



HAL
open science

Weak suitable solutions for 3D MHD equations for intermittent initial data

Pedro Gabriel Fernández-Dalgo, Oscar Jarrin

► **To cite this version:**

Pedro Gabriel Fernández-Dalgo, Oscar Jarrin. Weak suitable solutions for 3D MHD equations for intermittent initial data. 2020. hal-02490130

HAL Id: hal-02490130

<https://hal.science/hal-02490130v1>

Preprint submitted on 24 Feb 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Weak suitable solutions for 3D MHD equations for intermittent initial data

Pedro Gabriel Fernández-Dalgo^{*†}, Oscar Jarrín^{‡§}

Abstract

In this note, we extend some recent results on the local and global existence of solutions for 3D magneto-hydrodynamics equations to the more general setting of the intermittent initial data, which is characterized through a local Morrey space. This large initial data space was also exhibit in a contemporary work [3] in the context of 3D Navier-Stokes equations.

Keywords : MHD equations; Local Morrey spaces; Global weak solutions; Suitable solutions.

AMS classification : 35Q30, 76D05.

1 Introduction

In a recent work [9], P. Fernandez-Dalgo & P.G. Lemarié-Rieusset obtained *new energy controls* for the homogeneous and incompressible Navier-Stokes (NS) equations, which allowed them to develop a theory to construct weak solutions for initial data \mathbf{u}_0 belonging to the weighted space $L^2_{w_\gamma} = L^2(w_\gamma dx)$, where, for $0 < \gamma \leq 2$ we define $w_\gamma(x) = (1 + |x|)^{-\gamma}$. Moreover, this method also gives a *new proof* of the existence of discretely self-similar solutions.

This new approach has attired the interest in the research community and more recently, in the paper [3] written by Bradshaw, Tsai & Kukavika, the main theorem on global existence given in [9] is improved with respect to the

^{*}LaMME, Univ Evry, CNRS, Université Paris-Saclay, 91025, Evry, France

[†]e-mail : pedro.fernandez@univ-evry.fr

[‡]Dirección de Investigación y Desarrollo (DIDE), Universidad Técnica de Ambato, Ambato, Ecuador

[§]e-mail : or.jarrin@uta.edu.ec

initial data \mathbf{u}_0 which belongs to a *larger space* than the weighted Lebesgue space above. More precisely, the authors prove that if \mathbf{u}_0 verifies

$$\lim_{R \rightarrow +\infty} R^{-2} \int_{|x| \leq R} |\mathbf{u}_0(x)|^2 dx = 0,$$

then the (NS) system, with a zero forcing tensor, has a global solution.

Due to the structural similarity between the (NS) equations and the magneto-hydrodynamics equations (see equations (MHD) below) it is quite natural to extend those recent results obtained for the (NS) equations to the more general setting of the coupled magneto-hydrodynamics system which writes down as follows:

$$\text{(MHD)} \quad \left\{ \begin{array}{l} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} + (\mathbf{b} \cdot \nabla) \mathbf{b} - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{u} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{u} - \nabla q + \nabla \cdot \mathbb{G}, \\ \nabla \cdot \mathbf{u} = 0, \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \mathbf{b}(0, \cdot) = \mathbf{b}_0. \end{array} \right.$$

Here the fluid velocity $\mathbf{u} : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, the fluid magnetic field $\mathbf{b} : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, the fluid pressure $p : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and the term $q : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ (which appears in physical models considering Maxwell's displacement currents [1], [18]) are the unknowns. On the other hand, the data of the problem are given by the fluid velocity at $t = 0$: $\mathbf{u}_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$; the magnetic field at $t = 0$, $\mathbf{b}_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$; and the tensors $\mathbb{F} = (F_{i,j})_{1 \leq i,j \leq 3}$, $\mathbb{G} = (G_{i,j})_{1 \leq i,j \leq 3}$ (where $F_{i,j}, G_{i,j} : [0, +\infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$) whose divergences: $\nabla \cdot \mathbb{F}, \nabla \cdot \mathbb{G}$, represent volume forces applied to the fluids.

In the setting of this coupled system, in a previous work [7], we adapted the *energy controls* given in [9] for the (NS) equations to the (MHD) equations and this approach allowed us to establish the existence of discretely self-similar solutions for discretely self-similar initial data belonging to L^2_{loc} ; and moreover, the existence of global suitable weak solutions when the initial data $\mathbf{u}_0, \mathbf{b}_0$ belong to the weighted spaces $L^2_{w_\gamma}(\mathbb{R}^3)$, for $0 < \gamma \leq 2$, and the tensor forces \mathbb{F}, \mathbb{G} belong to the space $L^2((0, +\infty), L^2_{w_\gamma}(\mathbb{R}^3))$. For all the details see Theorem 1 and Theorem 2 in [7].

In this paper, we continue with the research program started in [7] for the (MHD) equations; and we *relax* the method developed in [9] to *enlarge* the initial data space. Indeed, following some ideas of [2] (for the (NS) equations)

we define $B_2(\mathbb{R}^3) \subset L^2_{\text{loc}}(\mathbb{R}^3)$ as the Banach space of all functions $u \in L^2_{\text{loc}}$ such that :

$$\|u\|_{B_2}^2 = \sup_{R \geq 1} R^{-2} \int_{|x| \leq R} |u|^2 dx < +\infty.$$

Moreover, we denote $B_2L^2(0, T)$ the Banach space defined as the space of all functions $u \in L^2_{\text{loc}}((0, T) \times \mathbb{R}^3)$ such that

$$\|u\|_{B_2L^2(0, T)}^2 = \sup_{R \geq 1} R^{-2} \int_{|x| \leq R} \int_0^T |u|^2 dt dx < +\infty.$$

In this framework, our main theorem reads as follows:

Theorem 1 *Let $0 < T < +\infty$. Let $\mathbf{u}_0, \mathbf{b}_0 \in B_2(\mathbb{R}^3)$ be divergence-free vector fields. Let \mathbb{F} and \mathbb{G} be tensors belonging to $B_2L^2(0, T)$. Then, there exists a time $0 < T_0 < T$ such that the system (MHD) has a solution $(\mathbf{u}, \mathbf{b}, p, q)$ which satisfies :*

- \mathbf{u}, \mathbf{b} belong to $L^\infty((0, T_0), B_2)$ and $\nabla \mathbf{u}, \nabla \mathbf{b}$ belong to $B_2L^2(0, T_0)$.
- The pressure p and the term q are related to $\mathbf{u}, \mathbf{b}, \mathbb{F}$ and \mathbb{G} by:

$$p = \sum_{1 \leq i, j \leq 3} \mathcal{R}_i \mathcal{R}_j (u_i u_j - b_i b_j - F_{i,j}) \quad \text{and} \quad q = - \sum_{1 \leq i, j \leq 3} \mathcal{R}_i \mathcal{R}_j (G_{i,j}),$$

where $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ denotes the Riesz transform.

- The map $t \in [0, T) \mapsto (\mathbf{u}(t, \cdot), \mathbf{b}(t, \cdot))$ is $*$ -weakly continuous from $[0, T)$ to $B_2(\mathbb{R}^3)$, and for all compact set $K \subset \mathbb{R}^3$ we have:

$$\lim_{t \rightarrow 0} \|(\mathbf{u}(t, \cdot) - \mathbf{u}_0, \mathbf{b}(t, \cdot) - \mathbf{b}_0)\|_{L^2(K)} = 0.$$

- The solution $(\mathbf{u}, \mathbf{b}, p, q)$ is suitable : there exists a non-negative locally finite measure μ on $(0, T) \times \mathbb{R}^3$ such that:

$$\begin{aligned} \partial_t \left(\frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2} \right) = & \Delta \left(\frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2} \right) - |\nabla \mathbf{u}|^2 - |\nabla \mathbf{b}|^2 - \nabla \cdot \left(\left[\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} + p \right] \mathbf{u} \right) \\ & + \nabla \cdot ([(\mathbf{u} \cdot \mathbf{b}) + q] \mathbf{b}) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) + \mathbf{b} \cdot (\nabla \cdot \mathbb{G}) - \mu. \end{aligned}$$

In particular we have the global control on the solution: for all $0 \leq t \leq T_0$,

$$\begin{aligned} \max\{ \|(\mathbf{u}, \mathbf{b})(t)\|_{B_2}^2, \|\nabla(\mathbf{u}, \mathbf{b})\|_{B_2L^2(0, T_0)}^2 \} & \leq C \|(\mathbf{u}_0, \mathbf{b}_0)\|_{B_2}^2 \\ + C \|(\mathbb{F}, \mathbb{G})\|_{B_2L^2(0, t)}^2 + C \int_0^t & \|(\mathbf{u}, \mathbf{b})(s)\|_{B_2}^2 + \|(\mathbf{u}, \mathbf{b})(s)\|_{B_2}^6 ds. \end{aligned} \quad (1)$$

- Finally, if the data verify:

$$\lim_{R \rightarrow +\infty} R^{-2} \int_{|x| \leq R} |\mathbf{u}_0(x)|^2 + |\mathbf{b}_0(x)|^2 dx = 0,$$

and

$$\lim_{R \rightarrow +\infty} R^{-2} \int_0^{+\infty} \int_{|x| \leq R} |\mathbb{F}(t, x)|^2 + |\mathbb{G}(t, x)|^2 dx ds = 0,$$

then $(\mathbf{u}, \mathbf{b}, p, q)$ is a global weak solution.

Remark 1.1 A vector field \mathbf{u} denotes the vector (u_1, u_2, u_3) and for a tensor $\mathbb{F} = (F_{i,j})$ we use $\nabla \cdot \mathbb{F}$ to denote the vector $(\sum_i \partial_i F_{i,1}, \sum_i \partial_i F_{i,2}, \sum_i \partial_i F_{i,3})$. Thus, if $\nabla \cdot \mathbf{u} = 0$ then we can write $(\mathbf{b} \cdot \nabla) \mathbf{u} = \nabla \cdot (\mathbf{b} \otimes \mathbf{u})$.

It is worth to make the following comments on this result. Remark first that we prove a global control on the solutions [\[1\]](#) which is not exhibited in [\[3\]](#). This new control is also valid for the (NS) equations (taking $\mathbf{b} = 0$, $\mathbf{b}_0 = 0$ and $\mathbb{G} = 0$ in the (MHD) system). On the other hand, it is interesting to note that the main difference between this result and our previous work [\[7\]](#) is that, in the more general setting of the space $B_2(\mathbb{R}^3)$, the control on the pressure p and the term q is a little more technical, and so the method seems not to be adaptable to study the existence of self-similar solutions of equations (MHD) as done in Theorem 2 in [\[7\]](#).

Getting back to the (NS) equations, the global existence and uniqueness of solutions for the 2D case with initial data $\mathbf{u}_0 \in B_2(\mathbb{R}^2)$ is an open problem proposed by A. Basson in [\[2\]](#). In further research, we think that it would be interesting to study this problem in the simplest and closest cases with an initial data in $\mathbf{u}_0 \in B_{2,0}(\mathbb{R}^2)$ (see Section [\[2\]](#) for a definition) or $\mathbf{u}_0 \in L_{w_\gamma}^2(\mathbb{R}^2)$ with $0 < \gamma \leq 2$.

