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► **To cite this version:**

Bruno Angles, Tuan Ngo Dac, Floric Tavares-Ribeiro. A class formula for admissible Anderson modules. 2020. hal-02490566v1

HAL Id: hal-02490566

<https://hal.science/hal-02490566v1>

Preprint submitted on 25 Feb 2020 (v1), last revised 16 Sep 2021 (v2)

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A CLASS FORMULA FOR ADMISSIBLE ANDERSON MODULES

BRUNO ANGLÈS, TUAN NGO DAC, AND FLORIC TAVARES RIBEIRO

ABSTRACT. In 2012 Taelman proved a class formula for Drinfeld $\mathbb{F}_q[\theta]$ -modules. For an arbitrary coefficient ring A , several deep but partial results in the direction of a class formula for Drinfeld A -modules have been obtained. In this paper, using a completely new approach based on the notion of Stark units, we establish the class formula for Drinfeld A -modules in full generality. Further we extend this class formula for A -finite Anderson modules and more generally for admissible Anderson modules.

CONTENTS

Introduction	1
1. Background	4
2. Anderson modules	8
3. Local factors	10
4. Class formula à la Taelman	13
5. Admissible Anderson modules	19
References	23

INTRODUCTION

The celebrated class number formula relates many important invariants of a number field (e.g. the ideal class number, the regulator of units) to a special value of its Dedekind zeta function. By analogy between number fields and function fields, Carlitz suggested to transport the classical results to the function field setting in positive characteristic. In 1935, Carlitz [13] succeeded to prove the first instance of the class formula for special Carlitz zeta values. These zeta values are attached to the field of rational functions equipped with the infinity place (i.e. when $A = \mathbb{F}_q[\theta]$) and intimately related to the so-called *Carlitz module* which is the first example of a Drinfeld module. In 2010, an important breakthrough was due to Taelman who gave an elegant and simple conjectural class formula for Drinfeld $\mathbb{F}_q[\theta]$ -modules (see [31], Conjecture 1). It states that the special value of the Goss L -function attached to a Drinfeld module at $s = 1$ is the product of a regulator term that is the covolume of the module of units and an algebraic term arising from a certain

Date: February 25, 2020.

2010 *Mathematics Subject Classification.* Primary 11G09; Secondary 11M38, 11R58.

Key words and phrases. Drinfeld modules, Anderson modules, L -series in characteristic p , class formula.

class module. Shortly after, Taelman [32] used the trace formula of Anderson [4] and proved this conjecture.

Following the fundamental work of Taelman, in recent years, the class formula has been rapidly developed in different directions. Fang [18] and Demeslay [15, 16] proved the class formula for Anderson $\mathbb{F}_q[\theta]$ -modules that are generalizations of Drinfeld $\mathbb{F}_q[\theta]$ -modules in higher dimensions. For general A , the class formula for Anderson A -modules is shown in particular cases and we refer the reader to [7, 9, 14, 34, 36]. Recently, motivated by the works of Böckle-Pink [11] and V. Lafforgue [22], Mornev [26] developed a theory of shtuka cohomology and proved the class formula for Drinfeld A -modules having good reduction everywhere.

In the present paper, for general A , we prove the class formula for Drinfeld A -modules in full generality, which generalizes the works of Taelman [32] and Mornev [26]. Our approach is completely different from the aforementioned works and based on the notion of Stark units developed in [7, 8, 10]. Further, we extend our proof to establish the class formula for admissible Anderson modules, in particular for A -finite Anderson modules.

Let us give now more precise statements of our results.

Throughout this paper, let K be a global function field over a finite field \mathbb{F}_q of characteristic p , having q elements (\mathbb{F}_q is algebraically closed in K). We fix a place ∞ of K of degree $d_\infty \geq 1$ and denote by A the ring of elements of K which are regular outside ∞ . The ∞ -adic completion K_∞ of K is equipped with the normalized ∞ -adic valuation $v_\infty : K_\infty \rightarrow \mathbb{Z} \cup \{+\infty\}$ and has residue field \mathbb{F}_∞ . The completion \mathbb{C}_∞ of a fixed algebraic closure \overline{K}_∞ of K_∞ comes with a unique valuation extending v_∞ , it will still be denoted by v_∞ . With the valuation v_∞ , we can define the following norm on \mathbb{C}_∞ : $|x| = q^{-v_\infty(x)}$. We define the Frobenius $\tau : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ as the \mathbb{F}_q -algebra homomorphism which sends x to x^q .

Let L/K be a finite extension and O_L be the integral closure of A in L . We set $L_\infty := L \otimes_K K_\infty$. Let E be an Anderson A -module (or an Anderson module for short) of dimension d defined over O_L , that means an \mathbb{F}_q -algebra homomorphism $E : A \rightarrow M_{d \times d}(O_L)\{\tau\}$ such that E is nonconstant and for all $a \in A$, if we write $E_a = \sum_{k \geq 0} E_{a,k} \tau^k$ with $E_{a,k} \in M_{d \times d}(O_L)$, then we require

$$(E_{a,0} - aI_d)^d = 0_d.$$

We denote by $\partial_E : A \rightarrow M_{d \times d}(L)$ the \mathbb{F}_q -algebra homomorphism given by $\partial_E(a) = E_{a,0}$ for $a \in A$. By definition, a Drinfeld A -module is just an Anderson module of dimension 1.

Let R be an O_L -algebra. We can define two A -module structures on R^d : the first one is denoted by $E(R)$ where A acts on R^d via E and the second one is denoted by $\text{Lie}_E(R)$ where A acts on R^d via ∂_E .

There exists a unique power series called the exponential series $\exp_E \in I_d + M_{d \times d}(L)\{\{\tau\}\}\tau$ such that

$$\exp_E \partial_E(a) = E_a \exp_E, \quad a \in A.$$

The exponential series \exp_E converges on $\text{Lie}_E(L_\infty)$. Taelman [31] introduced the unit module attached to E/O_L as follows:

$$U(E/O_L) := \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(O_L)\}.$$

By [31], $U(E/O_L)$ is in fact an A -lattice in $\mathrm{Lie}_E(L_\infty)$. In other words, $U(E/O_L)$ is a finitely generated A -module which is discrete and cocompact in $\mathrm{Lie}_E(L_\infty)$.

Following the pioneer work of Anderson [3] in which he introduced the analogue of cyclotomic units for the Carlitz module, the authors have introduced and developed the theory of Stark units for Anderson modules. Roughly speaking, they are units coming from the canonical deformation of Drinfeld modules in Tate algebras in the sense of Pellarin [28]. The germs of the concept of Stark units can be found in [5, 6]. The notion has been conceptualized in [10] for Drinfeld modules over $\mathbb{F}_q[\theta]$ and then further developed in the general context in [7, 8, 9].

The notion of Stark units is a powerful tool for investigating log-algebraicity identities. A log-algebraicity result consists of the construction of a specific unit in connection with special L -values of a Drinfeld module. The notion of log-algebraicity is first introduced by Thakur (see [34, 35]). It has been notably developed by Anderson [2] and has become a very active topic in function field arithmetic. We note that the units obtained by log-algebraicity theorems turn out to be always Stark units.

More precisely, the Stark units attached to E/O_L are units coming from the canonical z -deformation \tilde{E} of E . By definition, let z be an indeterminate with the rule $\tau z = z\tau$ and we denote by \tilde{E} the homomorphism of $\mathbb{F}_q(z)$ -algebras $\tilde{E} : \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} A \rightarrow M_{d \times d}(\mathbb{F}_q(z) \otimes_{\mathbb{F}_q} O_L)\{\tau\}$ such that

$$\tilde{E}_a = \sum_{k \geq 0} z^k E_{a,k} \tau^k, \quad a \in A.$$

We write $\exp_E = \sum_{i \geq 0} E_i \tau^i$ with $E_i \in M_{d \times d}(L)$ and set $\exp_{\tilde{E}} := \sum_{i \geq 0} z^i E_i \tau^i$. Then one sees easily that $\exp_{\tilde{E}} \partial_E(a) = \tilde{E}_a \exp_{\tilde{E}}$ for all $a \in A$. Further, $\exp_{\tilde{E}}$ converges on $\mathrm{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty))$ where $\mathbb{T}_z(L_\infty)$ is the ∞ -adic completion of $L_\infty[z]$. Thus we can define

$$U(\tilde{E}/O_L[z]) := \{x \in \mathrm{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(O_L[z])\}.$$

Let $\mathrm{ev} : \mathrm{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \mathrm{Lie}_E(L_\infty)$ be the evaluation at $z = 1$. We define the module of Stark units as follows:

$$U_{St}(E/O_L) := \mathrm{ev}(U(\tilde{E}/O_L[z])).$$

One can prove that $U_{St}(E/O_L)$ is a sub- A -lattice of $U(E/O_L)$ in $\mathrm{Lie}_E(L_\infty)$. Consequently, we get a regulator term $[\mathrm{Lie}_E(O_L) : U_{St}(E/O_L)]_A$ which is an invertible A -lattice in K_∞ .

Next, we introduce the special L -value attached to E/O_L . For any prime \mathfrak{P} of O_L , we define the local factor

$$Z_{\mathfrak{P}}(E/O_L) := \frac{\mathrm{Fitt}_A(\mathrm{Lie}_E(O_L/\mathfrak{P}O_L))}{\mathrm{Fitt}_A(E(O_L/\mathfrak{P}O_L))}$$

which is an A -lattice in K_∞ . We would like to deal with the infinite product of A -lattices in K_∞ :

$$L(E/O_L) = \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L)$$

where the product runs over the set of primes \mathfrak{P} of O_L . The rough idea is to reduce this infinite product of A -lattices to an infinite product of elements in K_∞ . The precise idea is given in Definition 4.1. In particular, if E is an *admissible*

Anderson module, then for all except finitely many primes \mathfrak{P} of O_L , the local factor $Z_{\mathfrak{P}}(E/O_L)$ is a principal lattice generated by an element $z_{\mathfrak{P}} \in K_{\infty}$ and we require that the infinite product $\prod_{\mathfrak{P}} z_{\mathfrak{P}}$ converges.

One of our main results is the following theorem:

Theorem A (Theorem 4.5). Let E be an admissible Anderson A -module defined over O_L . Then we have the class formula:

$$[\mathrm{Lie}_E(O_L) : U_{\mathrm{st}}(E/O_L)]_A = L(E/O_L).$$

As we mention above, the notion of Stark units is conceived to prove log-algebraicity identities. It is quite unexpected that it can be applied to prove the previous class formula.

As an application, inspired by [14, 19, 30], we will show that any A -finite Anderson module is admissible (see Theorem 5.7). In particular, the class formula holds for A -finite Anderson modules (see Theorem 5.8). As an immediate consequence, the class formula holds for Drinfeld A -modules since Drinfeld A -modules are always A -finite. This theorem generalizes the works of Taelman [32] and Mornev [26].

Theorem B (Corollary 5.9). Let ϕ be a Drinfeld A -module defined over O_L . Then $L(\phi/O_L)$ is well-defined and is a principal A -lattice in K_{∞} . Furthermore we have the following class formula:

$$[O_L : U_{\mathrm{st}}(\phi/O_L)]_A = [O_L : U(\phi/O_L)]_A \mathrm{Fitt}_A(H(\phi/O_L)) = L(\phi/O_L).$$

Acknowledgements. The second author (T. ND.) was partially supported by ANR Grant COLOSS ANR-19-CE40-0015-02.

