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Generalization of Rellich-Kondrachov theorem and trace compactness for fractal boundaries

Anna Rozanova-Pierrat*

Abstract

We present a survey of recent results from the functional analysis that allow to solve PDEs in a large class of domains with irregular boundaries. We extend the previously introduced concept of admissible domains with a d -set boundary to domains with boundaries carrying measures that are not necessarily Ahlfors d -regular. We prove generalizations of the Rellich-Kondrachov theorem and the compactness of the trace operator and obtain uniqueness and existence results for weak solutions to Poisson boundary value problems with Robin boundary conditions. We observe the usual properties of the associated spectral problem.

Keywords: fractal boundaries, compact operators, d -set, trace and extension operators, Rellich-Kondrachov theorem.

1 Introduction

From the theory of partial differential equations it is known that the irregularity of the boundary of the considered domain can be a serious obstacle even for the proof of the existence of a weak solution. In this paper we are interested to identify a class of domains with irregular boundaries for which we still have weak well-posedness for elliptic problems. Here we consider a specific problem, namely the Poisson equation with homogeneous Robin boundary conditions,

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{with } \alpha > 0 \text{ on } \partial\Omega. \end{cases} \quad (1)$$

To provide a rigorous weak formulation of this problem it is important to be able to integrate by parts and to have a well-defined trace operator for $\partial\Omega$. For smooth or Lipschitz boundaries $\partial\Omega$ this is classical and well-known (for sufficiently smooth boundary see Raviart-Thomas [36], for the Lipschitz case see Marschall [33] and [16, 35]). If $\partial\Omega$ is Lipschitz, then the normal unit vector ν to the boundary $\partial\Omega$ exists almost everywhere, the trace operator $\text{Tr} : W^{1,2}(\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ is linear, continuous and surjective, [31, 33, 16, 35], and has a linear right continuous inverse, *i.e.* an extension operator $E : H^{\frac{1}{2}}(\partial\Omega) \rightarrow W^{1,2}(\Omega)$ is such that $\text{Tr}(E(u)) = u$.

Moreover, for $u, v \in W^{1,2}(\Omega)$ with $\Delta u \in L^2(\Omega)$ the usual Green formula holds in the sense that

$$\int_{\Omega} \nabla u \nabla v dx = \left\langle \frac{\partial u}{\partial \nu}, \text{Tr} v \right\rangle_{(H^{\frac{1}{2}}(\partial\Omega))', H^{\frac{1}{2}}(\partial\Omega)} - \int_{\Omega} \nabla v \nabla u dx. \quad (2)$$

This formula understands the existence of the normal derivative of u on $\partial\Omega$ as the existence of a linear continuous form on $H^{\frac{1}{2}}(\partial\Omega)$, where $H^{\frac{1}{2}}(\partial\Omega)$ is the image of $W^{1,2}(\Omega)$ for a Lipschitz domain Ω by the trace operator. The dual space $(H^{\frac{1}{2}}(\partial\Omega))'$ is usually denoted by $H^{-\frac{1}{2}}(\partial\Omega)$.

*CentraleSupélec, Université Paris-Saclay, France (correspondence, anna.rozanova-pierrat@centralesupelec.fr).

For Lipschitz domains it is also possible to give a definition in the weak sense of the divergence operator for vector valued functions (see for instance Theorem 2.5 § 2 [15]) and to introduce the usual formula of integration by parts for all u and v from $W^{1,2}(\Omega)$ in the following weak sense

$$\langle uv_i, v \rangle_{(H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega))} := \int_{\Omega} \frac{\partial u}{\partial x_i} v dx + \int_{\Omega} u \frac{\partial v}{\partial x_i} dx \quad i = 1, \dots, n, \quad (3)$$

where by uv_i is denoted the linear continuous functional on $H^{\frac{1}{2}}(\partial\Omega)$.

