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by

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Abstract. — Let p be a prime number, and let K be a number field. For $p = 2$, assume moreover K totally imaginary. In this note we prove the existence of asymptotically good extensions L/K of cohomological dimension 2 in which infinitely many primes split completely. Our result is inspired by a recent work of Hajir, Maire, and Ramakrishna [**7**].

Let K be a number field, and let L/K be an infinite unramified extension. Denote by $\mathscr{S}_{\mathrm{L/K}}$ the set of prime ideals of K that split completely in L/K. In [**8**] Ihara proved that $\displaystyle\sum_{\mathfrak{p}\in\mathscr{S}_{\mathrm{L/K}}} \frac{\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})} < \infty$, and raised the following interesting question: are there L/K for which $\mathscr{S}_{\mathrm{L/K}}$ is infinite ? This question was recently answered in the positive by Hajir, Maire, and Ramakrishna in [**7**]. In fact, infinite unramified extensions L/K are some special cases of infinite extensions for which the root discriminants $\mathrm{rd}_{\mathrm{F}} := |\mathrm{Disc}_{\mathrm{F}}|^{1/[\mathrm{F}:\mathbb{Q}]}$ are bounded, where the number fields F vary in L/K, and $\mathrm{Disc}_{\mathrm{F}}$ is the discriminant of F. Such extensions are called *asymptotically good*, and it is now well-known that in such extensions the inequality of Ihara involving $\mathscr{S}_{\mathrm{L/K}}$ still holds (see for example [**16**], or [**13**] for the study of such extensions).

Pro-p extensions of number fields with restricted ramification allow us to exhibit asymptotically good extensions. Let p be a prime number, and let S be a finite set of prime ideals of K coprime to p (more precisely each $\mathfrak{p} \in S$ is such that $|\mathscr{O}_{\mathrm{K}}/\mathfrak{p}| \equiv 1 (\mathrm{mod}\ p)$); the set S is called *tame*. Let K_S the maximal pro-p extension of K unramified outside S, put $\mathrm{G}_S = \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$. In K_S/K the root discriminants are bounded by some constant depending on the discriminant of K and the norm of the places of S (see for example [**6**, Lemma 5]). Moreover thanks to Golod-Shafarevich criterion, it is well-known that K_S/K is infinite when $|S|$ is large as compared to $[\mathrm{K} : \mathbb{Q}]$ (see for example [**14**, Chapter X, §10, Theorem 10.10.1]), and then asymptotically good. *E.g.* for $p > 2$, $\mathbb{Q}_S/\mathbb{Q}$ is infinite when $|S| \geqslant 4$. In [**7**] the authors showed that when S is large, there exist infinite subextension

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L/K of K_S/K for which the set $\mathscr{S}_{\mathrm{L/K}}$ is infinite. But they give no information about the structure of Gal(L/K). Here we prove:

Theorem A. — *Let p be a prime number, and let* K *be a number field. For $p = 2$ assume* K *totally imaginary. Let T and S_0 be two disjoint finite sets of prime ideals of* K *where S_0 is tame. Then for infinitely many finite sets S of tame prime ideals of* K *containing S_0 there exist an infinite pro-p extension* L/K *in K_S/K such that*

(i) the set $\mathscr{S}_{\mathrm{L/K}}$ of places that split completely in L/K *contains T;*
(ii) the set $\mathscr{S}_{\mathrm{L/K}}$ is infinite;
(iii) the pro-p group $\mathrm{G} = \mathrm{Gal}(\mathrm{L/K})$ *is of cohomological dimension* 2*;*
(iv) the minimal number of relations of G *is infinite, i.e.* $\dim H^2(\mathrm{G}, \mathbb{F}_p) = \infty$*;*
(v) for each $\mathfrak{p} \in S$, the local extension $\mathrm{L}_\mathfrak{p}/\mathrm{K}_\mathfrak{p}$ is maximal, i.e. isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_p$;
(vi) the Poincaré series of the algebra $\mathbb{F}_p[\![\mathrm{G}]\!]$, endowed with the graduation from the ideal of augmentation, is equal to $\left(1 - dt + rt^2 + t^3 \sum_{n\geqslant 0} t^n\right)^{-1}$, where $d = \dim \mathrm{G}_S$, and where r is explicit, depending on K, S, T.

Remark 1. — *We will see that the pro-p group* G *of Theorem A is mild in the terminology of Anick* [**2**]*. See also Labute* [**10**] *for arithmetic contexts.*

The proof uses various tools.
The first one is the strategy developed initially by Labute [**10**], then by Labute-Mináč [**11**], Schmidt [**15**], Forré [**4**] etc. for studying the cohomological dimension of a pro-p group G, through the notion of strongly free sets introduced by Anick [**1**]. By following the approach of Forré [**4**], we refine this idea when the minimal number of relations of G is infinite.
This key idea is associated to a result of Schmidt [**15**] that shows that the pro-p group G_S is of cohomological dimension 2 for some well-chosen S; the proof of Schmidt involves the cup-product $H^1(\mathrm{G}_S, \mathbb{F}_p) \cup H^1(\mathrm{G}_S, \mathbb{F}_p)$. Here we use the translation of this cup-product in the polynomial algebra, due to Forré. In particular, this allows us to choose infinitely many Frobenius in G_S such that the family of the highest terms of these plus the highest terms of the relations of G_S, is combinatorially free (see §1.1.3 and Definition 1.2).
We conclude by cutting the tower K_S/K by all these Frobenius: this is the strategy of [**7**].

This note contains two sections. In §1 we recall the results we need regarding pro-p groups, graded algebras, and arithmetic of pro-p extensions with restricted ramification. In §2 we start with an example when $\mathrm{K} = \mathbb{Q}$, and prove the main result.

Notations.
Let p be a prime number.
• If V is a $\mathbb{F}_p$-vector space we denote by $\dim \mathrm{V}$ its dimension over $\mathbb{F}_p$.
• For a pro-p group G, we denote by $H^i(\mathrm{G})$ the cohomology group $H^i(\mathrm{G}, \mathbb{F}_p)$. The p-rank of G, which is equal to $\dim H^1(\mathrm{G})$, is noted $d_p\mathrm{G}$.

1. The results we need

1.1. On pro-p groups. — For this section we refer to [**3**], [**9**, Chapters 5,6 and 7], and [**4**]. Take a prime number p.