This paper is organized as follows. In Section [\[2\]](#) we state some useful tools on the local Morrey spaces. Section [\[3\]](#) is devoted to some *a priori* estimates and stability results on the (MHD) equations, which will allow us to prove our main result in the last Section [\[4\]](#).

2 The local Morrey space B_γ^p

In order to understand how Theorem [\[1\]](#) generalizes the results obtained by [\[9\]](#), we recall some useful results obtained in [\[8\]](#). We consider the space \mathbb{R}^d only in this section.

Definition 2.1 Let $\gamma \geq 0$ and $1 < p < +\infty$. We denote $B_\gamma^p(\mathbb{R}^d)$ the Banach space of all functions $u \in L_{\text{loc}}^p(\mathbb{R}^d)$ such that :

$$\|u\|_{B_\gamma^p} = \sup_{R \geq 1} \left(\frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p dx \right)^{1/p} < +\infty.$$

Moreover, for $0 < T \leq +\infty$, $B_\gamma^p L^p(0, T)$ is the Banach space of all functions $u \in (L_t^p L_x^p)_{\text{loc}}([0, T] \times \mathbb{R}^d)$ such that

$$\|u\|_{B_\gamma^p L^p(0, T)} = \sup_{R \geq 1} \left(\frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u(t, x)|^p dx dt \right)^{1/p} < +\infty.$$

In what follows, we will denote $B_\gamma^p(\mathbb{R}^d) = B_\gamma^p$ and $B_2^2 = B_2$.

Also, the space $B_{\gamma,0}^p$ is defined as the subspace of all functions $u \in B_\gamma^p$ such that $\lim_{R \rightarrow +\infty} \frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p dx = 0$; and similar, $B_{\gamma,0}^p L^p(0, T)$ is the subspace of all functions $u \in B_\gamma^p L^p(0, T)$ such that $\lim_{R \rightarrow +\infty} \frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u(t, x)|^p dx dt = 0$.

The following result shows how B_γ^p is strongly lied with the weighted spaces $L_{w_\gamma}^p = L^p(w_\gamma dx)$ (where $w_\gamma = (1 + |x|)^{-\gamma}$) considered in [7] and [9].

Lemma 2.1 Consider $\gamma \geq 0$ and let $\gamma < \delta < +\infty$. We have the continuous embedding

$$L_{w_\gamma}^p \subset B_{\gamma,0}^p \subset B_\gamma^p \subset L_{w_\delta}^p.$$

Moreover, for all $0 < T \leq +\infty$ we have:

$$L^p((0, T), L_{w_\gamma}^p) \subset B_{\gamma,0}^p L^p(0, T) \subset B_\gamma^p L^p(0, T) \subset L^p((0, T), L_{w_\delta}^p).$$

Proof. Only the embedding $L^p((0, T), L_{w_\gamma}^p) \subset B_{\gamma,0}^p L^p(0, T)$ is not proved in [8] and we prove it. Let $\lambda > 1$ and $n \in \mathbb{N}$, let $u_n(t, x) = u(t, \lambda^n x)$. We have:

$$\begin{aligned} \sup_{R \geq 1} \frac{1}{(\lambda^n R)^\gamma} \int_0^T \int_{|x| \leq \lambda^n R} |u(t, x)|^p dx dt &= \sup_{R \geq 1} \frac{\lambda^{(d-\gamma)n}}{R^\gamma} \int_0^T \int_{|x| \leq R} |u(t, \lambda^n x)|^p dx dt \\ &= \lambda^{(d-\gamma)n} \|u_n\|_{B_\gamma^p L^p(0, T)}^p \leq C \lambda^{(d-\gamma)n} \|u_n\|_{L^p L_{w_\gamma}^p}^p \leq C \int_0^T \int |u(s, x)|^p \frac{1}{(\lambda^n + |x|)^\gamma} dx dt, \end{aligned}$$

and we conclude by dominated convergence. \diamond

Thereafter, we have the following result involving the interpolation theory of Banach spaces:

Theorem 2 ([8]) *The space B_γ^p can be obtained by interpolation: for all $0 < \gamma < \delta < \infty$ we have $B_\gamma^p = [L^p, L_{w_\delta}^p]_{\frac{\gamma}{\delta}, \infty}$; and the norms $\|\cdot\|_{B_\gamma^p}$ and $\|\cdot\|_{[L^p, L_{w_\delta}^p]_{\frac{\gamma}{\delta}, \infty}}$ are equivalent.*

This theorem has a useful corollary and in order to state it we need first the following result on the Muckenhoupt weights (see [10] for a definition).

Lemma 2.2 (Muckenhoupt weights, [9]) *If $0 < \delta < d$ and $1 < p < +\infty$. Then, $w_\delta(x) = (1 + |x|)^{-\delta}$ belongs to the Muckenhoupt class $\mathcal{A}_p(\mathbb{R}^3)$. Moreover we have:*

- The Riesz transforms R_j are bounded on $L_{w_\gamma}^p$: $\|R_j f\|_{L_{w_\gamma}^p} \leq C_{p,\delta} \|f\|_{L_{w_\gamma}^p}$
- The Hardy–Littlewood maximal function operator is bounded on $L_{w_\gamma}^p$:

$$\|\mathcal{M}_f\|_{L_{w_\gamma}^p} \leq C_{p,\delta} \|f\|_{L_{w_\gamma}^p}.$$

With this lemma at hand, the next important corollary of Theorem [2] follows:

Corollary 2.1 *If $0 < \delta < d$ and $1 < p < +\infty$, then we have:*

- The Riesz transforms R_j are bounded on B_δ^p : $\|R_j f\|_{B_\delta^p} \leq C_{p,\delta} \|f\|_{B_\delta^p}$
- The Hardy–Littlewood maximal function operator is bounded on B_δ^p :

$$\|\mathcal{M}_f\|_{B_\delta^p} \leq C_{p,\delta} \|f\|_{B_\delta^p}.$$

Proof. Remark that Theorem [2] implies $B_\delta^p = [L^p, L_{w_{\delta_0}}^p]_{\frac{\delta}{\delta_0}, \infty}$, for some $\delta < \delta_0 < d$. So, we conclude directly by Lemma [2.2]. \diamond

3 Some results for the (MHD^*) system

Our main theorem bases on the two following results for the equations:

$$(MHD^*) \left\{ \begin{array}{l} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{v} \cdot \nabla) \mathbf{u} + (\mathbf{c} \cdot \nabla) \mathbf{b} - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{v} \cdot \nabla) \mathbf{b} + (\mathbf{c} \cdot \nabla) \mathbf{u} - \nabla q + \nabla \cdot \mathbb{G}, \\ \nabla \cdot \mathbf{u} = 0, \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \mathbf{b}(0, \cdot) = \mathbf{b}_0. \end{array} \right.$$

In this system, the functions (\mathbf{v}, \mathbf{c}) are defined as follows:

- when we will consider the (MHD) equations we have $(\mathbf{v}, \mathbf{c}) = (\mathbf{u}, \mathbf{b})$.
- when we will consider the regularized (MHD) equations we have $(\mathbf{v}, \mathbf{c}) = (\mathbf{u} * \theta_\epsilon, \mathbf{b} * \theta_\epsilon)$, where, for $0 < \epsilon < 1$ and for a fixed, non-negative and radially non increasing test function $\theta \in \mathcal{D}(\mathbb{R}^3)$ which is equals to 0 for $|x| \geq 1$ and $\int \theta dx = 1$; we define $\theta_\epsilon(x) = \frac{1}{\epsilon^3} \theta(x/\epsilon)$.

3.1 A priori estimates

Theorem 3 *Let $0 < T < +\infty$. Let $\mathbf{u}_0, \mathbf{b}_0 \in B_2$ be a divergence-free vector fields and let \mathbb{F}, \mathbb{G} be tensors such that $\mathbb{F}, \mathbb{G} \in B_2 L^2(0, T)$. Moreover, let $(\mathbf{u}, \mathbf{b}, p, q)$ be a solution of the problem (MHD*).*

We suppose that:

- \mathbf{u}, \mathbf{b} belongs to $L^\infty((0, T), B_2)$ and $\nabla \mathbf{u}, \nabla \mathbf{b}$ belongs to $B_2 L^2(0, T)$.
- The pressure p and the term q are related to $\mathbf{u}, \mathbf{b}, \mathbb{F}$ and \mathbb{G} by

$$p = \sum_{1 \leq i, j \leq 3} \mathcal{R}_i \mathcal{R}_j (v_i u_j - c_i b_j - F_{i,j}) \quad \text{and} \quad q = \sum_{1 \leq i, j \leq 3} \mathcal{R}_i \mathcal{R}_j (v_i b_j - c_j u_i - G_{ij}).$$

- The map $t \in [0, T) \mapsto \mathbf{u}(t, \cdot)$ is $*$ -weakly continuous from $[0, T)$ to B_2 , and for all compact set $K \subset \mathbb{R}^3$ we have:

$$\lim_{t \rightarrow 0} \|(\mathbf{u}(t, \cdot) - \mathbf{u}_0, \mathbf{b}(t, \cdot) - \mathbf{b}_0)\|_{L^2(K)} = 0.$$

- The solution $(\mathbf{u}, \mathbf{b}, p, q)$ is suitable : there exists a non-negative locally finite measure μ on $(0, T) \times \mathbb{R}^3$ such that

$$\begin{aligned} \partial_t \left(\frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2} \right) &= \Delta \left(\frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2} \right) - |\nabla \mathbf{u}|^2 - |\nabla \mathbf{b}|^2 - \nabla \cdot \left(\left(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} \right) \mathbf{v} + p \mathbf{u} \right) \\ &+ \nabla \cdot ((\mathbf{u} \cdot \mathbf{b}) \mathbf{c} + q \mathbf{b}) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) + \mathbf{b} \cdot (\nabla \cdot \mathbb{G}) - \mu. \end{aligned} \tag{2}$$

Then, exists a constant $C \geq 1$, which does not depend on T , and not on $\mathbf{u}_0, \mathbf{b}_0, \mathbf{u}, \mathbf{b}, \mathbb{F}, \mathbb{G}$ nor ϵ , such that:

- We have the following control on $[0, T)$:

$$\begin{aligned} \max\{\|(\mathbf{u}, \mathbf{b})(t)\|_{B_2}^2, \|\nabla(\mathbf{u}, \mathbf{b})\|_{B_2 L^2(0,t)}^2\} &\leq C \|(\mathbf{u}_0, \mathbf{b}_0)\|_{B_2}^2 \\ &+ C \|(\mathbb{F}, \mathbb{G})\|_{B_2 L^2(0,t)}^2 + C \int_0^t \|(\mathbf{u}, \mathbf{b})(s)\|_{B_2}^2 + \|(\mathbf{u}, \mathbf{b})(s)\|_{B_2}^6 ds. \end{aligned} \tag{3}$$

- Moreover, if $T_0 < T$ is small enough:

$$C \left(1 + \|(\mathbf{u}_0, \mathbf{b}_0)\|_{B_2}^2 + \|(\mathbb{F}, \mathbb{G})\|_{B_2 L^2(0, T_0)}^2 \right)^2 T_0 \leq 1,$$

then the following control respect to the data holds:

$$\begin{aligned} & \sup_{0 \leq t \leq T_0} \max\{ \|(\mathbf{u}, \mathbf{b})(t, \cdot)\|_{B_2}^2, \|\nabla(\mathbf{u}, \mathbf{b})\|_{B_2 L^2(0, t)}^2 \} \\ & \leq C \left(1 + \|(\mathbf{u}_0, \mathbf{b}_0)\|_{B_2}^2 + \|(\mathbb{F}, \mathbb{G})\|_{B_2 L^2(0, T_0)}^2 \right). \end{aligned} \quad (4)$$

Proof. In this proof, we will focus only in the case $(\mathbf{v}, \mathbf{c}) = (\mathbf{u} * \theta_\varepsilon, \mathbf{b} * \theta_\varepsilon)$ (the case $(\mathbf{v}, \mathbf{c}) = (\mathbf{u}, \mathbf{b})$ can be treated in a similar way). The proof of this theorem follows similar ideas of the proof of Theorem 3 in [7] and we will only detail the main computations.

