

A note on harmonious sets

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Abstract

A flaw in *Algebraic numbers and harmonic analysis*, Elsevier (1972), is corrected.

1 A wrong lemma is revisited

If Λ is a harmonious set and if F is a finite set, then $F \cup \Lambda$ is still harmonious. This is Theorem II, page 45 of [3]. Unfortunately the proof given in [3] is wrong. This wrong proof was based on Lemma 5, page 45. But Lemma 5 in Chapter 2 of [3] is doubtful. A correct proof of Theorem II is given in this note. It bridges the gap with some remarkable results by Nikolai Bogolyubov and Erling Følner [1], [2].

The following notations will be used. A locally compact abelian (l.c.a.) group is denoted by G . The given topology on G is \mathcal{T}_0 . The Bohr compactification of G is \tilde{G} . Then G is a dense subgroup of \tilde{G} and the Bohr almost periodic functions on G are the restriction to G of the continuous functions on \tilde{G} . The topology on G which is induced by the topology of \tilde{G} is denoted by \mathcal{T} . This topology \mathcal{T} is the weakest topology on G for which the Bohr almost periodic functions are continuous. In particular \mathcal{T} is weaker than \mathcal{T}_0 . If E and F are two subsets of G , $E - F$ denotes the set of all differences $x - y$, $x \in E, y \in F$. Similarly for $E + F$. A subset E of G is closed for the topology \mathcal{T} if and only if $E = G \cap K$ where $K \subset \tilde{G}$ is a compact set. If it is the case we have $E = G \cap K_0$ where K_0 is the closure of E in \tilde{G} . If $V \subset G$ is a compact set for the topology \mathcal{T}_0 then $E + V$ is closed for the topology \mathcal{T} and $E + V = G \cap (K_0 + V)$. If E is closed for the topology \mathcal{T} this is not necessarily the case for $E - E$. In particular even if E is closed for the topology \mathcal{T} and if K_0 is the closure of E in \tilde{G} , $K_0 - K_0$ is in general larger than the closure of $E - E$ in \tilde{G} .

If $0 \leq \epsilon < 2$ the ϵ -dual of a set $\Lambda \subset \mathbb{R}^n$ is the set $\Lambda_\epsilon^* \subset \mathbb{R}^n$ defined by

$$\Lambda_\epsilon^* = \{x; |\exp(2\pi i x \cdot y) - 1| \leq \epsilon, \forall y \in \Lambda\}. \quad (1)$$

This definition extends naturally to the general case of a l.c.a. group Γ . Following Lev Pontryagin the dual group of G is the multiplicative group $\Gamma = G^*$ consisting of all continuous characters on G . A character χ on G is a homomorphism $\chi : G \mapsto \mathbb{T}$ where \mathbb{T} is the multiplicative group $\{z; |z| = 1\}$. Then the dual group of Γ is G . If $0 \leq \epsilon < 2$ the ϵ -dual of a set $\Lambda \subset \Gamma$ is the set $\Lambda_\epsilon^* \subset G$ defined by

$$\Lambda_\epsilon^* = \{\chi \in G; |\chi(y) - 1| \leq \epsilon, \forall y \in \Lambda\}. \quad (2)$$

We have $\Lambda_\epsilon^* = -\Lambda_\epsilon^*$, Λ_ϵ^* is closed in G for the topology \mathcal{T} , and

$$\Lambda_\epsilon^* \pm \Lambda_\epsilon^* \subset \Lambda_{2\epsilon}^*. \quad (3)$$

A Bohr set is defined by $B(F, \epsilon) = F_\epsilon^*$ where $F \subset \Gamma$ is finite and $\epsilon > 0$. Bohr sets are fundamental neighborhoods of 0 for the topology \mathcal{T} . A set $E \subset G$ is relatively dense in G if there exists a compact set K such that $K + E = G$. A set $\Lambda \subset \Gamma$ is harmonious if for any positive ϵ the set $\Lambda_\epsilon^* \subset G$ is relatively dense in G . A harmonious set is uniformly discrete [3]: there is a neighborhood V of 0 such that for $\lambda \in \Lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$ we have $(\lambda + V) \cap (\lambda' + V) = \emptyset$. Let H the additive subgroup of Γ generated by Λ . Then Λ is harmonious if for any homomorphism $\chi : H \mapsto \mathbb{T}$ and any positive ϵ there exists a continuous character ξ on Γ such that $\sup_{x \in \Lambda} |\chi(x) - \xi(x)| \leq \epsilon$.