1.1.1. Minimal presentation and cohomological dimension. — Let G be a pro-p group of finite rank d, and let $1 \to \mathrm{R} \to \mathrm{F} \to \mathrm{G} \to 1$ be a minimal presentation of G by a free pro-p group F. Let $\mathscr{F} := \{\rho_i\}_{i\in I}$ be an $\mathbb{F}_p$-basis of $\mathrm{R}/\mathrm{R}^p[\mathrm{F},\mathrm{R}]$; observe that I is not necessarily finite. The algebra $\Lambda_{\mathrm{G}} := \mathbb{F}_p[\![\mathrm{G}]\!]$ acts on $\mathrm{R}/\mathrm{R}^p[\mathrm{R},\mathrm{R}]$, and by Nakayama's lemma the ρ_i's generate topologically $\mathrm{R}/\mathrm{R}^p[\mathrm{R},\mathrm{R}]$ as Λ_{G}-module (see for example [**3**, Corollary 1.5]).

Let us recall the definition of the cohomological dimension $\mathrm{cd}(\mathrm{G})$ of G: it is the smallest integer n (eventually $n = \infty$) such that $H^i(\mathrm{G}) = 0$ for every $i \geqslant n+1$.

Theorem 1.1. — *The following assertions are equivalent:*

(i) $\mathrm{cd}(\mathrm{G}) \leqslant 2$*;*

(ii) $\mathrm{R}/\mathrm{R}^p[\mathrm{R},\mathrm{R}]$ *is a free compact* Λ_{G}*-module;*

(iii) $\mathrm{R}/\mathrm{R}^p[\mathrm{R},\mathrm{R}] \simeq \prod_I \Lambda_{\mathrm{G}}$.

Moreover, $\dim H^2(\mathrm{G}) = |I|$.

Proof. — See [**3**, Corollary 5.3] or [**9**, Chapter 7, §7.3, Theorem 7.7]. □

We are going to translate conditions of Theorem 1.1 in the algebra $\mathbb{F}_p^{nc}[\![X_1, \cdots, X_d]\!]$.

1.1.2. Filtred and graded algebras. — The results of this section can be found in [**1**].

• Let
$$\mathrm{E} = \mathbb{F}_p^{nc}[\![X_1, \cdots, X_d]\!]$$
be the algebra of noncommutative series in $X_1, \cdots, X_d$ with coefficients in $\mathbb{F}_p$. We consider now noncommutative multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, with $\alpha_i \in \{1, \cdots, d\}$, and we denote by X_α the monomial element of the form $X_\alpha = X_{\alpha_1} \cdots X_{\alpha_n}$. We endow each X_i with the degree 1; the degree $\deg(X_\alpha)$ of X_α is $|\alpha|$.

For $Z = \sum_\alpha a_\alpha X_\alpha$, the quantity $\omega(Z) = \min_{a_\alpha \neq 0}\{\deg(X_\alpha)\}$ is the valuation of Z, with the convention that $\omega(0) = \infty$. For $n \geqslant 0$, put $\mathrm{E}_n = \{Z \in \mathrm{E}, \omega(Z) \geqslant n\}$. Observe that E_1 is the augmentation ideal of E: this is the two-sided ideal of E topologically generated by the X_i's. The algebra E is filtered by the E_n's and its graded algebra Grad(E) is then:
$$\mathrm{Grad(E)} = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} \mathrm{E}_n/\mathrm{E}_{n+1} \simeq \mathbb{F}_p^{nc}[X_1, \ldots, X_d].$$
In other words Grad(E) is isomorphic to the noncommutative polynomial algebra $\mathrm{A} := \mathbb{F}_p^{nc}[X_1, \ldots, X_d]$, where each X_i is endowed with the formal degree 1. Let $\mathrm{A}_n = \{z \in \mathrm{A}, \omega(z) \geqslant n\}$ be the gradation of A; observe that A_1 is the augmentation ideal of A.

• Let $X_\alpha, X_{\alpha'}$ be two monomials (viewed in E or in A). The element X_α is a *submonomial* of $X_{\alpha'}$, if $X_{\alpha'} = X_\beta X_\alpha X_{\beta'}$, with $X_\beta, X_{\beta'}$ two monomials of A.

Definition 1.2. — A family $\mathscr{F} = \{X_{\alpha^{(i)}}\}_{i\in I}$ of monomials of A is combinatorially free if for all $i \neq j$:

(i) $X_{\alpha^{(i)}}$ is not a submonomial of $X_{\alpha^{(j)}}$,

(ii) if $X_{\alpha^{(i)}} = X_\alpha X_\beta$ and $X_{\alpha^{(j)}} = X_{\alpha'} X_{\beta'}$, then $X_\alpha \neq X_{\alpha'}$, with $X_\alpha, X_\beta, X_{\alpha'}, X_{\beta'}$ non-trivial monomials, *i.e.* $\neq 1$.

The monomials may be endowed with a total order $<$ as follows.

First let us consider the natural ordering $<'$ defined by: $X_1 <' X_2 <' \cdots <' X_d$.

Let X_α and X_β two monomials, we say that $X_\alpha > X_\beta$, if $\omega(X_\alpha) < \omega(X_\beta)$; if X_α and X_β have the same valuation, we use the lexicographic order induced by $<'$.

Now, let $Z = \sum_\alpha a_\alpha X_\alpha$ be a nonzero element of E, with $a_\alpha \in \mathbb{F}_p$. Then $\widehat{Z} := \max\{X_\alpha,\ a_\alpha \neq 0\}$ is the *highest term* respecting the order $<$.

• Let $\mathscr{F} := \{Z_i\}_{i\in I}$ be a locally finite graded subset of A_1 generating C as two-sided A-ideal: $\mathrm{C} = \mathrm{A}\mathscr{F}\mathrm{A}$. Observe that I is countable. Let $\mathrm{B} := \mathrm{A}/\mathrm{C}$ be the quotient endowed with the quotient gradation; we denote by $P_\mathrm{B}(t) = \sum_{n\in\mathbb{Z}_{\geqslant 0}} \dim(\mathrm{B}_n/\mathrm{B}_{n+1}) \cdot t^n$ the Poincaré series of B. Observe that the family $\mathscr{F}$ generates the B-module C/CA_1.