We start by proving the global control (3). The idea is to apply the energy balance (2) to a suitable test function. Let $0 < t_0 < t_1 < T$. We consider a function α_{η, t_0, t_1} which converges almost everywhere to $\mathbf{1}_{[t_0, t_1]}$ and such that $\partial_t \alpha_{\eta, t_0, t_1}$ is the difference between two identity approximations, the first one in t_0 and the second one in t_1 . For this, we take a non-decreasing function $\alpha \in \mathcal{C}^\infty(\mathbb{R})$ which is equals to 0 on $(-\infty, \frac{1}{2})$ and is equals to 1 on $(1, +\infty)$. Then, for $0 < \eta < \min(\frac{t_0}{2}, T - t_1)$ we set the function $\alpha_{\eta, t_0, t_1}(t) = \alpha(\frac{t - t_0}{\eta}) - \alpha(\frac{t - t_1}{\eta})$. On the other hand, we consider a non-negative function $\phi \in \mathcal{D}(\mathbb{R}^3)$ which is equals to 1 for $|x| \leq 1/2$ and is equals to 0 for $|x| \geq 1$; and for $R \geq 1$ we set $\phi_R(x) = \phi(\frac{x}{R})$.

Thus, by the energy balance (2) we can write

$$\begin{aligned} & - \iint \frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} \partial_t \alpha_{\eta, t_0, t_1} \phi_R dx ds + \iint |\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2 \alpha_{\eta, t_0, t_1} \phi_R dx ds \\ & \leq \iint \frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2} \alpha_{\eta, t_0, t_1} \Delta \phi_R dx ds \\ & \quad + \sum_{i=1}^3 \iint \left[\left(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} \right) v_i + p u_i \right] \alpha_{\eta, t_0, t_1} \partial_i \phi_R dx ds \\ & \quad + \sum_{i=1}^3 \iint [(\mathbf{u} \cdot \mathbf{b}) c_i + q b_i] \alpha_{\eta, t_0, t_1} \partial_i \phi_R dx ds \\ & \quad - \sum_{1 \leq i, j \leq 3} \left(\iint F_{i,j} u_j \alpha_{\eta, t_0, t_1} \partial_i \phi_R dx ds + \iint F_{i,j} \partial_i u_j \alpha_{\eta, t_0, t_1} \phi_R dx ds \right) \\ & \quad - \sum_{1 \leq i, j \leq 3} \left(\iint G_{i,j} b_j \alpha_{\eta, t_0, t_1} \partial_i \phi_R dx ds + \iint G_{i,j} \partial_i b_j \alpha_{\eta, t_0, t_1} \phi_R dx ds \right), \end{aligned}$$

and taking the limit when η goes to 0, by the dominated convergence theorem we obtain (when the limit in the left side is well-defined):

$$\begin{aligned}
& - \lim_{\eta \rightarrow 0} \iint \frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} \partial_t \alpha_{\eta, t_0, t_1} \phi_R dx ds + \int_{t_0}^{t_1} \int |\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2 \phi_R dx ds \\
& \leq \int_{t_0}^{t_1} \int \frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2} \Delta \phi_R dx ds \\
& \quad + \sum_{i=1}^3 \int_{t_0}^{t_1} \int \left[\left(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} \right) v_i + p u_i \right] \partial_i \phi_R dx ds \\
& \quad + \sum_{i=1}^3 \int_{t_0}^{t_1} \int [(\mathbf{u} \cdot \mathbf{b}) c_i + q b_i] \partial_i \phi_R dx ds \\
& \quad - \sum_{1 \leq i, j \leq 3} \left(\int_{t_0}^{t_1} \int F_{i,j} u_j \partial_i \phi_R dx ds + \int_{t_0}^{t_1} \int F_{i,j} \partial_i u_j \phi_R dx ds \right) \\
& \quad - \sum_{1 \leq i, j \leq 3} \left(\int_{t_0}^{t_1} \int G_{i,j} b_j \partial_i \phi_R dx ds + \int_{t_0}^{t_1} \int G_{i,j} \partial_i b_j \phi_R dx ds \right).
\end{aligned}$$

We define now the quantity

$$A_R(t) = \int (|\mathbf{u}(t, x)|^2 + |\mathbf{b}(t, x)|^2) \phi_R(x) dx,$$

hence, if t_0 and t_1 are Lebesgue points of $A_R(t)$ and moreover, due to the fact that

$$- \iint \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} \right) \partial_t \alpha_{\eta, t_0, t_1} \phi_R dx ds = - \frac{1}{2} \int \partial_t \alpha_{\eta, t_0, t_1} A_R(s) ds,$$

we have

$$\lim_{\eta \rightarrow 0} - \iint \left(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} \right) \partial_t \alpha_{\eta, t_0, t_1} \phi_R dx ds = \frac{1}{2} (A_R(t_1) - A_R(t_0)).$$

Then, since ϕ_R is a support compact function we can let t_0 go to 0 and thus we can replace t_0 by 0 in this inequality. Moreover, if we let t_1 go to t , then by the *-weak continuity we have $A_R(t) \leq \lim_{t_1 \rightarrow t} A_R(t_1)$, and thus we

may replace t_1 by $t \in (0, T)$. In this way, for every $t \in (0, T)$ we can write:

$$\begin{aligned}
& \int \frac{|\mathbf{u}(t, x)|^2 + |\mathbf{b}(t, x)|^2}{2} \phi_R dx + \int_0^t \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2) \phi_R ds dx \\
& \leq \int \frac{|\mathbf{u}_0(x)|^2 + |\mathbf{b}_0(x)|^2}{2} \phi_R dx + \int_0^t \int \frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2} \Delta \phi_R ds dx \\
& \quad + \sum_{i=1}^3 \int_0^t \int \left[\left(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2} \right) v_i + p u_i \right] \partial_i \phi_R dx ds \\
& \quad + \sum_{i=1}^3 \int_0^t \int [(\mathbf{u} \cdot \mathbf{b}) c_i + q b_i] \partial_i \phi_R dx ds \\
& \quad - \sum_{1 \leq i, j \leq 3} \left(\int_0^t \int F_{i,j} u_j \partial_i \phi_R dx ds + \int_0^t \int F_{i,j} \partial_i u_j \phi_R dx ds \right) \\
& \quad - \sum_{1 \leq i, j \leq 3} \left(\int_0^t \int G_{i,j} b_j \partial_i \phi_R dx ds + \int_0^t \int G_{i,j} \partial_i b_j \phi_R dx ds \right).
\end{aligned} \tag{5}$$

In this inequality, we still need to estimate the terms in the right-hand side. For the second term, as $R \geq 1$ we write

$$\frac{1}{R^2} \int (|\mathbf{u}|^2 + |\mathbf{b}|^2) \Delta \phi_R dx \leq \frac{C}{R^4} \int_{B(0,R)} (|\mathbf{u}|^2 + |\mathbf{b}|^2) dx \leq C(\|\mathbf{u}\|_{B_2}^2 + \|\mathbf{b}\|_{B_2}^2).$$

The third and fourth terms are estimates as follows. We consider first the expressions where the pressure terms p and q do not appear. Using the Hölder inequalities and the Sobolev embeddings we have:

$$\begin{aligned}
\sum_{i=1}^3 \int \frac{(\mathbf{u} \cdot \mathbf{b})}{2} (b_i * \theta_\epsilon) \partial_i \phi_R dx & \leq \|\mathbf{u}\|_{L^{\frac{12}{5}}(B(0,R))} \|\mathbf{b}\|_{L^{\frac{12}{5}}(B(0,R))} \|\mathbf{b} * \theta_\epsilon\|_{L^6(B(0,R))} \|\nabla \phi_R\|_{L^\infty} \\
& \leq \frac{C}{R} \|\mathbf{u}\|_{L^2(B(0,R))}^{3/4} \|\mathbf{u}\|_{L^6(B(0,R))}^{1/4} \|\mathbf{b}\|_{L^2(B(0,R))}^{3/4} \|\mathbf{b}\|_{L^6(B(0,R+1))}^{5/4} \\
& \leq \frac{C}{R} \|\mathbf{b}\|_{L^2(B(0,R))}^{3/4} \|\mathbf{u}\|_{L^2(B(0,R))}^{3/4} U^{1/4} B^{5/4},
\end{aligned}$$

where we have denoted the quantities

$$U = \left(\int |\phi_{2R} \nabla \mathbf{u}|^2 dx \right)^{1/2} + \left(\int_{|x| \leq 2R} |\mathbf{u}|^2 dx \right)^{1/2}$$

and

$$B = \left(\int |\phi_{2(R+1)} \nabla \mathbf{b}|^2 dx \right)^{1/2} + \left(\int_{|x| \leq 2(R+1)} |\mathbf{b}|^2 dx \right)^{1/2}.$$

Thus, we can write (by the Young's inequalities for products with $1 = \frac{1}{8} + \frac{1}{8} + \frac{5}{8}$):

$$\begin{aligned} & \frac{1}{R^2} \sum_{i=1}^3 \int \frac{(\mathbf{u} \cdot \mathbf{b})}{2} (b_i * \theta_\epsilon) \partial_i \phi_R dx \\ & \leq C \left(\frac{\|\mathbf{u}\|_{L^2(B(0,R))}}{R} \right)^{3/4} \left(\frac{\|\mathbf{b}\|_{L^2(B(0,R))}}{R} \right)^{3/4} \left(\frac{U}{R} \right)^{1/4} \left(\frac{B}{R} \right)^{5/4} \\ & \leq C \|(\mathbf{u}, \mathbf{b})\|_{B_2}^6 + C \|(\mathbf{u}, \mathbf{b})\|_{B_2}^2 + \frac{C_0}{R^2} \int |\phi_{2R} \nabla \mathbf{u}|^2 + |\phi_{2(R+1)} \nabla \mathbf{b}|^2 dx \end{aligned}$$

where $C_0 > 0$ is an arbitrarily small constant.

Now, in order to estimate the expressions where the pressure terms p and q appear, we need the following technical lemma which will be proved at the end of this section.