Our goal is to prove the following theorem:

Theorem 1.1. *If Γ is a l.c.a. group, if $\Lambda \subset \Gamma$ is harmonious and if $F \subset \Gamma$ is finite, then $\Lambda \cup F$ is still harmonious.*

Two proofs of Theorem 1.1 are given. They rely on the additive properties of relatively dense sets with respect to the topology \mathcal{T} .

Lemma 1.1. *Let $M \subset G$ be a relatively dense subset which is closed for the topology \mathcal{T} . Let $\Omega \subset \tilde{G}$ be a neighborhood of 0 for the topology \mathcal{T} . Then $\Lambda = (M - M) \cap \Omega$ is relatively dense in G .*

In Lemma 5 of [3] the assumption that M is closed for the topology \mathcal{T} was forgotten. Before proving Lemma 1.1 let us give an important corollary.

Theorem 1.2. *Let $\Lambda \subset \Gamma$ be a discrete set of points. Let us assume that for any positive ϵ there exists a finite subset F_ϵ of Λ such that the ϵ -dual M_ϵ of $\Lambda \setminus F_\epsilon$ is relatively dense. Then Λ is harmonious.*

We first prove Theorem 1.2. It suffices to show that for any positive ϵ the 2ϵ -dual $\Lambda_{2\epsilon}^*$ of Λ is relatively dense. Let η be a positive real number. The η -dual of $\Lambda \setminus F_\epsilon$ is denoted by $Q_{(\eta,\epsilon)}$ and the 2ϵ -dual of F_ϵ is denoted by Ω_ϵ . On one hand Ω_ϵ is a neighborhood of 0 for the topology \mathcal{T} . On the other hand $Q_{(\eta,\epsilon)}$ is closed for the topology \mathcal{T} and $Q_{(\epsilon,\epsilon)} - Q_{(\epsilon,\epsilon)} \subset Q_{(2\epsilon,\epsilon)}$. By assumption $M_\epsilon = Q_{(\epsilon,\epsilon)}$ is relatively dense. We now apply Lemma 1.1 to M_ϵ and conclude that $Q_{(2\epsilon,\epsilon)} \cap \Omega_\epsilon$ is relatively dense. Finally $Q_{(2\epsilon,\epsilon)} \cap \Omega_\epsilon = \Lambda_{2\epsilon}^*$ which ends the proof. Theorem 1.2 obviously implies Theorem 1.1.

If G is a locally compact abelian group a subset $M \subset G$ is relatively dense with respect to k elements if there exists a finite set F with k elements such that $M + F = G$. If M is relatively dense in the usual sense then for every neighborhood V of 0 in G there exists an integer k such that $M + V$ is relatively dense with respect to k elements. Indeed we know that there exists a compact set K such that $M + K = G$ and there exists a finite set F such that $K \subset F + V$. Then $M + V + F = G$.

Lemma 1.1 is a corollary of Lemma 1.2. The notations used in Lemma 1.1 are kept here.

Lemma 1.2. *If $M \subset G$ is relatively dense in G and is closed for the topology \mathcal{T} then for every $V \subset G$ which is a neighborhood of 0 for \mathcal{T}_0 , $M - M + V$ is a neighborhood of 0 for the topology \mathcal{T} .*

Let us show that Lemma 2.1 implies Lemma 1.1. We begin with two simple observations.

