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NÉRON BLOWUPS AND LOW-DEGREE COHOMOLOGICAL APPLICATIONS

BY ARNAUD MAYEUX, TIMO RICHARZ AND MATTHIEU ROMAGNY

ABSTRACT. We define dilatations of general schemes and study their basic properties. Dilatations of group schemes are –in favorable cases– again group schemes, called Néron blowups. We give two applications to their cohomology in degree zero (integral points) and degree one (torsors): we prove a canonical Moy-Prasad isomorphism that identifies the graded pieces in the congruent filtration of G with the graded pieces in its Lie algebra \mathfrak{g} , and we show that many level structures on moduli stacks of G -bundles are encoded in torsors under Néron blowups of G .

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1. INTRODUCTION

1.1. Motivation and goals. Néron blowups (or dilatations) provide a tool to modify group schemes over the fibers of a given Cartier divisor on the base. Classically, their integral points over discrete valuation rings appear as congruence subgroups of reductive groups over fields, see [Ana73, §2.1.2], [WW80, p. 551], [BLR90, §3.2], [PY06, §§7.2–7.4] and [Yu15, §2.8]. Over two-dimensional base schemes Néron blowups also appear in [PZ13, p. 175].

This note extends the theory of dilatations to general schemes which –in the case of group schemes– are also called Néron blowups. We give two applications involving their cohomology in degree zero (integral points) and in degree one (torsors). Our results concerning their integral points lead to a general form of an isomorphism of Moy and Prasad, frequently used in representation theory. Our results concerning their torsors show that these naturally encode many level structures on moduli stacks of bundles. This is used to construct integral models of moduli stacks of shtukas with level structures as in [Dri87] and [Var04] which for parahoric level structures might be seen as function field analogues of the integral models of Shimura varieties in [KP18].

1.2. Results. Let S be a scheme. Let S_0 be an effective Cartier divisor on S , i.e., a closed subscheme which is locally defined by a single non-zero divisor. We denote by $\text{Sch}_S^{S_0\text{-reg}}$ the full subcategory of schemes $T \rightarrow S$ such that $T|_{S_0} := T \times_S S_0$ is an effective Cartier divisor on T . This category contains all flat schemes over S . For a group scheme $G \rightarrow S$ together with a closed subgroup $H \subset G|_{S_0}$ over S_0 , we define the contravariant functor $\mathcal{G}: \text{Sch}_S^{S_0\text{-reg}} \rightarrow \text{Groups}$ given by all morphisms of S -schemes $T \rightarrow G$ such that the restriction $T|_{S_0} \rightarrow G|_{S_0}$ factors through H .

Theorem. (1) *The functor \mathcal{G} is representable by an open subscheme of the full scheme-theoretic blowup of G in H . The structure morphism $\mathcal{G} \rightarrow S$ is an object in $\text{Sch}_S^{S_0\text{-reg}}$. (2.2, 2.3, 2.5)*

(2) *The canonical map $\mathcal{G} \rightarrow G$ is affine. Its restriction over $S \setminus S_0$ induces an isomorphism $\mathcal{G}|_{S \setminus S_0} \cong G|_{S \setminus S_0}$. Its restriction over S_0 factors as $\mathcal{G}|_{S_0} \rightarrow H \subset G|_{S_0}$. (2.3, 3.1)*

(3) *If $H \rightarrow S_0$ has connected fibres and $H \subset G|_{S_0}$ is regularly immersed, then $\mathcal{G}|_{S_0} \rightarrow S_0$ has*

connected fibres. (2.14, 3.2)

(4) If $G \rightarrow S$, $H \rightarrow S_0$ are flat and (locally) of finite presentation and $H \subset G|_{S_0}$ is regularly immersed, then $\mathcal{G} \rightarrow S$ is flat and (locally) of finite presentation. If both $G \rightarrow S$, $H \rightarrow S_0$ are smooth, then $\mathcal{G} \rightarrow S$ is smooth. (2.14, 3.2)

(5) Assume that $\mathcal{G} \rightarrow S$ is flat. Then its formation commutes with base change $S' \rightarrow S$ in $\text{Sch}_S^{S_0\text{-reg}}$, and it carries the structure of a group scheme such that the canonical map $\mathcal{G} \rightarrow G$ is a morphism of S -group schemes. (2.6, 3.2)

(6) Assume that $G \rightarrow S$ is flat, finitely presented and $H \rightarrow S_0$ is flat, regularly immersed in $G|_{S_0}$. There is a canonical exact sequence of S_0 -group schemes $1 \rightarrow V \rightarrow \mathcal{G}|_{S_0} \rightarrow H \rightarrow 1$ where V is the vector bundle given by restriction to the unit section of an explicit twist of the normal bundle of H in $G|_{S_0}$. If H lifts to a flat S -subgroup scheme of G , this sequence splits canonically. (3.5)

We call $\mathcal{G} \rightarrow S$ the *Néron blowup* (or *dilatation*) of G in H along S_0 . Note that $\mathcal{G} \rightarrow S$ is a group object in $\text{Sch}_S^{S_0\text{-reg}}$ by (1), but that $\mathcal{G} \rightarrow S$ is a group scheme only if the self products $\mathcal{G} \times_S \mathcal{G}$ and $\mathcal{G} \times_S \mathcal{G} \times_S \mathcal{G}$ are objects in $\text{Sch}_S^{S_0\text{-reg}}$ which holds for example in (5), cf. §3.1 for details. If S is the spectrum of a discrete valuation ring and if S_0 is defined by the vanishing of a uniformizer, then $\mathcal{G} \rightarrow S$ is the group scheme constructed in [Ana73, §2.1.2], [WW80, p. 551], [BLR90, §3.2], [PY06, §§7.2–7.4] and [Yu15, §2.8]. For an example of Néron blowups over two-dimensional base schemes we refer to [PZ13, p. 175], cf. also Example 3.3.

We point out that most of the foundations of the study of dilatations can be settled in an absolute setting for schemes. That is, we initially develop the theory of affine blowups (or dilatations) for closed subschemes Z in a scheme X along a divisor D . It is only later that we specialize to relative schemes (over some base S , with the divisor coming from the base) and then further to group schemes.

The applications we give originate from a sheaf-theoretic viewpoint on Néron blowups. Write $j: S_0 \hookrightarrow S$ the closed immersion of the Cartier divisor, and assume that $G \rightarrow S$ and $H \rightarrow S_0$ are flat, finitely presented groups. In this context, the dilatation $\mathcal{G} \rightarrow G$ sits in an exact sequence of sheaves of pointed sets on the small syntomic site of S (Lemma 3.7):

$$1 \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow j_*(G_0/H) \longrightarrow 1,$$

where $G_0 := G|_{S_0}$. If $G \rightarrow S$ and $H \rightarrow S_0$ are smooth, then the sequence is exact as a sequence of sheaves on the small étale site of S . Considering the associated sequence on global sections, we obtain the following theorem which generalizes and unifies several results found in the literature (Remark 4.4) under the name of *Moy-Prasad isomorphisms*.

Corollary. *Let r, s be integers such that $0 \leq r/2 \leq s \leq r$. Let (\mathcal{O}, π) be a henselian pair where $\pi \subset \mathcal{O}$ is an invertible ideal. Let G be a smooth, separated, finitely presented \mathcal{O} -group scheme and G_r the r -th iterated dilatation of the unit section. Let \mathfrak{g}_r be the Lie algebra of G_r . If \mathcal{O} is local or G is affine, then there is a canonical isomorphism $G_s(\mathcal{O})/G_r(\mathcal{O}) \xrightarrow{\sim} \mathfrak{g}_s(\mathcal{O})/\mathfrak{g}_r(\mathcal{O})$. (4.3)*

As another application, we are interested in comparing \mathcal{G} -torsors with G -torsors. In light of the above short exact sequence of sheaves, there is an equivalence between the category of \mathcal{G} -torsors and the category of G -torsors equipped with a section of their pushforward along $G \rightarrow j_*(G_0/H)$, see [Gi71, Chap. III, §3.2, Prop. 3.2.1].

This has consequences for moduli of torsors over curves. We thus specialize to the following setting, cf. §4.2.1. Assume that X is a smooth, projective, geometrically irreducible curve over a field k with a Cartier divisor $N \subset X$, that $G \rightarrow X$ is a smooth, affine group scheme and that $H \rightarrow N$ is a smooth closed subgroup scheme of $G|_N$. In this case, the Néron blowup $\mathcal{G} \rightarrow X$ is a smooth, affine group scheme. Let Bun_G (resp. $\text{Bun}_{\mathcal{G}}$) denote the moduli stack of G -torsors (resp. \mathcal{G} -torsors) on X . This is a quasi-separated, smooth algebraic stack locally of finite type over k (cf. e.g. [He10, Prop. 1]). Pushforward of torsors along $\mathcal{G} \rightarrow G$ induces a morphism $\text{Bun}_{\mathcal{G}} \rightarrow \text{Bun}_G$, $\mathcal{E} \mapsto \mathcal{E} \times^{\mathcal{G}} G$. We also consider the stack $\text{Bun}_{(G,H,N)}$ of G -torsors on X with level- (H,N) -structures, cf. Definition

4.5. Its k -points parametrize pairs (\mathcal{E}, β) consisting of a G -torsor $\mathcal{E} \rightarrow X$ and a section β of the fppf quotient $(\mathcal{E}|_N/H) \rightarrow N$, i.e., β is a reduction of $\mathcal{E}|_N$ to an H -torsor.

Corollary. *There is an equivalence of k -stacks*

$$\mathrm{Bun}_{\mathcal{G}} \xrightarrow{\cong} \mathrm{Bun}_{(G,H,N)}, \quad \mathcal{E} \mapsto (\mathcal{E} \times^{\mathcal{G}} G, \beta_{\mathrm{can}}),$$

where β_{can} denotes the canonical reduction induced from the factorization $\mathcal{G}|_N \rightarrow H \subset G|_N$ given by (2) in the Theorem. (4.7, 4.8)

If $H = \{1\}$ is trivial, then $\mathrm{Bun}_{(G,H,N)}$ is the moduli stack of G -torsors equipped with level- N -structures. If $G \rightarrow X$ is reductive, if N is reduced and if H is a parabolic subgroup in $G|_N$, then $\mathrm{Bun}_{(G,H,N)}$ is the moduli stack of G -torsors with quasi-parabolic structures as in [LS97]. In this case the restriction of \mathcal{G} to the completed local rings of X are parahoric group schemes in the sense of [BT84] and the previous corollary was pointed out in [PR10, §2.a.]. Thus, many level structures are encoded in torsors under Néron blowups. This construction is also compatible with the adelic viewpoint, cf. Corollary 4.11.

Now assume that k is a finite field. As a consequence of the corollary one naturally obtains integral models of moduli stacks of G -shtukas on X with level structures over N as in [Dri87] for $G = \mathrm{GL}_n$ and in [Var04] for general G . We thank Alexis Bouthier for informing us that this was already pointed out in [NN08, §2.4] for congruence level. General properties of moduli stacks of shtukas for smooth, affine group schemes are studied in [AH], [AHab19] and [Br]. In §4.2.2 below, we make the connection between G -shtukas with level structures as in [Dri87, Var04, Laf18] and \mathcal{G} -shtukas as in [AH, AHab19, Br]. We expect the point of view of \mathcal{G} -shtukas, as opposed to G -shtukas with level structures, to be fruitful for investigations also outside the case of parahoric level structures.

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2. DILATATIONS

Dilatations (or affine blowups) are spectra of affine blowup algebras [StaPro, 052Q]. We start with the basic definitions and some properties as needed later.

2.1. **Definition.** Fix a scheme X . Let $Z \subset D$ be closed subschemes in X , and assume that D is locally principal. Denote by $\mathcal{J} \subset \mathcal{I}$ the associated quasi-coherent sheaves of ideals in \mathcal{O}_X so that $Z = V(\mathcal{I}) \subset V(\mathcal{J}) = D$.

Definition 2.1. We use the following terminology:

- (1) The *affine blowup algebra* of \mathcal{O}_X in \mathcal{I} along \mathcal{J} is the quasi-coherent sheaf of \mathcal{O}_X -algebras

$$\mathcal{O}_X[\frac{\mathcal{I}}{\mathcal{J}}] \stackrel{\mathrm{def}}{=} \left[(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X)[\mathcal{J}^{-1}] \right]_{\mathrm{deg}=0},$$

where $\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X = \mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \dots$ denotes the Rees algebra.

- (2) The *dilatation* (or *affine blowup*) of X in Z along D is the X -affine scheme

$$\mathrm{Bl}_Z^D X \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_X[\frac{\mathcal{I}}{\mathcal{J}}]).$$

The subscheme Z , or the pair (Z, D) , is called the *center* of the dilatation.

Here we view $\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X$ as a quasi-coherent $\mathbb{Z}_{\geq 0}$ -graded \mathcal{O}_X -algebra. If $\mathcal{J} = (b)$ is principal with $b \in \Gamma(X, \mathcal{J})$, then $(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X)[\mathcal{J}^{-1}] := (\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X)[b^{-1}]$ which is well-defined independently of the choice of generators of \mathcal{J} . If \mathcal{J} is only locally principal, then the localization $(\mathrm{Bl}_{\mathcal{I}}\mathcal{O}_X)[\mathcal{J}^{-1}]$ is defined by glueing. It inherits a grading by giving local generators of \mathcal{J} degree 1. Finally, the affine blowup algebra $\mathcal{O}_X[\frac{\mathcal{I}}{\mathcal{J}}]$ is the subsheaf of degree 0 elements in this localization.