1. BACKGROUND

1.1. Notation. We keep the notation in the Introduction. Further, we fix a uniformizer π of K_{∞} and identify $K_{\infty} = \mathbb{F}_{\infty}((\pi))$.

Letting z be an indeterminate, we will consider a field k which is either \mathbb{F}_q or $\mathbb{F}_q(z)$. We endow $k \otimes_{\mathbb{F}_q} K$ with the topology coming from the discrete topology on k and the v_{∞} -adic topology on K , and consider its completion

$$k_{\infty} := k \widehat{\otimes}_{\mathbb{F}_q} K_{\infty}.$$

Note that $k_{\infty} = K_{\infty} = \mathbb{F}_{\infty}((\pi))$ if $k = \mathbb{F}_q$ and $k_{\infty} = \mathbb{F}_{\infty}(z)((\pi))$ if $k = \mathbb{F}_q(z)$.

We still denote by v_{∞} the ∞ -adic Gauss valuation on $\mathbb{C}_{\infty}(z)$. Let F be a subfield of \mathbb{C}_{∞} and complete for v_{∞} , we denote by $\mathbb{T}_z(F)$ the completion of $F[z]$ for v_{∞} and \widetilde{F} the completion of $F(z)$ for v_{∞} . Then the $\mathbb{F}_q(z)$ -vector space generated by $\mathbb{T}_z(F)$ is dense in \widetilde{F} . Observe that $\mathbb{T}_z(K_{\infty}) = \mathbb{F}_{\infty}[z)((\pi))$.

Let L/K be a finite extension and $L_{\infty} := L \otimes_K K_{\infty}$. Let O_L be the integral closure of A in L . We set $\widetilde{L}_{\infty} := L \otimes_K \widetilde{K}_{\infty}$. Let \widetilde{O}_L be the $\mathbb{F}_q(z)$ -vector space generated by O_L in \widetilde{L}_{∞} .

1.2. Fitting ideals. We recall basic facts on the theory of Fitting ideals. The standard references are the appendix to [25] and [17, 23, 27]. We will call here *Fitting ideal* what is usually called in the theory the initial or zero-th Fitting ideal.

We consider a commutative ring R and M a finitely presented R -module. Note that if R is Noetherian, this is equivalent to M being finitely generated. If

$$R^a \longrightarrow R^b \longrightarrow M \longrightarrow 0$$

is a presentation of M , and if X is the matrix of the map $R^a \rightarrow R^b$ then one defines $\text{Fitt}_R(M)$ to be the ideal of R generated by all the $b \times b$ minors of X if $b \leq a$, and $\text{Fitt}_R(M) = 0$ if $b > a$. This is independent of the choice of presentation for M . Note that if M is torsion, one always has $b \leq a$.

We list below basic properties of Fitting ideals.

- (1) If $M \simeq M_1 \times M_2$ is a direct product of finitely presented R -modules, then $\text{Fitt}_R(M) = \text{Fitt}_R(M_1) \cdot \text{Fitt}_R(M_2)$.
- (2) One always has $\text{Fitt}_R(M) \subset \text{Ann}_R(M)$, and if M is generated by b elements, then $\text{Ann}_R(M)^b \subset \text{Fitt}_R(M)$.
- (3) If $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ is exact, then $\text{Fitt}_R(M_1) \cdot \text{Fitt}_R(M_2) \subset \text{Fitt}_R(M)$.
- (4) If $I \subset R$ is an ideal, then $\text{Fitt}_{R/I}(M/IM)$ is the image of $\text{Fitt}_R(M)$ in R/I .
- (5) More generally, if $R \rightarrow R'$ is a ring homomorphism, then $\text{Fitt}_{R'}(R' \otimes_R M) = R' \otimes_R \text{Fitt}_R(M)$.

All the assertions above can be found in [25], except the last one which appears in [17], Corollary 20.5 as an immediate consequence of the right exactness of the tensor product.

In the case where R is a Dedekind ring, the structure theorem asserts that if M is a finitely generated and torsion R -module, then there exist ideals I_1, \dots, I_n of R such that $M \simeq R/I_1 \times \dots \times R/I_n$. We deduce that $\text{Fitt}_R(M) = \prod_{i=1}^n I_i$. Moreover, Fitting ideals are multiplicative in exact sequences. That is, if $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ is exact, then $\text{Fitt}_R(M_1) \cdot \text{Fitt}_R(M_2) = \text{Fitt}_R(M)$. This can be deduced from [12], Chapter VII, Section 4 n.5, Proposition 10.

1.3. Ratio of co-volumes. We define the ratio of co-volumes by following the ideas presented in [12], Chapter VII, Section 4 n.6. The reader can compare this section with the original construction of [32] or with [7], Section 2.3.

In this section, k is either \mathbb{F}_q or $\mathbb{F}_q(z)$ and we recall that the field $k_\infty := k \widehat{\otimes}_{\mathbb{F}_q} K_\infty$ identifies with $k((\pi))$. If V is a finite dimensional k_∞ -vector space, then it is endowed with the natural topology coming from k_∞ . We set $kA := k \otimes_{\mathbb{F}_q} A$.

Definition 1.1. Let V be a finite dimensional k_∞ -vector space. A sub- kA -module M of V is a kA -lattice in V if M is discrete in V and M generates V over k_∞ .

Lemma 1.2. *Let V be a finite dimensional k_∞ -vector space, and let M be a sub- kA -module of V . Then M is discrete in V if and only if there exist $e_1, \dots, e_m \in M$ which are k_∞ -linearly independent and such that the quotient $M / \bigoplus_{i=1}^m kAe_i$ is a finite dimensional k -vector space.*

Proof. Let $\theta \in A \setminus \mathbb{F}_q$. If we write $A = \bigoplus_{j=1}^s \mathbb{F}_q[\theta]f_j$, then

$$k_\infty = \bigoplus_{j=1}^s k \left(\begin{pmatrix} 1 \\ \theta \end{pmatrix} \right) f_j.$$

Let us assume that M is discrete in V . Let W be the k_∞ -vector space generated by M , then there exist $e_1, \dots, e_m \in M$ such that

$$W = \bigoplus_{i=1}^m k_\infty e_i = \bigoplus_{i,j} k \left(\left(\frac{1}{\theta} \right) \right) e_i f_j.$$

We put

$$N = \bigoplus_{i=1}^m k A e_i = \bigoplus_{i,j} k[\theta] e_i f_j.$$

Since M is discrete in W , there exists an integer $N \geq 1$ such that

$$M \cap \bigoplus_{i,j} \frac{1}{\theta^N} k \left[\left[\frac{1}{\theta} \right] \right] e_i f_j = \{0\}.$$

Now observe that we have an injection of k -vector spaces

$$\frac{M}{N} \hookrightarrow \frac{W}{N \oplus \bigoplus_{i,j} \frac{1}{\theta^N} k \left[\left[\frac{1}{\theta} \right] \right] e_i f_j}.$$

Since $k \left(\left(\frac{1}{\theta} \right) \right) = k[\theta] \oplus \frac{1}{\theta} k \left[\left[\frac{1}{\theta} \right] \right]$, the k -vector space $\frac{W}{N \oplus \bigoplus_{i,j} \frac{1}{\theta^N} k \left[\left[\frac{1}{\theta} \right] \right] e_i f_j}$ is finite dimensional. Therefore the quotient $\frac{M}{\bigoplus_{i=1}^m k A e_i}$ is a finite dimensional k -vector space.

Conversely, let us assume that there exist $e_1, \dots, e_m \in M$ which are k_∞ -linearly independent and such that the quotient $\frac{M}{\bigoplus_{i=1}^m k A e_i}$ is a finite dimensional k -vector space. Then there exists $a \in k[\theta] \setminus \{0\}$ such that

$$aM \subset \bigoplus_{i=1}^m k A e_i.$$

Let $m = \deg_\theta a$. It follows that

$$M \cap \bigoplus_{i,j} \frac{1}{\theta^{m+1}} k \left[\left[\frac{1}{\theta} \right] \right] e_i f_j = \{0\}.$$

We deduce that M is discrete in V . □

As an immediate consequence, we obtain the following proposition.

Proposition 1.3. *Let V be a finite dimensional k_∞ -vector space, and let M be a sub- kA -module of V . The following assertions are equivalent :*

- (i) M is a kA -lattice in V .
- (ii) M is discrete in V , finitely generated as a kA -module such that

$$\dim_{\text{Frac}(kA)} M \otimes_{kA} \text{Frac}(kA) = \dim_{k_\infty} V.$$

- (iii) *There exists a k_∞ -basis (e_1, \dots, e_n) of V such that for all $i \in \{1, \dots, n\}$, $e_i \in M$, and the quotient $\frac{M}{\bigoplus_{i=1}^n k A e_i}$ is a finite dimensional k -vector space.*

We can now proceed with the definition of ratio of co-volumes of lattices.

Let us start with the case where V is a k_∞ -vector space of dimension one. We write $V = k_\infty e$. Let M be a kA -lattice in V . By Proposition 1.3, there exist a nonzero fractional ideal I_M of kA and $\alpha_M \in k_\infty^\times$ such that $M = \alpha_M I_M e$. Letting M, M' be two kA -lattices in V , we define

$$[M' : M]_{kA} := \alpha_M \alpha_{M'}^{-1} I_M I_{M'}^{-1}.$$

We see that $[M' : M]_{kA}$ is well-defined and does not depend on the choice of e . Furthermore, if $M \subset M'$, then we have

$$[M' : M]_{kA} = \text{Fitt}_{kA}(M'/M).$$

We also observe that, if M_1, M_2, M_3 are three kA -lattices in V , then

$$[M_1 : M_3]_{kA} = [M_1 : M_2]_{kA} [M_2 : M_3]_{kA}.$$

Let us deal now with the general case. We put $\dim_{k_\infty} V = n \geq 1$. Then $\wedge^n V$ is a k_∞ -vector space of dimension one and we denote by $\wedge^n : V^n \rightarrow \wedge^n V$ the natural k_∞ -linear map. Let M, M' be two kA -lattices in V . We denote by N and N' the respective images of M^n and M'^n in $\wedge^n V$ via the map \wedge^n . By Proposition 1.3, both N and N' are kA -lattices in $\wedge^n V$. We set

$$[M' : M]_{kA} := [N' : N]_{kA}.$$

Since kA is a Dedekind domain, it follows that there exist two k_∞ -bases (e_1, \dots, e_n) and (f_1, \dots, f_n) of V such that

$$M = \bigoplus_{i=1}^{n-1} kAe_i \oplus Ie_n, \quad M' = \bigoplus_{i=1}^{n-1} kAf_i \oplus Jf_n,$$

where I, J are two nonzero fractional ideals of kA . Let $\sigma : V \rightarrow V$ be the k_∞ -linear isomorphism of V such that $\sigma(e_i) = f_i$. Then