Lemma 1.3. *Any set $V \subset G$ which is open for the topology \mathcal{T} is relatively dense in G*

Lemma 1.4. *If $M \subset G$ and if there exists a compact set K such that $M + K$ is relatively dense, then M is relatively dense.*

We are ready to show that Lemma 2.1 implies Lemma 1.1. Let $M_2 = (M - M) \cap \Omega$ and let us assume that V is a compact neighborhood of 0 for the topology \mathcal{T}_0 . Without losing generality we can assume $V \subset \Omega$. Then $M_2 + V = (M - M + V) \cap \Omega$. By Lemma 2.1 $M - M + V$ is a neighborhood of 0 for the topology \mathcal{T} . Therefore $(M - M + V) \cap \Omega$ is also a neighborhood of 0 for the topology \mathcal{T} . Finally $M_2 + V$ is relatively dense and so is M_2 .

We now prove Lemma 1.2 under the following form:

Lemma 1.5. *If $M \subset G$ is relatively dense in G and is closed for the topology \mathcal{T} then for every $V \subset G$ which is a neighborhood of 0 for \mathcal{T}_0 , there exists a $x_0 \in G$ such that $M + V - x_0$ is a neighborhood of 0 for the topology \mathcal{T} .*

Lemma 1.5 implies Lemma 1.2. Without losing generality it can be assumed that $V = W - W$ where $W \subset G$ is a compact neighborhood of 0 for \mathcal{T}_0 . We know that $\Omega = M + W - x_0$ is a neighborhood of 0. In particular $0 = m + w - x_0$ where $m \in M$ and $w \in W$. Therefore $\Omega = M - m + W - w$ is a neighborhood of 0 for the topology \mathcal{T} which ends the proof of Lemma 1.2. We now prove Lemma 1.5. Without losing generality it can be assumed that $V = W - W$ where $W \subset G$ is a compact neighborhood of 0 for \mathcal{T}_0 . Since M is relatively dense there exists a finite set F such that $M + W + F = G$. Let $K = \overline{M}$ be closure of M in \mathcal{G} . Then $\overline{M + W + F} = K + W + F = \mathcal{G}$. Therefore $K + W + y_0$, one of the compact sets $K + W + y$, $y \in F$, has a non empty interior in \mathcal{G} . Finally there is a Bohr set $B = B(F, \epsilon)$ and a $\chi_0 \in \mathcal{G}$ such that $(B + \chi_0) \subset K + W$. Let $x_0 \in G \cap (B(F, \epsilon/2) + \chi_0)$. We have $(B(F, \epsilon/2) + x_0) \subset (B + \chi_0)$ and $G \cap (B(F, \epsilon/2) + x_0) \subset G \cap (B + \chi_0) \subset M + W$. Finally $M + W - x_0$, $x_0 \in M + W$, contains a Bohr set and Lemma 1.5 is proved.

2 Bogolyobov's approach

The second improvement on Lemma 1.1 was discovered by Nikolai Bogolyubov's work and by Erling Følner [1], [2]. They proved that Lemma 1.1 remains valid when M is any relatively dense set at the expense of replacing a simple difference by an iterated difference. This important improvement is not needed in the proof of Theorem 1.1. Here is Følner's theorem.

Theorem 2.1. *If $M \subset G$ is relatively dense with respect to k elements and if $V \subset G$ is a neighborhood of 0 in G for the topology \mathcal{T}_0 , then the set $M_4 = M - M + M - M + V$ is a neighborhood of 0 for the topology \mathcal{T} .*

As above Følner's theorem implies the following results.

Corollary 2.1. *If $M \subset G$ is relatively dense and if Ω is a neighborhood of 0 for the topology \mathcal{T} , then the set $(M - M + M - M) \cap \Omega$ is relatively dense.*

Ironically my erroneous Lemma 5 of [3] was close to be true.

References

- [1] E. Følner. Generalization of a theorem of Bogoliuboff to topological abelian groups. With an appendix on Banach mean values in non-abelian groups, Math. Scand. 2 (1954), 518.

- [2] E. Følner. Note on a generalization of a theorem of Bogoliuboff, Math. Scand. 2 (1954), 224226.
- [3] Y. Meyer. Algebraic numbers and harmonic analysis. Elsevier (1972)