If X is affine, we denote $B := \Gamma(X, \mathcal{O}_X)$, $I := \Gamma(X, \mathcal{I})$, $J := \Gamma(X, \mathcal{J})$ and $\mathrm{Bl}_I B := \Gamma(X, \mathrm{Bl}_{\mathcal{I}} \mathcal{O}_X) = \bigoplus_{n \geq 0} I^n$. Moreover, if $J = (b)$ is principal, then $B[\frac{I}{b}] := \Gamma(X, \mathcal{O}_X[\frac{\mathcal{I}}{b}])$ is the algebra whose elements are equivalence classes of fractions x/b^n with $x \in I^n$, where two representatives $x/b^n, y/b^m$ with $x \in I^n, y \in I^m$ define the same element in $B[\frac{I}{b}]$ if and only if there exists an integer $l \geq 0$ such that

$$(2.1) \quad b^l(b^m x - b^n y) = 0 \quad \text{inside } B.$$

By [StaPro, 07Z3],

$$(2.2) \quad \text{the image of } b \text{ in } B[\frac{I}{b}] \text{ is a non-zero divisor,}$$

$$(2.3) \quad bB[\frac{I}{b}] = IB[\frac{I}{b}], \text{ and}$$

$$(2.4) \quad B[\frac{I}{b}][b^{-1}] = B[b^{-1}].$$

In particular, the ring $B[\frac{I}{b}]$ is the B -subalgebra of $B[b^{-1}]$ generated by fractions x/b with $x \in I$.

2.2. Basic properties. We proceed with the notation from §2.1. The following results generalize [BLR90, §3.2, Prop. 1].

Lemma 2.2. *The affine blowup $\mathrm{Bl}_Z^D X$ is the open subscheme of the blowup $\mathrm{Bl}_Z X = \mathrm{Proj}(\mathrm{Bl}_{\mathcal{I}} \mathcal{O}_X)$ defined by the complement of $V_+(\mathcal{J})$.*

Proof. Our claim is Zariski local on X . We reduce to the case where $X = \mathrm{Spec}(B)$ is affine and $J = (b)$ is principal. Then $B[\frac{I}{b}]$ is the homogenous localization of $B \oplus I \oplus I^2 \oplus \dots$ at $b \in I$ viewed as an element in degree 1, cf. [StaPro, 052Q]. This shows that $\mathrm{Spec}(B[\frac{I}{b}])$ is the complement of $V_+(b)$ in $\mathrm{Proj}(\mathrm{Bl}_I B)$. \square

Lemma 2.3. *As closed subschemes of $\mathrm{Bl}_Z^D X$, one has*

$$\mathrm{Bl}_Z^D X \times_X Z = \mathrm{Bl}_Z^D X \times_X D,$$

which is an effective Cartier divisor on $\mathrm{Bl}_Z^D X$.

Proof. Our claim is Zariski local on X . We reduce to the case where $X = \mathrm{Spec}(B)$ is affine and $J = (b)$ is principal. We have to show that $bB[\frac{I}{b}] = IB[\frac{I}{b}]$, and that b is a non-zero divisor in $B[\frac{I}{b}]$. This is (2.2) and (2.3) above. \square

When D is a Cartier divisor, we can also realize $\mathrm{Bl}_Z^D X$ as a closed subscheme of the affine projecting cone. Recall that classically, this cone is defined as the relative spectrum of the Rees algebra $\mathrm{Bl}_{\mathcal{I}} \mathcal{O}_X$, so the blowup and its affine cone are complementary to each other in the completed projective cone; see [EGA2, §8.3]. However, twisting the blowup algebras with the invertible ideal sheaf \mathcal{J} gives rise to different embeddings. Indeed, according to [EGA2, §8.1.1] we have a canonical isomorphic presentation of the usual blowup as $\mathrm{Bl}_Z X = \mathrm{Proj}(\bigoplus_{n \geq 0} \mathcal{I}^n \otimes \mathcal{J}^{-n})$. Here we define the *affine projecting cone* of $\mathrm{Bl}_Z X$ (with respect to the chosen presentation) as

$$\mathbf{C}_Z X \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\bigoplus_{n \geq 0} \mathcal{I}^n \otimes \mathcal{J}^{-n}).$$

Lemma 2.4. *If D is a Cartier divisor, the affine blowup $\mathrm{Bl}_Z^D X$ is the closed subscheme of the affine projecting cone $\mathbf{C}_Z X$ defined by the equation $\varrho - 1$, where $\varrho \in \mathcal{I} \otimes \mathcal{J}^{-1}$ is the image of 1 under the inclusion $\mathcal{O}_X = \mathcal{J} \otimes \mathcal{J}^{-1} \subset \mathcal{I} \otimes \mathcal{J}^{-1}$.*

Proof. Let $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n \otimes \mathcal{J}^{-n}$. There is a surjective morphism of sheaves of algebras $\mathcal{A} \rightarrow \mathcal{O}_X[\frac{\mathcal{I}}{\mathcal{J}}]$ defined by mapping a local section $i \otimes j^{-1}$ in degree 1 to i/j . To check that $\varrho - 1$ goes to 0 and generates the kernel, we may work locally on some affine open subscheme $U \subset X$ where the sheaf \mathcal{J} is generated by a section b . Let $t = b^\vee$ be the generator for \mathcal{J}^{-1} , dual to b . Let $B = \Gamma(X, \mathcal{O}_X)$ and $I = \Gamma(X, \mathcal{I})$. Then the map $\mathcal{A}(U) \rightarrow \mathcal{O}_X[\frac{\mathcal{I}}{\mathcal{J}}](U)$ is given by

$$(\bigoplus_{n \geq 0} I^n t^n) \longrightarrow B[\frac{I}{b}], \quad \sum_{n \geq 0} i_n t^n \mapsto \sum_{n \geq 0} i_n / b^n.$$

This induces an isomorphism $(\bigoplus_{n \geq 0} I^n t^n) / (bt - 1) \xrightarrow{\sim} B[\frac{I}{b}]$. \square

2.3. Universal property. In this text we will use regular immersions in a possibly non-Noetherian setting where the reference [EGA4.4, §16.9 and §19] is inadequate. In this case we refer to the Stacks Project [StaPro]. There, four notions of regularity are studied: by decreasing order of strength, *regular*, *Koszul-regular*, *H_1 -regular*, *quasi-regular* (see Sections 067M and 0638 in *loc. cit.*). The useful ones for us are the first (which is regularity in its classical meaning) and the third: an H_1 -regular sequence is a sequence whose Koszul complex has no homology in degree 1. In [StaPro] several results are stated under the weakest H_1 -regular assumption. For simplicity we will state our results for regular immersions, although all of them hold also for H_1 -regular immersions.

Note that regularity and H_1 -regularity coincide for sequences composed of one element x , because for them the Koszul complex has length one and the homology group in degree 1 is just the x -torsion. In particular, for locally principal subschemes the three notions regular, Koszul-regular and H_1 -regular are equivalent.

Let us denote by $\text{Sch}_X^{D\text{-reg}}$ the full subcategory of schemes $T \rightarrow X$ such that $T \times_X D \subset T$ is regularly immersed, or equivalently is an effective Cartier divisor (possibly the empty set) on T . If $T' \rightarrow T$ is flat and $T \rightarrow X$ is an object in this category, so is the composition $T' \rightarrow T \rightarrow X$. In particular, the category $\text{Sch}_X^{D\text{-reg}}$ can be equipped with the fpqc/fppf/étale/Zariski Grothendieck topology so that the notion of sheaves is well-defined.

As $\text{Bl}_Z^D X \rightarrow X$ defines an object in $\text{Sch}_X^{D\text{-reg}}$ by Lemma 2.3, the contravariant functor

$$(2.5) \quad \text{Sch}_X^{D\text{-reg}} \rightarrow \text{Sets}, \quad (T \rightarrow X) \mapsto \text{Hom}_{X\text{-Sch}}(T, \text{Bl}_Z^D X)$$

together with $\text{id}_{\text{Bl}_Z^D X}$ determines $\text{Bl}_Z^D X \rightarrow X$ uniquely up to unique isomorphism. The next proposition gives the universal property of dilatations.

Proposition 2.5. *The affine blowup $\text{Bl}_Z^D X \rightarrow X$ represents the contravariant functor $\text{Sch}_X^{D\text{-reg}} \rightarrow \text{Sets}$ given by*

$$(2.6) \quad (f: T \rightarrow X) \longmapsto \begin{cases} \{*\}, & \text{if } f|_{T \times_X D} \text{ factors through } Z \subset X; \\ \emptyset, & \text{else.} \end{cases}$$

Proof. Let F be the functor defined by (2.6). If $T \rightarrow \text{Bl}_Z^D X$ is a map of X -schemes, then the structure map $T \rightarrow X$ restricted to $T \times_X D$ factors through $Z \subset X$ by Lemma 2.3. This defines a map

$$(2.7) \quad \text{Hom}_{X\text{-Sch}}(-, \text{Bl}_Z^D X) \longrightarrow F$$

of contravariant functors $\text{Sch}_X^{D\text{-reg}} \rightarrow \text{Sets}$. We want to show that (2.7) is bijective when evaluated at an object $T \rightarrow X$ in $\text{Sch}_X^{D\text{-reg}}$. As (2.7) is a morphism of Zariski sheaves, we reduce to the case where both $X = \text{Spec}(B)$, $T = \text{Spec}(R)$ are affine and $J = (b)$ is principal.

For injectivity, let $g, g': B[\frac{I}{b}] \rightarrow R$ be two B -algebra maps. We need to show $g = g'$. Indeed, since $B[b^{-1}] = B[\frac{I}{b}][b^{-1}]$ by (2.4), we get $g[b^{-1}] = g'[b^{-1}]$. As b is a non-zero divisor in R by assumption, this implies $g = g'$.

For surjectivity, consider an element in $F(\text{Spec}(R))$ which corresponds to a ring morphism $g: B \rightarrow R$ such that I is contained in the kernel of $B \rightarrow R \rightarrow R/bR$. We need to show that g extends (necessarily unique) to an B -algebra morphism $\tilde{g}: B[\frac{I}{b}] \rightarrow R$. Let $[x/b^n]$, $x \in I^n$ be a class in $B[\frac{I}{b}]$. Since $g(I^n) \subset (b^n)$ in R , the b -torsion freeness of R implies that there is a unique element $r = r(x, n) \in R$ such that $g(x) = b^n \cdot r$. We define $\tilde{g}([x/b^n]) := r(x, n)$. This is well-defined: If y/b^m , $y \in I^m$ is another representative of $[x/b^n]$, then applying g to equation (2.1) yields $b^m g(x) = b^n g(y)$ in R . It follows that $r(x, n) = r(y, m)$. Thus, \tilde{g} is well-defined. Similarly, one checks that \tilde{g} defines a morphism of B -algebras. \square

2.4. Functoriality. Let $Z' \subset D' \subset X'$ be another triple as in §2.1. A morphism $X' \rightarrow X$ such that its restriction to D' (resp. Z') factors through D (resp. Z) induces a unique morphism $\text{Bl}_{Z'}^{D'} X' \rightarrow$

$\mathrm{Bl}_Z^D X$ such that the following diagram of schemes

$$\begin{array}{ccc} \mathrm{Bl}_{Z'}^{D'} X' & \longrightarrow & \mathrm{Bl}_Z^D X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X. \end{array}$$

commutes. Indeed, the existence of $\mathrm{Bl}_{Z'}^{D'} X' \rightarrow \mathrm{Bl}_Z^D X$ follows directly from Definition 2.1. The uniqueness can be tested Zariski locally on X and X' where it follows from (2.2) and (2.4).