$$N = Ie_1 \wedge \dots \wedge e_n,$$

$$N' = \det(\sigma)Je_1 \wedge \dots \wedge e_n.$$

Thus we get

$$[M' : M]_{kA} = \det(\sigma)^{-1}IJ^{-1}.$$

Note that the set of kA -lattices in k_∞ is naturally an abelian group, and

$$[M : M']_{kA} = [M' : M]_{kA}^{-1}.$$

Finally, if M_1, M_2, M_3 are three kA -lattices in V , then

$$[M_1 : M_3]_{kA} = [M_1 : M_2]_{kA} [M_2 : M_3]_{kA}.$$

Proposition 1.4. *If M and M' are two kA -lattices of a finite dimensional k_∞ -vector space V , with $M' \subset M$, then M/M' is torsion and*

$$[M : M']_{kA} = \text{Fitt}_{kA}(M/M').$$

Proof. It is clear that a quotient of lattices is torsion. By Proposition 1.3, M is contained in a lattice L which is free over kA . It implies that

$$0 \longrightarrow M \longrightarrow L \longrightarrow L/M \longrightarrow 0$$

is a presentation of L/M . If (e_1, \dots, e_n) is a kA -basis of L , and N is the image of M^n in $\wedge^n V$, it is equal to $Ie_1 \wedge \dots \wedge e_n$ where I is the ideal of kA generated by the $n \times n$ determinants of elements of M written in the basis (e_1, \dots, e_n) . Thus the definitions of $[L : M]_{kA}$ and $\text{Fitt}_{kA}(L/M)$ coincide. Moreover, we have an exact sequence of finitely generated and torsion kA -modules

$$0 \longrightarrow M/M' \longrightarrow L/M' \longrightarrow L/M \longrightarrow 0$$

so that

$$\begin{aligned} \text{Fitt}_{kA}(M/M') &= \text{Fitt}_{kA}(L/M') \text{Fitt}_{kA}(L/M)^{-1} \\ &= [L : M']_{kA} [L : M]_{kA}^{-1} \\ &= [M : M']_{kA}. \end{aligned}$$

The proof is finished. \square

We obtain immediately the following corollary:

Corollary 1.5. *Let V be a finite dimensional k_∞ -vector space and let M, M' be two kA -lattices in V . Let $\sigma : V \rightarrow V$ be a k_∞ -linear isomorphism such that $\sigma(M) \subset M'$. Then*

$$[M' : M]_{kA} = \det \sigma^{-1} \cdot \text{Fitt}_{kA} \left(\frac{M'}{\sigma(M)} \right).$$

2. ANDERSON MODULES

Let $d \geq 1$ be an integer. For an \mathbb{F}_q -algebra R , let $M = (a_{ij})_{1 \leq i, j \leq d} \in M_{d \times d}(R)$ be a $d \times d$ matrix with coefficients in R . If $k \geq 0$ is an integer, we set $M^{(k)}$ to be the matrix whose ij -entry is given by $(a_{i,j})^{(k)} := (a_{i,j}^{q^k})$. We denote by $M_{d \times d}(R)\{\tau\}$ the non-commutative ring of twisted polynomials in τ with coefficients in $M_{d \times d}(R)$ equipped with the usual addition and the commutation rule

$$\tau^k M = M^{(k)} \tau^k, \quad k \in \mathbb{N}.$$

Now let R be a field extension of \mathbb{F}_q equipped with an \mathbb{F}_q -algebra homomorphism $\iota : A \rightarrow R$. An *Anderson A -module* (or an *Anderson module* for short) of dimension d over R is an \mathbb{F}_q -algebra homomorphism $E : A \rightarrow M_{d \times d}(R)\{\tau\}$ such that

1) For all $a \in A$, if we write $E_a = \sum_{k \geq 0} E_{a,k} \tau^k$, $E_{a,k} \in M_{d \times d}(R)$, then we require

$$(E_{a,0} - \iota(a)I_d)^d = 0_d.$$

2) There exists $a \in A$ such that $E_a \neq E_{a,0}$.

We denote by $\partial_E : A \rightarrow M_{d \times d}(R)$ the \mathbb{F}_q -algebra homomorphism given by $\partial_E(a) = E_{a,0}$ for $a \in A$.

By definition, a *Drinfeld module* over R is an Anderson module of dimension 1 over R .

Let E be an Anderson module of dimension d over R as above and let B be an R -algebra. We can define two A -module structures on B^d . The first one is denoted by $E(B)$ where A acts on B^d via E :

$$a \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = \sum_{k \geq 0} E_{a,k} \begin{pmatrix} m_1^{q^k} \\ \vdots \\ m_d^{q^k} \end{pmatrix}, \quad \text{for } a \in A, \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} \in B^d.$$

The second one is denoted by $\text{Lie}_E(B)$ where A acts on B^d via ∂_E :

$$a \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = E_{a,0} \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix}, \quad \text{for } a \in A, \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} \in B^d.$$

2.1. Setup. From now on, let L/K be a finite extension and O_L be the integral closure of A in L . We set $L_\infty := L \otimes_K K_\infty$. We view L as contained in L_∞ via the diagonal embedding $x \mapsto x \otimes 1$.

We consider now an Anderson module E defined over O_L , that means $E_a \in M_{d \times d}(O_L)\{\tau\}$ for all $a \in A$. It follows that we have an \mathbb{F}_q -algebra homomorphism $\partial_E : A \rightarrow M_{d \times d}(O_L)$ which extends uniquely to a continuous map $\partial_E : K_\infty \rightarrow M_{d \times d}(L_\infty)$ (see for example [16], Section 2.1.2).

Let $M_{d \times d}(L)\{\{\tau\}\}$ be the non-commutative ring of twisted power series in τ with coefficient in $M_{d \times d}(L)$. One can show that there exist unique power series $\exp_E, \log_E \in I_d + M_{d \times d}(L)\{\{\tau\}\}\tau$ ([20], Theorem 5.9.6) satisfying the following equalities in $M_{d \times d}(L)\{\{\tau\}\}$:

$$\begin{aligned} \exp_E \partial_E(a) &= E_a \exp_E, \quad a \in A, \\ \log_E E_a &= \partial_E(a) \log_E, \quad a \in A, \\ \exp_E \log_E &= \log_E \exp_E = I_d. \end{aligned}$$

These power series \exp_E and \log_E are called *the exponential series* and *the logarithm series* attached to E . The exponential series \exp_E converges on $\text{Lie}_E(L_\infty)$ (see for example [20], Section 5.9, and the original article of Anderson [1]). In particular, \exp_E induces a homomorphism of A -modules

$$\exp_E : \text{Lie}_E(L_\infty) \rightarrow E(L_\infty).$$

We should stress that the logarithm series \log_E does not converge everywhere but only on a neighborhood of 0.

Following Taelman [31], we introduce *the unit module* attached to E/O_L as follows:

$$U(E/O_L) := \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(O_L)\}.$$

By [31], one can easily deduce that $U(E/O_L)$ is in fact an A -lattice in $\text{Lie}_E(L_\infty)$. Since $U(E/O_L) = \exp_E^{-1}(E(O_L))$, we deduce that the exponential series \exp_E induces an exact sequence of A -modules:

$$0 \rightarrow U(E/O_L) \rightarrow \text{Lie}_E(L_\infty) \rightarrow \frac{E(L_\infty)}{E(O_L)} \rightarrow \frac{E(L_\infty)}{E(O_L) + \exp_E(\text{Lie}_E(L_\infty))} \rightarrow 0.$$

One can show (see [31], Theorem 1) that

$$H(E/O_L) := \frac{E(L_\infty)}{E(O_L) + \exp_E(\text{Lie}_E(L_\infty))}$$

is a finite A -module which is called *the class module* attached to E/O_L .

2.2. The z -deformation of an Anderson module.

We keep the previous notation. Let z be an indeterminate over K_∞ and let $\mathbb{T}_z(K_\infty)$ be the Tate algebra in the variable z with coefficients in K_∞ . We set

$$\mathbb{T}_z(L_\infty) := L_\infty \otimes_{K_\infty} \mathbb{T}_z(K_\infty).$$

The map $\tau : L_\infty \rightarrow L_\infty, x \mapsto x^q$, extends uniquely into a continuous homomorphism of $\mathbb{F}_q[z]$ -algebras $\tau : \mathbb{T}_z(L_\infty) \rightarrow \mathbb{T}_z(L_\infty)$.

Recall that \widetilde{O}_L is the $\mathbb{F}_q(z)$ -vector space generated by O_L in \widetilde{L}_∞ . In particular, $\widetilde{A} = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} A$. Then the z -deformation of the Anderson module E denoted by \widetilde{E} is defined to be the homomorphism of $\mathbb{F}_q(z)$ -algebras $\widetilde{E} : \widetilde{A} \rightarrow M_{d \times d}(\widetilde{O}_L)\{\{\tau\}\}$ such that

$$\widetilde{E}_a = \sum_{k \geq 0} z^k E_{a,k} \tau^k, \quad a \in A.$$

There exists a unique element $\exp_{\widetilde{E}} \in I_d + \tau M_{d \times d}(L[z])\{\{\tau\}\}$ such that

$$\exp_{\widetilde{E}} \partial_E(a) = \widetilde{E}_a \exp_{\widetilde{E}}, \quad a \in A.$$

If $\exp_E = \sum_{i \geq 0} E_i \tau^i$, $E_i \in M_{d \times d}(L)$, one can show that $\exp_{\tilde{E}} = \sum_{i \geq 0} z^i E_i \tau^i$. In particular, $\exp_{\tilde{E}}$ converges on $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty))$ and induces a homomorphism of $A[z]$ -modules

$$\exp_{\tilde{E}} : \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \tilde{E}(\mathbb{T}_z(L_\infty)).$$

We define

$$U(\tilde{E}/O_L[z]) := \{x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(O_L[z])\},$$

$$U(\tilde{E}/\tilde{O}_L) := \{x \in \text{Lie}_{\tilde{E}}(\tilde{L}_\infty) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\tilde{O}_L)\}.$$

We can show (see [10], Proposition 1) that $U(\tilde{E}/O_L[z])$ is a finitely generated $A[z]$ -module in $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty))$ and $U(\tilde{E}/\tilde{O}_L)$ is an \tilde{A} -lattice in $\text{Lie}_{\tilde{E}}(\tilde{L}_\infty)$. Further, $U(\tilde{E}/\tilde{O}_L)$ is the $\mathbb{F}_q(z)$ -vector space generated by $U(\tilde{E}/O_L[z])$.