Thanks to the classical results of Calderon-Stein [9, 38] it is known that every Lipschitz domain Ω is an extension domain for the Sobolev space $W^{k,p}(\Omega)$ with $1 \leq p \leq \infty$, $k \in \mathbb{N}^*$, we briefly recall the definition of this notion

Definition 1. ($W^{k,p}$ -extension domains) A domain $\Omega \subset \mathbb{R}^n$ is called a $W^{k,p}$ -extension domain ($k \in \mathbb{N}^*$) if there exists a bounded linear extension operator $E : W^{k,p}(\Omega) \rightarrow W_p^k(\mathbb{R}^n)$, i.e. a linear operator assigning a function $v = Eu \in W_p^k(\mathbb{R}^n)$ with $v|_{\Omega} = u$ to any $u \in W^{k,p}(\Omega)$, such that

$$\|v\|_{W_p^k(\mathbb{R}^n)} \leq C \|u\|_{W^{k,p}(\Omega)} \quad \text{with a universal constant } C > 0.$$

This result was generalized by Jones [21] in the framework of (ε, ∞) -domains which give an optimal class of extension domains in \mathbb{R}^2 , but not in \mathbb{R}^3 . The optimal class of extension domains for $p > 1$ in \mathbb{R}^n was found more recently by Hajlas, Koskela and Tuominen [17]. These results are discussed in Section 2, where we give all definitions.

Thanks to the results in [41, 27, 28, 6, 5] it is possible to generalize the trace operator to cases of more irregular boundaries, such as d -sets, and even to sets without a fixed dimension [23, 19]. The definition of the trace for a regular distribution and different image spaces leading to different Green formulas are presented in Section 3.

But to be able to ensure the weak well-posedness of problem (1) and to discuss the associated spectral problem for $-\Delta$, we also need the compactness of the inclusion $W^{1,2}(\Omega)$ in $L^2(\Omega)$ and the compactness of the trace operator, now considered as an operator from $W^{1,2}(\Omega)$ to $L^2(\partial\Omega)$.

Thanks to [13] Theorem V.4.17 it is known that if a domain Ω has a continuous boundary (in the sense of graphs, see [13] Definition V.4.1) then $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. General d -set boundaries with $d > n - 1$, as for instance a von Koch curve, do not satisfy the assumption to have a continuous boundary. In our article [5] we proved this fact in the framework of admissible domains with a d -set boundary. Here we prove it also for more general boundaries as in [22, 23] (see Section 4). This may be seen as an update to the concept of admissible domains introduced first in [5]. We follow the same idea as in [5] and introduce the class of all Sobolev extension domains with boundaries on which one can define a surjective linear continuous trace operator with linear continuous right inverse. To emphasize their "extension nature", we call these domains Sobolev admissible domains (see Definition 7).

The most common examples of Sobolev admissible domains are domains with regular or Lipschitz boundaries, with a d -set boundaries, such as Von Koch fractals or with a "mixed" boundary (for instance, the boundary of a cylindrical domain in \mathbb{R}^3 with the base being a snowflake domain in \mathbb{R}^2 as considered for the Koch snowflake base in [30, 11]).

The generalization of the Kondrachov-Rellich theorem in the framework of Sobolev admissible domains allows to extend the compactness studies of the trace from [4] and to update the results of [5] (see Section 5): for a Sobolev admissible domain with a compact boundary the trace operator mapping from $W^{1,2}(\Omega)$ to $L^2(\partial\Omega)$ is compact.

Thus, as for the usual Lipschitz bounded case, the problem (1) is weakly well-posed and the corresponding spectral problem has a countable number of eigenvalues going to $+\infty$ with the eigenfunctions forming an orthogonal basis in $W^{1,2}(\Omega)$ which becomes an orthonormal basis in $L^2(\Omega)$ by the classical Hilbert-Schmidt theorem for compact self-adjoint operators on a Hilbert space (see Section 6).