Lemma 3.1 *Within the hypothesis of Theorem [3](#), the terms p and q belong $L_{\text{loc}}^{3/2}$. Moreover, there exist an arbitrarily small constant $C_0 > 0$ and a constant $C > 0$, which do not depend on T , \mathbf{u} , \mathbf{b} , \mathbf{u}_0 , \mathbf{b}_0 , \mathbb{F} , \mathbb{G} nor ϵ ; such that for all $R \geq 1$ and for all $0 \leq t \leq T$ we have:*

$$\begin{aligned} & \frac{1}{R^2} \sum_{i=1}^3 \int_0^t \int (p u_i + q b_i) \partial_i \phi_R ds dx \\ & \leq C \|(\mathbb{F}, \mathbb{G})\|_{B_2 L^2(0,t)}^2 + C \int_0^t \|(\mathbf{u}, \mathbf{b})(s)\|_{B_2}^2 + \|(\mathbf{u}, \mathbf{b})(s)\|_{B_2}^6 \\ & \quad + \frac{C_0}{R^2} \int \int_0^t |\varphi_{2(5R+1)} \nabla \mathbf{u}|^2 + |\varphi_{2(5R+1)} \nabla \mathbf{b}|^2 dx. \end{aligned}$$

Finally, the fifth and sixth terms (which involve the tensor forces \mathbb{F} and \mathbb{G}) are easily estimate as follows. We will write down only the estimates for \mathbb{F} since the estimates for \mathbb{G} are completely similar:

$$\left| \frac{1}{R^2} \sum_{1 \leq i,j \leq 3} \int_0^t \int F_{i,j} (\partial_i u_j) \phi_R dx ds \right| \leq C \|\mathbb{F}\|_{B_2 L^2(0,t)}^2 + \frac{C_0}{R^2} \int_0^t \int_{|x| < R} |\nabla \mathbf{u}|^2 dx ds,$$

and

$$\left| \frac{1}{R^2} \sum_{1 \leq i,j \leq 3} \int_0^t \int F_{i,j} u_i \partial_j (\phi_R) dx ds \right| \leq C \|\mathbb{F}\|_{B_2 L^2(0,t)}^2 + C \int_0^t \|(\mathbf{u}, \mathbf{b})(s)\|_{B_2}^2 ds.$$

where $C_0 > 0$ always denote a small enough constant.

Once we dispose of all these estimates, we are able to write

$$\begin{aligned}
& \int \left(\frac{|\mathbf{u}(t, x)|^2}{2} + \frac{|\mathbf{b}(t, x)|^2}{2} \right) \phi_R dx + \int_0^t \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2) \phi_R ds dx \\
& \leq \int \left(\frac{|\mathbf{u}(0, x)|^2}{2} + \frac{|\mathbf{b}(0, x)|^2}{2} \right) \phi_R dx + C \|(\mathbb{F}, \mathbb{G})\|_{B_2 L^2(0, t)}^2 ds \\
& \quad + C \int_0^t \|(\mathbf{u}, \mathbf{b})(s, \cdot)\|_{B_2}^2 + \|(\mathbf{u}, \mathbf{b})(s, \cdot)\|_{B_2}^6 ds \\
& \quad + \frac{C_0}{R^2} \int \int_0^t |\varphi_{2(5R+1)} \nabla \mathbf{u}|^2 + |\varphi_{2(5R+1)} \nabla \mathbf{b}|^2 dx,
\end{aligned}$$

where the desired energy control (3) follows. To finish this proof, the estimate (4) follows directly from (3) and the Lemma 3.1 in (7) (see the proof of Corollary 3.3, page 17, for all the details). \diamond

Proof of Lemma 3.1. As in the proof of the theorem above, we will consider only the case $(\mathbf{v}, \mathbf{c}) = (\mathbf{u} * \theta_\varepsilon, \mathbf{b} * \theta_\varepsilon)$. Moreover, we will focus only on the expression which involves the pressure p , since the computations for the other expression, where the term q appears, are completely similar.

We write $\frac{1}{R^2} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |pu_k| |\partial_k \phi_R| dx ds \leq \frac{c}{R^3} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |pu_k| dx ds$, and recalling that $p = \sum_{1 \leq i, j \leq 3} \mathcal{R}_i \mathcal{R}_j ((u_i * \theta_\varepsilon) u_j - (b_i * \theta_\varepsilon) b_j - F_{i, j})$, the last expression allow us to write

$$\begin{aligned}
& \frac{1}{R^2} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |pu_k| |\partial_k \phi_R| dx ds \\
& \leq \frac{c}{R^3} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |u_k \sum_{i, j=1}^3 \mathcal{R}_i \mathcal{R}_j ((u_i * \theta_\varepsilon) u_j)| dx ds \\
& \quad + \frac{c}{R^3} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |u_k \sum_{i, j=1}^3 \mathcal{R}_i \mathcal{R}_j ((b_i * \theta_\varepsilon) b_j - F_{i, j})| dx ds,
\end{aligned}$$

and since we have the same information on \mathbf{u} and \mathbf{b} it is enough to study the last term above. For $R \geq 1$ we define the following expressions:

$$p_1 = \sum_{i, j} \mathcal{R}_i \mathcal{R}_j (\mathbf{1}_{|y| < 5R} (\theta_\varepsilon * b_i) b_j), \quad p_2 = - \sum_{i, j} \mathcal{R}_i \mathcal{R}_j (\mathbf{1}_{|y| \geq 5R} (\theta_\varepsilon * b_i) b_j),$$

and

$$p_3 = - \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbf{1}_{|y| < 5R} F_{i,j}), \quad p_4 = \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbf{1}_{|y| \geq 5R} F_{i,j}),$$

and then, by the Young's inequalities (for products), we have

$$\begin{aligned} & \frac{c}{R^3} \sum_{k=1}^3 \int_0^t \int_{|x| \leq R} |u_k \sum_{i,j=1}^3 \mathcal{R}_i \mathcal{R}_j ((b_i * \theta_\varepsilon) b_j - F_{i,j})| dx ds \\ & \leq \frac{C}{R^3} \int_0^t \int_{|x| \leq R} (|p_1|^{3/2} + |p_2|^{3/2} + |\mathbf{u}|^3 + |p_3|^2 + |p_4|^2 + |\mathbf{u}|^2) dx ds, \end{aligned}$$

where we will study each term separately.

To study p_1 , by the continuity of \mathcal{R}_i on $L^{\frac{3}{2}}(\mathbb{R}^3)$, since the test function θ_ε verifies $\int \theta_\varepsilon(x) dx = 1$ and $\text{supp}(\theta_\varepsilon) \subset B(0, 1)$ and moreover, by the Fubini's theorem we can write

$$\begin{aligned} \int_{|x| \leq R} |p_1|^{3/2} dx & \leq C \int |p_1|^{3/2} dx \leq C \int |(\mathbf{1}_{|x| < 5R} (\theta_\varepsilon * \mathbf{b}) \otimes \mathbf{b})|^{3/2} dx \\ & \leq C \left(\int |\mathbf{1}_{|x| < 5R} (\theta_\varepsilon * \mathbf{b})|^3 dx \right)^{1/2} \left(\int |\mathbf{1}_{|y| < 5R} \mathbf{b}|^3 dx \right)^{1/2} \\ & \leq C \left(\int_{|x| \leq 5R} \int_{|x-z| \leq 1} \theta_\varepsilon(x-z) |\mathbf{b}(z)|^3 dz dx \right)^{1/2} \left(\int |(\mathbf{1}_{|y| < 5R} \mathbf{b})|^3 dx \right)^{1/2} \\ & \leq C \left(\int_{|x| \leq 5R} \int_{|z| \leq 5R+1} \theta_\varepsilon(x-z) |\mathbf{b}(z)|^3 dz dx \right)^{1/2} \left(\int |(\mathbf{1}_{|y| < 5R} \mathbf{b})|^3 dx \right)^{1/2} \\ & \leq C \int_{|z| \leq 5R+1} |\mathbf{b}|^3 dz. \end{aligned}$$

With this estimate at hand, we see that

$$\int_{|x| \leq R} |\mathbf{u}|^3 + |p_1|^{3/2} dx \leq C \int_{|x| \leq 5R+1} |\mathbf{u}|^3 + |\mathbf{b}|^3 dx,$$

and using the Sobolev embedding we write

$$\begin{aligned} & \frac{C}{R^3} \int_{|x| \leq 5R+1} |\mathbf{u}|^3 dx \leq \frac{C}{R^3} \|\mathbf{u}\|_{L^2(B(0,5R+1))}^{3/2} \|\mathbf{u}\|_{L^6(B(0,5R+1))}^{3/2} \\ & \leq \frac{C}{R^{3/2}} \|\mathbf{u}\|_{L^2(B(0,5R+1))}^{3/2} \left(\left(\frac{1}{R^2} \int |\phi_{2(5R+1)} \nabla \mathbf{u}|^2 dx \right)^{1/2} + \left(\frac{1}{R^2} \int_{|x| \leq 2(5R+1)} |\mathbf{u}|^2 dx \right)^{1/2} \right)^{3/2} \\ & \leq C \|\mathbf{u}\|_{B_2}^6 + C_0 \|\mathbf{u}\|_{B_2}^2 + \frac{C_0}{R^2} \int |\phi_{2(5R+1)} \nabla \mathbf{u}|^2 dx, \end{aligned}$$

where $C_0 > 0$ is an arbitrarily small constant. Similar bounds works for \mathbf{b} .

We study now the term p_2 . Remark first that there exist a constant $C > 0$ (which does not depend on $R > 1$) such that for all $|x| \leq R$ and all $|y| \geq 5R$, the kernel $\mathbb{K}_{i,j}$ of the operator $\mathcal{R}_i \mathcal{R}_j$ verifies $|\mathbb{K}_{i,j}(x-y)| \leq \frac{C}{|y|^3}$ (see [10] for a proof) and then we write:

$$\begin{aligned}
& \left(\int_{|x| \leq R} |p_2|^{3/2} dx \right)^{2/3} \\
& \leq C \sum_{i,j} \left(\int_{|x| \leq R} \left(\int_{|y| \geq 5R} |\mathbb{K}_{i,j}(x-y)| |(\theta_\epsilon * b_i)(y) b_j(y)| \mathbf{1}_{|y| \geq 5R} dy \right)^{3/2} dx \right)^{2/3} \\
& \leq C \left(\int_{|x| \leq R} \left(\int_{|y| \geq 5R} \frac{1}{|y|^3} |(\theta_\epsilon * \mathbf{b}) \otimes \mathbf{b}| dy \right)^{3/2} dx \right)^{2/3} \\
& \leq CR^2 \int_{|y| \geq 5R} \frac{1}{|y|^3} |(\theta_\epsilon * \mathbf{b}) \otimes \mathbf{b}| dy \\
& \leq CR^2 \left(\int_{|y| \geq 5R} \frac{1}{|y|^3} |\theta_\epsilon * \mathbf{b}|^2 dy \right)^{1/2} \left(\int_{|y| \geq 5R} \frac{1}{|y|^3} |\mathbf{b}|^2 dy \right)^{1/2} \\
& \leq CR^2 \left(\int_{|y| \geq 5R} \frac{1}{|y|^3} \int_{|y-z| < 1} \theta_\epsilon(y-z) |\mathbf{b}(z)|^2 dz dy \right)^{1/2} \left(\int_{|y| \geq 5R} \frac{1}{|y|^3} |\mathbf{b}|^2 dy \right)^{1/2} \\
& \leq CR^2 \left(\int_{|y| \geq 5R} \int_{|z| \geq 5R-1} \frac{1}{|z|^3} \theta_\epsilon(y-z) |\mathbf{b}(z)|^2 dz dy \right)^{1/2} \left(\int_{|y| \geq 5R} \frac{1}{|y|^3} |\mathbf{b}|^2 dy \right)^{1/2} \\
& \leq CR^2 \int_{|z| \geq 5R-1} \frac{1}{|z|^3} |\mathbf{b}|^2 dz.
\end{aligned}$$