We set

$$H(\tilde{E}/O_L[z]) := \frac{\tilde{E}(\mathbb{T}_z(L_\infty))}{\tilde{E}(O_L[z]) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)))},$$

$$H(\tilde{E}/\tilde{O}_L) := \frac{\tilde{E}(\tilde{L}_\infty)}{\tilde{E}(\tilde{O}_L) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\tilde{L}_\infty))}.$$

Proposition 2.1. *The $A[z]$ -module $H(\tilde{E}/O_L[z])$ is a finitely generated and torsion $\mathbb{F}_q[z]$ -module, with no z -torsion. In particular,*

$$H(\tilde{E}/\tilde{O}_L) = 0.$$

Proof. The proof is similar to that of [8], Theorem 3.3 for Anderson $\mathbb{F}_q[\theta]$ -modules (see also [10], Proposition 2 and [7], Section 2.2 for Drinfeld modules). \square

Let $\text{ev} : \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \text{Lie}_E(L_\infty)$ be the evaluation at $z = 1$. Observe that ev induces a short exact sequence of A -modules:

$$0 \rightarrow (z-1) \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \xrightarrow{\text{ev}} \text{Lie}_E(L_\infty) \rightarrow 0.$$

We define

$$U_{St}(E/\tilde{O}_L) := \text{ev}(U(\tilde{E}/O_L[z])).$$

We observe that $U_{St}(E/\tilde{O}_L) \subset U(E/\tilde{O}_L)$. The A -module $U_{St}(E/\tilde{O}_L)$ is called *the module of Stark units* attached to E/\tilde{O}_L . One can prove that $U_{St}(E/\tilde{O}_L)$ is an A -lattice in $\text{Lie}_E(L_\infty)$ and that we have (see [7, 8, 10]):

$$(2.1) \quad \text{Fitt}_A \left(\frac{U(E/\tilde{O}_L)}{U_{St}(E/\tilde{O}_L)} \right) = \text{Fitt}_A(H(E/\tilde{O}_L)).$$

3. LOCAL FACTORS

3.1. Local factors. For all nonzero ideal \mathcal{I} in O_L , we define

$$Z_{\mathcal{I}}(E/O_L) := \frac{\text{Fitt}_A(\text{Lie}_E(O_L/\mathcal{I}O_L))}{\text{Fitt}_A(E(O_L/\mathcal{I}O_L))} \subset K_\infty,$$

$$Z_{\mathcal{I}}(\tilde{E}/\tilde{O}_L) := \frac{\text{Fitt}_{\tilde{A}}(\text{Lie}_{\tilde{E}}(\tilde{O}_L/\mathcal{I}\tilde{O}_L))}{\text{Fitt}_{\tilde{A}}(\tilde{E}(\tilde{O}_L/\mathcal{I}\tilde{O}_L))} \subset \tilde{K}_\infty.$$

If \mathfrak{P} is a nonzero prime ideal of O_L , $Z_{\mathfrak{P}}$ will be called the local factor at \mathfrak{P} . For an ideal I of A , we will write Z_I instead of Z_{IO_L} .

Lemma 3.1. *Let I and J be two nonzero coprime ideals of O_L . Then we have*

$$\begin{aligned} Z_{IJ}(E/O_L) &= Z_I(E/O_L) Z_J(E/O_L), \\ Z_{IJ}(\tilde{E}/\tilde{O}_L) &= Z_I(\tilde{E}/\tilde{O}_L) Z_J(\tilde{E}/\tilde{O}_L). \end{aligned}$$

Proof. By the Chinese Remainder Theorem, we have

$$\begin{aligned} \text{Fitt}_A(\text{Lie}_E(O_L/IJO_L)) &= \text{Fitt}_A(\text{Lie}_E(O_L/IO_L)) \text{Fitt}_A(\text{Lie}_E(O_L/JO_L)) \\ \text{Fitt}_A(E(O_L/IJO_L)) &= \text{Fitt}_A(E(O_L/IO_L)) \text{Fitt}_A(E(O_L/JO_L)). \end{aligned}$$

It follows that

$$Z_{IJ}(E/O_L) = Z_I(E/O_L) Z_J(E/O_L).$$

Similarly, we get the second equality. \square

Lemma 3.2. *Let I be a non zero ideal of O_L and $n \geq 1$. Then we have*

$$\begin{aligned} Z_I(E/O_L) &= Z_{I^n}(E/O_L), \\ Z_I(\tilde{E}/\tilde{O}_L) &= Z_{I^n}(\tilde{E}/\tilde{O}_L). \end{aligned}$$

Proof. We give a proof by induction for the first equality since the second one can be proved similarly. The case $n = 1$ is clear. Next, the short exact sequence

$$0 \rightarrow I^n/I^{n+1} \rightarrow O_L/I^{n+1} \rightarrow O_L/I^n \rightarrow 0$$

gives

$$\text{Fitt}_A(\text{Lie}_E(O_L/I^{n+1})) = \text{Fitt}_A(\text{Lie}_E(I^n/I^{n+1})) \text{Fitt}_A(\text{Lie}_E(O_L/I^n)),$$

$$\text{Fitt}_A(E(O_L/I^{n+1})) = \text{Fitt}_A(E(I^n/I^{n+1})) \text{Fitt}_A(E(O_L/I^n)).$$

For any $x \in I^n$ and $a \in A$, we know that $E_a(x) \equiv \partial_E(a)x \pmod{I^{qn}}$. Thus $E(I^n/I^{n+1}) \simeq \text{Lie}_E(I^n/I^{n+1})$ as A -modules. It follows that $\text{Fitt}_A(E(I^n/I^{n+1})) = \text{Fitt}_A(\text{Lie}_E(I^n/I^{n+1}))$. Thus we get

$$Z_{I^n}(E/O_L) = Z_{I^{n+1}}(E/O_L)$$

as desired. \square

The following proposition is an immediate consequence of Lemmas 3.1 and 3.2:

Proposition 3.3. *Let I be a nonzero ideal in O_L . Then*

$$\begin{aligned} Z_I(E/O_L) &= \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L), \\ Z_I(\tilde{E}/\tilde{O}_L) &= \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L), \end{aligned}$$

where the product runs over all the primes \mathfrak{P} of O_L containing I .

3.2. Elimination of local factors. Let $I = (\xi)$ be a principal ideal of A . Then there exists a unique Anderson A -module E' over O_L satisfying for all $a \in A$, $\xi E'_a = E_a \xi$. For $a \in A$, if $E_a = \sum_{i=0}^r E_{a,i} \tau^i$, then $E'_a = \sum_{i=0}^r \xi^{q^i-1} E_{a,i} \tau^i$. Note that for all $a \in A$, $\partial_E(a) = \partial_{E'}(a)$.

Lemma 3.4. *Let $I = (\xi)$ be a principal ideal in A and E' be the Anderson A -module over O_L satisfying $\xi E'_a = E_a \xi$ for all $a \in A$. Then $IU(E'/O_L)$ is a sub- A -module of $U(E/O_L)$. Further we have*

$$Z_I(E/O_L) = [U(E'/O_L) : U(E/O_L)]_A \frac{\text{Fitt}_A(H(E/O_L))}{\text{Fitt}_A(H(E'/O_L))}.$$

Proof. We denote by d the dimension of E . Because of the unicity of exponential series, we see that

$$\exp_{E'} = \xi^{-1} \exp_E \xi.$$

We deduce that

$$IU(E'/O_L) = \xi U(E'/O_L) = \{x \in \text{Lie}_E(L_\infty), \exp_E(x) \in \xi O_L^d\}.$$

And thus $IU(E'/O_L)$ is a sub- A -module of $U(E/O_L)$. We deduce also that \exp_E induces an injection $\exp_E : U(E/O_L)/IU(E'/O_L) \hookrightarrow O_L^d/\xi O_L^d$. We therefore get a short exact sequence of torsion A -modules:

$$0 \longrightarrow \frac{U(E/O_L)}{IU(E'/O_L)} \longrightarrow E \left(\frac{O_L}{\xi O_L} \right) \longrightarrow \frac{O_L^d + \exp_E(\text{Lie}_E(L_\infty))}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} \longrightarrow 0$$

where A acts on the third term via E .

We deduce that

$$\text{Fitt}_A(E(O_L/IO_L)) = \text{Fitt}_A \left(\frac{U(E/O_L)}{IU(E'/O_L)} \right) \text{Fitt}_A \left(\frac{O_L^d + \exp_E(\text{Lie}_E(L_\infty))}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} \right).$$

Since $U(E'/O_L)$ et $\text{Lie}_E(O_L)$ have the same A -ranks, we get

$$\begin{aligned} \text{Fitt}_A \left(\frac{U(E/O_L)}{IU(E'/O_L)} \right) &= [U(E/O_L) : U(E'/O_L)]_A \text{Fitt}_A \left(\frac{U(E'/O_L)}{IU(E'/O_L)} \right) \\ &= [U(E/O_L) : U(E'/O_L)]_A \text{Fitt}_A(\text{Lie}_E(O_L/IO_L)). \end{aligned}$$

Further, we have

$$\begin{aligned} &\text{Fitt}_A \left(\frac{O_L^d + \exp_E(\text{Lie}_E(L_\infty))}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} \right) \\ &= \text{Fitt}_A \left(\frac{L_\infty^d}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} \right) \text{Fitt}_A(H(E/O_L))^{-1}. \end{aligned}$$

Since $\exp_{E'} = \xi^{-1} \exp_E \xi$ and $\xi L_\infty^d = L_\infty^d$, we get isomorphisms of A -modules:

$$\frac{L_\infty^d}{\xi O_L^d + \exp_E(\text{Lie}_E(L_\infty))} = \frac{L_\infty^d}{\xi O_L^d + \xi \exp_{E'}(\text{Lie}_E(L_\infty))} \simeq H(E'/O_L).$$

The Lemma follows immediately. \square

Corollary 3.5. *Let $\xi \in A$ be a nonzero element. Denote by $\mathfrak{P}_1, \dots, \mathfrak{P}_n$ the prime ideals in O_L containing ξ . Let E' be the Anderson A -module over O_L satisfying for all $a \in A$, $\xi E'_a = E_a \xi$. Then*

$$\prod_{i=1}^n Z_{\mathfrak{P}_i}(E/O_L) = [U(E'/O_L) : U(E/O_L)]_A \frac{\text{Fitt}_A(H(E/O_L))}{\text{Fitt}_A(H(E'/O_L))}.$$

Proof. This is an immediate consequence of Proposition 3.3 and Lemma 3.4. \square

By similar arguments and Proposition 2.1, we obtain the z -version of Corollary 3.5:

Proposition 3.6. *Let $\xi \in A$ be a nonzero element. Denote by $\mathfrak{P}_1, \dots, \mathfrak{P}_n$ the prime ideals in O_L containing ξ . Let E' be the Anderson A -module over O_L satisfying for all $a \in A$, $\xi E'_a = E_a \xi$. Then we have*

$$\prod_{i=1}^n Z_{\mathfrak{P}_i}(\tilde{E}/\tilde{O}_L) = \left[U(\tilde{E}'/\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}.$$

We also need the following lemma.

Lemma 3.7. *Let $\xi \in A$ be a nonzero element and E' be the Anderson A -module over O_L satisfying for all $a \in A$, $\xi E'_a = E_a \xi$. Then for a prime \mathfrak{P} in O_L containing ξ , we have $Z_{\mathfrak{P}}(E'/O_L) = A$ and $Z_{\mathfrak{P}}(\tilde{E}'/\tilde{O}_L) = \tilde{A}$. Otherwise, we have $Z_{\mathfrak{P}}(E'/O_L) = Z_{\mathfrak{P}}(E/O_L)$ and $Z_{\mathfrak{P}}(\tilde{E}'/\tilde{O}_L) = Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$.*

Proof. Suppose first that \mathfrak{P} is a prime ideal in O_L containing ξ . It follows that if $a \in A$, then $E'_a \in a + \xi O_L[\tau]$. Thus for all $x \in O_L$ and $a \in A$, we have $E'_a(x) \equiv ax \pmod{\mathfrak{P}O_L}$. We deduce that $Z_{\mathfrak{P}}(E'/O_L) = A$ and $Z_{\mathfrak{P}}(\tilde{E}'/\tilde{O}_L) = \tilde{A}$.