2.5. Base change. Now let $X' \rightarrow X$ be a map of schemes, and denote by $Z' \subset D' \subset X'$ the preimage of $Z \subset D \subset X$. Then $D' \subset X'$ is locally principal so that the affine blow $\mathrm{Bl}_{Z'}^{D'} X' \rightarrow X'$ is well-defined. By §2.4 there is a canonical morphism of X' -schemes

$$(2.8) \quad \mathrm{Bl}_{Z'}^{D'} X' \longrightarrow \mathrm{Bl}_Z^D X \times_X X'.$$

Lemma 2.6. *If $\mathrm{Bl}_Z^D X \times_X X' \rightarrow X'$ is an object of $\mathrm{Sch}_{X'}^{D', \text{-reg}}$, then (2.8) is an isomorphism.*

Proof. Our claim is Zariski local on X and X' . We reduce to the case where both $X = \mathrm{Spec}(B)$, $X' = \mathrm{Spec}(B')$ are affine, and $J = (b)$ is principal. We denote $Z' = \mathrm{Spec}(B'/I')$ and $D' = \mathrm{Spec}(B'/J')$. Then $J' = (b')$ is principal as well where b' is the image of b under $B \rightarrow B'$. We need to show that the map of B' -algebras

$$B' \otimes_B B\left[\frac{I}{b}\right] \longrightarrow B'\left[\frac{I'}{b'}\right]$$

is an isomorphism. However, this map is surjective with kernel the b' -torsion elements [StaPro, 0BIP]. As b' is a non-zero divisor in $B' \otimes_B B\left[\frac{I}{b}\right]$ by assumption, the lemma follows. \square

Corollary 2.7. *If the morphism $X' \rightarrow X$ is flat and has some property \mathcal{P} which is stable under base change, then $\mathrm{Bl}_{Z'}^{D'} X' \rightarrow \mathrm{Bl}_Z^D X$ is flat and has \mathcal{P} .*

Proof. Since flatness is stable under base change the projection $p: \mathrm{Bl}_Z^D X \times_X X' \rightarrow \mathrm{Bl}_Z^D X$ is flat and has property \mathcal{P} . By Lemma 2.6, it is enough to check that the closed subscheme $\mathrm{Bl}_Z^D X \times_X D'$ defines an effective Cartier divisor on $\mathrm{Bl}_Z^D X \times_X X'$. But this closed subscheme is the preimage of the effective Cartier divisor $\mathrm{Bl}_Z^D X \times_X D$ under the flat map p , and hence is an effective Cartier divisor as well. \square

2.6. Exceptional divisor. For closed subschemes $Z \subset D$ in X with D locally principal, we saw in Lemma 2.3 that the preimage of the center $\mathrm{Bl}_Z^D X \times_X Z = \mathrm{Bl}_Z^D X \times_X D$ is an effective Cartier divisor in $\mathrm{Bl}_Z^D X$. It is called the *exceptional divisor* of the affine blowup.

In order to describe it, as before we denote by \mathcal{I} and \mathcal{J} the sheaves of ideals of Z and D in \mathcal{O}_X . Also we let $\mathcal{C}_{Z/D} = \mathcal{I}/(\mathcal{I}^2 + \mathcal{J})$ and $\mathcal{N}_{Z/D} = \mathcal{C}_{Z/D}^\vee$ be the conormal and normal sheaves of Z in D .

Proposition 2.8. *Assume that $D \subset X$ is an effective Cartier divisor, and $Z \subset D$ is a regular immersion. Write $\mathcal{J}_Z := \mathcal{J}|_Z$.*

- (1) *The exceptional divisor $\mathrm{Bl}_Z^D X \times_X Z \rightarrow Z$ is a vector bundle, Zariski locally over Z isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \rightarrow Z$.*
- (2) *If $H^1(Z, \mathcal{N}_{Z/D} \otimes \mathcal{J}_Z) = 0$ (for example if Z is affine), then $\mathrm{Bl}_Z^D X \times_X Z \rightarrow Z$ is globally isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \rightarrow Z$.*
- (3) *If Z is a transversal intersection in the sense that there is a cartesian square of closed subschemes whose vertical maps are regular immersions*

$$\begin{array}{ccc} W & \hookrightarrow & X \\ \uparrow & \square & \uparrow \\ Z & \hookrightarrow & D \end{array}$$

then $\mathrm{Bl}_Z^D X \times_X Z \rightarrow Z$ is globally and canonically isomorphic to $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \rightarrow Z$.

Proof. Using Lemma 2.4 we can view the affine blowup $\text{Bl}_Z^D X$ as the closed subscheme with equation $\varrho - 1 = 0$ of the affine projecting cone $C_Z X = \text{Spec}(\mathcal{A})$ with $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n \otimes \mathcal{J}^{-n}$. First we compute the preimage $C_Z X \times_X Z$. We have:

$$\mathcal{A} \otimes \mathcal{O}_X / \mathcal{I} = \bigoplus_{k \geq 0} (\mathcal{I}^k / \mathcal{I}^{k+1}) \otimes \mathcal{J}^{-k}.$$

Since the immersions $Z \subset D \subset X$ are regular, the conormal sheaf $\mathcal{C}_{Z/X} = \mathcal{I} / \mathcal{I}^2$ is finite locally free, and the canonical surjective morphism of \mathcal{O}_X -algebras

$$\text{Sym}_{\mathcal{O}_X}^\bullet(\mathcal{C}_{Z/X}) \longrightarrow \text{Gr}_{\mathcal{I}}^\bullet(\mathcal{O}_X)$$

is an isomorphism, that is, $\text{Sym}^k(\mathcal{C}_{Z/X}) \rightarrow \mathcal{I}^k / \mathcal{I}^{k+1}$ is an isomorphism for each $k \geq 0$. Taking into account that \mathcal{J} is an invertible sheaf, we obtain

$$\text{Sym}^k(\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1}) \xrightarrow{\simeq} (\mathcal{I}^k / \mathcal{I}^{k+1}) \otimes \mathcal{J}^{-k}.$$

It follows that $\text{Spec}(\mathcal{A}) = \mathbb{V}(\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1})$ and that $\text{Bl}_Z^D X \times_X Z$ is the closed subscheme cut out by $\varrho - 1$ inside the latter. Now consider the sequence of conormal sheaves:

$$0 \longrightarrow \mathcal{C}_{D/X}|_Z \longrightarrow \mathcal{C}_{Z/X} \longrightarrow \mathcal{C}_{Z/D} \longrightarrow 0.$$

By [EGA4.4, Prop. 16.9.13] or [StaPro, 063N] in the non-Noetherian case, this sequence is exact and locally split (beware that $\mathcal{C}_{Z/X}$ is denoted $\mathcal{N}_{Z/X}$ in the former reference). Using the fact that $\mathcal{C}_{D/X}|_Z \otimes \mathcal{J}_Z^{-1} \simeq \mathcal{O}_Z$ is freely generated by ϱ as a subsheaf of $\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1}$, we obtain an extension

$$(2.9) \quad 0 \longrightarrow \varrho \mathcal{O}_Z \longrightarrow \mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1} \longrightarrow \mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1} \longrightarrow 0.$$

Now we consider the three cases listed in the proposition.

(1) Since $\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}$ is locally free, locally over Z we can choose a splitting of the exact sequence (2.9) of conormal sheaves:

$$\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1} = \varrho \mathcal{O}_Z \oplus \mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}.$$

Mapping $\varrho \mapsto 1$ yields a morphism of \mathcal{O}_X -modules

$$\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1} \longrightarrow \mathcal{O}_Z \oplus (\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}) \subset \text{Sym}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1})$$

which extends to a surjection of algebras with kernel $(\varrho - 1)$:

$$\text{Sym}(\mathcal{C}_{Z/X} \otimes \mathcal{J}_Z^{-1}) \longrightarrow \text{Sym}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}).$$

This identifies $\text{Spec}(\mathcal{B} \otimes \mathcal{O}_X / \mathcal{I})$ with the vector bundle $\mathbb{V}(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1})$, locally over Z .

(2) The exact sequence defines a class in $\text{Ext}_{\mathcal{O}_X}^1(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}, \mathcal{O}_Z)$. Because the conormal sheaf is locally free, we have:

$$\text{Ext}_{\mathcal{O}_Z}^1(\mathcal{C}_{Z/D} \otimes \mathcal{J}_Z^{-1}, \mathcal{O}_Z) \simeq \text{Ext}_{\mathcal{O}_Z}^1(\mathcal{O}_Z, \mathcal{C}_{Z/D}^\vee \otimes \mathcal{J}_Z) \simeq H^1(Z, \mathcal{N}_{Z/D} \otimes \mathcal{J}_Z).$$

By assumption this vanishes and we obtain a global splitting. From this one concludes as before.

(3) If Z is the transversal intersection of W and D , then we have two exact, locally split sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}_{D/X}|_Z & & \mathcal{C}_{Z/W} & \longrightarrow & 0 \\ & & \searrow & \mathcal{C}_{Z/X} & \nearrow & & \\ & & & \mathcal{C}_{Z/X} & & & \\ & & \mathcal{C}_{W/X}|_Z & \dashrightarrow & \mathcal{C}_{Z/D} & \longrightarrow & 0. \end{array}$$

We claim that the dashed arrow is an isomorphism. To see this, write $\mathcal{I}, \mathcal{J}, \mathcal{K}$ the defining ideals of Z, D, W . The composition $\mathcal{C}_{W/X}|_Z \rightarrow \mathcal{C}_{Z/D}$ is the following map:

$$\mathcal{K} / (\mathcal{K}^2 + \mathcal{I}\mathcal{K}) \longrightarrow \mathcal{I} / (\mathcal{I}^2 + \mathcal{J}).$$

From the fact that $\mathcal{I} = \mathcal{J} + \mathcal{K}$ we deduce:

- $\mathcal{K}^2 + \mathcal{I}\mathcal{K} = \mathcal{K}^2 + \mathcal{J}\mathcal{K}$, hence $\mathcal{K} / (\mathcal{K}^2 + \mathcal{I}\mathcal{K}) = \mathcal{K} / (\mathcal{K}^2 + \mathcal{J}\mathcal{K})$.

- $\mathcal{I}^2 = \mathcal{J}^2 + \mathcal{J}\mathcal{K} + \mathcal{K}^2$, hence $\mathcal{I}^2 + \mathcal{J} = \mathcal{K}^2 + \mathcal{J}$ and $\mathcal{I}/(\mathcal{I}^2 + \mathcal{J}) = (\mathcal{J} + \mathcal{K})/(\mathcal{K}^2 + \mathcal{J}) = \mathcal{K}/(\mathcal{K}^2 + \mathcal{J} \cap \mathcal{K})$.

Hence, the map above is an isomorphism if and only if $\mathcal{J}\mathcal{K} = \mathcal{J} \cap \mathcal{K}$, which holds because W cuts D transversally (this is another way of saying that a local equation for \mathcal{J} remains a nonzerodivisor in \mathcal{O}_W). This provides a canonical splitting $\mathcal{C}_{Z/X} = \mathcal{C}_{D/X}|_Z \oplus \mathcal{C}_{Z/D}$. One concludes as before. \square

2.7. Iterated dilatations. Here we study the behaviour of dilatations under iteration. Namely, we will prove that when the center Z of the affine blowup is a transversal intersection $W \cap D$, it can be dilated any finite number of times and the result of r dilatations can be seen as the dilatation of the single “thickened” center rZ (to be defined below) inside the multiple Cartier divisor rD . To make this precise, we first study the lifting of subschemes along an affine blowup.

Lemma 2.9. *Let $Z \subset D \subset X$ be closed subschemes with $D \subset X$ a Cartier divisor. Let $\iota: W \hookrightarrow X$ be an immersion such that $\iota^{-1}(Z) = \iota^{-1}(D)$ is a Cartier divisor in W . Set $X' := \text{Bl}_Z^D X$ and let $\iota': W \rightarrow X'$ be the lift of ι given by the universal property of the dilatation.*

- (1) *If ι is an open immersion, then ι' is an open immersion.*
- (2) *If ι is a closed immersion, then ι' is a closed immersion.*
- (3) *Write ι as the composition $W \hookrightarrow U \hookrightarrow X$ where $U \subset X$ is the largest open subscheme such that W is a closed subscheme of U . Let \mathcal{J} (resp. \mathcal{K}) be the ideal sheaf of D (resp. W) in U . Then ι' is the composition $W \hookrightarrow U' \hookrightarrow X'$ where $U' = X' \times_X U$ is the preimage of U and $W \hookrightarrow U'$ is a closed immersion with sheaf of ideals $\mathcal{K}\mathcal{O}_{U'} \otimes (\mathcal{J}\mathcal{O}_{U'})^{-1}$.*

Proof. (1) In this case ι is flat, and the formation of the dilatation commutes with base change. That is, the canonical morphism of W -schemes

$$\text{Bl}_{Z \cap W}^{D \cap W} W \longrightarrow X' \times_X W$$

is an isomorphism. But by the assumption $\text{Bl}_{Z \cap W}^{D \cap W} W \rightarrow W$ is an isomorphism, and this identifies $W \rightarrow X'$ with the preimage of $W \hookrightarrow X$ in X' .

(2) Let $\mathcal{K} \subset \mathcal{O}_X$ be the ideal sheaf of W . We will prove that $\iota': W \rightarrow X'$ is a closed immersion with ideal sheaf $\mathcal{K}\mathcal{O}_{X'} \otimes (\mathcal{J}\mathcal{O}_{X'})^{-1}$. First of all ι' is automatically a monomorphism of schemes, and a proper map because ι is proper and $X' \rightarrow X$ is separated. Therefore ι' is a closed immersion by [EGA4.4, Cor. 18.12.6]. The computation of the ideal sheaf is a local matter so we can suppose that $X = \text{Spec}(A)$ is affine and the ideal sheaf \mathcal{J} is generated by a section b . We write $I, J, K \subset A$ the ideals defining Z, D, W and $t := b^\vee$ the generator of \mathcal{J}^{-1} , dual to b . The assumptions of the lemma mean that $I + K = J + K$ and b is a nonzerodivisor in A/K . From Lemma 2.4, we know that X' is the spectrum of the ring

$$A' = (\oplus_{e \geq 0} I^e t^e)/(bt - 1).$$

In the present local situation, the map $\iota': W \rightarrow X'$ is given by a lifting of $\iota^\sharp: A \rightarrow A/K$ to a map $(\iota')^\sharp: A' \rightarrow A/K$. Since A' is generated by It as an A -algebra, this map is determined by the formula $(\iota')^\sharp(it) = j^\sharp(a)$, for all $i \in I$ written $i = ab + k \in I \subset bA + K$. In particular we see that $(\iota')^\sharp(Kt) = 0$. Now working modulo $(bt - 1) + Kt$ in the ring $C = \oplus_{e \geq 0} I^e t^e$, we have:

$$It \subset btA + Kt \equiv A + Kt \equiv A$$

which sits in the degree 0 part of C , whence a surjection $A \hookrightarrow C \rightarrow A'/KtA'$. Moreover $bKt \equiv K$ implies that K in degree 0 belongs to the ideal generated by Kt , hence finally $A'/KtA' \xrightarrow{\sim} A/K$ as desired.

