mathbb{F}_q[t].$$

*Proof.* By restriction, we can consider  $E$  as an Anderson  $\mathbb{F}_q[t]$ -module. The first part is proved in [28], Theorem 2 in the case of the Carlitz module. However, the same arguments apply for all Anderson  $\mathbb{F}_q[t]$ -modules. Combining with (2.1), we obtain the second part.  $\square$

**Theorem 4.7.** *Let  $E$  be an admissible Anderson module defined over  $O_L$ . Then we have the class formula*

$$\left[ \text{Lie}_E(O_L) : U_{\text{St}}(E/O_L) \right]_A = L(E/O_L).$$

*Proof.* By Corollary 3.5 and Lemma 3.7, we can suppose that for all primes  $\mathfrak{P}$  in  $O_L$ , there exists an element  $x_{\mathfrak{P}} \in A[z]$  such that  $\frac{x_{\mathfrak{P}}(0)}{x_{\mathfrak{P}}(1)}$  generates  $Z_{\mathfrak{P}}(E/O_L)$  and  $\frac{x_{\mathfrak{P}}(0)}{x_{\mathfrak{P}}}$  generates  $Z_{\mathfrak{P}}(\widetilde{E}/\widetilde{O}_L)$ . Further, we have

$$x(z) = \prod_{\mathfrak{P}} \frac{x_{\mathfrak{P}}(0)}{x_{\mathfrak{P}}} \in \mathbb{T}_z(K_\infty).$$

First we show that we have an inclusion

$$(4.4) \quad \left[ \text{Lie}_E(O_L) : U_{\text{St}}(E/O_L) \right]_A \subset L(E/O_L).$$

We now fix a basis  $\mathcal{B}$  of  $\mathrm{Lie}_E(L_\infty)$  over  $K_\infty$ . This is still a basis of  $\mathrm{Lie}_{\widetilde{E}}(\widetilde{L}_\infty)$  over  $\widetilde{K}_\infty$ , and a basis of  $\mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty))$  over  $\mathbb{T}_z(K_\infty)$ . Therefore we can use this basis to compute both  $[\mathrm{Lie}_{\widetilde{E}}(\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L)]_{\widetilde{A}}$  and  $[\mathrm{Lie}_E(O_L) : U(E/O_L)]_A$ . To simplify the notation, we identify here  $A$  and  $\iota(A)$ . We obtain

$$\left[ \widetilde{A}\mathcal{B} : \mathrm{Lie}_E(\widetilde{O}_L) \right]_{\widetilde{A}} = \mathbb{F}_q(z) [\mathcal{A}\mathcal{B} : \mathrm{Lie}_E(O_L)]_A.$$

Recall that  $U(\widetilde{E}/\widetilde{O}_L) = \mathbb{F}_q(z)U(\widetilde{E}/O_L[z])$ . It implies that if we consider a determinant of elements of  $U_{\mathrm{St}}(E/O_L)$  in the basis  $\mathcal{B}$ , it comes by evaluation at  $z = 1$  from the determinant of elements in  $U(\widetilde{E}/O_L[z])$  and thus belong to  $[\mathcal{A}\mathcal{B} : \mathrm{Lie}_E(O_L)]_A^{-1}x(1)$ . It follows immediately that  $[\mathrm{Lie}_E(O_L) : U_{\mathrm{St}}(E/O_L)]_A \subset L(E/O_L)$ .

Next we claim that the inclusion (4.4) is in fact an equality. We write  $L(E/O_L) = uA$  where  $u$  is a unit in  $K_\infty^\times$ . By (4.4), we have

$$\frac{1}{u} [\mathrm{Lie}_E(O_L) : U_{\mathrm{St}}(E/O_L)]_A \subset A.$$

Thus there exists an integer  $h \geq 1$  such that

$$\frac{1}{u^h} [\mathrm{Lie}_E(O_L) : U_{\mathrm{St}}(E/O_L)]_A^h = \alpha A, \quad \alpha \in A \setminus \{0\}.$$

Let  $t \in A \setminus \mathbb{F}_q$ . By Lemma 4.6, we have

$$[\mathrm{Lie}_E(O_L) : U_{\mathrm{St}}(E/O_L)]_{\mathbb{F}_q[t]} = \alpha_t \mathbb{F}_q[t], \quad \text{with } v_\infty(\alpha_t) = 0.$$

It follows that  $N_{K/\mathbb{F}_q(t)}(\alpha) \in \mathbb{F}_q^\times$ . Thus  $\alpha \in \mathbb{F}_q^\times$  and we get

$$[\mathrm{Lie}_E(O_L) : U_{\mathrm{St}}(E/O_L)]_A = L(E/O_L)$$

as required.  $\square$

**Remark 4.8.** We can generalize [28], Theorem 2 to the  $\mathbb{F}_q(z)$ -context. In fact, the proof of this theorem carries over without modification. By the same arguments as above, we deduce that the inclusion (4.1) is also an equality, i.e., the class formula over  $\mathbb{F}_q(z)$  holds for admissible Anderson modules.

## 5. ADMISSIBLE ANDERSON MODULES

In this section, we first prove that all Anderson modules are admissible if  $A$  is a polynomial ring. Then we use this result to show that the admissibility criterion is equivalent to an *a priori* weaker condition.

**5.1. Anderson  $\mathbb{F}_q[t]$ -modules.** The proof of the following lemma is inspired by that of [30], Lemma 7.1.

**Lemma 5.1.** *Let  $\mathbb{F}/\mathbb{F}_q$  be a finite extension of degree  $m$  and let  $d \geq 1$  be an integer. Let  $f \in M_{d \times d}(\mathbb{F}[z])\{\tau\}$  such that*

$$f = \sum_{i=0}^r A_i z^i \tau^i, \quad A_i \in M_{d \times d}(\mathbb{F}), r \geq 1.$$

*We put  $\chi(X, z) = \det_{\mathbb{F}_q(z)}(X \mathrm{Id} - f|_{\mathbb{F}(z)^d}) \in \mathbb{F}_q[X, z]$ . Then*

- (i)  $\chi(X, z) \in \mathbb{F}_q[X, z^m]$ ,
- (ii)  $\deg_X(\chi(X, z) - \chi(X, 0)) \leq \deg_X(\chi(X, z)) - \frac{m}{r}$ .