We consider now the case where \mathfrak{P} is not a prime ideal in A containing ξ . The desired equality is an immediate consequence of the fact that the multiplication by ξ is an isomorphism of $O_L/\mathfrak{P}O_L$. \square

4. CLASS FORMULA À LA TAE LMAN

We keep the notation of Section 3. Recall that L/K is a finite extension, O_L denotes the integral closure of A in L and E is an Anderson module defined over O_L . Then \tilde{E} denotes the z -deformation of E and we have defined local factors attached to E and \tilde{E} in Section 3.

4.1. Admissible Anderson modules.

Definition 4.1. We say that an Anderson module E defined over O_L is *admissible* if the following conditions are satisfied:

(P) There exists a finite set S of primes of A such that for all prime \mathfrak{P} of O_L with $\mathfrak{P} \notin S$, there exists $x_{\mathfrak{P}} \in A[z]$ such that

$$\text{Fitt}_{\tilde{A}}(\tilde{E}(\tilde{O}_L/\mathfrak{P}\tilde{O}_L)) = x_{\mathfrak{P}}\tilde{A},$$

and

$$\begin{aligned} \text{Fitt}_A(\text{Lie}_E(O_L/\mathfrak{P}O_L)) &= x_{\mathfrak{P}}(0)A, \\ \text{Fitt}_A(E(O_L/\mathfrak{P}O_L)) &= x_{\mathfrak{P}}(1)A. \end{aligned}$$

(C) The infinite product

$$\prod_{\mathfrak{p} \notin S} \frac{x_{\mathfrak{p}}(0)}{x_{\mathfrak{p}}}$$

converges in \widetilde{K}_{∞} .

Let E be an admissible Anderson module defined over O_L as above. We set

$$x := \prod_{\mathfrak{p} \notin S} \frac{x_{\mathfrak{p}}(0)}{x_{\mathfrak{p}}} \in \widetilde{K}_{\infty},$$

$$y := \prod_{\mathfrak{p} \notin S} \frac{x_{\mathfrak{p}}(0)}{x_{\mathfrak{p}}(1)} \in K_{\infty}.$$

Then we define

$$L(\widetilde{E}/\widetilde{O}_L) := (x) \prod_{\mathfrak{p} \in S} Z_{\mathfrak{p}}(\widetilde{E}/\widetilde{O}_L) \subset \widetilde{K}_{\infty},$$

$$L(E/O_L) := (y) \prod_{\mathfrak{p} \in S} Z_{\mathfrak{p}}(E/O_L) \subset K_{\infty}.$$

Further, there exists a finite set $S' \subset S$ of prime of O_L such that $\prod_{\mathfrak{p} \notin S'} \frac{x_{\mathfrak{p}}(0)}{x_{\mathfrak{p}}}$ converges in $\mathbb{T}_z(K_{\infty})$.

4.2. A partial class formula over $\mathbb{F}_q(z)$ for admissible Anderson modules.
This section is devoted to the proof of the class formula over $\mathbb{F}_q(z)$ for admissible Anderson modules.

Theorem 4.2. *Let E be an admissible Anderson module defined over O_L . Then we have the partial class formula:*

$$(4.1) \quad \left[\text{Lie}_{\widetilde{E}}(\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L) \right]_{\widetilde{A}} \subset L(\widetilde{E}/\widetilde{O}_L).$$

The lattice $\left[\text{Lie}_{\widetilde{E}}(\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L) \right]_{\widetilde{A}}$ is also called the *regulator* of $U(\widetilde{E}/\widetilde{O}_L)$. This partial class formula thus states, roughly speaking, that the regulator of the z -units is a sub- \widetilde{A} -lattice of the L -function lattice.

Before giving a proof of the above Theorem, we need some preparatory results. We denote by $\mathbb{F}_q[z]_{(z)}$ the localization of $\mathbb{F}_q[z]$ at z .

Lemma 4.3. *Let E be an Anderson module of dimension d defined over O_L . We write $\exp_E = \sum_{k \geq 0} e_k \tau^k$. Let $n \geq 1$ be an integer such that for all $0 \leq k \leq n-1$, the coefficient e_k is integral, i.e. $e_k \in M_{d \times d}(O_L)$. Then any non zero element in $U(\widetilde{E}/\widetilde{O}_L)$ can be written in the form $z^a(\alpha + z^n \beta)$ with $a \in \mathbb{Z}$, $\alpha \in (\mathbb{F}_q[z]_{(z)} O_L[z])^d$ not divisible by z and $\beta \in (\mathbb{F}_q[z]_{(z)} \mathbb{T}_z(L_{\infty}))^d$.*

Conversely, for any $\alpha \in (\mathbb{F}_q[z]_{(z)} O_L[z])^d$, there exists $\beta \in (\mathbb{F}_q[z]_{(z)} \mathbb{T}_z(L_{\infty}))^d$ such that $(\alpha + z^n \beta) \in \mathbb{F}_q[z]_{(z)} U(\widetilde{E}/O_L[z])$.

Proof. We write $\log_E = \sum_{k \geq 0} l_k \tau^k$ with $l_k \in L$. Since $e_k \in M_{d \times d}(O_L)$ for all $0 \leq k \leq n-1$, it follows that for $0 \leq k \leq n-1$, we also have $l_k \in M_{d \times d}(O_L)$. Therefore, if $u \in U(\widetilde{E}/O_L[z])$, then we have

$$u \equiv \log_{\widetilde{E}}(\exp_{\widetilde{E}}(u)) \pmod{z^n L_{\infty}[[z]]},$$

and thus

$$u \in (O_L[z])^d \pmod{z^n \mathbb{T}_z(L_\infty)}.$$

Moreover, if $u \in (\mathbb{F}_q[z]_{(z)} \mathbb{T}_z(L_\infty))^d$ such that $zu \in (\mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z]))^d$, then $u \in (\mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z]))^d$. Thus, if $u \in U(\tilde{E}/O_L[z])$ is nonzero, then there is $a \in \mathbb{N}$, $\alpha \in (O_L[z])^d$ not divisible by z and $\beta \in (\mathbb{T}_z(L_\infty))^d$ such that $u = z^a(\alpha + z^n\beta)$.

Since $U(\tilde{E}/\tilde{O}_L)$ is the $\mathbb{F}_q(z)$ -vector space generated by $U(\tilde{E}/O_L[z])$, we deduce that any element $u \in U(\tilde{E}/\tilde{O}_L)$ can be written in the form $u = z^a(\alpha + z^n\beta)$ with $a \in \mathbb{Z}$, $\alpha \in (\mathbb{F}_q[z]_{(z)} O_L[z])^d$ not divisible by z , and $\beta \in (\mathbb{F}_q[z]_{(z)} \mathbb{T}_z(L_\infty))^d$.

Conversely, let $\alpha \in (O_L[z])^d$. By the assumption on the coefficients e_k ($0 \leq k \leq n$) of \exp_E , there exist $x \in (O_L[z])^d$ and $y \in (\mathbb{T}_z(L_\infty))^d$ such that $\exp_{\tilde{E}}(\alpha) = x + z^n y$. By Proposition 2.1, there is $f \in \mathbb{F}_q[z]$ not divisible by z , $y_1 \in (O_L[z])^d$ and $y_2 \in (\mathbb{T}_z(L_\infty))^d$ such that $fy = y_1 + \exp_{\tilde{E}}(y_2)$. Therefore, $f\alpha - z^n y_2 \in U(\tilde{E}/O_L[z])$. This concludes the proof. \square

Proposition 4.4. *We keep the hypothesis of Lemma 4.3. Then any non zero element in $\left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}$ can be written in the form $z^a(\alpha + z^n\beta)$ with $a \in \mathbb{Z}$, $\alpha \in \mathbb{F}_q[z]_{(z)} A$ not divisible by z and $\beta \in \mathbb{F}_q[z]_{(z)} \mathbb{T}_z(K_\infty)$.*

Proof. The lattice $\left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}$ can be computed in the following way: we fix a K -basis \mathcal{B} of $\text{Lie}_E(L)$ in $\text{Lie}_E(O_L)$. If we set $M := A\mathcal{B}$, M is a sub- A -lattice of $\text{Lie}_E(O_L)$. It follows that

$$\left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}} = \left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : \tilde{M} \right]_{\tilde{A}} \left[\tilde{M} : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}.$$

Since

$$\left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : \tilde{M} \right]_{\tilde{A}} = \mathbb{F}_q(z) \left[\text{Lie}_E(O_L) : M \right]_A,$$

we are reduced to computing $\left[\tilde{M} : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}$, that is, determinants of vectors of elements of $U(\tilde{E}/\tilde{O}_L)$ in the basis \mathcal{B} .

First we remark that because of Lemma 4.3, for all $m \in O_L^d = \text{Lie}_E(O_L)$, we can find a unit in $\mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z])$ which is congruent to m modulo z^n . Thus the evaluation $z \mapsto 0$ induces a surjective homomorphism of A -modules $\mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z]) \rightarrow \text{Lie}_E(O_L)$. Since $\text{Lie}_E(O_L)$ is finitely generated and torsion free over the Dedekind domain A , it is projective. This implies that there is a section to the above evaluation map, i.e. a homomorphism of A -modules $\iota : \text{Lie}_E(O_L) \rightarrow \mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z])$ such that for all $m \in \text{Lie}_E(O_L)$, $\iota(m) \equiv m \pmod{z}$.

By induction, we can prove that for all $b \in \mathbb{N}$, for all $u \in \mathbb{F}_q[z]_{(z)} U(\tilde{E}/O_L[z])$, there exist $m_0, \dots, m_{b-1} \in \iota(\text{Lie}_E(O_L))$ such that

$$\sum_{j=0}^{b-1} z^j \iota(m_j) \equiv u \pmod{z^b}.$$

We consider now a vector $u = (u_1, \dots, u_s) \in U(\tilde{E}/\tilde{O}_L)^s$ where s is the rank of O_L^d over A , and compute its determinant in the basis \mathcal{B} . Since $U(\tilde{E}/\tilde{O}_L)$ is the $\mathbb{F}_q(z)$ -vector space generated by $U(\tilde{E}/O_L[z])$, we can suppose $(u_1, \dots, u_s) \in$

$(\mathbb{F}_q[z]_{(z)}U(\tilde{E}/O_L[z]))^s$. We also suppose that $\det_{\mathcal{B}}(u)$ is not zero and let a be the biggest integer such that z^a divides $\det_{\mathcal{B}}(u)$.