- (3) This is the conjunction of (1) and (2). \square

In view of the preceding lemma, if we fix a closed subscheme $i: W \hookrightarrow X$ such that $i^{-1}(Z) = i^{-1}(D)$ is a Cartier divisor in W , we will be able to lift W to the dilatation and hence iterate the process. So we place ourselves in the following situation.

Assumption 2.10. *The schemes $Z \subset D \subset X$ sit in a cartesian diagram of closed subschemes*

$$(\mathcal{D}) : \begin{array}{ccc} W & \xhookrightarrow{i} & X \\ \uparrow & \square & \uparrow \\ Z & \hookrightarrow & D \end{array}$$

such that the vertical maps are Cartier divisor inclusions.

In this situation we can construct a sequence of dilatations

$$\dots \longrightarrow X_r \longrightarrow X_{r-1} \longrightarrow \dots \longrightarrow X_1 \longrightarrow X_0 = X$$

and closed immersions $i_r: W \hookrightarrow X_r$, as follows. We let $\mathcal{D}_0 = \mathcal{D}$, $X_0 = X$, $D_0 = D$, and $i_0 = i: W \hookrightarrow X_0$. Let $u_1: X_1 \rightarrow X_0$ be the dilatation of Z in (X_0, D_0) , and D_1 the preimage of D_0 in X_1 . Since \mathcal{D}_0 is a cartesian diagram of closed subschemes, we have $i_0^{-1}(Z) = i_0^{-1}(D) = X$ which is a Cartier divisor in W . So by the universal property and Lemma 2.9, there is a closed immersion $i_1: W \rightarrow X_1$ lifting i_0 . Moreover,

$$(i_1)^{-1}(D_1) = (i_1)^{-1}(u_1^{-1}(D)) = i^{-1}(D) = Z.$$

That is, we again have a cartesian diagram

$$(\mathcal{D}_1) : \begin{array}{ccc} W & \xhookrightarrow{i_1} & X_1 \\ \uparrow & \square & \uparrow \\ Z & \hookrightarrow & D_1 \end{array}$$

where the vertical maps are Cartier divisor inclusions. Our sequence is obtained by iterating this construction.

Lemma 2.11. *Under Assumption 2.10, denote by $\mathcal{I}, \mathcal{J}, \mathcal{K}$ the ideal sheaves of Z, D, W in \mathcal{O}_X . Let rD be the r -th multiple of D as a Cartier divisor, and $rZ := W \cap rD$. Then the dilatation $v_r: X'_r \rightarrow X$ of (rZ, rD) in X is characterized as being universal among all morphisms $V \rightarrow X$ with the following two properties:*

- (i) $\mathcal{J}\mathcal{O}_V$ is an invertible sheaf,
- (ii) $\mathcal{K}\mathcal{O}_V$ is divisible by $\mathcal{J}^r\mathcal{O}_V$, that is we have $\mathcal{K}\mathcal{O}_V = \mathcal{J}^r\mathcal{O}_V \cdot \mathcal{K}_r$ for some sheaf of ideals $\mathcal{K}_r \subset \mathcal{O}_V$.

Proof. The defining properties of the dilatation v_r say that it is universal among morphisms $V \rightarrow X$ such that $rZ \times_X V = rD \times_X V$ is a Cartier divisor. Since the ideal sheaves of rD and rZ are \mathcal{J}^r and $\mathcal{J}^r + \mathcal{K}$ respectively, these properties mean that the ideal $\mathcal{J}^r\mathcal{O}_V$ is invertible and $\mathcal{J}^r\mathcal{O}_V = (\mathcal{J}^r + \mathcal{K})\mathcal{O}_V$. But the properties “ \mathcal{J} is invertible” and “ \mathcal{J}^r is invertible” are equivalent, as follows from the isomorphism between the blow-up of \mathcal{J} and the blow-up of \mathcal{J}^r , see [EGA2, Def. 8.1.3]. This takes care of (i). Besides, $\mathcal{J}^r\mathcal{O}_V = (\mathcal{J}^r + \mathcal{K})\mathcal{O}_V$ means that $\mathcal{K}\mathcal{O}_V \subset \mathcal{J}^r\mathcal{O}_V$ and in the situation where $\mathcal{J}\mathcal{O}_V$ is invertible, this is the same as saying that

$$\mathcal{K}\mathcal{O}_V = \mathcal{J}^r\mathcal{O}_V \cdot \mathcal{K}_r$$

with $\mathcal{K}_r = (\mathcal{K}\mathcal{O}_V : \mathcal{J}^r\mathcal{O}_V)$ as an ideal of \mathcal{O}_V . (Note that in this case $(\mathcal{K}\mathcal{O}_V : \mathcal{J}^r\mathcal{O}_V) \simeq (\mathcal{K} \otimes \mathcal{J}^{-r})\mathcal{O}_V$ as an \mathcal{O}_V -module.) This takes care of (ii). \square

Proposition 2.12. *In the situation of Assumption 2.10, let*

$$\dots \longrightarrow X_r \longrightarrow X_{r-1} \longrightarrow \dots \longrightarrow X_1 \longrightarrow X_0 = X$$

be the sequence of dilatations constructed above. Let rD be the r -th multiple of D as a Cartier divisor, and $rZ := W \cap rD$. Then the composition $X_r \rightarrow X$ is the dilatation of (rZ, rD) inside X .

Proof. According to Lemma 2.11, dilating (rZ, rD) means making \mathcal{J} invertible and \mathcal{K} divisible by \mathcal{J}^r , all of this in a universal way. This can be done by steps:

- make \mathcal{J} invertible and make $\mathcal{K}_0 = \mathcal{K}$ divisible by \mathcal{J} ;

- keep \mathcal{J} invertible and make $\mathcal{K}_1 := (\mathcal{K}_0 : \mathcal{J}) \simeq \mathcal{K} \otimes \mathcal{J}^{-1}$ divisible by \mathcal{J} ;
- keep \mathcal{J} invertible and make $\mathcal{K}_2 := (\mathcal{K}_1 : \mathcal{J}) \simeq \mathcal{K}_1 \otimes \mathcal{J}^{-1} \simeq \mathcal{K} \otimes \mathcal{J}^{-2}$ divisible by \mathcal{J} ; etc, and finally
- keep \mathcal{J} invertible and make $\mathcal{K}_{r-1} := (\mathcal{K}_{r-2} : \mathcal{J}) \simeq \mathcal{K} \otimes \mathcal{J}^{-(r-1)}$ divisible by \mathcal{J} .

In view of Lemma 2.9, these steps amount to:

- dilate Z in (X, D) ,
- dilate Z in (X_1, D_1) ,
- dilate Z in (X_2, D_2) , etc. until
- dilate Z in (X_{r-1}, D_{r-1}) .

In this way we see the equivalence between the dilatation of the thick pair (rZ, rD) and the sequence of dilatations of Z constructed after 2.10. \square

2.8. Flatness and smoothness. Flatness and smoothness properties of blowups are discussed in [EGA4.4, §19.4]. Here we need slightly different versions. We proceed with the notation from §2.1. We assume further that there exists a scheme S under X together with a locally principal closed subscheme $S_0 \subset S$ fitting into a commutative diagram of schemes

$$(2.10) \quad \begin{array}{ccccc} Z & \longrightarrow & D & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S_0 & \longrightarrow & S, \end{array}$$

where the square is cartesian, that is $D \rightarrow X_0 := X \times_S S_0$ is an isomorphism.

Lemma 2.13. *Assume that S_0 is an effective Cartier divisor in S . Let $f : Y \rightarrow S$ be a morphism of schemes such that $Y_0 := Y \times_S S_0$ is a Cartier divisor in Y . Assume that both restriction of f above $S \setminus S_0$ and S_0 are flat. If one of the following holds:*

- (i) S, Y are locally noetherian,
- (ii) $Y \rightarrow S$ is locally of finite presentation,

then f is flat.

Proof. Since by assumption u is flat at all points above the open subscheme $S \setminus S_0$, it is enough to prove that u is flat at all points $y \in Y$ lying above a point $s \in S_0$.

In case (i), the local criterion for flatness [EGA3.1, Chap. O, 10.2.2] (cf. also [StaPro, 00ML]) shows that $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{Y,y}$ is flat and we are done.

In case (ii), we may localize around y and s and hence assume that Y and S are affine and small enough so that the ideal sheaf of S_0 in S is generated by an element $f \in A = \Gamma(S, \mathcal{O}_S)$. We write $A = \text{colim} A_i$ as the union of its subrings of finite type over \mathbb{Z} . In each A_i , the element f is a nonzerodivisor. Write $S_i := \text{Spec}(A_i)$ and $S_{i,0} := \text{Spec}(A_i/f)$. Using the results of [EGA4.3, Chap. IV, §8] on limits of schemes, we can find an index i and a morphism of finite presentation $Y_i \rightarrow S_i$ such that $Y_{i,0} := Y_i \times_{S_i} S_{i,0}$ is a Cartier divisor in Y_i and $Y_{i,0} \rightarrow S_{i,0}$ is flat, and such that the situation (S, S_0, Y) is a pullback of $(S_i, S_{i,0}, Y_i)$ by $S \rightarrow S_i$. More in detail, using the following results we find indices which we increase at each step in order to have all the conditions simultaneously met: use [EGA4.3, Thm. 8.8.2] to find morphisms $Y_i \rightarrow S_i$ and $Y_{i,0} \rightarrow S_{i,0}$, use [EGA4.3, Cor. 8.8.2.5] to make $Y_i \times_{S_i} S_{i,0}$ and $Y_{i,0}$ isomorphic over $S_{i,0}$, use [EGA4.3, Thm. 11.2.6] to ensure $Y_{i,0} \rightarrow S_{i,0}$ flat, and use [EGA4.3, Prop. 8.5.6] to ensure that f is a nonzerodivisor in \mathcal{O}_{Y_i} , i.e. $Y_{i,0} \subset Y_i$ is a Cartier divisor. Since A_i is noetherian, for $(S_i, S_{i,0}, Y_i)$ we can apply case (i) and the result follows by base change. \square

For our conventions on regular immersions, the reader is referred back to §2.3.

Proposition 2.14. *Assume that S_0 is an effective Cartier divisor on S .*

- (1) *If $Z \subset D$ is regular, then $\text{Bl}_Z^D X \rightarrow X$ is of finite presentation.*
- (2) *If $Z \subset D$ is regular, the fibers of $\text{Bl}_Z^D X \times_S S_0 \rightarrow S_0$ are connected (resp. irreducible, geometrically connected, geometrically irreducible) if and only if the fibers of $Z \rightarrow S_0$ are.*

- (3) If $X \rightarrow S$ is flat and if moreover one of the following holds:
- (i) $Z \subset D$ is regular, $Z \rightarrow S_0$ is flat and S, X are locally Noetherian,
 - (ii) $Z \subset D$ is regular, $Z \rightarrow S_0$ is flat and $X \rightarrow S$ is locally of finite presentation,
 - (iii) the local rings of S are valuation rings,
- then $\mathrm{Bl}_Z^D X \rightarrow S$ is flat.
- (4) If both $X \rightarrow S, Z \rightarrow S_0$ are smooth, then $\mathrm{Bl}_Z^D X \rightarrow S$ is smooth.

Proof. For (1) recall that the blowup of a regularly immersed subscheme has an explicit structure, where generating relations between local generators of the blown up ideal are the obvious ones, in finite number; see [StaPro, 0BIQ]. This shows that $\mathrm{Bl}_Z^D X \rightarrow X$ is locally of finite presentation. Being also affine, it is of finite presentation.

For (2) about connectedness and irreducibility, recall from Proposition 2.8 (1) that the exceptional divisor is a vector bundle over Z . Since vector bundles are submersions, the elementary topological lemma [EGA4.2, Lem. 4.4.2] (cf. also [StaPro, 0377]) gives the assertion.