*Proof.* Let  $V = \mathbb{F}(z)^d$  and  $W = \mathbb{F}(z) \otimes_{\mathbb{F}_q(z)} V$ . We let  $\tau$  act on  $W$  via  $1 \otimes \tau$ , and  $f$  act on  $W$  via  $1 \otimes f$ . Thus we get

$$\chi(X, z) = \det_{\mathbb{F}(z)}(X\text{Id} - f|_W).$$

Let us write  $V^m = V_1 \oplus \dots \oplus V_m$ , where  $V_i = V$  as an  $\mathbb{F}_q(z)$ -vector space and  $\mathbb{F}$  acts on  $V_i$  by

$$l.v = l^{q^{i-1}}v, \quad \text{for all } l \in \mathbb{F}, v \in V_i.$$

We also denote by  $\sigma : V^m \rightarrow V^m$  the map given by

$$(5.1) \quad \sigma(v_1, \dots, v_m) := (\tau(v_m), \tau(v_1), \dots, \tau(v_{m-1})).$$

Note that  $\sigma$  is an isomorphism of  $\mathbb{F}(z)$ -vector spaces. Let  $g : V^m \rightarrow V^m$  be the map defined by

$$g(v_1, \dots, v_m) := \sum_{i=0}^r A_i z^i \sigma^i(v_1, \dots, v_m),$$

where for  $A \in M_{d \times d}(\mathbb{F})$ ,  $(v_1, \dots, v_m) \in V^m$ , we put  $A(v_1, \dots, v_m) = (Av_1, \dots, Av_m)$ . We observe that  $g$  is  $\mathbb{F}(z)$ -linear.

Now, let  $\psi : W \rightarrow V^m$  be the  $\mathbb{F}_q(z)$ -linear map defined by

$$\psi(l \otimes s) := (ls, l^q s, \dots, l^{q^{m-1}} s), \quad \text{for all } l \in \mathbb{F}, v \in V.$$

Then  $\psi$  is an isomorphism of  $\mathbb{F}(z)$ -vector spaces. Further, we have

$$\begin{aligned} \psi \circ \tau &= \sigma \circ \psi, \\ \psi \circ f &= g \circ \psi. \end{aligned}$$

We deduce

$$\chi(X, z) = \det_{\mathbb{F}(z)}(X\text{Id} - g|_{V^m}).$$

Let  $(e_1, \dots, e_d)$  be the canonical  $\mathbb{F}(z)$ -basis of  $V$ . For  $1 \leq i \leq d, 1 \leq j \leq m$ , let  $e_{ij}$  be the following element of  $V^m$ :  $(0, \dots, 0, e_i, 0, \dots, 0)$  where  $e_i$  appears in the  $j$ th position. Then  $\mathcal{B} := (e_{11}, \dots, e_{d1}, \dots, e_{1m}, \dots, e_{dm})$  is an  $\mathbb{F}(z)$ -basis of  $V^m$ . We observe that

$$\sigma^k(e_{ij}) = e_{i(j+k)},$$

so that

$$g(e_{ij}) = \sum_{k=0}^r z^k A_k e_{i(j+k)}.$$

Thus the matrix of  $g$  in the basis  $\mathcal{B}$  is given by the block matrix:

$$\left( \sum_{k \equiv i-j \pmod{m}} z^k A_k^{(1-i)} \right)_{1 \leq i \leq m, 1 \leq j \leq m}$$

where for  $B = (b_{ij}) \in M_{d \times d}(\mathbb{F})$  and  $h \in \mathbb{Z}$ , we recall that  $B^{(h)} = (b_{ij}^q)$ . Thus, by expanding the determinant of the matrix of  $X\text{Id} - g$  in the basis  $\mathcal{B}$ , we deduce that  $\chi(X, z) \in \mathbb{F}_q[X, z^m]$ . Note also that the degree in  $z$  of the coefficients of the matrix of  $X\text{Id} - g$  in the basis  $\mathcal{B}$  are bounded by  $r$ . Thus, since  $\chi(X, z) \in \mathbb{F}_q[X, z^m]$ , we get

$$\deg_X(\chi(X, z) - \chi(X, 0)) \leq \deg_X(\chi(X, z)) - \frac{m}{r}.$$

□

**Theorem 5.2.** *We suppose that  $A = \mathbb{F}_q[t]$ . Let  $E$  be an Anderson  $A$ -module defined over  $O_L$ . Then  $E$  is admissible.*

*Proof.* Since  $A = \mathbb{F}_q[t]$ , we have  $\tilde{A} = \mathbb{F}_q(z)[t]$ . Let  $\mathfrak{P}$  be a non-zero prime in  $O_L$ . We put  $\mathbb{F}_{\mathfrak{P}} = O_L/\mathfrak{P}$ . Then

$$\text{Fitt}_{\tilde{A}}\left(\tilde{E}(\mathbb{F}_{\mathfrak{P}}(z))\right) = \det_{\mathbb{F}_q(z)[X]} \left( X\text{Id} - E_t|_{\mathbb{F}_{\mathfrak{P}}(z)[X]} \right) \Big|_{X=t} \tilde{A}.$$

Since  $\mathbb{F}_{\mathfrak{P}}[z, X]$  is a finitely generated and free  $\mathbb{F}_q[z, X]$ -module, we deduce that

$$\det_{\mathbb{F}_q(z)[X]} \left( X\text{Id} - E_t|_{\mathbb{F}_{\mathfrak{P}}(z)[X]} \right) \Big|_{X=t} \in \mathbb{F}_q[z, X].$$

It follows that

$$\text{Fitt}_{\tilde{A}}\left(\tilde{E}(\mathbb{F}_{\mathfrak{P}}(z))\right) = x_{\mathfrak{P}}\tilde{A},$$

where  $x_{\mathfrak{P}}$  is a monic polynomial in  $t$  with coefficients in  $\mathbb{F}_q[z]$ . We observe that

$$\begin{aligned} \text{Fitt}_A(\text{Lie}_E(\mathbb{F}_{\mathfrak{P}})) &= x_{\mathfrak{P}}(0)A, \\ \text{Fitt}_A(E(\mathbb{F}_{\mathfrak{P}})) &= x_{\mathfrak{P}}(1)A. \end{aligned}$$

It remains to prove that the infinite product  $\prod_{\mathfrak{P}} \frac{x_{\mathfrak{P}}(0)}{x_{\mathfrak{P}}}$  converges in  $\mathbb{F}_q(z)$  ( $(\frac{1}{t})$ ) which is a consequence of Lemma 5.1, Part (ii).  $\square$

As an immediate consequence of the above theorem combined with Theorem 4.7, we recover the class formula for Anderson  $\mathbb{F}_q[t]$ -modules which was already proved in [18] and [15, 16].