By the previous discussion, for $1 \leq i \leq s$, there exist $m_{i,0}, \dots, m_{i,a+n-1} \in \iota(\text{Lie}_E(O_L))$ such that

$$\sum_{j=0}^{a+n-1} z^j \iota(m_{i,j}) \equiv u_i \pmod{z^{a+n}}.$$

We set $u'_i := \sum_{j=0}^{a+n-1} z^j \iota(m_{i,j}) \in \mathbb{F}_q[z]_{(z)}U(\tilde{E}/O_L[z])$. It follows that

$$\det_{\mathcal{B}}(u) \equiv \det_{\mathcal{B}}(u') \pmod{z^{a+n}}$$

and

$$\det_{\mathcal{B}}(u') = \sum_{j=0}^{a+n-1} z^j \sum_{i_1+\dots+i_s=j} \det_{\mathcal{B}}(\iota(m_{1,i_1}), \dots, \iota(m_{s,i_s})) \pmod{z^{a+n}}.$$

By induction on j , we prove that for all $0 \leq j < a$,

$$\sum_{i_1+\dots+i_s=j} \det_{\mathcal{B}}(\iota(m_{1,i_1}), \dots, \iota(m_{s,i_s})) = 0.$$

In fact, its evaluation at $z = 0$ is 0, and this is equal to $\sum_{i_1+\dots+i_s=j} \det_{\mathcal{B}}(m_{1,i_1}, \dots, m_{s,i_s})$. Thus

$$\sum_{i_1+\dots+i_s=j} \det_{\mathcal{B}}(\iota(m_{1,i_1}), \dots, \iota(m_{s,i_s})) = 0.$$

We obtain

$$\det_{\mathcal{B}}(u') = \sum_{j=a}^{a+n-1} z^j \sum_{i_1+\dots+i_s=j} \det_{\mathcal{B}}(\iota(m_{1,i_1}), \dots, \iota(m_{s,i_s})) \pmod{z^{a+n}}.$$

Thus, again by Lemma 4.3, if $a \leq j < a+n$, the coefficient of z^j of this determinant belongs to $[M : \text{Lie}_E(O_L)]_A$, as desired.

The same arguments obviously apply to sums of determinants, and this concludes the proof. \square

We are now ready to prove the main theorem of this Section.

Proof of Theorem 4.2. We divide the proof into two steps.

Step 1. We first show that Theorem 4.2 holds for any admissible Anderson module E defined over O_L satisfying the following condition: for all primes \mathfrak{P} of O_L , $Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$ is principal.

In fact, we claim that there exists a sequence $\xi_n \in A \setminus \{0\}$, $n \geq 0$ such that:

- (i) for all $n \in \mathbb{N}$, ξ_{n+1} divides ξ_n ,
- (ii) for any prime P in A , there exists an $n \in \mathbb{N}$ such that $\xi_n \in P$,
- (iii) for all $n \in \mathbb{N}$, $E^{[n]}$ denotes the Anderson module such that $\xi_n E_a^{[n]} = E_a \xi_n$ for all $a \in A$, we denote $\exp_{E^{[n]}} = \sum_{k \geq 0} e_k^{[n]} \tau^k$, then for all $0 \leq k \leq n-1$, the coefficient $e_k^{[n]}$ is integral: $e_k^{[n]} \in M_{d \times d}(O_L)$.

In fact, recall that $\exp_E = \sum_{k \geq 0} e_k \tau^k$. Then we have

$$\exp_{E^{[n]}} = \xi_n^{-1} \exp_E \xi_n = \sum_{k \geq 0} \xi_n^{q^k - 1} e_k \tau^k,$$

that is, $e_k^{[n]} = \xi_n^{q^k - 1} e_k$. Thus we can choose $\xi_n \in A \setminus \{0\}$, $n \geq 0$ such that Conditions (i) and (iii) are satisfied. Condition (ii) is verified by requiring $\xi_n \in P$ for all P of degree lower than n .

We now write

$$\begin{aligned} & \left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}} \\ &= \left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}^{[n]}/\tilde{O}_L) \right]_{\tilde{A}} \left[U(\tilde{E}^{[n]}/\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}. \end{aligned}$$

By Proposition 3.6, we have

$$\left[U(\tilde{E}^{[n]}/\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}} = \prod_{\xi_n \in \mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$$

so that

$$\left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}} = \left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}^{[n]}/\tilde{O}_L) \right]_{\tilde{A}} \prod_{\xi_n \in \mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L).$$

Let $y_n \in A[z]$ denote the generator of $\prod_{\xi_n \in \mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$ given by the admissibility condition and y its limit, which generates $\prod_{\mathfrak{P}} Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$. For all non zero $u \in \left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}/\tilde{O}_L) \right]_{\tilde{A}}$, there exists $x_n \in \left[\text{Lie}_{\tilde{E}}(\tilde{O}_L) : U(\tilde{E}^{[n]}/\tilde{O}_L) \right]_{\tilde{A}}$ such that $u = x_n y_n$. Thus x_n converges to a certain element x and $u = xy$. By Proposition 4.4, x_n is of the form $z^{a_n}(\alpha_n + z^n \beta_n)$ with $a_n \in \mathbb{Z}$, $\alpha_n \in \mathbb{F}_q[z]_{(z)} A$ not divisible by z and $\beta_n \in \mathbb{F}_q[z]_{(z)} \mathbb{T}_z(K_\infty)$. It implies that the sequence $(a_n)_{n \in \mathbb{N}}$ becomes stationary. Moreover, for all n_0 , the sequence $(x_n \bmod z^{n_0})_{n \geq n_0}$ converges to $x \bmod z^{n_0}$ in $\mathbb{F}_q[z]_{(z)} A / (z^{n_0})$, which is discrete. Thus, the coefficients of x must belong to A , and this implies that the limit x belongs to \tilde{A} . Thus we obtain the proof of Theorem 4.2 for E verifying $Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$ is principal for all primes \mathfrak{P} of O_L .

Step 2. We now deal with the general case. Let E be an admissible Anderson module defined over O_L . Then there exists a finite set S of primes of A such that $Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$ is principal for all primes \mathfrak{P} of O_L with $\mathfrak{P} \notin S$. We choose a nonzero element ξ of A such that ξ belongs to for all primes \mathfrak{P} of O_L with $\mathfrak{P} \in S$.

Let E' be the Anderson A -module over O_L satisfying for all $a \in A$, $\xi E'_a = E_a \xi$. By Lemma 3.7, for any prime \mathfrak{P} in O_L containing ξ , we have $Z_{\mathfrak{P}}(\tilde{E}'/\tilde{O}_L) = \tilde{A}$. Otherwise, we have $Z_{\mathfrak{P}}(\tilde{E}'/\tilde{O}_L) = Z_{\mathfrak{P}}(\tilde{E}/\tilde{O}_L)$. In particular,

$$L(\tilde{E}/\tilde{O}_L) = L(\tilde{E}'/\tilde{O}_L) \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L)$$

where the product runs over the primes \mathfrak{P} in O_L containing ξ .

Further, E' is admissible and for all primes \mathfrak{P} of O_L , $Z_{\mathfrak{P}}(\tilde{E}'/\tilde{O}_L)$ is principal. By Step 1, Theorem 4.2 holds for E' and we get

$$\left[\text{Lie}_{\tilde{E}'}(\tilde{O}_L) : U(\tilde{E}'/\tilde{O}_L) \right]_{\tilde{A}} \subset L(\tilde{E}'/\tilde{O}_L).$$

Therefore, by Proposition 3.6, we deduce

$$\begin{aligned} \left[\text{Lie}_{\widetilde{E}}(\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L) \right]_{\widetilde{A}} &= \left[\text{Lie}_{\widetilde{E}'}(\widetilde{O}_L) : U(\widetilde{E}'/\widetilde{O}_L) \right]_{\widetilde{A}} \left[U(\widetilde{E}'/\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L) \right]_{\widetilde{A}} \\ &= \left[\text{Lie}_{\widetilde{E}'}(\widetilde{O}_L) : U(\widetilde{E}'/\widetilde{O}_L) \right]_{\widetilde{A}} \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L) \\ &\subset L(\widetilde{E}'/\widetilde{O}_L) \prod_{\mathfrak{P}} Z_{\mathfrak{P}}(E/O_L) = L(\widetilde{E}/\widetilde{O}_L) \end{aligned}$$

as required. \square

4.3. A class formula for admissible Anderson modules. We now apply the class formula over $\mathbb{F}_q(z)$ to deduce a class formula for admissible Anderson modules.

Theorem 4.5. *Let E be an admissible Anderson module defined over O_L . Then we have the class formula*

$$[\text{Lie}_E(O_L) : U_{\text{st}}(E/O_L)]_A = L(E/O_L).$$

Proof. By Corollary 3.5 and Lemma 3.7, we can suppose that for all primes \mathfrak{P} of O_L , there exists an element $x_{\mathfrak{P}} \in A[z]$ such that $\frac{x_{\mathfrak{P}}(0)}{x_{\mathfrak{P}}(1)}$ generates $Z_{\mathfrak{P}}(E/O_L)$ and $\frac{x_{\mathfrak{P}}(0)}{x_{\mathfrak{P}}}$ generates $Z_{\mathfrak{P}}(\widetilde{E}/\widetilde{O}_L)$. Further, we have

$$x(z) = \prod_{\mathfrak{P}} \frac{x_{\mathfrak{P}}(0)}{x_{\mathfrak{P}}} \in \mathbb{T}_z(K_{\infty}).$$

First we show that we have an inclusion

$$(4.2) \quad [\text{Lie}_E(O_L) : U_{\text{st}}(E/O_L)]_A \subset L(E/O_L).$$

In fact, let us fix now a basis \mathcal{B} of $\text{Lie}_E(L_{\infty})$ over K_{∞} . This is still a basis of $\text{Lie}_E(\widetilde{L}_{\infty})$ over \widetilde{K}_{∞} , and a basis of $\text{Lie}_E(\mathbb{T}_z(L_{\infty}))$ over $\mathbb{T}_z(K_{\infty})$. Therefore we can use this basis to compute both $[\text{Lie}_{\widetilde{E}}(\widetilde{O}_L) : U(\widetilde{E}/\widetilde{O}_L)]_{\widetilde{A}}$ and $[\text{Lie}_E(O_L) : U(E/O_L)]_A$. It follows that

$$\left[\widetilde{A}\mathcal{B} : \text{Lie}_E(\widetilde{O}_L) \right]_{\widetilde{A}} = \mathbb{F}_q(z) [\mathcal{A}\mathcal{B} : \text{Lie}_E(O_L)]_A.$$

Recall that $U(\widetilde{E}/\widetilde{O}_L) = \mathbb{F}_q(z)U(\widetilde{E}/O_L[z])$. It implies that if we consider a determinant of elements of $U_{\text{st}}(E/O_L)$ in the basis \mathcal{B} , it comes by evaluation at $z = 1$ from the determinant of elements in $U(\widetilde{E}/O_L[z])$ and thus belong to $[\mathcal{A}\mathcal{B} : \text{Lie}_E(O_L)]_A^{-1} x(1)$. It follows immediately that $[\text{Lie}_E(O_L) : U_{\text{st}}(E/O_L)]_A \subset L(E/O_L)$.

Next we claim that the inclusion (4.2) is in fact an equality. We write $L(E/O_L) = uA$ where u is a unit in K_{∞}^{\times} . By (4.2), we have

$$\frac{1}{u} [\text{Lie}_E(O_L) : U_{\text{st}}(E/O_L)]_A \subset A.$$

Thus there exists an integer $h \geq 1$ such that

$$\frac{1}{u^h} [\text{Lie}_E(O_L) : U_{\text{st}}(E/O_L)]_A^h = \alpha A, \quad \alpha \in A \setminus \{0\}.$$

Let $\theta \in A \setminus \mathbb{F}_q$. By a similar proof to that of [31], Theorem 2, we have

$$[\text{Lie}_E(O_L) : U_{\text{st}}(E/O_L)]_{\mathbb{F}_q[\theta]} = \alpha_{\theta} \mathbb{F}_q[\theta], \quad \text{with } v_{\infty}(\alpha_{\theta}) = 0.$$

It follows that $N_{K/\mathbb{F}_q(\theta)}(\alpha) \in \mathbb{F}_q^\times$. Thus $\alpha \in \mathbb{F}_q^\times$ and we get

$$[\mathrm{Lie}_E(O_L) : U_{\mathrm{st}}(E/O_L)]_A = L(E/O_L)$$

as required. \square

Remark 4.6. We can generalize [31], Theorem 2 to the $\mathbb{F}_q(z)$ -context. In fact, the proof of this theorem carries over without modification. By the same arguments as above, we deduce that the inclusion (4.1) is also an equality, i.e. the class formula over $\mathbb{F}_q(z)$ holds for admissible Anderson modules.