For (3)(i)-(ii), we apply Lemma 2.13 to $Y := \mathrm{Bl}_Z^D X$. The preimage of S_0 under the affine blowup $f: Y \rightarrow S$ is equal to $\mathrm{Bl}_Z^D X \times_X D = \mathrm{Bl}_Z^D X \times_X Z$ by Lemma 2.3. This implies that the restriction $f|_{f^{-1}(S \setminus S_0)}$ is equal to $X \setminus D \rightarrow S \setminus S_0$ which is flat by assumption. It remains to show flatness in points of $\mathrm{Bl}_Z^D X$ lying over S_0 . For this note that the restriction $f|_{f^{-1}(S_0)}$ factors as

$$(2.11) \quad \mathrm{Bl}_Z^D X \times_X D = \mathrm{Bl}_Z^D X \times_X Z \longrightarrow Z \longrightarrow S_0,$$

where the first map is smooth by Proposition 2.8 and the second map is flat by assumption. Then Lemma 2.13 applies and gives flatness of $Y \rightarrow S$.

For (3)(iii) we can work locally at a point of S and hence assume that S is the spectrum of a valuation ring R . We use the fact that flat R -modules are the same as torsionfree R -modules. Locally over an open subscheme $\mathrm{Spec}(B) \subset X$, the Rees algebra $\mathrm{Bl}_I B = B[It]$ is a subalgebra of the polynomial algebra $B[t]$ and the affine blowup algebra is a localization of the latter. It follows that if B is R -torsionfree then the affine blowup algebra also, hence it is flat.

For (4) assume that $X \rightarrow S$ and $Z \rightarrow S_0$ are smooth. Then (4) follows from [EGA4.4, Thm. 17.5.1] (cf. also [StaPro, 01V8]) once we know that $\mathrm{Bl}_Z^D X \rightarrow S$ is locally of finite presentation, flat and has smooth fibers. Applying [EGA4.4, Prop. 19.2.4] to the commutative triangle in (2.10) we see that $Z \subset D$ is regularly immersed. Therefore, $\mathrm{Bl}_Z^D X \rightarrow S$ is flat and locally of finite presentation by parts (1) and (3). The smoothness of the fiber over points in $S \setminus S_0$ is clear, and follows from (2.11) over points in S_0 . This proves (4). \square

3. NÉRON BLOWUPS

We extend the theory of Néron blowups of affine group schemes over discrete valuation rings as in [Ana73, 2.1.2], [WW80, p. 551], [BLR90, §3.2], [Yu15, §2.8] and [PY06, §§7.2–7.4] to group schemes over arbitrary bases.

3.1. Definition. Let S be a scheme, and let $G \rightarrow S$ be a group scheme. Let $S_0 \subset S$ be a locally principal closed subscheme, and consider the base change $G_0 := G \times_S S_0$. Let $H \subset G_0$ be a closed subgroup scheme over S_0 . Let $\mathcal{G} := \mathrm{Bl}_H^{G_0} G \rightarrow G$ be the dilatation of G in H along the locally principal, closed subscheme $G_0 \subset G$ in the sense of Definition 2.1. In this case, we also call $\mathcal{G} \rightarrow S$ the *Néron blowup of G in H (along S_0)*. We denote by $\mathcal{G}_0 := \mathcal{G} \times_S S_0 \rightarrow S_0$ its exceptional divisor.

Let $\mathrm{Sch}_S^{S_0\text{-reg}}$ be the full subcategory of schemes $T \rightarrow S$ such that $T_0 := T \times_S S_0$ defines an effective Cartier divisor on T . By Lemma 2.3 the structure morphism $\mathcal{G} \rightarrow S$ defines an object in $\mathrm{Sch}_S^{S_0\text{-reg}}$.

Lemma 3.1. *Let $\mathcal{G} \rightarrow S$ be the Néron blowup of G in H along S_0 .*

- (1) *The scheme $\mathcal{G} \rightarrow S$ represents the contravariant functor $\mathrm{Sch}_S^{S_0\text{-reg}} \rightarrow \mathrm{Sets}$ given for $T \rightarrow S$ by the set of all S -morphisms $T \rightarrow G$ such that the induced morphism $T_0 \rightarrow G_0$ factors through $H \subset G_0$.*

- (2) The map $\mathcal{G} \rightarrow G$ is affine. Its restriction over $S \setminus S_0$ induces an isomorphism $\mathcal{G}|_{S \setminus S_0} \cong G|_{S \setminus S_0}$. Its restriction over S_0 factors as $\mathcal{G}_0 \rightarrow H \subset G_0$.

Proof. Part (1) is a reformulation of Proposition 2.5, and (2) is immediate from Lemmas 2.2 and 2.3. \square

By virtue of Lemma 3.1 (1) the (forgetful) map $\mathcal{G} \rightarrow G$ defines a subgroup functor when restricted to the category $\text{Sch}_S^{S_0\text{-reg}}$. As $\mathcal{G} \rightarrow S$ is an object in $\text{Sch}_S^{S_0\text{-reg}}$, it is a group object in this category. Here we note that products in the category $\text{Sch}_S^{S_0\text{-reg}}$ exist and are computed as $\text{Bl}_{S_0}(X_1 \times_S X_2)$ by the universal property of the blowup [StaPro, 085U]. This is the closed subscheme of $X_1 \times_S X_2$ which is locally defined by the ideal of a -torsion elements for a local equation a of S_0 in S . In particular, if $\mathcal{G} \rightarrow S$ is flat, then it is equipped with the structure of a group scheme such that $\mathcal{G} \rightarrow G$ is a morphism of S -group schemes.

3.2. Properties. We continue with the notation of §3.1. Additionally assume that S_0 is an effective Cartier divisor in S . Again recall our conventions on regular immersions from §2.3. The following summarizes the main properties of Néron blowups.

Theorem 3.2. *Let $\mathcal{G} \rightarrow G$ be the Néron blowup of G in H along S_0 .*

- (1) *If $G \rightarrow S$ is (quasi-)affine, then $\mathcal{G} \rightarrow S$ is (quasi-)affine.*
- (2) *If $G \rightarrow S$ is (locally) of finite presentation and $H \subset G_0$ is regular, then $\mathcal{G} \rightarrow S$ is (locally) of finite presentation.*
- (3) *If $H \rightarrow S_0$ has connected fibres and $H \subset G_0$ is regular, then $\mathcal{G} \times_S S_0 \rightarrow S_0$ has connected fibres.*
- (4) *Assume that $G \rightarrow S$ is flat and one of the following holds:*
 - (i) *$H \subset G_0$ is regular, $H \rightarrow S_0$ is flat and S, G are locally Noetherian,*
 - (ii) *$H \subset G_0$ is regular, $H \rightarrow S_0$ is flat and $G \rightarrow S$ is locally of finite presentation,*
 - (iii) *the local rings of S are valuation rings,**then $\mathcal{G} \rightarrow S$ is flat.*
- (5) *If both $G \rightarrow S, H \rightarrow S_0$ are smooth, then $\mathcal{G} \rightarrow S$ is smooth.*
- (6) *Assume that $\mathcal{G} \rightarrow S$ is flat. If $S' \rightarrow S$ is a scheme such that $S'_0 := S' \times_S S_0$ is an effective Cartier divisor on S' , then the base change $\mathcal{G} \times_S S' \rightarrow S'$ is the Néron blowup of $G \times_S S'$ in $H \times_{S_0} S'_0$ along S'_0 .*

In cases (4) and (5), the map $\mathcal{G} \rightarrow S$ is a group scheme.

Proof. The map $\mathcal{G} \rightarrow G$ is affine by Lemma 3.1 (2) which implies (1). Items (2) to (5) are a direct transcription of Proposition 2.14, noting for (3) that for schemes equipped with a section the properties “with connected fibers” and “with geometrically connected fibers” are equivalent [EGA4.2, Cor. 4.5.14] (cf. also [StaPro, 04KV]). Part (6) follows from Lemma 2.6, noting that the preimage of S'_0 under the flat map $\mathcal{G} \times_S S' \rightarrow S'$ defines an effective Cartier divisor. \square

Example 3.3. Let $G_0 \rightarrow \text{Spec}(\mathbb{Z})$ be a Chevalley group scheme, that is, a split reductive group scheme with connected fibers [Co14, §6.4] (note that some authors call this a Demazure group scheme, keeping the term Chevalley group scheme for split semisimple groups). Consider the base change $G := G_0 \times_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}}^1$ to the affine line. Let $S_0 = \text{Spec}(\mathbb{Z})$ considered as the effective Cartier divisor defined by the zero section of $S = \mathbb{A}_{\mathbb{Z}}^1$. Let $P_0 \subset G_0$ be a parabolic subgroup. By Theorem 3.2 (3) and (5), the Néron blowup $\mathcal{G} \rightarrow \mathbb{A}_{\mathbb{Z}}^1$ of G in P_0 is a smooth, affine group scheme with connected fibers. In fact, it is an easy special case of the group schemes constructed in [PZ13, §4] and [Lou, §2]. Let ϖ denote a global coordinate on $\mathbb{A}_{\mathbb{Z}}^1$. By Theorem 3.2 (6) the base changes have the following properties:

- (1) *If k is any field, then $\mathcal{G}(k[[\varpi]])$ is the subgroup of those elements in $G(k[[\varpi]]) = G_0(k[[\varpi]])$ whose reduction modulo ϖ lies in $P_0(k)$.*
- (2) *If p is any prime number, then $\mathcal{G}_{\varpi \rightarrow p}(\mathbb{Z}_p)$ is subgroup of those elements in $G_{\varpi \rightarrow p}(\mathbb{Z}_p) = G_0(\mathbb{Z}_p)$ whose reduction modulo p lies in $P_0(\mathbb{F}_p)$.*

In other words the respective base changes $\mathcal{G} \times_{\mathbb{A}_{\mathbb{Z}}^1} \text{Spec}(k[[\varpi]])$ and $\mathcal{G} \times_{\mathbb{A}_{\mathbb{Z}}^1, \varpi \mapsto p} \text{Spec}(\mathbb{Z}_p)$ are parahoric group schemes in the sense of [BT84], cf. also [PZ13, Cor. 4.2] and [Lou, §2.6]. Thus, the Néron blowup $\mathcal{G} \rightarrow \mathbb{A}_{\mathbb{Z}}^1$ can be viewed as a family of parahoric group schemes.

3.3. Group structure on the exceptional divisor. We continue with the notation of §3.1. In this subsection we take up the description of the exceptional divisor from Proposition 2.8, in the context of group schemes: we explain the interplay between the ambient group structure and the vector bundle structure on the exceptional divisor.

In the sequel, for any scheme X we write $\Gamma(X) = H^0(X, \mathcal{O}_X)$ its ring of global functions.

Lemma 3.4. *Assume that S is affine. Let \mathcal{G} be an S -group scheme and $\mathcal{G}_0 := \mathcal{G} \times_S S_0$. Denote by $i: \mathcal{G}_0 \hookrightarrow \mathcal{G}$ the closed immersion and \mathcal{K} the corresponding ideal sheaf. Let $m, \text{pr}_1, \text{pr}_2: \mathcal{G} \times_S \mathcal{G} \rightarrow \mathcal{G}$ be the multiplication and the projections, with corresponding morphisms:*

$$\begin{aligned} m^\#, \text{pr}_1^\#, \text{pr}_2^\#: \Gamma(\mathcal{G}) &\rightarrow \Gamma(\mathcal{G} \times_S \mathcal{G}) \\ (i \times i)^\#: \Gamma(\mathcal{G} \times_S \mathcal{G}) &\longrightarrow \Gamma(\mathcal{G}_0 \times_{S_0} \mathcal{G}_0). \end{aligned}$$

If $\delta := m^\# - \text{pr}_1^\# - \text{pr}_2^\#$, we have $\delta(H^0(\mathcal{G}, \mathcal{K})) \subset \ker((i \times i)^\#)$.

Proof. Each of the maps $f \in \{m, \text{pr}_1, \text{pr}_2\}$ fits in a commutative diagram:

$$\begin{array}{ccc} \mathcal{G}_0 \times_{S_0} \mathcal{G}_0 & \xrightarrow{i \times i} & \mathcal{G} \times_S \mathcal{G} \\ f \downarrow & & \downarrow f \\ \mathcal{G}_0 & \xrightarrow{i} & \mathcal{G}. \end{array}$$

Since $H^0(\mathcal{G}, \mathcal{K})$ is the kernel of the map $i^\#: \Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{G}_0)$, by taking global sections we obtain $((i \times i)^\# f^\#)(H^0(\mathcal{G}, \mathcal{K})) = 0$. \square

We recall that for a group scheme $G \rightarrow S$ with unit section $e: S \rightarrow G$, the Lie algebra $\text{Lie}(G/S)$ is the S -group scheme $\mathbb{V}(e^* \Omega_{G/S}^1)$.