**5.2. Reduction from  $A$  to  $\mathbb{F}_q[t]$ .** In this section we use the result of the previous section on Anderson  $\mathbb{F}_q[t]$ -modules to show that the admissibility conditions (P) and (C) from Definition 4.1 can be weakened by a new condition close to (P). In particular, under this condition, the convergence (i.e., Condition (C)) is automatic, which was suggested by the referee.

To do so, we start by proving the following technical result which, as in the proof of Lemma 5.1, uses an idea from [30], Lemma 7.1. For  $m \geq 1$ , we will denote by  $A\mathbb{F}_q(z^m)$  the  $\mathbb{F}_q(z^m)$ -submodule of  $A$  generated by  $A$ .

**Proposition 5.3.** *Let  $\mathbb{F}/\mathbb{F}_q$  be a finite extension of degree  $m$  and let  $E/\mathbb{F}$  be an Anderson  $A$ -module of dimension  $d$ . Let  $\tilde{E} : A \rightarrow M_{d \times d}(\mathbb{F}[z])\{\tau\}$  be the canonical  $z$ -deformation of  $E$ . Then, there exists an ideal  $I$  of  $A\mathbb{F}_q(z^m)$  such that*

$$\text{Fitt}_{\tilde{A}}\left(\tilde{E}(\mathbb{F}(z))\right) = I\tilde{A}.$$

*Proof.* Let  $V = \mathbb{F}[z]^d$  which is an  $A[z]$ -module via  $\tilde{E}$ . Let  $A_{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{F}_q} A$ . We set

$$W := \mathbb{F} \otimes_{\mathbb{F}_q} V,$$

which is an  $A_{\mathbb{F}}[z]$ -module. As in the proof of Lemma 5.1, we have an isomorphism of  $\mathbb{F}$ -vector spaces  $\psi : W \rightarrow V^m$  and an automorphism  $\sigma$  of  $V^m$  defined by (5.1) satisfying  $\psi \circ (1 \otimes \tau) = \sigma \circ \psi$ . If  $a \in A$  and  $\tilde{E}_a = \sum_{i=0}^{r_a} A_i z^i \tau^i$ , then we define  $F_a : V^m \rightarrow V^m$  the homomorphism of  $\mathbb{F}[z]$ -algebras given by

$$F_a(v_1, \dots, v_m) := \sum_{i=0}^{r_a} A_i z^i \sigma^i(v_1, \dots, v_m).$$

We deduce that  $\psi$  is an isomorphism of  $A_{\mathbb{F}}[z]$ -modules.

We now set

$$U := \frac{V\{\tau\}}{(\tau^m - 1)V\{\tau\}},$$

which is a free  $\mathbb{F}_q[z]$ -module of rank  $m^2d$ . We view  $U$  as an  $\mathbb{F}$ -vector space via

$$l \cdot \sum_{i=1}^m v_i \tau^{i-1} := \sum_{i=1}^m v_i l^{q^{i-1}} \tau^{i-1}, \quad \text{for all } l \in \mathbb{F}, v_1, \dots, v_m \in V.$$

We also let  $A$  act on  $U$  via  $\mathbb{F}[z]$ -linear maps given by

$$a \cdot u \equiv \tilde{E}_a u \pmod{\tau^m - 1}, \quad \text{for all } a \in A, u \in U.$$

Thus, we have an isomorphism of  $A_{\mathbb{F}}[z]$ -modules

$$\begin{aligned} \rho : V^m &\rightarrow U \\ (v_1, \dots, v_m) &\mapsto \sum_{i=1}^m v_i \tau^{i-1}. \end{aligned}$$

Thus

$$\text{Fitt}_{A_{\mathbb{F}}[z]} W = \text{Fitt}_{A_{\mathbb{F}}[z]} V^m = \text{Fitt}_{A_{\mathbb{F}}[z]} U.$$

Let  $(e_1, \dots, e_d)$  be an  $\mathbb{F}[z]$ -basis of  $V = \mathbb{F}[z]^d$ . Then

$$U = \sum_{i=1}^d \sum_{j=0}^{m-1} A_{\mathbb{F}}[z] e_i \tau^j.$$

Thus we get a surjective homomorphism of  $A_{\mathbb{F}}[z]$ -modules

$$\pi : A_{\mathbb{F}}[z]^{md} \rightarrow U.$$

Let  $R = \text{Ker} \pi$ . We see that

$$R = \{(\alpha_{1,0}, \dots, \alpha_{d,0}, \dots, \alpha_{1,m-1}, \dots, \alpha_{d,m-1}) \in A_{\mathbb{F}}[z]^{md} \mid \sum_{i,j} \alpha_{i,j} \cdot e_i \tau^j = 0\}.$$

By definition of Fitting ideals (see Section 1.2),

$$\text{Fitt}_{A_{\mathbb{F}}[z]} U = \sum_{a_1, \dots, a_{md} \in R} \det(a_1, \dots, a_{md}) A_{\mathbb{F}}[z].$$

We now set

$$R_0 := \{(\alpha_{1,0}, \dots, \alpha_{d,0}, \dots, \alpha_{1,m-1}, \dots, \alpha_{d,m-1}) \in R \mid \exists 0 \leq j_0 \leq m-1, \alpha_{i,j} \in z^{j+j_0} A_{\mathbb{F}}[z^m]\}.$$

We claim that

$$R = R_0 A_{\mathbb{F}}[z].$$

In fact, we work in  $V\{\tau\}$  and view  $V\{\tau\}$  as an  $\mathbb{F}$ -vector space via

$$l \cdot x := xl, \quad \text{for all } l \in \mathbb{F}, x \in V\{\tau\},$$

and as an  $A$ -module via

$$a \cdot x := \tilde{E}_a x, \quad \text{for all } a \in A, x \in V\{\tau\}, .$$

We naturally extend  $\tilde{E}$  to a homomorphism of  $\mathbb{F}$ -algebras  $A_{\mathbb{F}} \rightarrow \mathbb{F} \otimes_{\mathbb{F}_q} M_{d \times d}(\mathbb{F}[z])\{\tau\}$ .

