

A note on harmonious sets

Yves Meyer

▶ To cite this version:

Yves Meyer. A note on harmonious sets. 2020. hal-02489358

HAL Id: hal-02489358 https://hal.science/hal-02489358

Preprint submitted on 24 Feb 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A note on harmonious sets

Yves François Meyer CMLA, ENS-Cachan, CNRS, Université Paris-Saclay, France

Abstract

A flaw in Algebraic numbers and harmonic analysis, Elsevier (1972), is corrected.

1 A wrong lemma is revisited

If Λ is a harmonious set and if F is a finite set, then $F \cup \Lambda$ is still harmonious. This is Theorem II, page 45 of [3]. Unfortunately the proof given in [3] is wrong. This wrong proof was based on Lemma 5, page 45. But Lemma 5 in Chapter 2 of [3] is doubtful. A correct proof of Theorem II is given in this note. It bridges the gap with some remarkable results by Nikolaï Bogolyubov and Erling Følner [1], [2].

The following notations will be used. A locally compact abelian (l.c.a.) group is denoted by G. The given topology on G is \mathcal{T}_0 . The Bohr compactification of Γ is G. Then G is a dense subgroup of G and the Bohr almost periodic functions on G are the restriction to G of the continuous functions on G. The topology on G which is induced by the topology of G is denoted by \mathcal{T} . This topology \mathcal{T} is the weakest topology on G for which the Bohr almost periodic functions are continuous. In particular \mathcal{T} is weaker than \mathcal{T}_0 . If E and F are two subsets of G, E - F denotes the set of all differences $x-y, x \in E, y \in F$. Similarly for E+F. A subset E of G is closed for the topology \mathcal{T} if and only if $E = G \cap K$ where $K \subset \widetilde{G}$ is a compact set. If it is the case we have $E = G \cap K_0$ where K_0 is the closure of E in \widetilde{G} . If $V \subset G$ is a compact set for the topology \mathcal{T}_0 then E + V is closed for the topology \mathcal{T} and $E + V = G \cap (K_0 + V)$. If E is closed for the topology \mathcal{T} this is not necessarily the case for E - E. In particular even if E is closed for the topology \mathcal{T} and if K_0 is the is the closure of E in G, $K_0 - K_0$ is in general larger than the closure of E - E in G.

If $0 \leq \epsilon < 2$ the ϵ -dual of a set $\Lambda \subset \mathbb{R}^n$ is the set $\Lambda^*_{\epsilon} \subset \mathbb{R}^n$ defined by

$$\Lambda_{\epsilon}^* = \{x; |\exp(2\pi i x \cdot y) - 1| \le \epsilon, \ \forall y \in \Lambda\}.$$
(1)

This definition extends naturally to the general case of a l.c.a. group Γ . Following Lev Pontryagin the dual group of G is the multiplicative group $\Gamma = G^*$ consisting of all continuous characters on G. A character χ on G is a homomorphism $\chi : G \mapsto \mathbb{T}$ where \mathbb{T} is the multiplicative group $\{z; |z| = 1\}$. Then the dual group of Γ is G. If $0 \le \epsilon < 2$ the ϵ -dual of a set $\Lambda \subset \Gamma$ is the set $\Lambda_{\epsilon}^* \subset G$ defined by

$$\Lambda_{\epsilon}^* = \{ \chi \in G ; |\chi(y) - 1| \le \epsilon, \ \forall y \in \Lambda \}.$$

$$(2)$$

We have $\Lambda_{\epsilon}^* = -\Lambda_{\epsilon}^*$, Λ_{ϵ}^* is closed in G for the topology \mathcal{T} , and

$$\Lambda_{\epsilon}^* \pm \Lambda_{\epsilon}^* \subset \Lambda_{2\epsilon}^*. \tag{3}$$