With this estimate, and the fact that $B_2(\mathbb{R}^3) \subset L^2_{w^3}(\mathbb{R}^3)$, we finally obtain

$$\frac{C}{R^3} \int_{|y| \leq R} |p_2|^{3/2} dx \leq C \left(\int \frac{1}{(1+|z|)^3} |\mathbf{b}|^2 \right)^{3/2} \leq C \|\mathbf{b}\|_{B_2}^3.$$

It remains to estimate the terms p_3 and p_4 which involve the tensor \mathbb{F} . For p_3 , using the continuity of the Riesz transform \mathcal{R}_i on L^2 , we obtain directly:

$$\frac{c}{R^3} \int_0^t \int_{|x| \leq R} |p_3|^2 dx ds \leq \frac{C}{R^3} \sum_{i,j} \int_0^t \int_{|x| < 5R} |\mathbb{F}_{i,j}|^2 dx ds \leq C \|\mathbb{F}\|_{B_2 L^2(0,t)}^2.$$

For the term p_4 , remark first that we have

$$\begin{aligned}
\left(\int_{|x| \leq R} |p_4|^2 dx \right)^{1/2} &\leq C \sum_{i,j} \left(\int_{|x| \leq R} \left(\int_{|y| \geq 5R} |\mathbb{K}_{i,j}(x-y) \mathbb{F}_{i,j}| dy \right)^2 dx \right)^{1/2} \\
&\leq C \sum_{i,j} \left(\int_{|x| \leq R} \left(\int_{|y| \geq 5R} \frac{1}{|y|^3} |\mathbb{F}_{i,j}| dy \right)^2 dx \right)^{1/2} \\
&\leq C \sum_{i,j} R^{3/2} \int_{|y| \geq 5R} \frac{1}{|y|^3} |\mathbb{F}_{i,j}| dy,
\end{aligned}$$

and then, for $0 < \delta < 1$, and by the Hölder inequalities we can write:

$$\begin{aligned}
\frac{C}{R^3} \int_0^t \int_{|x| \leq R} |p_4|^2 dx ds &\leq C \sum_{i,j} \int_0^t \left(\int \frac{1}{(1+|x|)^3} |\mathbb{F}_{i,j}| dx \right)^2 ds \\
&\leq C \sum_{i,j} \int_0^t \int \frac{1}{(1+|x|)^{2+\delta}} |\mathbb{F}_{i,j}|^2 dx ds \\
&\leq C \sum_{i,j} \int \frac{1}{(1+|x|)^{2+\delta}} \int_0^t |\mathbb{F}_{i,j}|^2 ds dx \\
&\leq C \|\mathbb{F}\|_{B_2 L^2(0,t)}^2.
\end{aligned}$$

The lemma is proven. \diamond

3.2 A stability result

Theorem 4 *Let $0 < T < +\infty$. Let $\mathbf{u}_{0,n}, \mathbf{b}_{0,n}$ be divergence-free vector fields such that $(\mathbf{u}_{0,n}, \mathbf{b}_{0,n}) \in B_2$. Let \mathbb{F}_n and \mathbb{G}_n be tensors such that $(\mathbb{F}_n, \mathbb{G}_n) \in B_2 L^2(0, T)$. Let $(\mathbf{u}_n, \mathbf{b}_n, p_n, q_n)$ be a solution of the (MHD*) problem:*

$$\begin{cases} \partial_t \mathbf{u}_n = \Delta \mathbf{u}_n - (\mathbf{v}_n \cdot \nabla) \mathbf{u}_n + (\mathbf{c}_n \cdot \nabla) \mathbf{b}_n - \nabla p_n + \nabla \cdot \mathbb{F}_n, \\ \partial_t \mathbf{b}_n = \Delta \mathbf{b}_n - (\mathbf{v}_n \cdot \nabla) \mathbf{b}_n + (\mathbf{c}_n \cdot \nabla) \mathbf{u}_n - \nabla q_n + \nabla \cdot \mathbb{G}_n, \\ \nabla \cdot \mathbf{u}_n = 0, \quad \nabla \cdot \mathbf{b}_n = 0, \\ \mathbf{u}_n(0, \cdot) = \mathbf{u}_{0,n}, \quad \mathbf{b}_n(0, \cdot) = \mathbf{b}_{0,n}. \end{cases} \quad (6)$$

which verifies the same hypothesis of Theorem [3](#).

If $(\mathbf{u}_{0,n}, \mathbf{b}_{0,n})$ is strongly convergent to $(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})$ in B_2 , and if the sequence $(\mathbb{F}_n, \mathbb{G}_n)$ is strongly convergent to $(\mathbb{F}_\infty, \mathbb{G}_\infty)$ in $B_2 L^2(0, T)$; then there exists $(\mathbf{u}_\infty, \mathbf{b}_\infty, p_\infty, q_\infty)$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ with values in \mathbb{N} such that:

- $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges $*$ -weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^\infty((0, T), B_2)$, $(\nabla \mathbf{u}_{n_k}, \nabla \mathbf{b}_{n_k})$ converges weakly to $(\nabla \mathbf{u}_\infty, \nabla \mathbf{b}_\infty)$ in $B_2 L^2(0, T)$.
- $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^2_{\text{loc}}([0, T] \times \mathbb{R}^3)$.
- For $2 < \gamma < 5/2$, the sequence (p_{n_k}, q_{n_k}) converges weakly to (p_∞, q_∞) in $L^3((0, T), L^{6/5}_{w_{6\gamma}}) + L^2((0, T), L^2_{w_\gamma})$.

Moreover, $(\mathbf{u}_\infty, \mathbf{b}_\infty, p_\infty, q_\infty)$ is a solution of the problem (MHD*):

$$\begin{cases} \partial_t \mathbf{u}_\infty = \Delta \mathbf{u}_\infty - (\mathbf{u}_\infty \cdot \nabla) \mathbf{u}_\infty + (\mathbf{b}_\infty \cdot \nabla) \mathbf{b}_\infty - \nabla p_\infty + \nabla \cdot \mathbb{F}_\infty, \\ \partial_t \mathbf{b}_\infty = \Delta \mathbf{b}_\infty - (\mathbf{u}_\infty \cdot \nabla) \mathbf{b}_\infty + (\mathbf{b}_\infty \cdot \nabla) \mathbf{u}_\infty - \nabla q_\infty + \nabla \cdot \mathbb{G}_\infty, \\ \nabla \cdot \mathbf{u}_\infty = 0, \quad \nabla \cdot \mathbf{b}_\infty = 0, \\ \mathbf{u}_\infty(0, \cdot) = \mathbf{u}_{0,\infty}, \quad \mathbf{b}_\infty(0, \cdot) = \mathbf{b}_{0,\infty}, \end{cases} \quad (7)$$

and verifies all the hypothesis of Theorem [3](#).

Proof. We will verify that the sequence $(\mathbf{u}_n, \mathbf{b}_n)$ satisfy the hypothesis of the Rellich lemma (see Lemma 6 in [9](#)). Remark first that: since for $2 < \gamma$ we have that $\mathbf{u}_n, \mathbf{b}_n$ is bounded in $L^\infty((0, T), B_2) \subset L^\infty((0, T), L^2_{w_\gamma})$ and moreover, since we have that $\nabla \mathbf{u}_n, \nabla \mathbf{b}_n$ is bounded in $B_2 L^2(0, T) \subset L^2((0, T), L^2_{w_\gamma})$, then for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$ we have that $(\varphi \mathbf{u}_n, \varphi \mathbf{b}_n)$ are bounded in $L^2((0, T), H^1)$. On the other hand, for the pressure p_n and the term q_n we write $p_n = p_{n,1} + p_{n,2}$ with

$$p_{n,1} = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (v_{n,i} u_{n,j} - c_{n,i} b_{n,j}), \quad p_{n,2} = - \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (F_{n,i,j}),$$

and we write $q_n = q_{n,1} + q_{n,2}$ with

$$q_{n,1} = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (v_{n,i} b_{n,j} - c_{n,i} u_{n,j}), \quad q_{n,2} = - \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (G_{n,i,j}).$$

From now on we fix $\gamma \in (2, \frac{5}{2})$, and using the interpolation inequalities and the continuity of the Riesz transforms in the Lebesgue weighted spaces we get that the sequence $(p_{n,1}, q_{n,1})$ is bounded in $L^3((0, T), L^{6/5}_{w_{6\gamma}})$. Indeed, for the term $p_{n,1}$ recall that by Lemma [2.2](#) we have that for $0 < \gamma < 5/2$ the weight $w_{6\gamma/5}$ belongs to the Muckenhoupt class $\mathcal{A}_p(\mathbb{R}^3)$ (with $1 < p < +\infty$) and then we can write:

$$\begin{aligned} \left\| \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbf{u}_{n,i} \mathbf{u}_{n,j}) w_\gamma \right\|_{L^{6/5}} &\leq \|(\mathbf{u}_n \otimes \mathbf{u}_n) w_\gamma\|_{L^{6/5}} \leq \|\sqrt{w_\gamma} \mathbf{u}_n\|_{L^2}^{\frac{3}{2}} \|\sqrt{w_\gamma} \mathbf{u}_n\|_{L^6}^{\frac{1}{2}} \\ &\leq \|\sqrt{w_\gamma} \mathbf{u}\|_{L^2}^{\frac{3}{2}} (\|\sqrt{w_\gamma} \mathbf{u}\|_{L^2} + \|\sqrt{w_\gamma} \nabla \mathbf{u}\|_{L^2})^{\frac{1}{2}}. \end{aligned}$$

The term $q_{n,1}$ is estimated in a similar way. Moreover we have that the sequence and $(p_{n,2}, q_{n,2})$ is bounded in $L^2((0, T), L^2_{w_\gamma})$. With these information, by equation (6) we obtain that $(\varphi \partial_t \mathbf{u}_n, \varphi \partial_t \mathbf{b}_n)$ are bounded in the space $L^2 L^2 + L^2 W^{-1, 6/5} + L^2 H^{-1} \subset L^2((0, T), H^{-2})$. Thus, we can apply the Rellich lemma and there exists an increasing sequence $(n_k)_{k \in \mathbb{N}}$ in \mathbb{N} , and there exist a couple of functions $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ such that $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^2_{\text{loc}}([0, T] \times \mathbb{R}^3)$. We also have that $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k}) = (\mathbf{v}_{n_k} * \theta_{\epsilon_{n_k}}, \mathbf{c}_{n_k} * \theta_{\epsilon_{n_k}})$ converges strongly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^2_{\text{loc}}([0, T] \times \mathbb{R}^3)$.