***Theorem 1.3* (Anick).** — *Let $\mathscr{F} = \{Z_i\}_{i\in I}$ be a locally finite graded subset of A_1, and let* C *be a two-sided ideal of* A *generated by the Z_i's; put $\mathrm{B} = \mathrm{A}/\mathrm{C}$. For each i, let $X_{\alpha^{(i)}} := \widehat{Z}_i$ be the highest term of Z_i. If the family $\{X_{\alpha^{(i)}}\}_{i\in I}$ is combinatorially free, then*

(i) C/CA_1 is a free B*-module over the Z_i's, and*

(ii) $P_\mathrm{B}(t) = \big(1 - dt + \sum_{i\in I} t^{n_i}\big)^{-1}$, where $n_i = \omega(Z_i) = \omega(X_{\alpha^{(i)}})$.

Proof. — See [**1**, Theorems 2.6 and 3.2]. □

If C/CA_1 is a free B-module over the Z_i's, we say that the family $\mathscr{F} = \{Z_i\}_{i\in I}$ is *strongly free* (see [**1**]).

Example 1.4. — Take $d = 5$, and the lexicographic ordering $X_1 < X_2 < \cdots < X_5$. Let $a_n \geqslant 1$ be an increasing sequence, $n \geqslant 1$, and consider the family $\mathscr{F} = \{X_5X_3, X_4X_2, X_4X_3, X_5X_2, X_5X_1, X_5X_4^{a_n}X_1, n \geqslant 1\}$. Put $\mathrm{B} = \mathrm{A}/\mathrm{A}\mathscr{F}\mathrm{A}$. Then $\mathscr{F}$ is combinatorially free, and $P_\mathrm{B}(t) = \big(1 - 5t + t^2 \sum_{n\geqslant 1} t^{a_n}\big)^{-1}$.

1.1.3. Pro-p groups of cohomological dimension $\leqslant 2$ and polynomial algebra. — Let us conserve the notations of §1.1.1.
Let F be a free pro-p group on d generators $x_1, \cdots, x_d$. Let $\Lambda_\mathrm{F} := \mathbb{F}_p[\![\mathrm{F}]\!]$ be the complete algebra associated to F. Recall that $\mathbb{F}_p[\![\mathrm{F}]\!]$ is isomorphic to the Magnus algebra $\mathrm{E} = \mathbb{F}_p^{nc}[\![X_1, \cdots, X_d]\!]$; this isomorphism φ is given by $x_i \mapsto X_i + 1$ (see for example [**9**, Chapter 7, §7.6, Theorem 7.16]).
Let us endow E with the filtration and the ordering of §1.1.2. The filtered isomorphism $\varphi : \Lambda_\mathrm{F} \xrightarrow{\simeq} \mathrm{E}$ allows us to endow Λ_F with the valuation ω_F defined as follows: $\omega_\mathrm{F}(z) = \omega(\varphi(z))$. Observe that $\mathrm{E}_1 \simeq \mathrm{I}_\mathrm{F} : \ker(\Lambda_\mathrm{F} \to \mathbb{F}_p)$, that is E_1 is isomorphic to the augmentation ideal of Λ_F.
Take $x \in \mathrm{F}$, $x \neq 1$. Then the degree $\deg(x)$ of x is defined as $\deg(x) := \omega_\mathrm{F}(x-1) = \omega(\varphi(x-1))$. We denote by $\widehat{x}$ the highest term of $\varphi(x-1) \in \mathrm{E}$. Hence $\widehat{x}$ is a monomial.

Example 1.5. — Take $d \geqslant 3$ with the lexicographic ordering $X_1 < X_2 < X_3 < \cdots < X_d$.

(i) The highest term of $[x_1, [x_2^{p^n}, x_3]]$ is $X_3X_2^{p^n}X_1$.

(ii) Given $x, y \in \mathrm{F}$, let us write $f_x(y) = [x, y] \in \mathrm{F}$. Then the highest term of $f_{x_1} \circ f_{x_2}^{\circ n}(x_3)$ is $X_3X_2^nX_1$.

Let G be a pro-p group of p-rank d, and let $1 \to \mathrm{R} \to \mathrm{F} \to \mathrm{G} \to 1$ be a minimal presentation of G by F; this induces a filtered morphism $\theta : \Lambda_\mathrm{F} \to \Lambda_\mathrm{G}$. We now endow Λ_G with the induced valuation ω_G of ω_F as follows: for $z \in \Lambda_\mathrm{G}$, let us define

$$\omega_\mathrm{G}(z) = \max\{\omega_\mathrm{F}(z'), z' \in \Lambda_\mathrm{F}, \theta(z') = z\}.$$

Put $\mathrm{E}_{\mathrm{G},n} = \{z \in \Lambda_\mathrm{G}, \omega_\mathrm{G}(z) \geqslant n\}$, the filtration of Λ_G. Then $\mathrm{Grad}(\Lambda_\mathrm{G}) = \bigoplus_n \mathrm{E}_{\mathrm{G},n}/\mathrm{E}_{\mathrm{G},n+1}$ is the graded algebra of $\mathbb{F}_p[\![\mathrm{G}]\!]$ respecting the quotient gradation with $P_\mathrm{G}(t) = \sum_{n\geqslant 0} \dim \mathrm{E}_{\mathrm{G},n}/\mathrm{E}_{\mathrm{G},n+1} \cdot t^n$ as Poincaré series.

For $n \geqslant 1$, put $\mathrm{F}_n := \{x \in \mathrm{F}, \varphi(x-1) \in \mathrm{E}_n\}$, and $\mathrm{G}_n = \mathrm{F}_n\mathrm{R}/\mathrm{R}$. The sequences (F_n) and (G_n) are the Zassenhaus filtrations of F and G. The filtration $(\mathrm{E}_{\mathrm{G},n})$ corresponds also to the filtration coming from the augmentation ideal of Λ_G (see for example [**12**, Appendice A.3, Théorème 3.5]).