A Bohr set is defined by $B(F, \epsilon) = F_{\epsilon}^*$ where $F \subset \Gamma$ is finite and $\epsilon > 0$. Bohr sets are fundamental neighborhoods of 0 for the topology \mathcal{T} . A set $E \subset G$ is relatively dense in G if there exists a compact set K such that K + E = G. A set $\Lambda \subset \Gamma$ is harmonious if for any positive ϵ the set $\Lambda_{\epsilon}^* \subset G$ is relatively dense in G. A harmonious set is uniformly discrete [3]: there is a neighborhood V of 0 such that for $\lambda \in \Lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$ we have $(\lambda + V) \cap (\lambda' + V) = \emptyset$. Let H the additive subgroup of Γ generated by Λ . Then Λ is harmonious if for any homomorphism $\chi : H \mapsto \mathbb{T}$ and any positive ϵ there exists a continuous character ξ on Γ such that $\sup_{x \in \Lambda} |\chi(x) - \xi(x)| \leq \epsilon$.

Our goal is to prove the following theorem:

Theorem 1.1. If Γ is a l.c.a. group, if $\Lambda \subset \Gamma$ is harmonious and if $F \subset \Gamma$ is finite, then $\Lambda \cup F$ is still harmonious.

Two proofs of Theorem 1.1 are given. They rely on the additive properties of relatively dense sets with respect to the topology \mathcal{T} .

Lemma 1.1. Let $M \subset G$ be a relatively dense subset which is closed for the topology \mathcal{T} . Let $\Omega \subset \widetilde{G}$ be a neighborhood of 0 for the topology \mathcal{T} . Then $\Lambda = (M - M) \cap \Omega$ is relatively dense in G.

In Lemma 5 of [3] the assumption that M is closed for the topology \mathcal{T} was forgotten. Before proving Lemma 1.1 let us give an important corollary.

Theorem 1.2. Let $\Lambda \subset \Gamma$ be a discrete set of points. Let us assume that for any positive ϵ there exists a finite subset F_{ϵ} of Λ such that the ϵ -dual M_{ϵ} of $\Lambda \setminus F_{\epsilon}$ is relatively dense. Then Λ is harmonious. We first prove Theorem 1.2. It suffices to show that for any positive ϵ the 2ϵ -dual $\Lambda_{2\epsilon}^*$ of Λ is relatively dense. Let η be a positive real number. The η -dual of $\Lambda \setminus F_{\epsilon}$ is denoted by $Q_{(\eta,\epsilon)}$ and the 2ϵ -dual of F_{ϵ} is denoted by Ω_{ϵ} . On one hand Ω_{ϵ} is a neighborhood of 0 for the topology \mathcal{T} . On the other hand $Q_{(\eta,\epsilon)}$ is closed for the topology \mathcal{T} and $Q_{(\epsilon,\epsilon)} - Q_{(\epsilon,\epsilon)} \subset Q_{(2\epsilon,\epsilon)}$. By assumption $M_{\epsilon} = Q_{(\epsilon,\epsilon)}$ is relatively dense. We now apply Lemma 1.1 to M_{ϵ} and conclude that $Q_{(2\epsilon,\epsilon)} \cap \Omega_{\epsilon}$ is relatively dense. Finally $Q_{(2\epsilon,\epsilon)} \cap \Omega_{\epsilon} = \Lambda_{2\epsilon}^*$ which ends the proof. Theorem 1.2 obviously implies Theorem 1.1.

If G is a locally compact abelian group a subset $M \subset G$ is relatively dense with respect to k elements if there exists a finite set F with k elements such that M + F = G. If M is relatively dense in the usual sense then for every neighborhood V of 0 in G there exists an integer k such that M + Vis relatively dense with respect to k elements. Indeed we know that there exists a compact set K such that M + K = G and there exists a finite set F such that $K \subset F + V$. Then M + V + F = G.

Lemma 1.1 is a corollary of Lemma 1.2. The notations used in Lemma 1.1 are kept here.

Lemma 1.2. If $M \subset G$ is relatively dense in G and is closed for the topology \mathcal{T} then for every $V \subset G$ which is a neighborhood of 0 for \mathcal{T}_0 , M - M + V is a neighborhood of 0 for the topology \mathcal{T} .

Let us show that Lemma 2.1 implies Lemma 1.1. We begin with two simple observations.