As $(\mathbf{u}_n, \mathbf{b}_n)$ are bounded in $L^\infty((0, T), L^2_{w_\gamma})$ and $(\nabla \mathbf{u}_n, \nabla \mathbf{b}_n)$ are bounded in $L^2((0, T), L^2_{w_\gamma})$, we have that $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges *-weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^\infty((0, T), L^2_{w_\gamma})$, and $(\nabla \mathbf{u}_{n_k}, \nabla \mathbf{b}_{n_k})$ converges weakly to $(\nabla \mathbf{u}_\infty, \nabla \mathbf{b}_\infty)$ in $L^2((0, T), L^2_{w_\gamma})$. Moreover, by the Sobolev embeddings and the interpolation inequalities we have that $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^3((0, T), L^3_{w_{3\gamma/2}})$. Also $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k}) = (\mathbf{v}_{n_k} * \theta_{\epsilon_{n_k}}, \mathbf{c}_{n_k} * \theta_{\epsilon_{n_k}})$ converges weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^3((0, T), L^3_{w_{3\gamma/2}})$, since it is bounded in $L^3((0, T), L^3_{w_{3\gamma/2}})$. In particular, we may observe that the terms $v_{n_k, i} u_{n_k, j}$, $c_{n_k, i} b_{n_k, j}$, $v_{n_k, i} b_{n_k, j}$ and $c_{n_k, i} u_{n_k, j}$ are weakly convergent in $(L^{6/5} L^{6/5})_{\text{loc}}$ and thus in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$.

As those terms are bounded in $L^3((0, T), L^{6/5}_{w_{6\gamma}})$, they are weakly convergent in $L^3((0, T), L^{6/5}_{w_{6\gamma}})$; and defining $p_\infty = p_{\infty,1} + p_{\infty,2}$ with

$$p_{\infty,1} = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (v_{\infty, i} u_{\infty, j} - c_{\infty, i} b_{\infty, j}), \quad p_2 = - \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (F_{\infty, i, j}),$$

and $q_\infty = q_{\infty,1} + q_{\infty,2}$ with

$$q_{\infty,1} = \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (v_{\infty, i} b_{\infty, j} - c_{\infty, i} u_{\infty, j}), \quad q_2 = - \sum_{i=1}^3 \sum_{j=1}^3 R_i R_j (G_{\infty, i, j}),$$

we obtain that $(p_{n_k,1}, q_{n_k,1})$ are weakly convergent in $L^3((0, T), L^{6/5}_{w_{6\gamma}})$ to $(p_{\infty,1}, q_{\infty,1})$, and moreover, we get that $(p_{n_k,2}, q_{n_k,2})$ is strongly convergent in $L^2((0, T), L^2_{w_\gamma})$ to $(p_{\infty,2}, q_{\infty,2})$. So, we have that $(\mathbf{u}_\infty, \mathbf{b}_\infty, p_\infty, q_\infty)$ verify the three first equations in the system (MHD^*) in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$.

It remains to verify the conditions at the time $t = 0$. Remark that $(\partial_t \mathbf{u}_\infty, \partial_t \mathbf{b}_\infty)$ are locally in $L^2 H^{-2}$, and then $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ have representatives such that $t \mapsto (\mathbf{u}_\infty(t, \cdot), \mathbf{b}_\infty(t, \cdot))$ is continuous from $[0, T)$ to $\mathcal{D}'(\mathbb{R}^3)$ (hence *-weakly continuous from $[0, T)$ to B_2) and moreover, they coincide with

$\mathbf{u}_\infty(0, \cdot) + \int_0^t \partial_t \mathbf{u}_\infty ds$ and $\mathbf{b}_\infty(0, \cdot) + \int_0^t \partial_t \mathbf{b}_\infty ds$. Thus, in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$, we have that

$$\begin{aligned} \mathbf{u}_\infty(0, \cdot) + \int_0^t \partial_t \mathbf{u}_\infty ds &= \mathbf{u}_\infty = \lim_{n_k \rightarrow +\infty} \mathbf{u}_{n_k} = \lim_{n_k \rightarrow +\infty} \mathbf{u}_{n_k,0} + \int_0^t \partial_t \mathbf{u}_{n_k} ds \\ &= \mathbf{u}_{\infty,0} + \int_0^t \partial_t \mathbf{u}_\infty ds, \end{aligned}$$

which implies that $\mathbf{u}_\infty(0, \cdot) = \mathbf{u}_{\infty,0}$. Similar we have the identity $\mathbf{b}_\infty(0, \cdot) = \mathbf{b}_{\infty,0}$. We conclude that $(\mathbf{u}_\infty, \mathbf{b}_\infty, p_\infty, q_\infty)$ is a solution of the (MHD^*) equations.

Our next task is to verify the local energy equality. We define the quantity

$$\begin{aligned} A_{n_k} &= -\partial_t \left(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2} \right) + \Delta \left(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2} \right) - \nabla \cdot \left(\left(\frac{|\mathbf{u}_{n_k}|^2}{2} + \frac{|\mathbf{b}_{n_k}|^2}{2} \right) \mathbf{v}_{n_k} \right) \\ &\quad - \nabla \cdot (p_{n_k} \mathbf{u}_{n_k}) - \nabla \cdot (q_{n_k} \mathbf{b}_{n_k}) + \nabla \cdot ((\mathbf{u}_{n_k} \cdot \mathbf{b}_{n_k}) \mathbf{c}_{n_k}) \\ &\quad + \mathbf{u}_{n_k} \cdot (\nabla \cdot \mathbb{F}_{n_k}) + \mathbf{b}_{n_k} \cdot (\nabla \cdot \mathbb{G}_{n_k}). \end{aligned}$$

Remark that by the information on $(\mathbf{u}_n, \mathbf{b}_n)$ and by interpolation we have $(\mathbf{u}_n, \mathbf{b}_n)$ are bounded in $L^{10/3}((0, T), L^{10/3}_{w_{5\gamma/3}})$ and then $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ are locally bounded in $L_t^{10/3} L_x^{10/3}$ and locally strongly convergent in $L_t^2 L_x^2$. So, $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly in $(L_t^3 L_x^3)_{loc}$. Moreover, by Lemma [3.1](#) we have that (p_{n_k}, q_{n_k}) are locally bounded in $L_t^{3/2} L_x^{3/2}$. Thus the quantity A_{n_k} converges in the distributional sense to

$$\begin{aligned} A_\infty &= -\partial_t \left(\frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2} \right) + \Delta \left(\frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2} \right) - \nabla \cdot \left(\left(\frac{|\mathbf{u}_\infty|^2}{2} + \frac{|\mathbf{b}_\infty|^2}{2} \right) \mathbf{v}_\infty \right) \\ &\quad - \nabla \cdot (p_\infty \mathbf{u}_\infty) - \nabla \cdot (q_\infty \mathbf{b}_\infty) + \nabla \cdot ((\mathbf{u}_\infty \cdot \mathbf{b}_\infty) \mathbf{c}_\infty) \\ &\quad + \mathbf{u}_\infty \cdot (\nabla \cdot \mathbb{F}_\infty) + \mathbf{b}_\infty \cdot (\nabla \cdot \mathbb{G}_\infty). \end{aligned}$$

Moreover, recall that by hypothesis of this theorem there exist μ_{n_k} a non-negative locally finite measure on $(0, T) \times \mathbb{R}^3$ such that

$$\begin{aligned} \partial_t \left(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2} \right) &= \Delta \left(\frac{|\mathbf{u}_{n_k}|^2 + |\mathbf{b}_{n_k}|^2}{2} \right) - |\nabla \mathbf{u}_{n_k}|^2 - |\nabla \mathbf{b}_{n_k}|^2 \\ &\quad - \nabla \cdot \left(\left(\frac{|\mathbf{u}_{n_k}|^2}{2} + \frac{|\mathbf{b}_{n_k}|^2}{2} \right) \mathbf{v}_{n_k} \right) - \nabla \cdot (p_{n_k} \mathbf{u}_{n_k}) - \nabla \cdot (q_{n_k} \mathbf{b}_{n_k}) \\ &\quad + \nabla \cdot ((\mathbf{u}_{n_k} \cdot \mathbf{b}_{n_k}) \mathbf{c}_{n_k}) + \mathbf{u}_{n_k} \cdot (\nabla \cdot \mathbb{F}_{n_k}) + \mathbf{b}_{n_k} \cdot (\nabla \cdot \mathbb{G}_{n_k}) - \mu_{n_k}. \end{aligned}$$

Then, by definition of A_{n_k} we can write $A_{n_k} = |\nabla \mathbf{u}_{n_k}|^2 + |\nabla \mathbf{b}_{n_k}|^2 + \mu_{n_k}$, and thus we have $A_\infty = \lim_{n_k \rightarrow +\infty} |\nabla \mathbf{u}_{n_k}|^2 + |\nabla \mathbf{b}_{n_k}|^2 + \mu_{n_k}$.

Now, let $\Phi \in \mathcal{D}((0, T) \times \mathbb{R}^3)$ be a non-negative function. As $\sqrt{\Phi}(\nabla \mathbf{u}_{n_k} + \nabla \mathbf{b}_{n_k})$ is weakly convergent to $\sqrt{\Phi}(\nabla \mathbf{u}_\infty + \nabla \mathbf{b}_\infty)$ in $L_t^2 L_x^2$, we have

$$\begin{aligned} \iint A_\infty \Phi \, dx \, ds &= \lim_{n_k \rightarrow +\infty} \iint A_{n_k} \Phi \, dx \, ds \geq \limsup_{n_k \rightarrow +\infty} \iint (|\nabla \mathbf{u}_{n_k}|^2 + |\nabla \mathbf{b}_{n_k}|^2) \Phi \, dx \, ds \\ &\geq \iint (|\nabla \mathbf{u}_\infty|^2 + |\nabla \mathbf{b}_\infty|^2) \Phi \, dx \, ds. \end{aligned}$$

Thus, there exists a non-negative locally finite measure μ_∞ on $(0, T) \times \mathbb{R}^3$ such that $A_\infty = (|\nabla \mathbf{u}_\infty|^2 + |\nabla \mathbf{b}_\infty|^2) + \mu_\infty$, and then we obtain the desired local energy equality:

$$\begin{aligned} \partial_t \left(\frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2} \right) &= \Delta \left(\frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2} \right) - |\nabla \mathbf{u}_\infty|^2 - |\nabla \mathbf{b}_\infty|^2 \\ &- \nabla \cdot \left(\left(\frac{|\mathbf{u}_\infty|^2}{2} + \frac{|\mathbf{b}_\infty|^2}{2} \right) \mathbf{v}_\infty \right) - \nabla \cdot (p_\infty \mathbf{u}_\infty) - \nabla \cdot (q_\infty \mathbf{b}_\infty) \\ &+ \nabla \cdot ((\mathbf{u}_\infty \cdot \mathbf{b}_\infty) \mathbf{c}_\infty) + \mathbf{u}_\infty \cdot (\nabla \cdot \mathbb{F}_\infty) + \mathbf{b}_\infty \cdot (\nabla \cdot \mathbb{G}_\infty) - \mu_\infty. \end{aligned}$$