The paper is organized as follows. In Section 2 we present recent results on Sobolev extension domains. In Section 3 we firstly define the trace operator on a d -set in Subsection 3.1 and secondly, in Subsection 3.2 we provide analogous results in a framework that does not require the boundaries to be exactly d -dimensional. We finish the section by a generalization of the Green formula and the integration by parts formula within this framework in Subsection 3.3. Using the results on the trace and on the extension operators, we introduce the concept of Sobolev admissible domains in Section 4 and generalize the Rellich-Kondrachov theorem. In Section 5 we show the compactness of the trace operator considered as an operator mapping to $L^p(\partial\Omega)$. In Section 6 we apply these theorems to show the well-posedness of the Poisson problem (1) on the $W^{1,2}$ -Sobolev admissible domains.

2 Sobolev extension domains

As in [5] we start by recalling the classical results of Calderon-Stein [9, 38]: every Lipschitz domain Ω is an extension domain for $W^{k,p}(\Omega)$ with $1 \leq p \leq \infty$, $k \in \mathbb{N}^*$. This result was generalized by Jones [21] in the framework of (ε, δ) -domains:

Definition 2. (*(ε, δ) -domain [21, 25, 41]*) *An open connected subset Ω of \mathbb{R}^n is an (ε, δ) -domain, $\varepsilon > 0$, $0 < \delta \leq \infty$, if whenever $x, y \in \Omega$ and $|x - y| < \delta$, there is a rectifiable arc $\gamma \subset \Omega$ with length $\ell(\gamma)$ joining x to y and satisfying*

1. $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$ (thus locally quasiconvex) and
2. $d(z, \partial\Omega) \geq \varepsilon|x - z| \frac{|y-z|}{|x-y|}$ for $z \in \gamma$.

As the constant δ is allowed to equal $+\infty$. For $\delta = +\infty$ it is possible to avoid the local character of this definition and in this case Ω is said to be an (ε, ∞) -domain. Definition 2 without the second condition yields the definition of a locally quasiconvex domain. The second condition prohibits the boundary to collapse into thin structures. This is in stark contrast to the case of fractal trees, in fact, this is the reason why fractal trees [1] are not (ε, ∞) -domains.

The (ε, δ) -domains are also called locally uniform domains [18]. Actually, bounded locally uniform domains, or bounded (ε, δ) -domains, are equivalent (see [18] point 3.4) to the uniform domains, firstly defined by Martio and Sarvas in [34], for which there are no more restriction $|x - y| < \delta$ (see Definition 2).

Thanks to Jones [21], it is known that any (ε, δ) -domain in \mathbb{R}^n is a $W^{k,p}$ -extension domain for all $1 \leq p \leq \infty$ and $k \in \mathbb{N}^*$. Moreover, for a bounded finitely connected domain $\Omega \subset \mathbb{R}^2$, Jones [21] proved that

$$\begin{aligned} \Omega \text{ is a } W^{k,p}\text{-extension domain } (1 \leq p \leq \infty \text{ and } k \in \mathbb{N}^*) &\iff \\ \Omega \text{ is an } (\varepsilon, \infty)\text{-domain for some } \varepsilon > 0 &\iff \\ \text{the boundary } \partial\Omega \text{ consists of finite number of points and quasi-circles.} & \end{aligned}$$

However, it is no more true for $n \geq 3$, i.e. there are $W^{1,p}$ -extension domains which are not locally uniform [21] (in addition, an (ε, δ) -domain in \mathbb{R}^n with $n \geq 3$ is not necessary a quasi-sphere).

To discuss general properties of locally uniform domains, let us introduce Ahlfors d -regular sets, which are more currently called by d -sets:

Definition 3. (*Ahlfors d -regular set or d -set [25, 26, 41, 39]*) *Let F be a closed Borel non-empty subset of \mathbb{R}^n . The set F is called a d -set ($0 < d \leq n$) if there exists a d -measure μ on F , i.e. a positive Borel measure with support F ($\text{supp } \mu = F$) such that there exist constants $c_1, c_2 > 0$,*

$$c_1 r^d \leq \mu(\overline{B_r(x)}) \leq c_2 r^d, \quad \text{for } \forall x \in F, 0 < r \leq 1,$$

where $B_r(x) \subset \mathbb{R}^n$ denotes the Euclidean ball centered at x and of radius r .