Theorem 3.5. *With the notation of §3.1, assume that $G \rightarrow S$ is flat, finitely presented and $H \rightarrow S_0$ is flat, regularly immersed in G_0 . Let $\mathcal{G} \rightarrow G$ be the dilatation of G in H with exceptional fibre $\mathcal{G}_0 := \mathcal{G} \times_S S_0$. Let \mathcal{J} be the ideal sheaf of G_0 in G and $\mathcal{J}_H := \mathcal{J}|_H$. Let V be the restriction of the normal bundle $\mathbb{V}(\mathcal{C}_{H/G_0} \otimes \mathcal{J}_H^{-1}) \rightarrow H$ along the unit section $e_0: S_0 \rightarrow H$.*

- (1) *There is a canonical exact sequence of S_0 -group schemes $1 \rightarrow V \rightarrow \mathcal{G}_0 \rightarrow H \rightarrow 1$.*
- (2) *If H lifts to a flat S -subgroup scheme of G , this sequence splits canonically.*
- (3) *If $G \rightarrow S$ is smooth, separated and $\mathcal{G} \rightarrow G$ is the Néron blowup of the unit section of G , the exact sequence of (1) gives a canonical isomorphism of smooth S_0 -group schemes $\mathcal{G}_0 \xrightarrow{\sim} \text{Lie}(G_0/S_0)$.*

Proof. (1) Let $\mathcal{F} = \mathcal{C}_{H/G_0} \otimes \mathcal{J}_H^{-1}$. Proposition 2.8 provides an isomorphism of S_0 -schemes:

$$\psi: \mathcal{G} \times_G H \xrightarrow{\sim} \mathbb{V}(\mathcal{F}).$$

Let $K = \ker(\mathcal{G}_0 \rightarrow H)$. To obtain the exact sequence of the statement, it is enough to prove that the restriction of ψ along the unit section $e_0: S_0 \rightarrow H$ is an isomorphism of S_0 -group schemes

$$e_0^* \psi: K \xrightarrow{\sim} V.$$

For this we may localize around a point of S_0 , hence assume that S and S_0 are affine and small enough so that \mathcal{J} is trivial. Proving that $e_0^* \psi$ is a morphism of groups is equivalent to checking an equality between two morphisms $K \times_{S_0} K \rightarrow V$. Since $V = \text{Spec}(\text{Sym}(\mathcal{F}_0))$ where $\mathcal{F}_0 := e_0^* \mathcal{F}$, this is the same as checking an equality between two maps of $\Gamma(S_0)$ -modules $H^0(S_0, \mathcal{F}_0) \rightarrow \Gamma(K \times_{S_0} K)$. More precisely, since K is affine we have $\Gamma(K \times_{S_0} K) = \Gamma(K) \otimes_{\Gamma(S_0)} \Gamma(K)$ and what we have to check is that $m^\#(x) = x \otimes 1 + 1 \otimes x$ for all $x \in H^0(S_0, \mathcal{F}_0)$, with m the multiplication of K . That is, we want to prove that

$$\delta(H^0(S_0, \mathcal{F}_0)) = 0$$

where $\delta : \Gamma(K) \rightarrow \Gamma(K \times_{S_0} K)$ is defined by $\delta = m^\# - \text{pr}_1^\# - \text{pr}_2^\#$.

In order to prove this, let \mathcal{I} be the ideal sheaf of the closed immersion $H \hookrightarrow G$, and let $f^*\mathcal{I}$ be its preimage as a module under the dilatation morphism $f: \mathcal{G} \rightarrow G$. Consider the closed immersions $K \hookrightarrow \mathcal{G}_0$ and $i: \mathcal{G}_0 \hookrightarrow \mathcal{G}$, and the diagram:

$$\begin{array}{ccccc} H^0(\mathcal{G}, f^*\mathcal{I}) & \longrightarrow & \Gamma(\mathcal{G}) & \xrightarrow{\delta} & \Gamma(\mathcal{G} \times_S \mathcal{G}) \\ \downarrow & & \downarrow & & \downarrow (i \times i)^\# \\ & & & & \Gamma(\mathcal{G}_0 \times_{S_0} \mathcal{G}_0) \\ & & & & \downarrow \\ H^0(S_0, \mathcal{F}_0) & \longrightarrow & \Gamma(K) & \xrightarrow{\delta} & \Gamma(K \times_{S_0} K). \end{array}$$

We claim that the vertical map $H^0(\mathcal{G}, f^*\mathcal{I}) \rightarrow H^0(S_0, \mathcal{F}_0)$ is surjective. To prove this, let $e_{\mathcal{G}}: S \rightarrow \mathcal{G}$ be the unit section of \mathcal{G} and $j: S_0 \hookrightarrow S$ be the closed immersion, and decompose the said map as follows:

$$H^0(\mathcal{G}, f^*\mathcal{I}) \xrightarrow{e_{\mathcal{G}}^*} H^0(S, e^*\mathcal{I}) \xrightarrow{j^*} H^0(S_0, j^*e^*\mathcal{I}) \longrightarrow H^0(S_0, \mathcal{F}_0).$$

The first map is surjective because it has the section σ^* where $\sigma: \mathcal{G} \rightarrow S$ is the structure map. The second map is surjective because it is obtained by taking global sections on the affine scheme S of the surjective map of sheaves $e^*\mathcal{I} \rightarrow j_*j^*e^*\mathcal{I}$. To show that the third map is surjective, start from the surjection of sheaves $\mathcal{I}|_H \rightarrow \mathcal{F}$. Since pullback is right exact, this gives rise to a surjection $j^*e^*\mathcal{I} = e_0^*\mathcal{I}|_H \rightarrow e_0^*\mathcal{F} = \mathcal{F}_0$. Taking global sections on the affine scheme S_0 we obtain the desired surjection.

Now let $\mathcal{I}\mathcal{O}_{\mathcal{G}} = f^{-1}\mathcal{I} \cdot \mathcal{O}_{\mathcal{G}}$ be the preimage of \mathcal{I} as an ideal. Note that by property of the dilatation, we have $\mathcal{I}\mathcal{O}_{\mathcal{G}} = \mathcal{J}\mathcal{O}_{\mathcal{G}} =: \mathcal{K}$. Therefore, according to Lemma 3.4, we have

$$\delta(H^0(\mathcal{G}, \mathcal{I}\mathcal{O}_{\mathcal{G}})) = \delta(H^0(\mathcal{G}, \mathcal{K})) \subset \ker((i \times i)^\#).$$

Precomposing with the surjection $f^*\mathcal{I} \rightarrow \mathcal{I}\mathcal{O}_{\mathcal{G}}$, we find that $H^0(\mathcal{G}, f^*\mathcal{I})$ is mapped into $\ker((i \times i)^\#)$ by δ . As a result $H^0(\mathcal{G}, f^*\mathcal{I})$ goes to zero in $\Gamma(K \times_{S_0} K)$. Since $H^0(\mathcal{G}, \mathcal{I}\mathcal{O}_{\mathcal{G}}) \rightarrow H^0(S_0, \mathcal{F}_0)$ is surjective, the commutativity of the diagram implies that $\delta(H^0(S_0, \mathcal{F}_0)) = 0$ in $\Gamma(K \times_{S_0} K)$, as desired. Hence $e_0^*\psi: K \rightarrow V$ is an isomorphism of groups. This proves (1).

(2) If H lifts to a flat S -subgroup scheme $\tilde{H} \subset G$, then by the universal property of the dilatation the map $\tilde{H} \rightarrow G$ lifts to a map $\tilde{H} \rightarrow \mathcal{G}$. In restriction to S_0 this splits the short exact sequence previously obtained.

(3) Finally if $G \rightarrow S$ is smooth, the unit section is a regular immersion with conormal sheaf $\omega_{G/S} = e^*\Omega_{G/S}^1$. In restriction to S_0 the group V is the Lie algebra whence the canonical isomorphism $\mathcal{G}_0 \xrightarrow{\sim} \text{Lie}(G_0/S_0)$. \square

Remark 3.6. In the situation of Theorem 3.5 (2), the group H acts by conjugation on $V = \mathbb{V}(e_0^*\mathcal{C}_{H/G_0} \otimes \mathcal{J}_{S_0}^{-1})$. This additive action is in fact linear, and is none other than the ‘‘adjoint’’ representation of H on its normal bundle as in [SGA3.1, Exp. I, Prop. 6.8.6]. Here the normal sheaf $\mathcal{J}_{S_0}^{-1}$ comes from the base and is endowed with the trivial action; for simplicity we describe the action Zariski locally on S_0 and omit it from the notation. We start from the conormal sequence of the inclusions $\{1\} \subset H \subset G_0$:

$$1 \dashrightarrow e_0^*\mathcal{C}_{H/G_0} \longrightarrow \mathcal{C}_{1/G_0} \longrightarrow \mathcal{C}_{1/H} \longrightarrow 1.$$

The sequence is exact on the left if $\{1\} \rightarrow H$ is a regular immersion. Taking linear duals, we obtain an exact sequence of S_0 -group schemes:

$$1 \longrightarrow \text{Lie}(H/S_0) \longrightarrow \text{Lie}(G_0/S_0) \longrightarrow \mathbb{V}(e_0^*\mathcal{C}_{H/G_0}) \dashrightarrow 1.$$

The middle term $\text{Lie}(G_0/S_0)$ supports the adjoint action of G_0 . The restricted action of H leaves stable the terms $\text{Lie}(H/S_0)$ and $\mathbb{V}(e_0^*\mathcal{C}_{H/G_0})$, cf. [SGA3.1, Exp. III, Lem. 4.25]. The former is the

adjoint action of H on its Lie algebra, and the latter is the action on the normal bundle along the unit section.

3.4. Néron blowups as syntomic sheaves. We continue with the notation of §3.1. Additionally assume that $j: S_0 \hookrightarrow S$ is an effective Cartier divisor, that $G \rightarrow S$ is a flat, finitely presented group scheme and that $H \subset G_0 := G \times_S S_0$ is a flat, finitely presented closed S_0 -subgroup scheme. In this context, there is another viewpoint on the dilatation \mathcal{G} of G in H , namely as the kernel of a certain map of syntomic sheaves.

To explain this, let $f: G_0 \rightarrow G_0/H$ be the morphism to the fppf quotient sheaf, which by Artin's theorem ([Ar74, Cor. 6.3]) is representable by an algebraic space. By the structure theorem for algebraic group schemes (see [SGA3.1, Exp. VII_B, Cor. 5.5.1]) the morphisms $G \rightarrow S$ and $H \rightarrow S_0$ are syntomic. Since $f: G_0 \rightarrow G_0/H$ makes G_0 an H -torsor, it follows that f is syntomic also.

Lemma 3.7. *Let S_{syn} be the small syntomic site of S . Let $\eta: G \rightarrow j_*j^*G$ be the adjunction map in the category of sheaves on S_{syn} and consider the composition $v = (j_*f) \circ \eta$:*

$$G \xrightarrow{\eta} j_*j^*G = j_*G_0 \xrightarrow{j_*f} j_*(G_0/H).$$

Then the dilatation $\mathcal{G} \rightarrow G$ is the kernel of v . More precisely, we have an exact sequence of sheaves of pointed sets in S_{syn} :

$$1 \longrightarrow \mathcal{G} \longrightarrow G \xrightarrow{v} j_*(G_0/H) \longrightarrow 1.$$

If $G \rightarrow S$ and $H \rightarrow S_0$ are smooth, then the sequence is exact as a sequence of sheaves on the small étale site of S .

Proof. That $\mathcal{G} \rightarrow G$ is the kernel of v follows directly from the universal property of the dilatation, restricted to syntomic S -schemes. It remains to prove that the map of sheaves v is surjective. It is enough to prove that both maps η and j_*f are surjective. For η this is because if $T \rightarrow S$ is a syntomic morphism, any point $t: T \rightarrow j_*G_0$ lifts tautologically to G after the syntomic refinement $T' = G \times_S T \rightarrow T$. For j_*f , we start from a syntomic morphism $T \rightarrow S$ and a point $t: T \rightarrow j_*(G_0/H)$, that is a morphism $T_0 \rightarrow G_0/H$. Using that $G_0 \rightarrow G_0/H$ is syntomic, we can find as before a syntomic refinement $T'_0 \rightarrow T_0$ and a lift $T'_0 \rightarrow G_0$. Using that syntomic coverings lift across closed immersions (see [StaPro, 04E4]), there is a syntomic covering $T'' \rightarrow T$ such that T''_0 refines T'_0 . This provides a lift of t to j_*G_0 .

Finally if $G \rightarrow S$ and $H \rightarrow S_0$ are smooth, the existence of étale sections for smooth morphisms ([EGA4.4, Cor. 17.16.3]) and the possibility to lift étale coverings across closed immersions (see [StaPro, 04E4] again) show that the sequence is exact also in the étale topology. \square

4. APPLICATIONS

Here we give two applications in cohomological degree 0 and 1 of the theory developed so far: integral points and torsors. In §4.1 we consider integral points of Néron blowups and discuss the isomorphism relating the graded pieces of the congruent filtration of G to the graded pieces of its Lie algebra \mathfrak{g} . In §4.2 we discuss torsors under Néron blowups and apply this in §4.2.1 to the construction of level structures on moduli stacks of G -bundles on curves, and in §4.2.2 to the construction of integral models of moduli stacks of shtukas.

4.1. Integral points and the Moy-Prasad isomorphism. In this subsection we prove an isomorphism describing the graded pieces of the filtration by congruence subgroups on the integral points of reductive group schemes. For the benefit of the interested reader, we provide comments on the literature on this topic in Remark 4.4 below.

We start with the following lemma.