In order to finish this proof, it remains to prove the convergence to the initial data $(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})$. Once we dispose of this local energy equality, as in [\(5\)](#) we can write:

$$\begin{aligned} &\int \frac{|\mathbf{u}_n(t, x)|^2 + |\mathbf{b}_n(t, x)|^2}{2} \phi_R \, dx + \int_0^t \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2) \phi_R \, dx \, ds \\ &\leq \int \frac{|\mathbf{u}_{0,n}(x)|^2 + |\mathbf{b}_{0,n}(x)|^2}{2} \phi_R \, dx + \int_0^t \int \frac{|\mathbf{u}_n|^2 + |\mathbf{b}_n|^2}{2} \Delta \phi_R \, dx \, ds \\ &\quad + \sum_{i=1}^3 \int_0^t \int \left[\left(\frac{|\mathbf{u}_n|^2}{2} + \frac{|\mathbf{b}_n|^2}{2} \right) v_{n,i} + p_n u_{n,i} \right] \partial_i \phi_R \, dx \, ds \\ &\quad + \sum_{i=1}^3 \int_0^t \int [(\mathbf{u}_n \cdot \mathbf{b}_n) c_{n,i} + q_n b_{n,i}] \partial_i \phi_R \, dx \, ds \\ &\quad - \sum_{1 \leq i, j \leq 3} \left(\int_0^t \int F_{n,i,j} u_{n,j} \partial_i \phi_R \, dx \, ds + \int_0^t \int F_{n,i,j} \partial_i u_{n,j} \phi_R \, dx \, ds \right) \\ &\quad - \sum_{1 \leq i, j \leq 3} \left(\int_0^t \int G_{n,i,j} b_{n,j} \partial_i \phi_R \, dx \, ds + \int_0^t \int G_{n,i,j} \partial_i b_j \phi_R \, dx \, ds \right). \end{aligned}$$

Then we have:

$$\begin{aligned}
& \limsup_{n_k \rightarrow +\infty} \int \frac{|\mathbf{u}_{n_k}(t, x)|^2 + |\mathbf{b}_{n_k}(t, x)|^2}{2} \phi_R dx + \int_0^t \int (|\nabla \mathbf{u}_{n_k}|^2 + |\nabla \mathbf{b}_{n_k}|^2) \phi_R dx ds \\
& \leq \int \frac{|\mathbf{u}_0(x)|^2 + |\mathbf{b}_0(x)|^2}{2} \phi_R dx + \int_0^t \int \frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2} \Delta \phi_R dx ds \\
& \quad + \sum_{i=1}^3 \int_0^t \int \left[\left(\frac{|\mathbf{u}_\infty|^2}{2} + \frac{|\mathbf{b}_\infty|^2}{2} \right) v_{\infty, i} + p_\infty u_{\infty, i} \right] \partial_i \phi_R dx ds \\
& \quad + \sum_{i=1}^3 \int_0^t \int [(\mathbf{u}_\infty \cdot \mathbf{b}_\infty) c_{\infty, i} + q_\infty b_{\infty, i}] \partial_i \phi_R dx ds \\
& \quad - \sum_{1 \leq i, j \leq 3} \left(\int_0^t \int F_{\infty, i, j} u_{\infty, j} \partial_i \phi_R dx ds + \int_0^t \int F_{\infty, i, j} \partial_i u_{\infty, j} \phi_R dx ds \right) \\
& \quad - \sum_{1 \leq i, j \leq 3} \left(\int_0^t \int G_{\infty, i, j} b_{\infty, j} \partial_i \phi_R dx ds + \int_0^t \int G_{\infty, i, j} \partial_i b_j \phi_R dx ds \right).
\end{aligned}$$

Recalling that $\mathbf{u}_{n_k} = \mathbf{u}_{0, n_k} + \int_0^t \partial_t \mathbf{u}_{n_k} ds$, we may observe that $\mathbf{u}_{n_k}(t, \cdot)$ converges to $\mathbf{u}_\infty(t, \cdot)$ in $\mathcal{D}'(\mathbb{R}^3)$, hence, it converges weakly in $L^2_{\text{loc}}(\mathbb{R}^3)$ and we can write:

$$\int \frac{|\mathbf{u}_\infty(t, x)|^2}{2} \phi_R dx \leq \limsup_{n_k \rightarrow +\infty} \int \frac{|\mathbf{u}_{n_k}(t, x)|^2}{2} \phi_R dx.$$

Moreover, this weakly convergence gives

$$\int_0^t \int \frac{|\nabla \mathbf{u}_\infty(s, x)|^2}{2} \phi_R dx ds \leq \limsup_{n_k \rightarrow +\infty} \int_0^t \int \frac{|\nabla \mathbf{u}_{n_k}(s, x)|^2}{2} \phi_R dx ds,$$

and we have the same estimates for \mathbf{b}_∞ . In this way we get

$$\begin{aligned}
& \int \frac{|\mathbf{u}_\infty(t, x)|^2 + |\mathbf{b}_\infty(t, x)|^2}{2} \phi_R dx + \int_0^t \int (|\nabla \mathbf{u}_\infty|^2 + |\nabla \mathbf{b}_\infty|^2) \phi_R dx ds \\
& \leq \int \frac{|\mathbf{u}_0(x)|^2 + |\mathbf{b}_0(x)|^2}{2} \phi_R dx + \int_0^t \int \frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2} \Delta \phi_R dx ds \\
& \quad + \sum_{i=1}^3 \int_0^t \int \left[\left(\frac{|\mathbf{u}_\infty|^2}{2} + \frac{|\mathbf{b}_\infty|^2}{2} \right) v_{\infty, i} + p_\infty u_{\infty, i} \right] \partial_i \phi_R dx ds
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^3 \int_0^t \int [(\mathbf{u}_\infty \cdot \mathbf{b}_\infty) c_{\infty,i} + q_\infty b_{\infty,i}] \partial_i \phi_R dx ds \\
& - \sum_{1 \leq i, j \leq 3} \left(\int_0^t \int F_{\infty,i,j} u_{\infty,j} \partial_i \phi_R dx ds + \int_0^t \int F_{\infty,i,j} \partial_i u_{\infty,j} \phi_R dx ds \right) \\
& - \sum_{1 \leq i, j \leq 3} \left(\int_0^t \int G_{\infty,i,j} b_{\infty,j} \partial_i \phi_R dx ds + \int_0^t \int G_{\infty,i,j} \partial_i b_{\infty,j} \phi_R dx ds \right).
\end{aligned}$$

Finally, letting t go to 0, we have:

$$\limsup_{t \rightarrow 0} \|(\mathbf{u}_\infty, \mathbf{b}_\infty)(t, \cdot)\|_{L^2(\phi_R(x)dx)}^2 \leq \|(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})\|_{L^2(\phi_R(x)dx)}^2.$$

On the other hand, by weakly convergence we also have

$$\|(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})\|_{L^2(\phi_R(x)dx)}^2 \leq \liminf_{t \rightarrow 0} \|(\mathbf{u}_\infty, \mathbf{b}_\infty)(t, \cdot)\|_{L^2(\phi_R(x)dx)}^2.$$

Thus we have the strong convergence to initial data in the Hilbert space $L^2(\phi_R(x)dx)$.

4 Proof of Theorem 1

4.1 Local in time existence

Following the ideas of 7, for the given function $\phi_R(x) = \phi(\frac{x}{R})$ and the Leray's projector \mathbb{P} , we define $\mathbf{u}_{0,R} = \mathbb{P}(\phi_R \mathbf{u}_0)$, $\mathbf{b}_{0,R} = \mathbb{P}(\phi_R \mathbf{b}_0)$, $\mathbb{F}_R = \phi_R \mathbb{F}$, $\mathbb{G}_R = \phi_R \mathbb{G}$; and we consider the approximated problem $(MHD_{R,\epsilon})$:

$$\left\{ \begin{array}{l}
\partial_t \mathbf{u}_{R,\epsilon} = \Delta \mathbf{u}_{R,\epsilon} - ((\mathbf{u}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla) \mathbf{u}_{R,\epsilon} + ((\mathbf{b}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla) \mathbf{b}_{R,\epsilon} - \nabla p_{R,\epsilon} + \nabla \cdot \mathbb{F}_R, \\
\partial_t \mathbf{b}_{R,\epsilon} = \Delta \mathbf{b}_{R,\epsilon} - ((\mathbf{u}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla) \mathbf{b}_{R,\epsilon} + ((\mathbf{b}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla) \mathbf{u}_{R,\epsilon} - \nabla q_{R,\epsilon} + \nabla \cdot \mathbb{G}_R, \\
\nabla \cdot \mathbf{u}_{R,\epsilon} = 0, \nabla \cdot \mathbf{b}_{R,\epsilon} = 0, \\
\mathbf{u}_{R,\epsilon}(0, \cdot) = \mathbf{u}_{0,R}, \mathbf{b}_{R,\epsilon}(0, \cdot) = \mathbf{b}_{0,R}.
\end{array} \right.$$

By the Appendix in 7 (see the page 35) we know that $(MHD_{R,\epsilon})$ has a unique solution $(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon})$ in $L^\infty((0, +\infty), L^2) \cap L^2((0, +\infty), \dot{H}^1)$, and moreover, this solution belongs to $\mathcal{C}([0, +\infty), L^2)$ and it fulfills the hypothesis of the Theorem 3. Applying this result (for the case $(\mathbf{v}, \mathbf{c}) = (\mathbf{u} * \theta_\epsilon, \mathbf{b} * \theta_\epsilon)$) there exists a constant $C > 0$ such that for every time T_0 small enough:

$$C \left(1 + \|(\mathbf{u}_{0,R}, \mathbf{b}_{0,R})\|_{B_2}^2 + \|(\mathbb{F}_{R,\epsilon}, \mathbb{G}_{R,\epsilon})\|_{B_2 L^2(0, T_0)}^2 \right)^2 T_0 \leq 1,$$

we have the controls:

$$\sup_{0 \leq t \leq T_0} \|(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon})(t)\|_{B_2}^2 \leq C \left(1 + \|(\mathbf{u}_{0,R}, \mathbf{b}_{0,R})\|_{B_2}^2 + \|(\mathbb{F}_{R,\epsilon}, \mathbb{G}_{R,\epsilon})\|_{B_2 L^2(0, T_0)}^2 \right),$$

and

$$\|\nabla(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon})\|_{B_2 L^2(0, T_0)}^2 \leq C \left(1 + \|(\mathbf{u}_{0,R}, \mathbf{b}_{0,R})\|_{B_2}^2 + \|(\mathbb{F}_{R,\epsilon}, \mathbb{G}_{R,\epsilon})\|_{B_2 L^2(0, T_0)}^2 \right).$$

Then, in the setting of Theorem [4](#), we set $(\mathbf{u}_{0,n}, \mathbf{b}_{0,n}) = (\mathbf{u}_{0,R_n}, \mathbf{b}_{0,R_n})$, $\mathbb{F}_n = \mathbb{F}_{R_n}$, $\mathbb{G}_n = \mathbb{G}_{R_n}$ and $(\mathbf{u}_n, \mathbf{b}_n) = (\mathbf{u}_{R_n, \epsilon_n}, \mathbf{b}_{R_n, \epsilon_n})$; and letting $R_n \rightarrow +\infty$ and $\epsilon_n \rightarrow 0$ we find a local solution of the (MHD) equations which verifies the desired properties stated in Theorem [1](#).