Lemma 1.3. Any set $V \subset G$ which is open for the topology \mathcal{T} is relatively dense in G

Lemma 1.4. If $M \subset G$ and if there exists a compact set K such that M + K is relatively dense, then M is relatively dense.

We are ready to show that Lemma 2.1 implies Lemma 1.1. Let $M_2 = (M - M) \cap \Omega$ and let us assume that V is a compact neighborhood of 0 for the topology \mathcal{T}_0 . Without losing generality we can assume $V \subset \Omega$. Then $M_2 + V = (M - M + V) \cap \Omega$. By Lemma 2.1 M - M + V is a neighborhood of 0 for the topology \mathcal{T} . Therefore $(M - M + V) \cap \Omega$ is also a neighborhood of 0 for the topology \mathcal{T} . Finally $M_2 + V$ is relatively dense and so is M_2 .

We now prove Lemma 1.2 under the following form:

Lemma 1.5. If $M \subset G$ is relatively dense in G and is closed for the topology \mathcal{T} then for every $V \subset G$ which is a neighborhood of 0 for \mathcal{T}_0 , there exists a $x_0 \in G$ such that $M + V - x_0$ is a neighborhood of 0 for the topology \mathcal{T} .

Lemma 1.5 implies Lemma 1.2. Without losing generality it can be assumed that V = W - W where $W \subset G$ is a compact neighborhood of 0 for \mathcal{T}_0 . We know that $\Omega = M + W - x_0$ is a neighborhood of 0. In particular $0 = m + w - x_0$ where $m \in M$ and $w \in W$. Therefore $\Omega = M - m + W - w$ is a neighborhood of 0 for the topology \mathcal{T} which ends the proof of Lemma 1.2. We now prove Lemma 1.5. Without losing generality it can be assumed that V = W - W where $W \subset G$ is a compact neighborhood of 0 for \mathcal{T}_0 . Since M is relatively dense there exists a finite set F such that M + W + F = G. Let $K = \overline{M}$ be closure of M in \mathcal{G} . Then $\overline{M + W + F} = K + W + F = \mathcal{G}$. Therefore $K + W + y_0$, one of the compact sets $K + W + y, y \in F$, has a non empty interior in \mathcal{G} . Finally there is a Bohr set $B = B(F, \epsilon)$ and a $\chi_0 \in \mathcal{G}$ such that $(B + \chi_0) \subset K + W$. Let $x_0 \in G \cap (B(F, \epsilon/2) + \chi_0)$. We have $(B(F, \epsilon/2)) + x_0 \subset (B + \chi_0)$ and $G \cap (B(F, \epsilon/2) + x_0) \subset G \cap (B + \chi_0) \subset M + W$. Finally $M + W - x_0, x_0 \in M + W$, contains a Bohr set and Lemma 1.5 is proved.

2 Bogolyobov's approach

The second improvement on Lemma 1.1 was discovered by Nikolaï Bogolyubov's work and by Erling Følner [1], [2]. They proved that Lemma 1.1 remains valid when M is any relatively dense set at the expense of replacing a simple difference by an iterated difference. This important improvement is not needed in the proof of Theorem 1.1. Here is Følner's theorem.

Theorem 2.1. If $M \subset G$ is relatively dense with respect to k elements and if $V \subset G$ is a neighborhood of 0 in G for the topology \mathcal{T}_0 , then the set $M_4 = M - M + M - M + V$ is a neighborhood of 0 for the topology \mathcal{T} .

As above Følner's theorem implies the following results.

Corollary 2.1. If $M \subset G$ is relatively dense and if Ω is a neighborhood of 0 for the topology \mathcal{T} , then the set $(M - M + M - M) \cap \Omega$ is relatively dense.

Ironically my erroneous Lemma 5 of [3] was close to be true.

References

 E. Følner. Generalization of a theorem of Bogoliuboff to topological abelian groups. With an appendix on Banach mean values in non-abelian groups, Math. Scand. 2 (1954), 518.

- [2] E. Følner. Note on a generalization of a theorem of Bogoliuboff, Math. Scand. 2 (1954), 224226.
- [3] Y. Meyer. Algebraic numbers and harmonic analysis. Elsevier (1972)