5. ADMISSIBLE ANDERSON MODULES

This section is devoted to exhibit a large class of admissible Anderson A -modules including Drinfeld modules. The fact that Drinfeld modules are admissible was proved thanks to the works of Gekeler [19] and Debry [14] (see also [30]). We will extend these works and show that any A -finite Anderson module is admissible. Finally, when $A = \mathbb{F}_q[\theta]$, we recall the nontrivial fact that any Anderson $\mathbb{F}_q[\theta]$ -module is admissible.

5.1. Dual Dieudonné t -modules. Let F be a perfect field of positive characteristic containing \mathbb{F}_q . Let $\sigma : F \rightarrow F$ be the isomorphism of \mathbb{F}_q -algebras given by $\sigma(t) = t^{\frac{1}{q}}$. Then σ induces a continuous isomorphism of $\mathbb{F}_q((\frac{1}{t}))$ -algebras $\sigma : F((\frac{1}{t})) \simeq F((\frac{1}{t}))$. A dual Dieudonné t -module over F is a pair (V, σ) where V is a finite dimensional $F((\frac{1}{t}))$ -vector space and $\sigma : V \rightarrow V$ is a surjective $\mathbb{F}_q((\frac{1}{t}))$ -linear map such that

$$\sigma(\lambda v) = \sigma(\lambda)\sigma(v), \quad \text{for all } \lambda \in F\left(\left(\frac{1}{t}\right)\right), v \in V.$$

Let $\frac{s}{r} \in \mathbb{Q}$ with $s, r \in \mathbb{Z}$ such that $r > 1$ and $(s, r) = 1$. We denote by $V_{\frac{s}{r}}(F, \sigma)$ the dual Dieudonné t -module given by

$$\begin{aligned} V_{\frac{s}{r}}(F) &= \bigoplus_{i=1}^r F\left(\left(\frac{1}{t}\right)\right) e_i, \\ \sigma(e_i) &= e_{i+1}, \sigma(e_r) = t^s e_1, \quad i \in \{1, \dots, r-1\}. \end{aligned}$$

Proposition 5.1. *We assume further that F is algebraically closed. Let (V, σ) be a dual Dieudonné t -module over F . Then there exist $\alpha_1, \dots, \alpha_m \in \mathbb{Q}$ such that we have an isomorphism of dual Dieudonné t -modules*

$$V \simeq \bigoplus_{i=1}^m V_{\alpha_i}(F).$$

Furthermore, the $\frac{1}{t}$ -adic valuations of the roots of the characteristic polynomial of σ in any $F((\frac{1}{t}))$ -basis of V are $\{-\alpha_1, \dots, -\alpha_m\}$ each counted with multiplicity $\dim_{F((\frac{1}{t}))} V_{\alpha_i}$. We call $\{\alpha_1, \dots, \alpha_m\}$ the set of slopes of (V, σ) .

Proof. As was remarked in [29], proof of Proposition 5.1.3, the Proposition is a consequence of the results established in [24], appendix B, but working with $F\{\sigma\}$ instead of $F\{\tau\}$. \square

Remark 5.2. Maxim Mornev has informed us that he and Lenny Taelman have noticed that [29], Proposition 5.1.3 does not hold as stated: one needs the field to be perfect in addition to being separably closed. This hypothesis holds in the proposition above.

Lemma 5.3. *Let (V, σ) be a dual Dieudonné t -module over an algebraically closed field F . We assume that there exists a $F[t]$ -submodule M of V such that M is a finitely generated $F\{\sigma\}$ -module and M contains a $F((\frac{1}{t}))$ -basis of V . Then the slopes of (V, σ) are strictly positives.*

Proof. The proof is similar to that of [30], Proposition 8 (compare also to [21], Proposition 3.10 (a)). By Proposition 5.1, there exist $\alpha_1, \dots, \alpha_l \in \mathbb{Q}$ and l such that

$$V \simeq \bigoplus_{j=1}^l V_{\alpha_j}(F).$$

If (e_1, \dots, e_m) is the $F((\frac{1}{t}))$ -basis of V corresponding to this decomposition, then there exists an integer $N \geq 1$ such that for all $i \in \{1, \dots, m\}$, we have $\sigma^N(e_i) = t^{s_i} e_i$, $s_i \in \mathbb{Z}$. We write $M = \sum_{j=1}^s F\{\sigma^N\} u_j$ and

$$u_j = \sum_{i=1}^m \delta_{i,j} e_i, \quad \delta_{i,j} \in F\left(\left(\frac{1}{t}\right)\right).$$

It follows that

$$\sigma^{Nk}(u_j) = \sum_{i=1}^m \sigma^{Nk}(\delta_{i,j}) t^{ks_i} e_i.$$

Thus

$$\sum_{j=1}^s \sum_{k=0}^{\delta_j} l_{k,j} \sigma^{Nk}(u_j) = \sum_{i=1}^m \left(\sum_{j=1}^s \sum_{k=0}^{\delta_j} l_{k,j} \sigma^{Nk}(\delta_{i,j}) t^{ks_i} \right) e_i.$$

Observe that if $s_i \leq 0$, we would get

$$v_{\frac{1}{t}} \left(\sum_{j=1}^s \sum_{k=0}^{\delta_j} l_{k,j} \sigma^{Nk}(\delta_{i,j}) t^{ks_i} \right) \geq \text{Inf}\{v_{\frac{1}{t}}(\delta_{i,j}), j = 1, \dots, s\}.$$

Since V is the $F[[\frac{1}{t}]]$ -module generated by M , we obtain a contradiction. We conclude that $s_i > 0$ for $i = 1, \dots, m$. \square

5.2. A -finite Anderson modules. Let L/K be a finite extension and let $F = L^{\text{perf}}$. Let E be an Anderson A -module defined over O_L . Let O_F be the integral closure of A in F . Then for every prime \mathfrak{P} of O_L there exists a unique prime \mathfrak{Q} of F over \mathfrak{P} and the inclusion $O_L \subset O_F$ induces an isomorphism $O_L/\mathfrak{P} \simeq O_F/\mathfrak{Q}$.

Let $\tau : F \rightarrow F, x \mapsto x^q$ and $\sigma : F \rightarrow F, x \mapsto x^{\frac{1}{q}}$. Let $A_F = A \otimes_{\mathbb{F}_q} F$ and let $A_F\{\sigma\}$ be the skew polynomial ring

$$\sigma(a \otimes 1) = (a \otimes 1)\sigma, \quad \sigma(1 \otimes \alpha) = (1 \otimes \alpha^{\frac{1}{q}})\sigma, \quad \text{for all } a \in A.$$

We will recall the construction of the dual A -motive $M(E/F)$ attached to E . We refer the reader to [21], Sections 5.1-5.2 for more details. We set $M(E/F) = F\{\tau\}^n$. We consider $M(E/F)$ as a free $F\{\sigma\}$ -module where F acts on $M(E/F)$ via right multiplication and σ acts on $M(E/F)$ via right multiplication by τ . Also we view $M(E/F)$ as an A -module via left multiplication by E . Thus $M(E/F)$ is a left $A_F\{\sigma\}$ -module such that

- 1) $(a \otimes 1 - 1 \otimes a)^n M(E/F) \subset \sigma M(E/F)$ for all $a \in A$.
- 2) $M(E/F)$ is a free $F\{\sigma\}$ -module of rank n .

Following [21], Definition 5.7, we say that E is A -finite if $M(E/F)$ is a finitely generated A_F -module. Note that this definition does not depend on the choice of

F . In that case $M(E/F)$ is a projective A_F -module by [20], Lemma 5.4.10. By [21], Corollary 5.13, we know that a Drinfeld A -module is always A -finite.

For the rest of this Section, we assume that E is A -finite. Then there exists an integer $m(E)$ such that for any prime \mathfrak{P} of O_L such that $|\mathbb{F}_{\mathfrak{P}}| \geq m(E)$ we have that $E \pmod{\mathfrak{P}}$ is A -finite where $\mathbb{F}_{\mathfrak{P}} = O_L/\mathfrak{P}$.

Let \mathfrak{P} be such a prime of O_L . Let $V(E/\mathbb{F}_{\mathfrak{P}}) = M(E/\mathbb{F}_{\mathfrak{P}}) \otimes_A K$ which is a finite dimensional K -vector space. We still denote by σ the K -linear map induced by σ (which is the right multiplication by τ). Let $P_{\mathfrak{P}}(X) = \det_K(X \text{Id}_{V(E/\mathbb{F}_{\mathfrak{P}})} - \sigma)$. Then clearly $P_{\mathfrak{P}}(X) \in A[X]$.

Proposition 5.4. *Let $\lambda \in \mathbb{C}_{\infty}$ such that $P_{\mathfrak{P}}(\lambda) = 0$.*

- 1) *We have $v_{\infty}(\lambda) < 0$.*
- 2) *There exists a constant $c(E) \in \mathbb{N} \setminus \{0\}$ (i.e. independent of \mathfrak{P}) such that for all such λ , $v_{\infty}(\frac{1}{\lambda}) \geq \frac{1}{c(E)}$.*

Proof.

1) We choose $\theta \in A \setminus \mathbb{F}_q$ and observe that $A \otimes_{\mathbb{F}_q[\theta]} \mathbb{F}_q(\theta) = K$. Let $P(X)$ be the minimal polynomial of τ acting on the $\mathbb{F}_q(\theta)$ -vector space $M(E/\mathbb{F}_{\mathfrak{P}}) \otimes_{\mathbb{F}_q[\theta]} \mathbb{F}_q(\theta) = M(E/\mathbb{F}_{\mathfrak{P}}) \otimes_A K$. It suffices to prove that the roots of $P(X)$ have the desired property.

Let $\overline{\mathbb{F}}_{\mathfrak{P}}$ be an algebraic closure of $\mathbb{F}_{\mathfrak{P}}$. Let V be the dual Dieudonné θ -module

$$\overline{\mathbb{F}}_{\mathfrak{P}}^n \{\tau\} \widehat{\otimes}_{\mathbb{F}_q[\theta]} \mathbb{F}_q \left(\left(\frac{1}{\theta} \right) \right).$$

By Lemma 5.3, the slopes of (V, σ) are strictly positives. Thus the slopes of $(V, \sigma^{m_{\mathfrak{P}}})$ are strictly positives. Let (u_1, \dots, u_m) be an $\mathbb{F}_{\mathfrak{P}}[t]$ -basis of $M(E/\mathbb{F}_{\mathfrak{P}})$, then (u_1, \dots, u_m) is an $\overline{\mathbb{F}}_{\mathfrak{P}} \left(\left(\frac{1}{\theta} \right) \right)$ -basis of V . Furthermore if $G(X) \in \mathbb{F}_{\mathfrak{P}}[\theta][X]$ is the characteristic polynomial of the matrix of $\sigma^{m_{\mathfrak{P}}}$ in this particular basis, then by Proposition 5.1, the roots of $G(X)$ have ∞ -adic valuations that are strictly negative. This is also the case for $\prod_{j=1}^s \prod_{\delta \in \text{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_q)} \delta(G(X))$. But $P(X)$ divides $\prod_{j=1}^s \prod_{\delta \in \text{Gal}(\mathbb{F}_{\mathfrak{P}}/\mathbb{F}_q)} \delta(G(X))$. The first assertion follows.