For  $A = \sum_i l_i \otimes A_i \in \mathbb{F} \otimes_{\mathbb{F}_q} M_{d \times d}(\mathbb{F})$ , we set  $A^{(k)} := \sum_i l_i^q \otimes A_i$ . Let  $(\alpha_{1,0}, \dots, \alpha_{d,0}, \dots, \alpha_{1,m-1}, \dots, \alpha_{d,m-1}) \in R$ . We write

$$\alpha_{ij} = \sum_{k=0}^{s_{ij}} \alpha_{ijk} z^k, \quad \alpha_{ijk} \in A_{\mathbb{F}},$$

and

$$\tilde{E}_{\alpha_{ijk}} = \sum_{l=0}^{r_{ijk}} A_{ijkl} z^l \tau^l, \quad A_{ijkl} \in \mathbb{F} \otimes_{\mathbb{F}_q} M_{d \times d}(\mathbb{F}).$$

We have

$$\sum_{i,j} \alpha_{ij} \cdot e_i \tau^j = \sum_{i,j,k,l} A_{ijkl}^{(j)} z^{k+l} e_i \tau^{j+l} \in (\tau^m - 1)V\{\tau\}.$$

Thus, for  $0 \leq l_0 \leq m-1$ , we get

$$\sum_{\substack{i,j,k \\ l \equiv l_0 - j \pmod{m}}} A_{ijkl}^{(j)} z^{k+l} e_i = 0.$$

It implies that for  $0 \leq k_0 \leq m-1$ ,

$$\sum_{\substack{i,j \\ k \equiv k_0 + j \pmod{m}, \\ l \equiv l_0 - j \pmod{m}}} A_{ijkl}^{(j)} e_i = 0.$$

For  $0 \leq k_0 \leq m-1$ , and for any integers  $i, j$ , we put

$$\alpha_{ij}^{k_0} := \sum_{k \equiv k_0 + j \pmod{m}} \alpha_{ijk} z^k.$$

Then

$$\alpha_{ij} = \sum_{k=0}^{m-1} \alpha_{ij}^{k_0}.$$

Now, for  $k_0$  fixed, by our previous computations we get that  $\sum_{i,j} \alpha_{ij}^{k_0} \cdot e_i \tau^j$  is equal to

$$\sum_{\substack{i,j,l \\ k \equiv k_0 - j \pmod{m}}} A_{ijkl}^{(j)} z^{k+l} e_i \tau^j = \sum_{l_0=0}^{m-1} \sum_{\substack{i,j \\ l \equiv l_0 - j \pmod{m}, \\ k \equiv k_0 - j \pmod{m}}} A_{ijkl}^{(j)} z^{k+l} e_i \tau^j \in (\tau^m - 1)V\{\tau\}.$$

Thus  $(\alpha_{ij}^{k_0}) \in R_0$ . We conclude that  $R = R_0 A_{\mathbb{F}}[z]$ .

Now let  $(a_1, \dots, a_{md}) \in R_0$  then there exist integers  $j_1, \dots, j_{md}$  such that the matrix whose lines are  $(z^{j_1} a_1, \dots, z^{j_{md}} a_{md})$  is a block matrix of the form  $(A_{ij})_{1 \leq i, j \leq m}$  where each  $A_{ij} \in M_{d \times d}(A_{\mathbb{F}}[z])$  is such that the degrees of the monomials of their entries as polynomials in  $z$  are congruent to  $j - i \pmod{m}$ . Thus, as in the proof of Lemma 5.1, we get

$$z^{j_1 + \dots + j_{md}} \det(a_1, \dots, a_{md}) \in A_{\mathbb{F}}[z^m].$$

Let  $\tilde{A}_{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{F}_q} \tilde{A}$  and  $A_{\mathbb{F}} \mathbb{F}_q(z^m)$  be the  $\mathbb{F}_q(z^m)$ -submodule of  $\tilde{A}_{\mathbb{F}}$  generated by  $A_{\mathbb{F}}$ . We deduce from the previous computations that there exists an ideal  $J$  of  $A_{\mathbb{F}} \mathbb{F}_q(z^m)$  such that

$$\text{Fitt}_{\tilde{A}}(\tilde{E}(\mathbb{F}(z))) \tilde{A}_{\mathbb{F}} = J \tilde{A}_{\mathbb{F}}.$$

Observe that if  $I$  is an ideal of  $\tilde{A}$  and  $I_{\mathbb{F}} = I\tilde{A}_{\mathbb{F}} \subset \tilde{A}_{\mathbb{F}}$  then if we choose an  $\mathbb{F}_q$ -basis of  $\mathbb{F}$ , we can recover the elements of  $I$  as the coefficients in the chosen basis of the elements of  $I_{\mathbb{F}}$ . As a consequence, we obtain that there is an ideal  $I$  of  $A\mathbb{F}_q(z^m)$  such that

$$\text{Fitt}_{\tilde{A}}\left(\tilde{E}(\mathbb{F}(z))\right) = I\tilde{A}.$$

□

**Lemma 5.4.** *The natural map  $\text{Pic}(A) \rightarrow \text{Pic}(\tilde{A})$  is injective, and we have*

$$\text{Pic}(\tilde{A})_{\text{tors}} = \text{Pic}(A).$$

*In particular, for any integer  $m \geq 1$ , the natural map  $\text{Pic}(A\mathbb{F}_q(z^m)) \rightarrow \text{Pic}(\tilde{A})$  is injective.*

*Proof.* Let  $I$  be a non-zero ideal of  $A$  such that  $I\tilde{A}$  is principal. Then there exists  $F(z) \in A[z]$  without roots in  $\mathbb{F}_q$  such that

$$I\tilde{A} = F(z)\tilde{A}.$$

Let  $n \geq 1$  be any integer such that  $I^n$  is a principal ideal of  $A$ . We deduce that there exist  $a \in A$  and  $\delta \in \mathbb{F}_q(z)$  such that

$$a = F(z)^n \delta.$$

This implies that  $F(z)^n \in A$  and  $\delta \in \mathbb{F}_q^\times$ . Thus, since  $A$  is integrally closed in  $\tilde{A}$ , we deduce that  $F(z) \in A$  and  $I = F(z)A$ . We conclude that the natural map  $\text{Pic}(A) \rightarrow \text{Pic}(\tilde{A})$  is injective.