Theorem 1.6. — *Let $\mathscr{F} = \{\rho_i\}_{i\in I}$ be a family of generators $\mathrm{R}/\mathrm{R}^p[\mathrm{R},\mathrm{R}]$. For each $\in I$, let $X_{\alpha^{(i)}} = \{\widehat{\rho}_i\}_{i\in I} \in \mathrm{A}$ be the highest term of ρ_i. If $\{X_{\alpha^{(i)}}\}_{i\in I}$ is combinatorially free, then*

(i) $\mathrm{R}/\mathrm{R}^p[\mathrm{R},\mathrm{R}] \simeq \prod_{i\in I} \Lambda_\mathrm{G}$, *and* $\mathrm{cd}(\mathrm{G}) \leqslant 2$;

(ii) $P_\mathrm{G}(t) = \big(1 - dt + \sum_{i\in I} t^{n_i}\big)^{-1}$, *where* $d = d_p\mathrm{G}$, *and* $n_i = \deg(\rho_i) = \omega(X_{\alpha^{(i)}})$.

Proof. — When the set of indexes I is finite, this version can be found in [**4**]. We show here that the result also holds when I is infinite. First, observe that as $\{X_{\alpha^{(i)}}\}_{i\in I}$ is combinatorially free then I is countable infinite.

For $i \in I$, put $Y_i = \varphi(\rho_i - 1) \in \mathrm{E}_1$; $n_i = \omega(Y_i)$. Let $\mathrm{I}(\mathrm{R}) \subset \mathrm{E}_1$ be the closed two-sided ideal of E_1 topologically generated by the Y_i's, $i \in I$; one has $\ker(\theta) \simeq \mathrm{I}(\mathrm{R})$ (see for example [**9**, Chapter 7, §7.6, Theorem 7.17]). Let us recall now the topological G-isomorphism between $\mathrm{R}/\mathrm{R}^p[\mathrm{R},\mathrm{R}]$ and $\mathrm{I}(\mathrm{R})/\mathrm{I}(\mathrm{R})\mathrm{E}_1$ (see for example [**4**, Proposition 4.3]). We want to some informations on the G-module $\mathrm{R}/\mathrm{R}^p[\mathrm{R},\mathrm{R}]$, and then on $\mathrm{I}(\mathrm{R})/\mathrm{I}(\mathrm{R})\mathrm{E}_1$.

For $i \in I$, let $Z_i \in \mathrm{A}$ be the initial form of $Y_i \in \mathrm{E}_1$ defined as follows: let us write $Y_i = Z_{i,n_i} + Z_{i,n_i+1} + \cdots$, where $n_i = \omega(Y_i)$ and where $Z_{i,j}$ are homogeneous polynomial of degree j (eventually $Z_{i,j} = 0$); then put $Z_i = Z_{i,n_i}$. Observe that $\widehat{\rho}_i = \widehat{Y}_i = \widehat{Z}_i$.

Let C be the closed ideal of $\mathrm{A} = \mathbb{F}_p^{nc}[X_1, \cdots, X_d]$ generated by the family $\{Z_i\}_{i\in I}$. As the family $\{\widehat{\rho}_i\}_{i\in I}$ is combinatorially free then by Theorem 1.3 the family $\{Z_i\}_{i\in I}$ is strongly free. Put $\mathrm{B} = \mathrm{A}/\mathrm{C}$.

Proposition 1.7. — *One has* $\mathrm{C} = \mathrm{Grad}(\mathrm{I}(\mathrm{R})) \subset \mathrm{A}$. *In particular, as graded* A*-modules, one gets* $\mathrm{Grad}(\Lambda_\mathrm{G}) \simeq \mathrm{B}$, *and*

$$\mathrm{Grad}(\mathrm{I}(\mathrm{R})/\mathrm{I}(\mathrm{R})\mathrm{E}_1) \simeq \mathrm{C}/\mathrm{C}\mathrm{A}_1 \simeq \bigoplus_{i\in I} \mathrm{B}Z_i \simeq \bigoplus_{i\in I} \mathrm{B}[n_i],$$

where $\mathrm{B}[n_i]$ *means* B *as* A*-module with an* n_i*-shift filtration.*

Proof. — This is only a slightly generalization of the case I finite; see proof of [**4**, Theorem 3.7]. □

Then by Theorem 1.3 and Proposition 1.7 we firstly get

$$P_\mathrm{G}(t) = P_\mathrm{B}(t) = \big(1 - dt + \sum_{i\in I} t^{n_i}\big)^{-1}.$$

Consider now the continuous morphism

$$\Psi : \prod_{i\in I} \Lambda_\mathrm{G} \to \mathrm{I}(\mathrm{R})/\mathrm{I}(\mathrm{R})\mathrm{E}_1 \simeq \mathrm{R}/\mathrm{R}^p[\mathrm{R},\mathrm{R}],$$

sendind (a_i) to $\sum_i a_i Y_i \pmod{\mathrm{I(R)E_1}}$; as $n_i \to \infty$ with i, it is well-defined. Remember that $\Lambda_{\mathrm{G}} \simeq \mathrm{E/I(R)}$. Put $\mathrm{N} = \ker(\Psi)$.

Lemma 1.8. — *The map Ψ is surjective.*

Proof. — Put $\mathrm{W} = \{\sum_{i\in I} a_i Y_i, a_i \in \mathrm{E}\} \subset \mathrm{I(R)}$. Then

$$\begin{aligned}\mathrm{I(R)} &= \mathrm{WE}\\ &= \mathrm{W}\mathbb{F}_p + \mathrm{EWE_1} = \mathrm{W} + \mathrm{WE_1}.\end{aligned}$$

We conclude by observing that $\mathrm{WE_1} \subset \mathrm{I(R)E_1}$. □

Therefore one gets a sequence of filtered G-modules:

$$1 \to \mathrm{N} \to \prod_{i\in I} \Lambda_{\mathrm{G}}[n_i] \xrightarrow{\Psi} \mathrm{I(R)/I(R)E_1} \to 1.$$

This one induces the following sequence of graded A-modules:

$$0 \to \mathrm{Grad(N)} \to \mathrm{Grad}(\prod_{i\in I} \Lambda_{\mathrm{G}}[n_i]) \to \mathrm{Grad(I(R)/I(R)E_1)} \to 0.$$

For the surjectivity, use the fact that I is countable. Now as $n_i \to \infty$ with i, then

$$\mathrm{Grad}\big(\prod_{i\in I} \Lambda_{\mathrm{G}}[n_i]\big) = \mathrm{Grad}\big(\bigoplus_{i\in I} \Lambda_{\mathrm{G}}[n_i]\big) \simeq \bigoplus_{i\in I} \mathrm{B}[n_i].$$

By Proposition 1.7, we finally get that Ψ induces an isomorphism between $\mathrm{Grad}\big(\prod_{i\in I} \Lambda_{\mathrm{G}}[n_i]\big)$ and $\mathrm{Grad}\big(\mathrm{I(R)/I(R)E_1}\big)$, which implies $\mathrm{Grad(N)} = 0$, then $\mathrm{N} = 0$. Hence, as G-modules, $\prod_{i\in I} \Lambda_{\mathrm{G}} \simeq \mathrm{I(R)/I(R)E_1} \simeq \mathrm{R/R^p[R,R]}$, and we conclude with Theorem 1.1. □

Remark 1.9. — Conclusions of Theorem 1.6 also hold if $\{\widehat{\rho}_i\}_{i\in I}$ is strongly free.