4.2 Global in time existence

Let $\lambda > 1$. For $n \in \mathbb{N}$ we consider the (MHD) equations with initial value

$$(\mathbf{u}_{0,n}, \mathbf{b}_{0,n}) = (\lambda^n \mathbf{u}_0(\lambda^n \cdot), \lambda^n \mathbf{b}_0(\lambda^n \cdot)),$$

and the forcing tensors

$$(\mathbb{F}_n, \mathbb{G}_n) = (\lambda^{2n} \mathbb{F}(\lambda^{2n} \cdot, \lambda^n \cdot), \lambda^{2n} \mathbb{G}(\lambda^{2n} \cdot, \lambda^n \cdot)).$$

Then, by the local in time existence proved above, there exists a solution $(\mathbf{v}_n, \mathbf{c}_n)$ on $(0, T_n)$, with

$$C \left(1 + \|(\mathbf{v}_{0,n}, \mathbf{c}_{0,n})\|_{B_2}^2 + \|(\mathbb{F}_n, \mathbb{G}_n)\|_{B_2 L^2(0, T_n)}^2 \right)^2 T_n = 1.$$

Remark also that by the well-known scaling properties of the (MHD) equations we have

$$(\mathbf{v}_n(t, x), \mathbf{c}_n(t, x)) = (\lambda^n \mathbf{u}_n(\lambda^{2n} t, \lambda^n x), \lambda^n \mathbf{b}_n(\lambda^{2n} t, \lambda^n x)),$$

where $(\mathbf{u}_n, \mathbf{b}_n)$ is a solution of the (MHD) on $(0, \lambda^{2n} T_n)$ associated with the initial data $(\mathbf{u}_0, \mathbf{b}_0)$ and then forcing tensors \mathbb{F} and \mathbb{G} .

At this point, we need the following simple remark which will be proved at the end of this section.

Remark 4.1 *If $\mathbf{u}_0, \mathbf{b}_0 \in B_{2,0}$ and $\mathbb{F}, \mathbb{G} \in B_{2,0} L^2(0, +\infty)$, then for all $\lambda > 1$ we have:*

$$\lim_{n \rightarrow +\infty} \frac{\lambda^n}{1 + \|(\mathbf{v}_{0,n}, \mathbf{c}_{0,n})\|_{B_2}^2 + \|(\mathbb{F}_n, \mathbb{G}_n)\|_{B_2 L^2}^2} = +\infty.$$

Thus, for $\lambda > 1$ fix we have $\lim_{n \rightarrow +\infty} \lambda^{2n} T_n = +\infty$. Then, for $T > 0$, let n_T such that $\lambda^{2n} T_n > T$ for $n \geq n_T$, then $(\mathbf{u}_n, \mathbf{b}_n)$ is a solution of the (MHD) equations on $(0, T)$.

We set now $(\mathbf{w}_n(t, x), \mathbf{d}_n(t, x)) = (\lambda^{n_T} \mathbf{u}_n(\lambda^{2n_T} t, \lambda^{n_T} x), \lambda^{n_T} \mathbf{b}_n(\lambda^{2n_T} t, \lambda^{n_T} x))$, where we observe that for $n \geq n_T$ the couple $(\mathbf{w}_n, \mathbf{d}_n)$ is a solution of (MHD) equations on $(0, \lambda^{-2n_T} T)$ with initial value $(\mathbf{v}_{0, n_T}, \mathbf{c}_{0, n_T})$ and forcing tensor $(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})$. But, since we have $\lambda^{-2n_T} T \leq T_{n_T}$, then we obtain

$$C \left(1 + \|(\mathbf{v}_{0, n_T}, \mathbf{c}_{0, n_T})\|_{B_2}^2 + \|(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})\|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2 \right)^2 \lambda^{-2n_T} T \leq 1,$$

and thus, by Theorem [3](#) we are able to write:

$$\sup_{0 \leq t \leq \lambda^{-2n_T} T} \|(\mathbf{w}_n, \mathbf{d}_n)(t, \cdot)\|_{L_{w_\gamma}^2}^2 \leq C(1 + \|(\mathbf{v}_{0, n_T}, \mathbf{c}_{0, n_T})\|_{B_2}^2 + \|(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})\|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2),$$

and

$$\|\nabla(\mathbf{w}_n, \mathbf{d}_n)\|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2 \leq C(1 + \|(\mathbf{v}_{0, n_T}, \mathbf{c}_{0, n_T})\|_{B_2}^2 + \|(\mathbb{F}_{n_T}, \mathbb{G}_{n_T})\|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2).$$

From these estimates we get the following uniforms controls for \mathbf{u}_n and \mathbf{b}_n :

$$\|(\mathbf{w}_n, \mathbf{d}_n)(t)\|_{B_2}^2 \geq \lambda^{n_T} \|(\mathbf{u}_n, \mathbf{b}_n)(\lambda^{2n_T} t, \cdot)\|_{B_2}^2,$$

and

$$\|\nabla(\mathbf{w}_n, \mathbf{d}_n)\|_{B_2 L^2(0, \lambda^{-2n_T} T)}^2 \geq \lambda^{n_T} \|\nabla(\mathbf{u}_n, \mathbf{b}_n)\|_{B_2 L^2(0, T)}^2.$$

In order to finish this proof, observe that we have controlled uniformly $\mathbf{u}_n, \mathbf{b}_n$ and $\nabla \mathbf{u}_n, \nabla \mathbf{b}_n$ on $(0, T)$ for $n \geq n_T$. Then, we may apply Theorem [4](#) to obtain a solution on $(0, T)$. As $T > 0$ is an arbitrary time, we can use a diagonal argument to obtain a solution \mathbf{u}, \mathbf{b} on $(0, +\infty)$. Finally, the control for the solution $(\mathbf{u}, \mathbf{b}, p, q)$ on $(0, +\infty)$ is given by Theorem [3](#). \diamond

Proof of Remark [4.1](#). It is enough to detail the computations for the functions $\mathbf{u}_{0, n}$ and \mathbb{F}_n since the computations for $\mathbf{b}_{0, n}$ and \mathbb{G}_n follows the same lines.

It is straightforward to see that we have

$$\frac{\|\mathbf{v}_{0, n}\|_{B_2}^2}{\lambda^n} = \sup_{R \geq 1} \frac{1}{\lambda^n R^2} \int_{|x| \leq R} |\lambda^n \mathbf{u}_0(\lambda^n x)|^2 dx = \sup_{R \geq 1} \frac{1}{(\lambda^n R)^2} \int_{|x| \leq \lambda^n R} |\mathbf{u}_0(x)|^2 dx,$$

and

$$\lim_{P \rightarrow +\infty} \sup_{R \geq P} \frac{1}{(\lambda^n R)^2} \int_{|x| \leq \lambda^n R} |\mathbf{u}_0(x)|^2 dx = \lim_{R \rightarrow +\infty} \frac{1}{R^2} \int_{|x| \leq R} |\mathbf{u}_0(x)|^2 dx = 0.$$

Moreover, remark that we have:

$$\begin{aligned} \frac{\|\mathbb{F}_n\|_{B_2 L^2(0, +\infty)}^2}{\lambda^n} &= \sup_{R \geq 1} \frac{1}{\lambda^n R^2} \int_0^{+\infty} \int_{|x| \leq R} |\lambda^{2n} \mathbb{F}(\lambda^{2n} t, \lambda^n x)|^2 dx ds \\ &= \sup_{R \geq 1} \frac{1}{(\lambda^n R)^2} \int_0^{+\infty} \int_{|x| \leq \lambda^n R} |\mathbb{F}(t, x)|^2 dx, \end{aligned}$$

and

$$\lim_{P \rightarrow +\infty} \sup_{R \geq P} \frac{1}{R^2} \int_0^{+\infty} \int_{|x| \leq R} |\mathbb{F}(t, x)|^2 dx ds = \lim_{R \rightarrow +\infty} \frac{1}{R^2} \int_0^{+\infty} \int_{|x| \leq R} |\mathbb{F}(t, x)|^2 dx ds = 0.$$

◇

Acknowledgement. We are very grateful to Pierre Gilles Lemarié-Rieusset for his encouragement in the study of the MHD equations through his interesting new methods developed for the Navier-Stokes equations. Moreover, we are grateful for his remarkably useful remarks to achieve some technical difficulties in this work.

References

- [1] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge (2000).
- [2] A. Basson, *Solutions spatialement homogènes adaptées des équations de Navier–Stokes*, Thèse, Université d’Évry (2006).
- [3] Z. Bradshaw, Kukavica I. and T.P. Tsai, *Existence of global weak solutions to the Navier-Stokes equations in weighted spaces*, arXiv:1910.06929v1 (2019).
- [4] Z. Bradshaw and T.P. Tsai, *Global existence, Regularity and Uniqueness of infinite energy solutions to the Navier-Stokes equations*, Communications in Partial Differential Equations (2019).
- [5] Z. Bradshaw and T.P. Tsai, *Discretely self-similar solutions to the Navier-Stokes equations with data in L^2_{loc}* , to appear in Analysis and PDE (2019).

- [6] D. Chae and J. Wolf, *Existence of discretely self-similar solutions to the Navier-Stokes equations for initial value in $L^2_{\text{loc}}(\mathbb{R}^3)$* , Ann. Inst. H. Poincaré Anal. Non Linéaire 35:1019–1039 (2018).
- [7] P.G. Fernández-Dalgo, O. Jarrín, *Discretely self-similar solutions for 3D MHD equations and global weak solutions in weighted L^2 spaces*, preprint (2019).
- [8] P.G. Fernández-Dalgo, P.G. Lemarié-Rieusset, *Characterisation of the pressure term in the incompressible Navier–Stokes equations on the whole space*, preprint arXiv:2001.10436 (2020).
- [9] P.G. Fernández-Dalgo, P.G. Lemarié-Rieusset, *Weak solutions for Navier–Stokes equations with initial data in weighted L^2 spaces.*, preprint (2019).
- [10] L. Grafakos, *Modern harmonic analysis (2nd ed.)*, Springer (2009).
- [11] H. Jia and V. Šverák, *Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions*, Invent. Math. 196: 233–265 (2014).
- [12] N. Kikuchi and G. Seregin, *Weak solutions to the Cauchy problem for the Navier–Stokes equations satisfying the local energy inequality*, in Nonlinear equations and spectral theory. Amer. Math. Soc. Transl. Ser. 2, 220, M.S. Birman and N.N. Uraltseva eds., 141–164 (2007).
- [13] CC. Lai, *Forward Discretely Self-similar Solutions of the MHD Equations and the Viscoelastic Navier–Stokes Equations with Damping* J. Math. Fluid Mech (2019).
- [14] P.G. Lemarié-Rieusset, *Solutions faibles d’énergie infinie pour les équations de Navier–Stokes dans \mathbb{R}^3* , C. R. Acad. Sci. Paris, Serie I. 328: 1133–1138 (1999).
- [15] P.G. Lemarié-Rieusset, *Recent developments in the Navier–Stokes problem*, CRC Press (2002).
- [16] P.G. Lemarié-Rieusset, *The Navier–Stokes problem in the 21st century*, Chapman & Hall/CRC, (2016).
- [17] J. Leray, *Essai sur le mouvement d’un fluide visqueux emplissant l’espace*, Acta Math. 63:193–248 (1934).

- [18] J. A. Shercliff, *A Textbook of Magnetohydrodynamics*, Pergamon Press, Oxford, (1965).
- [19] J. Zhang and T. Zhang *Global existence of discretely self-similar solutions to the generalized MHD system in Besov space* Journal of Mathematical Physics (2019)