Let  $X/\mathbb{F}_q$  be the projective non-singular curve associated to  $K/\mathbb{F}_q$  and let  $J/\mathbb{F}_q$  be the jacobian variety associated to  $X$ . Then we have isomorphisms of groups

$$\text{Cl}^0(K) \simeq J(\mathbb{F}_q),$$

$$\text{Cl}^0(K(z)) \simeq J(\mathbb{F}_q(z)).$$

By the Mordell-Weil Theorem we know that  $J(\mathbb{F}_q(z))$  is finitely generated and therefore  $J(\mathbb{F}_q(z))_{\text{tors}}$  is a finite abelian group. Let  $P \in J(\mathbb{F}_q(z))_{\text{tors}}$  and let  $\phi(P)$  be the point obtained by elevating to its  $q$ th power every coordinate of  $P$ . Since  $J(\mathbb{F}_q(z))_{\text{tors}}$  is a finite group, there exist integers  $n \neq m$  such that  $\phi^m(P) = \phi^n(P)$ . Thus  $P \in J(\mathbb{F}_q) \cap J(\mathbb{F}_q(z)) = J(\mathbb{F}_q)$ . Therefore  $J(\mathbb{F}_q(z))_{\text{tors}} = J(\mathbb{F}_q)$ . Then the natural map  $\text{Cl}^0(K) \rightarrow \text{Cl}^0(K(z))$  is injective and we have  $\text{Cl}^0(K(z))_{\text{tors}} = \text{Cl}^0(K)$ . We have two compatible exact sequences

$$0 \rightarrow \text{Cl}^0(K) \rightarrow \text{Pic}(A) \rightarrow \frac{\mathbb{Z}}{d_\infty \mathbb{Z}} \rightarrow 0,$$

$$0 \rightarrow \text{Cl}^0(K(z))_{\text{tors}} \rightarrow \text{Pic}(\tilde{A})_{\text{tors}} \rightarrow \frac{\mathbb{Z}}{d_\infty \mathbb{Z}} \rightarrow 0.$$

Thus

$$\text{Pic}(\tilde{A})_{\text{tors}} = \text{Pic}(A).$$

Let  $m \geq 1$  be an integer and let  $I$  be an ideal of  $\tilde{A}\mathbb{F}_q(z^m)$  such that  $I\tilde{A}$  is principal. We deduce that  $I^m$  is a principal ideal of  $\tilde{A}\mathbb{F}_q(z^m)$  by taking the ideal norm map from  $K(z)$  to  $K(z^m)$ . Thus  $I = J\delta\tilde{A}\mathbb{F}_q(z^m)$  for some ideal  $J$  of  $A$ , and  $\delta \in K(z^m)$ . But  $J\tilde{A}$  is principal, thus  $J$  must be a principal ideal of  $A$ . Therefore  $I$  is a principal

ideal of  $\tilde{A}\mathbb{F}_q(z^m)$ . We conclude that the natural map  $\text{Pic}(\tilde{A}\mathbb{F}_q(z^m)) \rightarrow \text{Pic}(\tilde{A})$  is injective.  $\square$

**Lemma 5.5.** *Let  $F(z) \in K_\infty[z] \setminus \{0\}$  be a polynomial of degree less than some integer  $r \geq 1$ . We suppose that  $F(0) = 1$  and that there exists  $C \geq 1$  with  $v_\infty(F(z) - 1) \geq C$ . Then*

$$F(z) = \prod_{i=1}^r (1 - \alpha_i z), \quad \alpha_i \in \mathbb{C}_\infty, v_\infty(\alpha_i) \geq \frac{C}{r}.$$

*Proof.* Let  $\alpha \in \mathbb{C}_\infty^\times$  such that  $F(\alpha^{-1}) = 0$ . We write  $F(z) = 1 + \sum_{i=1}^r a_i z^i$  with  $a_i \in K_\infty, v_\infty(a_i) \geq C$ . Then

$$\sum_{i=1}^r \frac{a_i}{\alpha^i} = -1.$$

Then there exists an integer  $i$ ,  $1 \leq i \leq r$ , such that

$$v_\infty\left(\frac{a_i}{\alpha^i}\right) \leq 0.$$

It follows that

$$v_\infty(\alpha) \geq \frac{v_\infty(a_i)}{i} \geq \frac{C}{r}.$$

$\square$

We are now ready to prove the main result of this section.

**Theorem 5.6.** *Let  $E/O_L$  be an Anderson  $A$ -module where  $L/\iota(K)$  is a finite extension. We suppose that  $E$  satisfies the following condition:*

**Condition (P')**: *There exists a finite set  $S$  of primes in  $O_L$  such that for all primes  $\mathfrak{P}$  in  $O_L$  with  $\mathfrak{P} \notin S$ ,  $\text{Fitt}_{\tilde{A}}\left(\tilde{E}\left(\frac{O_L}{\mathfrak{P}}(z)\right)\right)$  is principal.*

*Then  $E$  is admissible.*

*Proof.* We first show that  $E$  satisfies Condition (C). In fact, we choose  $t \in A \setminus \mathbb{F}_q$  such that  $K/\mathbb{F}_q(t)$  is a finite separable extension. We then write

$$\tilde{E}_t = \sum_{i=0}^r A_i z^i \tau^i, \quad A_i \in M_{d \times d}(O_L).$$

Let  $\mathfrak{P}$  be a prime in  $O_L$  with  $\mathfrak{P} \notin S$ . We put  $\mathbb{F}_{\mathfrak{P}} = O_L/\mathfrak{P}$  and  $m_{\mathfrak{P}} = [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q]$ . By Lemmas 5.1 and 5.5,

$$\text{Fitt}_{\mathbb{F}_q(z)[t]}(\tilde{E}(\mathbb{F}_{\mathfrak{P}}(z))) = G_{\mathfrak{P}}(z)\mathbb{F}_q(z)[t],$$

where

$$\frac{G_{\mathfrak{P}}(z)}{G_{\mathfrak{P}}(0)} = \prod_{i=1}^r (1 - \alpha_i z^{m_{\mathfrak{P}}}), \quad \alpha_i \in \mathbb{C}_\infty, v_\infty(\alpha_i) \geq \frac{m_{\mathfrak{P}}}{r}.$$