Remark 1.10. — For references on graded and filtered modules, see also [**12**, Chapter I and II].

1.1.4. Cup-products and cohomological dimension. — Here we suppose now $p > 2$. Let G be a pro-p group of p-rank d which is not pro-p free. Recall that the cup product sends $H^1(\mathrm{G}) \otimes H^1(\mathrm{G})$ to $H^2(\mathrm{G})$. Labute in [**10**] gave a criterion involving cup-products so that $\mathrm{cd(G)} = 2$. This point of view has been developped by Forré in [**4**]. Let us recall it.

***Theorem 1.11* (Forré)**. — *Let $p > 2$ be a prime number. Let* G *be a finitely presented pro-p group which is not pro-p free. Suppose that $H^1(\mathrm{G}) = U \oplus V$ such that $U \cup U = 0$ and $U \cup V = H^2(\mathrm{G})$. Put $c = \dim V$. Then $\mathrm{cd(G)} = 2$, and* G *can be described by some relations $\rho_1, \cdots, \rho_r$ such that the highest term of each ρ_i can be written as $X_{t(i)}X_{s(i)}$ for some $s(i), t(i)$ such that $s(i) \leqslant c < t(i)$, and such that $(s(i), t(i)) \neq (s(j), t(j))$ for $i \neq j$.*

Proof. — See the proof of [**4**, Theorem 6.4, Corollary 6.6] with the choice of the ordering $X_1 < X_2 < \cdots < X_d$. □

Remark 1.12. — Observe that the family $\{X_{s(i)}X_{t(i)}\}_i$ of Theorem 1.11 is combinatorially free.

Before to present a corollary, let us make the following observation: given $n \geqslant 1$, thanks to Example 1.5, one may find some $x \in \mathrm{F}$ such that the highest term of x is like $X_k X_j^n X_i$ for $i < j < k$.

Corollary 1.13. — *Consider the situation of Theorem 1.11. Suppose $c \geqslant 2$. For some fixed $1 < i_0 \leqslant c < j_0 \leqslant d$, and $n \geqslant 1$, let $x_n \in \mathrm{F}$ of highest term $X_{j_0} X_{i_0}^n X_1$. Suppose moreover that $r < (d-c)(c-1)$. Then there exists (i_0, j_0) such that the family $\{\widehat{\rho_1}, \cdots, \widehat{\rho_r}, \widehat{x_n}, n \geqslant 1\}$ is combinatorially free. In particular for such (i_0, j_0):*

- *(i) the group quotient $\Gamma := \mathrm{F}/\langle \rho, \cdots, \rho_r, x_n, n \in \mathbb{Z}_{>0}\rangle^{\mathrm{Nor}}$ of G is of cohomological dimension 2;*
- *(ii) $\dim H^2(\Gamma, \mathbb{F}_p) = \infty$;*
- *(iii) The Poincaré series of Λ_Γ is $\big(1 - dt + rt^2 + t^3 \sum_{n \geqslant 0} t^n\big)^{-1}$.*

Proof. — Thanks to Theorem 1.11, for $i = 1, \cdots, r$, the highest term of ρ_i is of the form $X_{t(i)} X_{s(i)}$ for some $s(i) \leqslant c < t(i)$, and the family $\mathscr{E} := \{X_{t(1)} X_{s(1)}, \cdots, X_{t(r)} X_{s(r)}\}$ is combinatorially free. Now, as $r < (d-c)(c-1)$ and $c \geqslant 2$, we can find (i_0, j_0) such that $X_{j_0} X_{i_0}$ is not in $\mathscr{E}$, and then $\mathscr{E} \cup \{X_{j_0} X_{i_0}^n X_1, n \in \mathbb{Z}_{>0}\}$ is combinatorially free. Then apply Theorem 1.6. ☐

Remark 1.14. — In fact $r \leqslant (d-c)c - 2$ is sufficient. Indeed, with such condition one has $X_{j_0} X_{i_0} \notin \mathscr{E}$ for some $(i_0, j_0) \neq (1, r)$, $i_0 \leqslant c < j_0 \leqslant r$. Hence, if $i_0 \neq 1$ the family $\mathscr{E} \cup \{X_{j_0} X_{i_0}^n X_1, n \in \mathbb{Z}_{>0}\}$ is combinatorially free. Otherwise $j_0 \neq r$, and take $\mathscr{E} \cup \{X_r X_{j_0}^n X_{i_0}, n \in \mathbb{Z}_{>0}\}$.

1.2. Arithmetic backgrounds. — Let p be a prime number, and let K be a number field. For $p = 2$, assume K totally imaginary. Let S and T two disjoint finite sets of prime ideals of the ring of integers $\mathscr{O}_\mathrm{K}$ of K. We assume moreover that each $\mathfrak{p} \in S$ is such that $|\mathscr{O}_\mathrm{K}/\mathfrak{p}| \equiv 1 (\mathrm{mod}\ p)$; the set S is called tame. We denote by $\mathrm{Cl}_\mathrm{K}^T(p)$ the p-Sylow of the T-class group of K.
Let $\mathrm{K}_S^T/\mathrm{K}$ be the maximal pro-p extension of K unramified outside S and where each $\mathfrak{p} \in T$ splits completely in K_S/K; put $\mathrm{G}_S^T = \mathrm{Gal}(\mathrm{K}_S^T/\mathrm{K})$. As we recalled it in Introduction, when G_S^T is infinite, the extension $\mathrm{K}_S^T/\mathrm{K}$ is asymptotically good. Recall Shafarevich's formula (see for example [**5**, Chapter I, §, Theorem 4.6]):