Lemma 4.1. *Let \mathcal{O} be a ring and $\pi \subset \mathcal{O}$ an invertible ideal such that (\mathcal{O}, π) is a henselian pair. Let G be a smooth, separated, finitely presented \mathcal{O} -group scheme and $\mathcal{G} \rightarrow G$ the dilatation of the*

trivial subgroup over \mathcal{O}/π . If either \mathcal{O} is local or G is affine, then the exact sequence of Lemma 3.7 induces an exact sequence of groups:

$$1 \longrightarrow \mathcal{G}(\mathcal{O}) \longrightarrow G(\mathcal{O}) \longrightarrow G(\mathcal{O}/\pi) \longrightarrow 1.$$

Proof. Write $S = \text{Spec}(\mathcal{O})$, $S_0 = \text{Spec}(\mathcal{O}/\pi)$ and set $G_0 = G \times_S S_0$, $\mathcal{G}_0 = \mathcal{G} \times_S S_0$. Consider the short exact sequence of Lemma 3.7 on the étale site and take the global sections over S . It is then enough to prove that the map $G(\mathcal{O}) \rightarrow G(\mathcal{O}/\pi)$ is surjective. For this start with an (\mathcal{O}/π) -point of G , i.e. a section $u_0 : S_0 \rightarrow G_0$ to the map $G_0 \rightarrow S_0$. If either \mathcal{O} is local or G is affine, u_0 factors through an open affine subscheme $U \subset G$. In this situation, the classical existence result for lifting of sections for smooth schemes over a henselian local ring (as in for example [BLR90, 2.3/5]) extends to henselian pairs, see [Gr72, Thm. 1.8]. In this way we see that u_0 lifts to a section u of $G \rightarrow S$. \square

Remark 4.2. The same proof gives a similar result with the dilatation of a split smooth unipotent closed subgroup $H \subset G_0$, with the group $G(\mathcal{O}/\pi)$ replaced by the pointed set $(G_0/H)(\mathcal{O}/\pi)$. Indeed, such a group H is obtained by successive extensions of the additive group \mathbb{G}_{a,S_0} over $S_0 = \text{Spec}(\mathcal{O}/\pi)$. Since S_0 is affine, the (étale or syntomic) cohomology of \mathbb{G}_{a,S_0} vanishes, being the coherent cohomology of \mathcal{O}_{S_0} . Using induction one concludes that $H^1(S_0, H)$ is trivial. Now starting from a section u_0 of $G_0/H \rightarrow S_0$ and pulling back the map $G_0 \rightarrow G_0/H$ along u_0 , we obtain an H -torsor $G_0 \times_{G_0/H} S_0 \rightarrow S_0$. By the previous remarks this torsor has a section v_0 . The latter lifts to a section of G by the same argument as in the proof of the lemma.

Theorem 4.3. *Let r, s be integers such that $0 \leq r/2 \leq s \leq r$. Let (\mathcal{O}, π) be a henselian pair where $\pi \subset \mathcal{O}$ is an invertible ideal. Let G be a smooth, separated, finitely presented \mathcal{O} -group scheme and G_r the r -th iterated dilatation of the unit section. Let \mathfrak{g}_r be the Lie algebra of G_r . If \mathcal{O} is local or G is affine, then there is a canonical isomorphism:*

$$G_s(\mathcal{O})/G_r(\mathcal{O}) \xrightarrow{\sim} \mathfrak{g}_s(\mathcal{O})/\mathfrak{g}_r(\mathcal{O}).$$

Proof. The $(r-s)$ -th iterated dilatation of G_s is naturally G_r . But as we observed in Proposition 2.12, the group scheme G_r can also be seen as the dilatation of $\{1\}$ in (\mathcal{O}, π^r) . For an integer $n \geq 0$, we write $\mathcal{O}_n := \mathcal{O}/\pi^n$. Putting these remarks together, the previous lemma applied to the dilatation of the group scheme $G = G_s$ with respect to (\mathcal{O}, π^{r-s}) yields an isomorphism:

$$(4.1) \quad G_s(\mathcal{O})/G_r(\mathcal{O}) \xrightarrow{\sim} G_s(\mathcal{O}_{r-s}).$$

We now consider the statement of the theorem. If $s = 0$ we have $r = 0$, hence left-hand side and right-hand side are equal to $\{1\}$ and the result is clear. Therefore we may assume that $s > 0$. Theorem 3.5 applied to the dilatation of $\{1\}$ in (\mathcal{O}, π^s) provides a canonical isomorphism

$$G_s|_{\mathcal{O}_s} \xrightarrow{\sim} \text{Lie}(G|_{\mathcal{O}_s}).$$

Since the functor Lie is idempotent, taking Lie algebras on both sides we deduce a canonical isomorphism

$$G_s|_{\mathcal{O}_s} \xrightarrow{\sim} \mathfrak{g}_s|_{\mathcal{O}_s}.$$

Since $r-s \leq s$, the ring \mathcal{O}_{r-s} is an \mathcal{O}_s -algebra and we can take \mathcal{O}_{r-s} -valued points in the previous isomorphism to obtain:

$$G_s(\mathcal{O}_{r-s}) \xrightarrow{\sim} \mathfrak{g}_s(\mathcal{O}_{r-s}).$$

Using (4.1) once for G_s and once for \mathfrak{g}_s , we end up with

$$G_s(\mathcal{O})/G_r(\mathcal{O}) \xrightarrow{\sim} \mathfrak{g}_s(\mathcal{O})/\mathfrak{g}_r(\mathcal{O}),$$

which is the desired canonical isomorphism. \square

Remark 4.4. In the literature on rational points of reductive groups over non-archimedean local fields, results similar to the isomorphism of Theorem 4.3 appeared with restrictions on the indices r, s , on the group schemes involved or on the ground ring. See for instance [Ser68, Prop. 6 (b)], [Ne99, Prop. 3.9 and 3.10] for the multiplicative group, [Ho77, p. 442 line 1], [Mo82], [BK93, p. 22]

for general linear groups, [Sec04, p. 337] for general linear groups over division algebras, and [PR84, §2], [MP94, §2], [MP96], [Ad98, §1], [Yu01, §1] for general reductive groups. In these examples, the isomorphisms are defined at the level of rational points using ad hoc explicit formulas. These isomorphisms are sometimes called Moy-Prasad isomorphisms as a tribute to [MP94], read [Yu15, 0.4] and [De02, p. 242 lines 18-19] for informations. In the case of an affine, smooth group scheme over a discrete valuation ring, the isomorphism of Theorem 4.3 appears without proof in [Yu15, proof of Lemma 2.8].

4.2. Torsors and level structures. In this subsection we adopt notations more specific to the study of torsors over curves. Let X be a scheme, and let $N \subset X$ be an effective Cartier divisor. Let $G \rightarrow X$ be a smooth, finitely presented group scheme, and let $H \subset G|_N$ be an N -smooth closed subgroup. We denote by $\mathcal{G} \rightarrow G$ the Néron blowup of G in H (over N) which is a smooth, finitely presented X -group scheme by Theorem 3.2.

For a scheme $T \rightarrow X$, let $B\mathcal{G}(T)$ (resp. $BG(T)$) denote the groupoid of right G -torsors on T in the fppf topology. Here we note that every such torsor is representable by a smooth algebraic space (of finite presentation), and hence admits sections étale locally. Whenever convenient we may therefore work in the étale topology as opposed to the fppf topology.

Pushforward of torsors along $\mathcal{G} \rightarrow G$ induces a morphism of contravariant functors $\text{Sch}_X \rightarrow \text{Groupoids}$ given by

$$(4.2) \quad B\mathcal{G} \rightarrow BG, \quad \mathcal{E} \mapsto \mathcal{E} \times^{\mathcal{G}} G.$$

Definition 4.5. For a scheme $T \rightarrow X$, let $B(G, H, N)(T)$ be the groupoid whose objects are pairs (\mathcal{E}, β) where $\mathcal{E} \rightarrow T$ is a right fppf G -torsor and β is a section of the fppf quotient

$$(\mathcal{E}|_{T_N}/H|_{T_N}) \rightarrow T_N,$$

where $T_N := T \times_X N$, i.e., β is a reduction of $\mathcal{E}|_{T_N}$ to an H -torsor. Morphisms $(\mathcal{E}, \beta) \rightarrow (\mathcal{E}', \beta')$ are given by isomorphisms of torsors $\varphi: \mathcal{E} \cong \mathcal{E}'$ such that $\bar{\varphi} \circ \beta = \beta'$ where $\bar{\varphi}$ denotes the induced map on the quotients. Note that if $T_N = \emptyset$, then there is no condition on the compatibility of β and β' .

Each of the contravariant functors $\text{Sch}_X \rightarrow \text{Groupoids}$ induced by $B\mathcal{G}$, BG and $B(G, H, N)$ defines a stack over X in the fppf topology. We call $B(G, H, N)$ the *stack of G -torsors equipped with level- (H, N) -structures*.

Lemma 4.6. *The map (4.2) factors as a map of X -stacks*

$$(4.3) \quad B\mathcal{G} \rightarrow B(G, H, N) \rightarrow BG,$$

where the second arrow denotes the forgetful map.

Proof. By Lemma 3.1 (1) the map $\mathcal{G}|_N \rightarrow G|_N$ factors as $\mathcal{G}|_N \rightarrow H \subset G|_N$. Thus, given a \mathcal{G} -torsor $\mathcal{E} \rightarrow T$ we get the H -equivariant map

$$\mathcal{E} \times^{\mathcal{G}|_{T_N}} H|_{T_N} \subset \mathcal{E} \times^{\mathcal{G}|_{T_N}} G|_{T_N}.$$

Passing to the fppf quotient for the right H -action defines the section β_{can} . The association $\mathcal{E} \mapsto (\mathcal{E} \times^{\mathcal{G}} G, \beta_{\text{can}})$ induces the desired map $B\mathcal{G} \rightarrow B(G, H, N)$. \square

Proposition 4.7. *The map (4.3) induces an equivalence of contravariant functors $\text{Sch}_X^{N\text{-reg}} \rightarrow \text{Groupoids}$ given by*

$$B\mathcal{G} \xrightarrow{\simeq} B(G, H, N), \quad \mathcal{E} \mapsto (\mathcal{E} \times^{\mathcal{G}} G, \beta_{\text{can}}).$$

Proof. For $T \rightarrow X$ in $\text{Sch}_X^{N\text{-reg}}$, we need to show that $B\mathcal{G}(T) \rightarrow B(G, H, N)(T)$ is an equivalence of groupoids. Since $\mathcal{G} \rightarrow X$ is smooth, in particular flat, its formation commutes with base change along $T \rightarrow X$ by Theorem 3.2. Hence, we may reduce to the case where $T = X$. Now recall from Lemma 3.7 the exact sequence of sheaves of pointed sets on the étale site of X ,

$$1 \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow j_*(G|_N/H) \longrightarrow 1,$$

where $j: N \subset X$ denotes the inclusion. The desired equivalence is a consequence of [Gi71, Chap. III, §3.2, Prop. 3.2.1]. A quasi-inverse is given by pulling back a section $\beta: X \rightarrow j_*(G|_N/H)$ along the \mathcal{G} -torsor $G \rightarrow j_*(G|_N/H)$. Here we have used that by smoothness of the group schemes $G \rightarrow X$, $\mathcal{G} \rightarrow X$, $H \rightarrow N$ and consequently of the quotient $G|_N/H$ we are allowed to work with the étale topology as opposed to the fppf topology. \square

4.2.1. *Level structures on moduli stacks of bundles on curves.* We continue with the notation, and additionally assume that X is a smooth, projective, geometrically irreducible curve over a field k , and that $G \rightarrow X$ and hence $\mathcal{G} \rightarrow X$ is affine.

Let $\text{Bun}_G := \text{Res}_{X/k} B\mathcal{G}$ (resp. $\text{Bun}_{\mathcal{G}} := \text{Res}_{X/k} B\mathcal{G}$) be the moduli stack of G -torsors (resp. \mathcal{G} -torsors) on X ; here $\text{Res}_{X/k}$ stands for the Weil restriction along $X \rightarrow \text{Spec}(k)$. This is a quasi-separated, smooth algebraic stack locally of finite type over k , cf. e.g. [He10, Prop. 1]. Similarly, let $\text{Bun}_{(G,H,N)} := \text{Res}_{X/k} B(G,H,N)$ be the stack parametrizing G -torsors over X with level- (H,N) -structures as in Definition 4.5.

Theorem 4.8. *The map (4.3) induces equivalences of contravariant functors $\text{Sch}_k \rightarrow \text{Groupoids}$ given by*

$$\text{Bun}_{\mathcal{G}} \xrightarrow{\cong} \text{Bun}_{(G,H,N)}, \quad \mathcal{E} \mapsto (\mathcal{E} \times^{\mathcal{G}} G, \beta_{\text{can}})$$

Proof. For any k -scheme T , the projection $X \times_k T \rightarrow X$ is flat, and hence defines an object in $\text{Sch}_X^{N\text{-reg}}$. The theorem follows from Proposition 4.7. \square

Example 4.9. If $H = \{1\}$ is trivial, then $\text{Bun}_{(G,H,N)}$ is the moduli stack of G -torsors on X with level- N -structures. If $G \rightarrow X$ is split reductive, if N is reduced and if H a parabolic subgroup in $G|_N$, then $\text{Bun}_{(G,H,N)}$ is the moduli stack of G -torsors with quasi-parabolic structures in the sense of Laszlo-Sorger [LS97], cf. [PR10, §2.a.] and [He10, §1, Exam. (2)].