By Condition (P'), Proposition 5.3 and Lemma 5.4, there exists  $x_{\mathfrak{P}}(z) \in A[z^{m_{\mathfrak{P}}}]$  without roots in  $\overline{\mathbb{F}_q}$  such that

$$\text{Fitt}_{\tilde{A}}\left(\tilde{E}(\mathbb{F}_{\mathfrak{P}}(z))\right) = x_{\mathfrak{P}}(z)\tilde{A}.$$

Since  $\text{Fitt}_{\mathbb{F}_q(z)[t]}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}(z))) = N_{K(z)/\mathbb{F}_q(t)(z)}(x_{\mathfrak{p}}(z))\mathbb{F}_q(z)[t]$ , we get

$$\frac{G_{\mathfrak{p}}(z)}{G_{\mathfrak{p}}(0)} = N_{K(z)/\mathbb{F}_q(t)(z)}\left(\frac{x_{\mathfrak{p}}(z)}{x_{\mathfrak{p}}(0)}\right) = N_{\tilde{K}_{\infty}/\mathbb{F}_q(z)(\frac{1}{t})}\left(\frac{x_{\mathfrak{p}}(z)}{x_{\mathfrak{p}}(0)}\right).$$

Thus there exists an integer  $r'$ ,  $1 \leq r' \leq r$ , such that

$$\frac{x_{\mathfrak{p}}(z)}{x_{\mathfrak{p}}(0)} = \prod_{i=1}^{r'} (1 - \beta_i z^{m_{\mathfrak{p}}}), \quad \alpha_i \in \mathbb{C}_{\infty}, v_{\infty}(\beta_i) \geq \frac{m_{\mathfrak{p}}}{r}.$$

We deduce that  $E$  satisfies Condition (C), that is, the following infinite product converges

$$\prod_{\mathfrak{p} \notin S} \frac{x_{\mathfrak{p}}(z)}{x_{\mathfrak{p}}(0)} \in \mathbb{T}_z(K_{\infty})^{\times}.$$

To conclude, we will show that Condition (P) is satisfied with the  $x_{\mathfrak{p}}$ 's as above. We observe that

$$\begin{aligned} \text{Fitt}_{\tilde{A}}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}(z))) &= \text{Fitt}_{A[z]}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}[z]))\tilde{A}, \\ \text{Fitt}_A(E(\mathbb{F}_{\mathfrak{p}})) &= \text{Fitt}_{A[z]}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}[z]))|_{z=1}, \\ \text{Fitt}_A(\text{Lie}_E(\mathbb{F}_{\mathfrak{p}})) &= \text{Fitt}_{A[z]}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}[z]))|_{z=0}. \end{aligned}$$

Note that there exist  $G_1(z), \dots, G_s(z) \in x_{\mathfrak{p}}(z)A[z]$  such that

$$\text{Fitt}_{A[z]}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}[z])) = \sum_{i=1}^s G_i(z)A[z].$$

It follows that

$$\begin{aligned} \text{Fitt}_A(E(\mathbb{F}_{\mathfrak{p}})) &\subset x_{\mathfrak{p}}(1)A, \\ \text{Fitt}_A(\text{Lie}_E(\mathbb{F}_{\mathfrak{p}})) &\subset x_{\mathfrak{p}}(0)A. \end{aligned}$$

If  $I$  is a non-zero ideal of  $A$  (resp.  $\tilde{A}$ ), then we set  $\deg I = \dim_{\mathbb{F}_q}(A/I)$  (resp.  $\deg I = \dim_{\mathbb{F}_q(z)}(\tilde{A}/I)$ ). Observe that if  $I = aA$  (resp.  $I = a\tilde{A}$ ), then  $\deg I = -d_{\infty}v_{\infty}(a)$ . Furthermore if  $I \subset J$  are non-zero ideals of  $A$  (resp.  $\tilde{A}$ ) such that  $\deg I = \deg J$ , then  $I = J$ . We know that

$$\deg \text{Fitt}_A(E(\mathbb{F}_{\mathfrak{p}})) = \deg \text{Fitt}_A(\text{Lie}_E(\mathbb{F}_{\mathfrak{p}})) = \deg \text{Fitt}_{\tilde{A}}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}(z))).$$

We note that

$$v_{\infty}(x_{\mathfrak{p}}(z)) = v_{\infty}(x_{\mathfrak{p}}(1)) = v_{\infty}(x_{\mathfrak{p}}(0)).$$

Putting altogether yields

$$\text{Fitt}_A(E(\mathbb{F}_{\mathfrak{p}})) = x_{\mathfrak{p}}(1)A,$$

$$\text{Fitt}_A(\text{Lie}_E(\mathbb{F}_{\mathfrak{p}})) = x_{\mathfrak{p}}(0)A.$$

This proves Condition (P) for  $E$ . We conclude that  $E$  is admissible.  $\square$

6.  $A$ -FINITE AND ABELIAN ANDERSON  $A$ -MODULES

In this section we prove that the Anderson  $A$ -modules which are either  $A$ -finite or abelian are admissible.

Let  $L/\iota(K)$  be a finite extension and let  $F = L^{\text{perf}}$ . Let  $O_F$  be the integral closure of  $\iota(A)$  in  $F$ . Then for any prime  $\mathfrak{P}$  in  $O_L$  there exists a unique prime  $\mathfrak{Q}$  in  $O_F$  over  $\mathfrak{P}$  and the inclusion  $O_L \subset O_F$  induces an isomorphism  $O_L/\mathfrak{P} \simeq O_F/\mathfrak{Q}$ .