$$d_p \mathrm{G}_S^T = |S| - (r_1 + r_2) - 1 - |T| + \delta_{\mathrm{K},p} + d_p \mathrm{V}_S^T/\mathrm{K}^{\times p},$$

where

$$\mathrm{V}_S^T = \{x \in \mathrm{K}^\times,\ x \in \mathrm{K}_\mathfrak{p}^p U_\mathfrak{p}\ \forall x \notin S \cup T,\ x \in \mathrm{K}_\mathfrak{p}^p\ \forall \mathfrak{p} \in S\},$$

and where $\delta_{\mathrm{K},p} = 1$ if K contains μ_p (the p-roots of 1), 0 otherwise. Here as usual, $\mathrm{K}_\mathfrak{p}$ is the completion of K at $\mathfrak{p}$, and $U_\mathfrak{p}$ is the group of the local units at $\mathfrak{p}$. Observe that if there is no p-extension of $\mathrm{K}(\mu_p)$ unramified outside T and p in which each prime of S splits completely, then $\mathrm{V}_S^T/\mathrm{K}^{\times\ p}$ is trivial: this is a Chebotarev condition type.

Schmidt in [**15**] showed that G_S^T may be *mild* following the terminology of Labute [**10**]. More precisely, he proved:

Theorem 1.15 **(Schmidt)**. — *Let K be a number field and let p be a prime number. For $p = 2$ suppose K totally imaginary. Let S_0 and T two disjoint finite sets of prime ideals of K with S_0 tame. Assume T sufficiently large such that $\mathrm{Cl}_\mathrm{K}^T(p)$ is trivial; when $\mu_\mathfrak{p} \subset \mathrm{K}$, assume moreover that T contains all prime ideals above p. Then there exist* *infinitely finite tame sets S containing S_0 such that $H^1(\mathrm{G}_S^T) = U \oplus V$ where the two subspaces U and V satisfy: (i) $U \cup U = 0$; (ii) $U \cup V = H^2(\mathrm{G}_S^T)$. Moreover, for such S and T one has* $\dim H^2(\mathrm{G}_S^T) = \dim H^1(\mathrm{G}_S^T) + r_1 + r_2 + |T| - 1$.

Theorem 1.15 is not presented in this form in [**15**], here we give the form we need: the result presented here can be found in the proof of Theorem 6.1 of [**15**].

At this level, let us compute the value of $c = \dim V$ of Theorem 1.15, following [**15**].
When $\mu_p \not\subset \mathrm{K}$ let us choose first a finite set S_0 of prime ideals of K, tame and disjoint from T, such that for every $\mathfrak{p} \in S_0$, one has

$$d_p \mathrm{G}_{S_0\backslash\{\mathfrak{p}\}}^T = |S_0| - r_1 - r_2 - |T| + \delta_{\mathrm{K},p},$$

which is equivalent by Shafarevich's formula to the triviality of $\mathrm{V}_{S_0\backslash\{\mathfrak{p}\}}^T/\mathrm{K}^{\times p}$.
When $\mu_p \subset \mathrm{K}$ let us choose S_0, finite, tame and disjoint from T, such that the set of the Frobenius at $\mathfrak{p}$ in G_T^{p-el} when $\mathfrak{p}$ varies in S_0, corresponds to the nontrivial elements of G_T^{p-el}, where G_T^{p-el} is the Galois group of the p-elementary abelian extension $\mathrm{K}_T^{p-el}/\mathrm{K}$ of K_T/K. Here one has also the triviality of $\mathrm{V}_{S_0\backslash\{\mathfrak{p}\}}^T/\mathrm{K}^{\times p}$.
The set S of Theorem 1.15 contains S_0, and is of size $2|S_0|$; the prime ideals $\mathfrak{p} \in S - S_0$ are choosen by respecting some global conditions, thanks to Chebotarev density theorem. Moreover $U = H^1(\mathrm{G}_{S_0}^T, \mathbb{F}_p)$, and the subspace V is such that $\dim V = c = |S_0|$. See [**15**, Proof of Theorem 6.1] for more details.
Now observe the following:

Lemma 1.16. — *Above the previous conditions, each prime $\mathfrak{p} \in S$ is ramified in the p-elementary abelian extension $\mathrm{K}_S^{T,p-el}/\mathrm{K}$ of $\mathrm{K}_S^T/\mathrm{K}$.*

Proof. — Observe first that if $S'' \subset S'$, then $\mathrm{V}_{S'}^T/\mathrm{K}^{\times p} \hookrightarrow \mathrm{V}_{S''}^T/\mathrm{K}^{\times p}$.
Hence thanks to the choice of S_0, it is not difficult to see the following: for every $\mathfrak{p} \in S$, $\mathrm{V}_{S\backslash\{\mathfrak{p}\}}^T/\mathrm{K}^{\times p}$ is trivial. Then by Shafarevich's formula, we get that

$$d_p \mathrm{G}_S^T = 1 + d_p \mathrm{G}_{S\backslash\{\mathfrak{p}\}}^T,$$

showing that $\mathfrak{p}$ is ramified in $\mathrm{K}_S^{T,p-el}/\mathrm{K}$. □

Put $\alpha_{\mathrm{K},T} = 3 + 2\sqrt{2 + r_1 + r_2 + |T|}$. In Theorem 1.15 one may take S sufficiently large so that $d = \dim H^1(\mathrm{G}_S^T, \mathbb{F}_p) > \alpha_{\mathrm{K},T}$.

Lemma 1.17. — *If $d > \alpha_{\mathrm{K},T}$, then $d + r_1 + r_2 + |T| - 1 < (d - c)(c - 1)$ for every $c \in [2, d]$.*

Proof. — Easy computation. □

Let us finish this part with an obvious observation thanks to class field theory.

Remark 1.18. — If G_S^T is not trivial and of cohomological dimension at most 2, then $\mathrm{cd}(\mathrm{G}_S^T) = 2$.

2. Example and proof

2.1. Example. — • Take $p > 2$, and $\mathrm{K} = \mathbb{Q}$. In this case the relations of the pro-p groups G_S are all local, and then not difficult to describe: this is the description due to Koch [**9**, Chapter 11, §11.4, Example 11.11].

Let ℓ be a prime number such that $p | \ell - 1$. Denote by $\mathbb{Q}_\ell$ the (unique) cyclic degree p-extension of $\mathbb{Q}$ unramified outside ℓ; the extension $\mathbb{Q}_\ell/\mathbb{Q}$ is totally ramified at ℓ.