We end this subsection by discussing Weil uniformizations. Let $|X| \subset X$ be the set of closed points, and let $\eta \in X$ be the generic point. We denote by $F = \kappa(\eta)$ the function field of X . For each $x \in |X|$, we let \mathcal{O}_x be the completed local ring at x with fraction field F_x and residue field $\kappa(x) = \mathcal{O}_x/\mathfrak{m}_x$. Let $\mathbb{A} := \prod'_{x \in |X|} F_x$ be the ring of adèles with subring of integral elements $\mathbb{O} = \prod_{x \in |X|} \mathcal{O}_x$. As in [N06, Lem. 1.1] or [Laf18, Rem. 8.21] one has the following result.

Proposition 4.10. *Assume that k is either a finite field or a separably closed field, and that $G \rightarrow X$ has connected fibers. Then there is an equivalence of groupoids*

$$(4.4) \quad \text{Bun}_G(k) \simeq \bigsqcup_{\gamma} G_{\gamma}(F) \backslash (G_{\gamma}(\mathbb{A})/G(\mathbb{O})),$$

where γ ranges over $\ker^1(F, G) := \ker(H_{\text{ét}}^1(F, G) \rightarrow \prod_{x \in |X|} H_{\text{ét}}^1(F_x, G))$, and where G_{γ} denotes the associated pure inner form of $G|_F$. The identification (4.4) is functorial in G among maps of X -group schemes which are isomorphisms in the generic fibre.

Proof. Under our assumptions, Lang's lemma implies that $H_{\text{ét}}^1(\mathcal{O}_x, G)$ is trivial for all $x \in |X|$: use that $H_{\text{ét}}^1(\kappa(x), G)$ is trivial because $G|_{\kappa(x)}$ smooth, affine, connected and $\kappa(x)$ is either finite or separably closed; then an approximation argument as in e.g. [RS20, Lem. A.4.3]. In particular, for every G -torsor $\mathcal{E} \rightarrow X$ the class of its generic fibre $[\mathcal{E}|_F]$ lies in $\ker^1(F, G)$. For each $\gamma \in \ker^1(F, G)$, we fix a G -torsor $\mathcal{E}_{\gamma}^0 \rightarrow \text{Spec}(F)$ of class γ . We denote by G_{γ} its group of automorphisms which is an inner form of G . We also fix an identification $G_{\gamma}(F_x) = G(F_x)$ for all $x \in |X|$, $\gamma \in \ker^1(F, G)$. In particular, $G_{\gamma}(\mathbb{A}) = G(\mathbb{A})$ so that the right hand quotient in (4.4) is well-defined. Now consider the groupoid

$$\Sigma_{\gamma} := \{(\mathcal{E}, \delta, (\epsilon_x)_{x \in |X|}) \mid \delta: \mathcal{E}|_F \simeq \mathcal{E}_{\gamma}^0, \quad \epsilon_x: \mathcal{E}^0 \simeq \mathcal{E}|_{\mathcal{O}_x}\}.$$

For each $x \in |X|$, we have

$$g_x := \delta|_{F_x} \circ \epsilon_x|_{F_x} \in \text{Aut}(\mathcal{E}_{\gamma}^0|_{F_x}) = G_{\gamma}(F_x) = G(F_x),$$

and further $g_x \in G(\mathcal{O}_x)$ for almost all $x \in |X|$. Thus, the collection $(g_x)_{x \in |X|}$ defines a point in $G(\mathbb{A}) = G_\gamma(\mathbb{A})$. In this way, we obtain an $G_\gamma(F) \times G(\mathbb{O})$ -equivariant map $\pi_\gamma: \Sigma_\gamma \rightarrow G_\gamma(\mathbb{A})$, and thus a commutative diagram of groupoids

$$\begin{array}{ccc} \bigsqcup_\gamma \Sigma_\gamma & \xrightarrow{\sqcup_\gamma \pi_\gamma} & \bigsqcup_\gamma G_\gamma(\mathbb{A}) \\ \downarrow & & \downarrow \\ \text{Bun}_G & \dashrightarrow & \bigsqcup_\gamma G_\gamma(F) \backslash (G_\gamma(\mathbb{A})/G(\mathbb{O})). \end{array}$$

As the vertical maps are disjoint unions of $G_\gamma(F) \times G(\mathbb{O})$ -torsors, the dashed arrow is fully faithful. Hence, it suffices to show that it is a bijection on isomorphism classes, i.e., a bijection of sets. We construct an inverse of the dashed arrow as follows: Given a representative $(g_x)_{x \in |X|} \in G_\gamma(\mathbb{A}) = G(\mathbb{A})$ of some class, there is a non-empty open subset $U \subset X$ such that $g_x \in G(\mathcal{O}_x)$ for all $x \in |U|$, and such that \mathcal{E}_γ^0 is defined over U . Let $X \setminus U = \{x_1, \dots, x_n\}$ for some $n \geq 0$. We define the associated G -torsor by gluing the torsor \mathcal{E}_γ^0 on U with the trivial G -torsor on

$$\text{Spec}(\mathcal{O}_{x_1}) \sqcup \dots \sqcup \text{Spec}(\mathcal{O}_{x_n})$$

using the elements g_{x_1}, \dots, g_{x_n} and the identification $G_\gamma(F_x) = G(F_x)$. The gluing is justified by the Beauville-Laszlo lemma [BL95], or alternatively [He10, Lem. 5]. This shows (4.4). From the construction of the map $\sqcup_\gamma \pi_\gamma$, one sees that (4.4) is functorial in G among generic isomorphisms. \square

Note that N defines an effective Cartier divisor on $\text{Spec}(\mathbb{O})$ so that the map of groups $\mathcal{G}(\mathbb{O}) \rightarrow G(\mathbb{O})$ is injective. As subgroups of $G(\mathbb{O})$ we have

$$(4.5) \quad \mathcal{G}(\mathbb{O}) = \ker(G(\mathbb{O}) \rightarrow G(\mathcal{O}_N) \rightarrow G(\mathcal{O}_N)/H(\mathcal{O}_N)),$$

where \mathcal{O}_N denotes the ring of functions on N viewed as a quotient ring $\mathbb{O}_X \rightarrow \mathcal{O}_N$.

Corollary 4.11. *Under the assumptions of Proposition 4.10, the Néron blowup $\mathcal{G} \rightarrow X$ is smooth, affine with connected fibers by Theorem 3.2, and there is a commutative diagram of groupoids*

$$\begin{array}{ccc} \text{Bun}_G(k) & \xrightarrow{\simeq} & \bigsqcup_\gamma G_\gamma(F) \backslash (G_\gamma(\mathbb{A})/\mathcal{G}(\mathbb{O})) \\ \downarrow & & \downarrow \\ \text{Bun}_G(k) & \xrightarrow{\simeq} & \bigsqcup_\gamma G_\gamma(F) \backslash (G_\gamma(\mathbb{A})/G(\mathbb{O})), \end{array}$$

identifying the vertical maps as the level maps.

Remark 4.12. Assume that k is either a finite field or an algebraically closed field, and that the generic fiber $G|_F$ is reductive. If $G|_F$ is either simply connected or split reductive, then $\ker^1(F, G)$ is trivial, cf. [Laf18, Rem. 8.21] and references therein.

4.2.2. *Integral models of moduli stacks of shtukas.* Here we point out that Theorem 4.8 immediately applies to construct certain integral models of moduli stacks of shtukas. We proceed with the notation of §4.2.1, and additionally assume that k is a finite field. Our presentation follows [Laf18, §§1-2].

For any partition $I = I_1 \sqcup \dots \sqcup I_r$, $r \in \mathbb{Z}_{\geq 1}$ of a finite index set, the *moduli stack of iterated G -shtukas* is the contravariant functor of groupoids $\text{Sch}_k \rightarrow \text{Groupoids}$ given by

$$(4.6) \quad \text{Sht}_{G, I_\bullet} \stackrel{\text{def}}{=} \left\{ \mathcal{E}_r \xrightarrow[\text{I}_r]{\alpha_r} \mathcal{E}_{r-1} \xrightarrow[\text{I}_{r-1}]{\alpha_{r-1}} \dots \xrightarrow[\text{I}_2]{\alpha_2} \mathcal{E}_1 \xrightarrow[\text{I}_1]{\alpha_1} \mathcal{E}_0 = \tau \mathcal{E}_r \right\},$$

where $\tau \mathcal{E} := (\text{id}_X \times \text{Frob}_{T/k})^* \mathcal{E}$ denotes the pullback under the relative Frobenius $\text{Frob}_{T/k}$. Here the dashed arrows in (4.6) indicate that the maps α_j between G -bundles are rationally defined. More precisely, $\text{Sht}_{G, I_\bullet}(T)$ classifies data $((\mathcal{E}_j)_{j=1, \dots, r}, \{x_i\}_{i \in I}, (\alpha_j)_{j=1, \dots, r})$ where $\mathcal{E}_j \in \text{Bun}_G(T)$ are torsors, $\{x_i\}_{i \in I} \in X^I(T)$ are points, and

$$\alpha_j: \mathcal{E}_j|_{X_T \setminus (\cup_{i \in I_j} \Gamma_{x_i})} \rightarrow \mathcal{E}_{j-1}|_{X_T \setminus (\cup_{i \in I_j} \Gamma_{x_i})}$$

are isomorphisms of torsors. Here $\Gamma_{x_i} \subset X_T$ denotes the graph of x_i viewed as a relative effective Cartier divisor on $X_T \rightarrow T$. We have a forgetful map $\mathrm{Sht}_{G, I_\bullet} \rightarrow X^I$. Similarly, we have the moduli stack $\mathrm{Sht}_{\mathcal{G}, I_\bullet} \rightarrow X^I$ defined by replacing G with \mathcal{G} . By [Var04] for split reductive groups and by [AH, Thm. 3.15] for general smooth, affine group schemes both stacks are ind-(Deligne-Mumford) stacks which are ind-(separated and of locally finite type) over k . Furthermore, pushforward of torsors along $\mathcal{G} \rightarrow G$ induces a map of X^I -stacks

$$(4.7) \quad \mathrm{Sht}_{\mathcal{G}, I_\bullet} \rightarrow \mathrm{Sht}_{G, I_\bullet},$$

cf. [Br]. We also consider the *moduli stack of iterated G -shtukas with level- (H, N) -structures*,

$$\mathrm{Sht}_{(G, H, N), I_\bullet} \rightarrow (X \setminus N)^I,$$

i.e., $\mathrm{Sht}_{(G, H, N), I_\bullet}(T)$ classifies data

$$((\mathcal{E}_j, \beta_j)_{j=1, \dots, r}, \{x_i\}_{i \in I}, (\alpha_j)_{j=1, \dots, r}),$$

where $(\mathcal{E}_j, \beta_j) \in \mathrm{Bun}_{(G, H, N)}(T)$ are G -torsors with a level- (H, N) -structure, $\{x_i\}_{i \in I} \in (X \setminus N)^I(T)$ are points, and

$$(4.8) \quad \alpha_j : (\mathcal{E}_j, \beta_j)|_{X_T \setminus (\cup_{i \in I_j} \Gamma_{x_i})} \rightarrow (\mathcal{E}_{j-1}, \beta_{j-1})|_{X_T \setminus (\cup_{i \in I_j} \Gamma_{x_i})}$$

are maps of G -torsors with a level- (H, N) -structure where $(\mathcal{E}_0, \beta_0) := (\tau \mathcal{E}_r, \tau \beta_r)$. We have a forgetful map of $(X \setminus N)^I$ -stacks

$$(4.9) \quad \mathrm{Sht}_{(G, H, N), I_\bullet} \rightarrow \mathrm{Sht}_{G, I_\bullet}|_{(X \setminus N)^I}.$$

Corollary 4.13. *Let G, H, N, \mathcal{G} and $I = I_1 \sqcup \dots \sqcup I_r$ be as above. Then the equivalence in Theorem 4.8 induces an equivalence*

$$\mathrm{Sht}_{\mathcal{G}, I_\bullet}|_{(X \setminus N)^I} \xrightarrow{\cong} \mathrm{Sht}_{(G, H, N), I_\bullet},$$

which is compatible with the maps (4.7) and (4.9).

Loosely speaking, $\mathrm{Sht}_{\mathcal{G}, I_\bullet} \rightarrow X^I$ is an integral model for $\mathrm{Sht}_{(G, H, N), I_\bullet} \rightarrow (X \setminus N)^I$, which however needs modification outside the case of parahoric group schemes $\mathcal{G} \rightarrow X$. Concretely, if the characteristic places $x_i, i \in I$ of the shtuka divide the level N , then there is simply no compatibility condition on the β_j 's in (4.8). Consequently, the fibers of the map (4.7) over such places are by [Br, Thm. 3.20] certain quotients of positive loop groups. In particular, these fibers are (in general) of strictly positive dimension, and furthermore are (in general) not proper if $\mathcal{G} \rightarrow X$ is not parahoric.

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