Let  $\tau : F \rightarrow F, x \mapsto x^q$  and  $\sigma : F \rightarrow F, x \mapsto x^{\frac{1}{q}}$ . Let  $A_F = A \otimes_{\mathbb{F}_q} F$  and let  $A_F\{\sigma\}$  be the skew polynomial ring

$$\sigma(a \otimes 1) = (a \otimes 1)\sigma, \quad \sigma(1 \otimes \alpha) = (1 \otimes \alpha^{\frac{1}{q}})\sigma, \quad \text{for all } a \in A.$$

**6.1.  $A$ -finite Anderson modules.** Let  $E$  be an Anderson  $A$ -module of dimension  $d$  defined over  $O_L$ . We recall the construction of the dual  $A$ -motive  $\mathcal{M}(E/F)$  attached to  $E$ . We refer the reader to [20], Sections 2.5.1-2.5.2 for more details. We set  $\mathcal{M}(E/F) = F\{\tau\}^d$ . We consider  $\mathcal{M}(E/F)$  as a free  $F\{\sigma\}$ -module where  $F$  acts on  $\mathcal{M}(E/F)$  via right multiplication and  $\sigma$  acts on  $\mathcal{M}(E/F)$  via right multiplication by  $\tau$ . Also we view  $\mathcal{M}(E/F)$  as an  $A$ -module via left multiplication by  $E$ . Thus  $\mathcal{M}(E/F)$  is a left  $A_F\{\sigma\}$ -module such that

- (1)  $(a \otimes 1 - 1 \otimes a)^d \mathcal{M}(E/F) \subset \sigma \mathcal{M}(E/F)$  for all  $a \in A$ .
- (2)  $\mathcal{M}(E/F)$  is a free  $F\{\sigma\}$ -module of rank  $d$ .

Following [20], Definition 2.5.9, we say that  $E$  is  $A$ -finite if  $\mathcal{M}(E/F)$  is a finitely generated  $A_F$ -module. Note that this definition does not depend on the choice of  $F$ . In that case  $\mathcal{M}(E/F)$  is a projective  $A_F$ -module by [19], Lemma 5.4.10.

If  $E$  is  $A$ -finite, then there exists an integer  $m(E)$  such that for any prime  $\mathfrak{P}$  in  $O_L$  such that  $|\mathbb{F}_{\mathfrak{P}}| \geq m(E)$  where  $\mathbb{F}_{\mathfrak{P}} = O_L/\mathfrak{P}$ , we have that  $E \pmod{\mathfrak{P}}$  is  $A$ -finite.

**Theorem 6.1.** *Let  $E$  be an Anderson module defined over  $O_L$  which is  $A$ -finite. Then  $E$  is admissible.*

*Proof.* Let  $\mathfrak{P}$  be a prime in  $O_L$  such that  $E \pmod{\mathfrak{P}}$  is  $A$ -finite. Let  $V(E/\mathbb{F}_{\mathfrak{P}}) = \mathcal{M}(E/\mathbb{F}_{\mathfrak{P}}) \otimes_A K$  which is a finite dimensional  $K$ -vector space. We still denote by  $\sigma$  the  $K$ -linear map induced by  $\sigma$  which is the right multiplication by  $\tau$ . Let  $P_{\mathfrak{P}}(X) = \det_K(X \text{Id}_{V(E/\mathbb{F}_{\mathfrak{P}})} - \sigma)$ . Then clearly  $P_{\mathfrak{P}}(X) \in A[X]$ .

By a direct computation, the right multiplication by  $\tau - z$  gives rise to an exact sequence of  $\tilde{A}$ -modules

$$(6.1) \quad 0 \rightarrow \mathcal{M}(E/\mathbb{F}_{\mathfrak{P}}(z)) \xrightarrow{\tau - z} \mathcal{M}(E/\mathbb{F}_{\mathfrak{P}}(z)) \rightarrow \tilde{E}(\mathbb{F}_{\mathfrak{P}}(z)) \rightarrow 0.$$

Thus by Corollary 1.6, we get

$$\text{Fitt}_{\tilde{A}} \left( \tilde{E}(\mathbb{F}_{\mathfrak{P}}(z)) \right) = P_{\mathfrak{P}}(z) \tilde{A}.$$

Consequently, the Theorem follows from Theorem 5.6.  $\square$

**6.2. Abelian Anderson modules.** Let  $E$  be an Anderson  $A$ -module of dimension  $d$  defined over  $O_L$ . We now recall the construction of the  $A$ -motive  $M(E/F)$  attached to  $E$ . We refer the reader to [20], Section 2.5.1 for more details. We set  $M(E/F) = F\{\tau\}^{1 \times d}$ , that is, elements of  $M(E/F)$  are row vectors of length  $d$ . We consider  $M(E/F)$  as a free  $F\{\tau\}$ -module where  $F$  acts on  $M(E/F)$  via left

multiplication and  $\tau$  acts on  $M(E/F)$  via left multiplication by  $\tau$ . Also we view  $M(E/F)$  as an  $A$ -module via right multiplication by  $E$ . Thus  $M(E/F)$  is a left  $A_F\{\tau\}$ -module such that

- (1)  $(a \otimes 1 - 1 \otimes a)^d M(E/F) \subset \tau M(E/F)$  for all  $a \in A$ .
- (2)  $M(E/F)$  is a free  $F\{\tau\}$ -module of rank  $d$ .

Following [20], Definition 2.5.5, we say that  $E$  is abelian if  $M(E/F)$  is a finitely generated  $A_F$ -module. Note that this definition does not depend on the choice of  $F$ . In that case  $M(E/F)$  is a projective  $A_F$ -module by [19], Lemma 5.4.10.

If  $E$  is abelian, then there exists an integer  $m(E)$  such that for any prime  $\mathfrak{P}$  in  $O_L$  such that  $|\mathbb{F}_{\mathfrak{P}}| \geq m(E)$  where  $\mathbb{F}_{\mathfrak{P}} = O_L/\mathfrak{P}$ , we have that  $E \pmod{\mathfrak{P}}$  is abelian.