Let $S = \{\ell_1, \cdots, \ell_d\}$ be d different prime numbers such that p divides each $\ell_i - 1$. The pro-p group G_S can be described by $x_1, \cdots, x_d$ generators, and $\rho_1, \cdots, \rho_d$ relations verifying:

$$\rho_i = \prod_{j \neq i} [x_i, x_j]^{a_j(i)} \bmod \mathrm{F}_3, \tag{1}$$

where $a_j(i) \in \mathbb{Z}/p\mathbb{Z}$; moreover the element x_i can be chosen such that it is a generator of the inertia group of ℓ_i. The element $a_j(i)$ is zero if and only the prime ℓ_i splits in $\mathbb{Q}_{\ell_j}/\mathbb{Q}$, which is equivalent to

$$\ell_i^{(\ell_j - 1)/p} \equiv 1 (\mathrm{mod}\ \ell_j).$$

• Typically take $p = 3$, and $S_0 = \{7, 13\}$, $T = \emptyset$. Then put $S = \{p_1, p_2, p_3, p_4, p_5\}$ with $p_1 = 31, p_2 = 19, p_3 = 13, p_4 = 337, p_5 = 7$. Then the highest terms of the relations (1), viewed in $\mathbb{F}_p^{nc}[X_1, \cdots, X_5]$, are $\widehat{\rho_1} = X_1X_3$, $\widehat{\rho_2} = X_2X_4$, $\widehat{\rho_3} = X_2X_3$, $\widehat{\rho_4} = X_1X_4$, $\widehat{\rho_5} = X_1X_5$. Hence as the $\widehat{\rho_i}$'s are combinatorially free, then G_S is of cohomological dimension 2 by Theorem 1.6.

Now for each $n \in \mathbb{Z}_{>0}$, let us choose a prime number p_n of $\mathbb{Z}$ such that the highest term in $\mathbb{F}_p^{nc}[\![X_1, \cdots, X_5]\!]$ of its Frobenius $\sigma_n \in \mathrm{G}_S$ is like $X_5X_4^nX_1$ (which is possible by Example 1.5 or Corollary 1.13, see next section). Then consider the maximal Galois subextension $\mathrm{L}/\mathbb{Q}$ of $\mathbb{Q}_S/\mathbb{Q}$ fixed by all the conjuguates of the τ_n's (this is the "cutting towers" strategy of [**7**]). Put $\mathrm{G} = \mathrm{Gal}(\mathrm{L}/\mathbb{Q})$. Then the pro-3 group G can be described by the generators $x_1, \cdots, x_5$, and the relations $\{\rho_1, \cdots, \rho_5, \tau_n, n \in \mathbb{Z}_{>0}\}$ (which is not *a priori* a minimal set). By construction all the p_n split totally in $\mathrm{L}/\mathbb{Q}$. Observe now that

$$\{\widehat{\rho_1}, \cdots, \widehat{\rho_5}, \widehat{\tau_n}, n \geqslant 1\} = \{X_5X_1, X_5X_2, X_4X_3, X_4X_2, X_5X_3, X_5X_4^nX_1, n \in \mathbb{Z}_{>0}\},$$

which is combinatorially free. By Theorem 1.6 the pro-3-group G is of cohomological dimension 2, $H^2(\mathrm{G})$ is infinite, and $\mathbb{F}_3[\![\mathrm{G}]\!]$ has $\left(1 - 5t + 5t^2 + t^3(1 + t + t^2 + \cdots)\right)^{-1}$ as Poincaré series.

2.2. Proof of the main result. — • Let $p > 2$ be a prime number, and let K be a number field. Let S_0 and T two finite disjoint sets of prime ideals of K, where S_0 is tame. Take T sufficiently large such that $\mathrm{Cl}_{\mathrm{K}}^T(p)$ is trivial. When K contains μ_p, assume moreover that T contains all p-adic prime ideals.

First take S containing S_0 as in Theorem 1.15, and sufficiently large such that $d > \alpha_{\mathrm{K},T}$. Put $\mathrm{G} = \mathrm{G}_S^T$. Here $r = \dim H^2(\mathrm{G}) = d + r_1 + r_2 - 1 + |T|$.

Let us start with a minimal presentation of G:

$$1 \longrightarrow \mathrm{R} \longrightarrow \mathrm{F} \xrightarrow{\varphi} \mathrm{G} \longrightarrow 1.$$

By Theorem 1.15 and Theorem 1.11 the quotient $\mathrm{R}/\mathrm{R}^p[\mathrm{F}, \mathrm{R}]$ may be generated as $\mathbb{F}_p$-vector spaces by some relations $\rho_1, \cdots, \rho_r$ such that the highest terms $\widehat{\rho_k}$ are like X_iX_j for some $i \leqslant c < j$, where $c = \dim V$. Observe that as G is FAb then $c \in [2, d-2]$.

Given $n \geqslant 1$, then the quotient $\mathrm{G}/\mathrm{G}_{n+1}$ is finite. Put $\mathrm{K}_{(n+1)} = (\mathrm{K}_S^T)^{\mathrm{G}_{n+1}}$. Let $a_n \in \mathbb{Z}_{>0}$ be an increasing sequence. Let $x_n \in \mathrm{F}_{a_n} \backslash \mathrm{F}_{a_n+1}$. By Chebotarev density theorem there exists some prime ideal $\mathfrak{p}_n \subset \mathscr{O}_\mathrm{K}$ such that $\sigma_{\mathfrak{p}_n}$ is conjuguate to x_n in $\mathrm{Gal}(\mathrm{K}_{(a_n+1)}/\mathrm{K})$. Here $\sigma_{\mathfrak{p}_n} \in \mathrm{G}$ denotes the Frobenius of $\mathfrak{p}_n$ in $\mathrm{K}_S^T/\mathrm{K}$. Now take $z_n \in \mathrm{F}$ such that $\varphi(z_n) = \sigma_{\mathfrak{p}_n}$. Hence

$$z_n \equiv \sigma_{\mathfrak{p}_n} \pmod{\mathrm{RF}_{a_n+1}}.$$

In other words, there exists $y_n \in \mathrm{F}_{a_n+1}$ and $r_n \in \mathrm{R}$ such that $z_n = \sigma_{\mathfrak{p}_n} y_n r_n$.