2) Let H be an algebraic closure of L . Let $P(X) \in H[t][X]$. Then there exists a finite extension H'/L such that $P(X) \in H'[t][X]$. Observe that the $\frac{1}{t}$ -adic Newton polygon of $P(X)$ is the same as that of $P(X) \pmod{\mathfrak{Q}}$ for almost all prime \mathfrak{Q} of $O_{H'}$. Part 2) is then a consequence of the proof of Part 1) and Proposition 5.1. \square

Lemma 5.5. *Let k be a field, $\mathbb{F}_q \subset k$, and we assume that \mathbb{F}_q is algebraically closed in k . Let $i \in \mathbb{Z}$. Let \mathbb{F}/\mathbb{F}_q be a finite extension of degree m and let $k_{\mathbb{F}} = k \otimes_{\mathbb{F}_q} \mathbb{F}$. Let V be a finite dimensional $k_{\mathbb{F}}$ -vector space and let $f : V \rightarrow V$ be a k -linear map such that*

$$f(\lambda.v) = \lambda^{q^i} . f(v), \quad \text{for all } \lambda \in \mathbb{F}, v \in V.$$

Then we have

$$\det_k(X \text{Id}_V - f) \in k[X^m].$$

Proof. We use similar arguments as those used in [14], proof of Proposition 3.6. Observe that $k_{\mathbb{F}}$ is a field. We have

$$f^m \in \text{End}_{k_{\mathbb{F}}}(V).$$

For $j = 1, \dots, m$, we denote by V_j the k -vector space V equipped with the $k_{\mathbb{F}}$ -vector space structure

$$\lambda.v := \lambda^{q^{ij}} v, \quad \text{for all } \lambda \in \mathbb{F}, v \in V.$$

Let $W = V_1 \oplus \dots \oplus V_m$. We define the $k_{\mathbb{F}}$ -linear map $\Psi : W \rightarrow W$ given by $(v_1, \dots, v_m) \mapsto (f(v_m), f(v_1), \dots, f(v_{m-1}))$. Let $\alpha : V \otimes_{\mathbb{F}_q} \mathbb{F} \rightarrow W$ be the isomorphism of $k_{\mathbb{F}}$ -vector spaces defined by $\alpha(v \otimes \lambda) = (v\lambda^{q^i}, v\lambda^{q^{2i}}, \dots, v\lambda^{q^{mi}})$. We get

$$\Psi \circ \alpha = \alpha \circ (f \otimes \text{Id}_{\mathbb{F}_{\mathfrak{p}}}).$$

It remains to apply [33], Lemma 7.2. The proof is finished. \square

Proposition 5.6. *There exists a constant $c(E) \in \mathbb{N}$ such that for every prime \mathfrak{p} of O_L such that $|\mathbb{F}_{\mathfrak{p}}| \geq m(E)$, we have*

$$v_{\infty} \left(\frac{P_{\mathfrak{p}}(X)}{P_{\mathfrak{p}}(0)} - 1 \right) \geq \frac{m_{\mathfrak{p}}}{c(E)},$$

where v_{∞} is the ∞ -adic Gauss valuation on $K[X]$.

Proof. We write $P_{\mathfrak{p}}(X) = \prod_j (X - \lambda_j)$, then

$$\frac{P_{\mathfrak{p}}(X)}{P_{\mathfrak{p}}(0)} = \prod_j \left(1 - \frac{X}{\lambda_j} \right) = 1 + \sum_{i \geq 1} \beta_i X^i, \quad \text{with } \beta_i \in K.$$

By Lemma 5.5, we have $P_{\mathfrak{p}}(X) \in A[X^{m_{\mathfrak{p}}}]$, and by Proposition 5.4, if $\beta_i \neq 0$, then we have $v_{\infty}(\beta_i) \geq \frac{m_{\mathfrak{p}}}{c(E)}$. \square

We state the main theorem of this Section.

Theorem 5.7. *Let E be an Anderson module defined over O_L which is A -finite. Then E is admissible.*

Proof. The right multiplication by τ induces an exact sequence of A -modules

$$0 \rightarrow M(E/\mathbb{F}_{\mathfrak{p}}) \xrightarrow{\tau} M(E/\mathbb{F}_{\mathfrak{p}}) \rightarrow \text{Lie}_E(\mathbb{F}_{\mathfrak{p}}) \rightarrow 0,$$

as well as an exact sequence of \tilde{A} -modules

$$0 \rightarrow M(E/\mathbb{F}_{\mathfrak{p}}(z)) \xrightarrow{\tau} M(E/\mathbb{F}_{\mathfrak{p}}(z)) \rightarrow \text{Lie}_{\tilde{E}}(\mathbb{F}_{\mathfrak{p}}(z)) \rightarrow 0.$$

We deduce that

$$\begin{aligned} \text{Fitt}_A(\text{Lie}_E(\mathbb{F}_{\mathfrak{p}})) &= P_{\mathfrak{p}}(0)A, \\ \text{Fitt}_{\tilde{A}}(\text{Lie}_{\tilde{E}}(\mathbb{F}_{\mathfrak{p}}(z))) &= P_{\mathfrak{p}}(0)\tilde{A}. \end{aligned}$$

Next, the right multiplication by $\tau - 1$ induces an exact sequence of A -modules

$$0 \rightarrow M(E/\mathbb{F}_{\mathfrak{p}}) \xrightarrow{\tau-1} M(E/\mathbb{F}_{\mathfrak{p}}) \rightarrow E(\mathbb{F}_{\mathfrak{p}}) \rightarrow 0.$$

It follows that

$$\text{Fitt}_A(E(\mathbb{F}_{\mathfrak{p}})) = P_{\mathfrak{p}}(1)A.$$

Finally, the right multiplication by $\tau - z$ gives rise to an exact sequence of \tilde{A} -modules

$$0 \rightarrow M(E/\mathbb{F}_{\mathfrak{p}}(z)) \xrightarrow{\tau-z} M(E/\mathbb{F}_{\mathfrak{p}}(z)) \rightarrow \tilde{E}(\mathbb{F}_{\mathfrak{p}}(z)) \rightarrow 0.$$

Thus we get

$$\text{Fitt}_{\tilde{A}}(\tilde{E}(\mathbb{F}_{\mathfrak{p}}(z))) = P_{\mathfrak{p}}(z)\tilde{A}.$$

Consequently, the Theorem follows from Proposition 5.6. \square

As a direct consequence of Theorems 4.5 and 5.7, we obtain the following theorem.

Theorem 5.8. *Let E be an A -finite Anderson module defined over O_L . Then we have a class formula*

$$[\mathrm{Lie}_E(O_L) : U_{\mathrm{st}}(E/O_L)]_A = L(E/O_L).$$

Since any Drinfeld module is A -finite (see [21], Corollary 5.15), by Theorem 5.8 and Equation (2.1), we obtain the class formula à la Taelman for Drinfeld modules, which extends known results [7, 14, 26, 32, 34, 36].

Corollary 5.9. *Let ϕ be a Drinfeld A -module defined over O_L . Then $L(\phi/O_L)$ is well-defined and is a principal A -lattice in K_∞ . Furthermore we have the following class formula:*

$$[O_L : U_{\mathrm{st}}(\phi/O_L)]_A = [O_L : U(\phi/O_L)]_A \mathrm{Fitt}_A(H(\phi/O_L)) = L(\phi/O_L).$$

5.3. Anderson $\mathbb{F}_q[\theta]$ -modules. When $A = \mathbb{F}_q[\theta]$, we would like to mention the nontrivial fact that any Anderson $\mathbb{F}_q[\theta]$ -module is admissible.

Theorem 5.10. *We suppose that $A = \mathbb{F}_q[\theta]$. Let E be an Anderson A -module defined over O_L . Then E is admissible.*

Proof. Since $A = \mathbb{F}_q[\theta]$, we have $\tilde{A} = \mathbb{F}_q(z)[\theta]$. Let \mathfrak{P} be a nonzero prime of O_L and recall that $\mathbb{F}_{\mathfrak{P}} = O_L/\mathfrak{P}$. Then

$$\mathrm{Fitt}_{\tilde{A}}\left(\tilde{E}(\mathbb{F}_{\mathfrak{P}}(z))\right) = \det_{\mathbb{F}_q(z)[X]} \left(X\mathrm{Id} - E_\theta|_{\mathbb{F}_{\mathfrak{P}}(z)[X]} \right) \Big|_{X=\theta} \tilde{A}.$$

Since $\mathbb{F}_{\mathfrak{P}}[z, X]$ is a finitely generated and free $\mathbb{F}_q[z, X]$ -module, we deduce that

$$\det_{\mathbb{F}_q(z)[X]} \left(X\mathrm{Id} - E_\theta|_{\mathbb{F}_{\mathfrak{P}}(z)[X]} \right) \Big|_{X=\theta} \in \mathbb{F}_q[z, X].$$

It follows that

$$\mathrm{Fitt}_{\tilde{A}}\left(\tilde{E}(\mathbb{F}_{\mathfrak{P}}(z))\right) = x_{\mathfrak{P}} \tilde{A},$$

where $x_{\mathfrak{P}}$ is a monic polynomial in θ with coefficients in $\mathbb{F}_q[z]$. Let us observe that

$$\mathrm{Fitt}_A(\mathrm{Lie}_E(\mathbb{F}_{\mathfrak{P}})) = x_{\mathfrak{P}}(0)A,$$

$$\mathrm{Fitt}_A(E(\mathbb{F}_{\mathfrak{P}})) = x_{\mathfrak{P}}(1)A.$$

It remains to prove that the infinite product $\prod_{\mathfrak{P}} \frac{x_{\mathfrak{P}}(0)}{x_{\mathfrak{P}}}$ converges in $\mathbb{F}_q(z) \left(\left(\frac{1}{\theta} \right) \right)$. In fact, it is the consequence of the adaptation in our context of [32], Theorem 3. We refer the reader to [15], proof of Proposition 3.5, for more details. \square

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UNIVERSITÉ DE CAEN NORMANDIE, LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, CNRS UMR 6139, CAMPUS II, BOULEVARD MARÉCHAL JUIN, B.P. 5186, 14032 CAEN CEDEX, FRANCE.

E-mail address: `bruno.angles@unicaen.fr`

CNRS - UNIVERSITÉ CLAUDE BERNARD LYON 1, INSTITUT CAMILLE JORDAN, UMR 5208, 43 BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE

E-mail address: `ngodac@math.univ-lyon1.fr`

UNIVERSITÉ DE CAEN NORMANDIE, LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, CNRS UMR 6139, CAMPUS II, BOULEVARD MARÉCHAL JUIN, B.P. 5186, 14032 CAEN CEDEX, FRANCE.

E-mail address: `floric.tavares-ribeiro@unicaen.fr`