**Lemma 6.2.** *Let  $k$  be a field and  $R$  be a Dedekind domain which is a  $k$ -algebra. If  $M$  is an  $R$ -module which is finite dimensional over  $k$ , then there exists an isomorphism of  $R$ -modules*

$$\mathrm{Hom}_k(M, k) \simeq M.$$

*Proof.* By the structure theorem of finitely generated torsion modules over Dedekind rings, we can suppose that  $M$  is of the form  $M = R/P^n$  where  $P$  is a non-zero prime in  $R$  and  $n \geq 1$ . But then  $M$  and  $\mathrm{Hom}_k(M, k)$  have the same dimension over  $k$  and we have both  $P^n \mathrm{Hom}_k(M, k) = \{0\}$  and  $P^{n-1} \mathrm{Hom}_k(M, k) \neq \{0\}$ . The result then follows again by the structure theorem.  $\square$

**Theorem 6.3.** *Let  $E$  be an Anderson module defined over  $O_L$  which is abelian. Then  $E$  is admissible.*

*Proof.* Let  $\mathfrak{P}$  be a prime in  $O_L$  such that  $E \pmod{\mathfrak{P}}$  is abelian. We put  $\mathbb{F}_{\mathfrak{P}} = O_L/\mathfrak{P}$  and  $m = [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q]$ . We define  $\widetilde{M}(E/\mathbb{F}_{\mathfrak{P}}) = \mathbb{F}_{\mathfrak{P}}(z)^d$  equipped with the following action of  $A$ : if  $a \in A$  and  $\widetilde{E}_a = \sum_{i=0}^k A_i z^i \tau^i$ , and  $X \in \widetilde{M}(E/\mathbb{F}_{\mathfrak{P}})$ , then

$$a.X = \sum_{i=0}^k {}^t A_i X z^i \tau^i.$$

We extend this action to an action of  $\widetilde{A}$  by  $\mathbb{F}_q(z)$ -linearity. Here we consider  $M(E/\mathbb{F}_{\mathfrak{P}})$  with column vectors instead of row vectors and twist the Frobenius map  $\tau$  with  $z$  as before.

We define  $K_z$  to be the cokernel of the left multiplication by  $\tau - z$  on  $\widetilde{M}(E/\mathbb{F}_{\mathfrak{P}})$ :

$$0 \longrightarrow \widetilde{M}(E/\mathbb{F}_{\mathfrak{P}}) \xrightarrow{\tau - z} \widetilde{M}(E/\mathbb{F}_{\mathfrak{P}}) \longrightarrow K_z \longrightarrow 0$$

Similarly to (6.1), a direct computation shows that  $K_z \simeq \mathbb{F}_{\mathfrak{P}}(z)^d$  with the action of  $a$  given by left multiplication by  $\widetilde{E}_a^* = \sum_{i=1}^k z^i \sigma^i({}^t A_i) \sigma^i$ .

We now fix an  $\mathbb{F}_q$ -basis  $\mathcal{B} = (e_1, \dots, e_m)$  of  $\mathbb{F}_{\mathfrak{P}}$  and define for all  $1 \leq i \leq m$  and  $1 \leq j \leq d$  the vector  $e_{ij} = (0, \dots, e_i, 0, \dots, 0)$  where  $e_i$  appears on the  $j$ th position. The family  $(e_{ij})_{ij}$  forms an  $\mathbb{F}_q(z)$ -basis  $\mathcal{B}' = (e_{11}, \dots, e_{1m}, e_{21}, \dots, e_{2m}, \dots, e_{dm})$  of  $\mathbb{F}_{\mathfrak{P}}(z)^d$ . Let us remark that we can, and will, choose  $\mathcal{B}$  so that for all  $i < m$ ,

$e_i = \tau^{i-1}e_1$ , and  $\tau^m e_1 = e_1$ . Then the matrix of  $\tau$  in  $\mathcal{B}$  is the permutation matrix

$$T = \begin{pmatrix} & & & 1 \\ & & & \\ & & \ddots & \\ & 1 & & \\ & & & & 1 \end{pmatrix}$$

and in the basis  $\mathcal{B}'$  of  $\mathbb{F}_{\mathfrak{p}}$ , it is the block matrix  $T' = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$ . If  $A = (a_{ij})_{1 \leq i, j \leq d} \in M_{d \times d}(\mathbb{F}_{\mathfrak{p}})$ , then its matrix in the basis  $\mathcal{B}'$  becomes a block matrix  $A' = (A_{ij})_{1 \leq i, j \leq d} \in M_{dm}(\mathbb{F}_q)$  where the  $A_{ij}$  are the matrices of the action of  $a_{ij}$  in  $\mathcal{B}$ . We see that the matrix of the action of  $\sigma^i({}^t A_i)\sigma^i$  in  $\mathcal{B}'$  is then  $T'^{-i} {}^t A'_i$ .

Since  $T^{-1} = {}^t T$ , it follows that  $T'^{-1} = {}^t T'$ . Therefore  $T'^{-i} {}^t A'_i = {}^t (A'_i T'^i)$ . We deduce that the action of  $A\mathbb{F}_q(z)$  on  $K_z$  is, in the basis  $\mathcal{B}'$ , the transposition of that of  $\mathbb{F}_{\mathfrak{p}}(z)^d$  via  $\tilde{E}$ .

By Corollary 1.6 and Lemma 6.2,

$$\begin{aligned} \det(\tau - z) &= \text{Fitt}_{\tilde{A}}(K_z) \\ &= \text{Fitt}_{\tilde{A}}\left(\text{Hom}_{\mathbb{F}_q(z)}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}(z)), \mathbb{F}_q(z))\right) \\ &= \text{Fitt}_{\tilde{A}}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}(z))). \end{aligned}$$

Then the Theorem follows immediately from Theorem 5.6.  $\square$

**6.3. Class formulas.** As a direct consequence of Theorems 4.7, 6.1 and 6.3, we obtain the following theorem.

**Theorem 6.4.** *Let  $E$  be an Anderson module defined over  $O_L$  which is either abelian or  $A$ -finite. Then we have the class formula*

$$[\text{Lie}_E(O_L) : U_{\text{St}}(E/O_L)]_A = L(E/O_L).$$

Since any Drinfeld module is both abelian and  $A$ -finite (see [20], Corollary 2.5.17), by Theorem 6.4 and Equation (2.1), we obtain the class formula à la Taelman for Drinfeld modules, which extends known results [7, 14, 24, 29, 31, 33].

**Corollary 6.5.** *Let  $\phi$  be a Drinfeld  $A$ -module defined over  $O_L$ . Then  $L(\phi/O_L)$  is well-defined and is a principal  $A$ -lattice in  $K_{\infty}$ . Furthermore we have the following class formula*

$$[O_L : U_{\text{St}}(\phi/O_L)]_A = [O_L : U(\phi/O_L)]_A \text{Fitt}_A(H(\phi/O_L)) = L(\phi/O_L).$$

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