Let $\Sigma = T \cup \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots\}$, and consider K_S^Σ the maximal pro-p extension of K unramified oustide S and where each primes $\mathfrak{p}_i$ of Σ splits completely. Put $\mathrm{G}_S^\Sigma = \mathrm{Gal}(\mathrm{K}_S^\Sigma/\mathrm{K})$. Then

$$\mathrm{G}_S^\Sigma \simeq \mathrm{G}/\langle \sigma_{\mathfrak{p}_n}, n \in \mathbb{Z}_{>0} \rangle^{\mathrm{Nor}}.$$

Here $\langle \sigma_{\mathfrak{p}_n}, n \in \mathbb{Z}_{>0} \rangle^{\mathrm{Nor}}$ is the normal closure of $\langle \sigma_{\mathfrak{p}_n}, n \in \mathbb{Z}_{>0} \rangle$ in G_S^Σ. Hence K_S^Σ satisfies (i) and (ii) of Theorem A. But observe now that

$$\mathrm{G}/\langle \sigma_{\mathfrak{p}_n}, n \in \mathbb{Z}_{>0} \rangle^{\mathrm{Nor}} \simeq \mathrm{F}/\langle \rho_1, \cdots, \rho_r, z_n, n \in \mathbb{Z}_{>0} \rangle^{\mathrm{Nor}} = \mathrm{F}/\langle \rho_1, \cdots, \rho_r, x_n y_n, n \in \mathbb{Z}_{>0} \rangle^{\mathrm{Nor}},$$

as $\sigma_{\mathfrak{p}_n}$ and x_n are conjugate.

Since the highest term of each $x_n y_n$ in $\mathrm{E} = \mathbb{F}_p^{nc}[\![X_1, \cdots, X_d]\!]$ is the same as the highest term of x_n, it suffices to choose the x_n's as in Corollary 1.13 which is possible: indeed as $d > \alpha_{\mathrm{K},T}$ then by Lemma 1.17 $r < (c-1)(d-c)$, for every $c \in [1, d-1]$. Thanks to Corollary 1.13, one gets (iii), (iv), and (vi) of Theorem A.

(v): by Lemma 1.16 each prime ideal $\mathfrak{p} \in S$ is ramified in $\mathrm{K}_S^{T,p-el}/\mathrm{K}$, showing that $\tau_\mathfrak{p} \in \mathrm{G}$ is not in $\mathrm{RF}^p[\mathrm{F},\mathrm{F}]$, where $\tau_\mathfrak{p}$ is a generator of the inertia group at $\mathfrak{p}$ in G. As the p-rank of G_S^Σ is the same as the p-rank of G, each prime $\mathfrak{p} \in S$ is ramified in K_S^Σ. But as G is without torsion (because $\mathrm{cd}(\mathrm{G}) = 2$), necessarly $\langle \tau_\mathfrak{p} \rangle \simeq \mathbb{Z}_p$, and the structure of local extensions forces $(\mathrm{K}_S^\Sigma)_\mathfrak{p}/\mathrm{K}_\mathfrak{p}$ to be maximal.

• Assume $p = 2$, and K be totally imaginary. Then Theorem 1.15 holds, but Theorem 1.11 does not. As explained by Forré in [**4**, Proof Theorem 6.4], one has to take two orderings to show that the highest terms of the relations $\rho_1, \cdots, \rho_r$ are strongly free. Now in this context the strategy of the approximation of elements x_n by some Frobenius as in Corollary 1.13 also applies. Then by following the proof of Theorem 6.4 in [**4**], and by choosing the x_n's as in the case $p \neq 2$, we observe that the initial forms of the new relations $\{\rho_1, \cdots, \rho_r, x_n, n \geqslant 1\}$ are still strongly free. We conclude by using Remark 1.9 of Theorem 1.6.

□

References

[1] D. J. Anick, *Non-commutative graded algebras and their Hilbert series*, J. of Algebra **78** n1 (1982), 120-140.

[2] D. J. Anick, *Inert sets and the Lie algebra associated to a group*, J. Algebra **111** (1987), 154-165.

[3] A. Brumer, *Pseudocompact algebras, profinite groups and class formations*, J. of Algebra **4** (1966), 442-470

[4] P. Forré, *Strongly free sequences and pro-p-groups of cohomological dimension* 2, J. reine u. angew. Math **658** (2011), 173-192.

[5] G. Gras, Class Field Theory, From Theory to practice, corr. 2nd ed., Springer Monographs in Mathematics, Springer (2005), xiii+507 pages.

[6] F. Hajir, C. Maire, *Tamely ramified towers and discriminant bounds for number fields*, Compositio Math. **128** (2001), 35-53.

[7] F. Hajir, C. Maire, R. Ramakrishna, *Cutting towers of number fields*, 2019, arXiv:1901.04354.

[8] Y. Ihara, *How many primes decompose completely in an infinite unramified Galois extension of a global field ?*, J. Math. Soc. Japon **35** (1983), no4, 693-709.

[9] H. Koch, Galois Theory of p-Extensions, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002.

[10] J. Labute, *Mild pro-p-groups and Galois groups of p-extensions of* $\mathbb{Q}$, J. reine u. angew. Math. **596** (2006), 155–182.

[11] J. Labute, J. Mináč, *Mild pro-2-groups and 2-extensions of* $\mathbb{Q}$ *with restricted ramification*, J. Algebra **332** (2011), 136–158.

[12] M. Lazard, *Groupes analytiques p-adiques*, IHES Publ. Math. **26** (1965), 389-603.

[13] P. Lebacque, *Quelques résultats effectifs concernant les invariants de Tsfasman-Vladut*, Ann. Inst. Fourier **65** (2015), no. 1, 63–99.

[14] J. Neukirch, A. Schmidt and K. Wingberg, Cohomology of Number Fields, GMW 323, Springer-Verlag Berlin Heidelberg, 2000.

[15] A. Schmidt, *Über Pro-p-Fundamentalgruppen markierter arithmetischer Kurven*, J. reine u. angew. Math. **640** (2010), 203-235.

[16] M. Tsfasman and S. Vladut, *Infinite global fields and the generalized Brauer-Siegel theorem.* Dedicated to Yuri I. Manin on the occasion of his 65th birthday, Mosc. Math. J. **2** (2002), no 2, 329-402.

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