As [25, Prop. 1, p 30] all d -measures on a fixed d -set F are equivalent, it is also possible to define a d -set by the d -dimensional Hausdorff measure m_d :

$$c_1 r^d \leq m_d(F \cap \overline{B_r(x)}) \leq c_2 r^d, \quad \text{for } \forall x \in F, 0 < r \leq 1$$

which in particular implies that F has Hausdorff dimension d in the neighborhood of each point of F [25, p.33].

If the boundary $\partial\Omega$ is a d -set endowed with the d -dimensional Hausdorff measure restricted to $\partial\Omega$, then we denote by $L^p(\partial\Omega, m_d)$ the Lebesgue space defined with respect to this measure with the norm

$$\|u\|_{L^p(\partial\Omega, m_d)} = \left(\int_{\partial\Omega} |u|^p dm_d \right)^{\frac{1}{p}}.$$

From [41], it is known that

- All (ε, δ) -domains in \mathbb{R}^n are n -sets (d -set with $d = n$):

$$\exists c > 0 \quad \forall x \in \overline{\Omega}, \forall r \in]0, \delta[\cap]0, 1[\quad \lambda(B_r(x) \cap \Omega) \geq C \lambda(B_r(x)) = cr^n,$$

where $\lambda(A)$ denotes the Lebesgue measure of a set A in \mathbb{R}^n . This property is also called the measure density condition [17]. Let us notice that an n -set Ω cannot be “thin” close to its boundary $\partial\Omega$, since it must all times contain a non trivial ball in its neighborhood.

- If Ω is an (ε, δ) -domain and $\partial\Omega$ is a d -set ($d < n$) then $\overline{\Omega} = \Omega \cup \partial\Omega$ is an n -set.

In particular, a Lipschitz domain Ω of \mathbb{R}^n is an (ε, δ) -domain and also an n -set [41]. But not every n -set is an (ε, δ) -domain: adding an in-going cusp to an (ε, δ) -domain we obtain an n -set which is not an (ε, δ) -domain anymore. Classical snowflake domains are examples of (ε, ∞) -domains with d -set boundary [10, 41], $d > n - 1$.

Recently, Hajłasz, Koskela and Tuominen [17] have proved that every $W^{k,p}$ -extension domain in \mathbb{R}^n for $1 \leq p < \infty$ and $k \geq 1$, $k \in \mathbb{N}$ is an n -set. In addition they proved the following statements:

Theorem 1. (i) A domain $\Omega \subset \mathbb{R}^n$ is a W_∞^1 -extension domain if and only if Ω is uniformly locally quasiconvex.

(ii) For $1 < p < \infty$, $k = 1, 2, \dots$ a domain $\Omega \subset \mathbb{R}^n$ is a W_p^k -extension domain if and only if Ω is an n -set and $W^{k,p}(\Omega) = C_p^k(\Omega)$ (in the sense of equivalent norms).

By $C_p^k(\Omega)$ is denoted the space of the fractional sharp maximal functions:

Definition 4. For a set $\Omega \subset \mathbb{R}^n$ of positive Lebesgue measure,

$$C_p^k(\Omega) = \{f \in L^p(\Omega)\}$$

$$f_{k,\Omega}^\sharp(x) = \sup_{r>0} r^{-k} \inf_{P \in \mathcal{P}^{k-1}} \frac{1}{\lambda(B_r(x))} \int_{B_r(x) \cap \Omega} |f - P| dy \in L^p(\Omega)$$

with the norm $\|f\|_{C_p^k(\Omega)} = \|f\|_{L^p(\Omega)} + \|f_{k,\Omega}^\sharp\|_{L^p(\Omega)}$. By \mathcal{P}^{k-1} we denote the space of polynomials of the order $k - 1$.

From [21] and [17] we immediately obtain the following, [5]

Corollary 1. Let Ω be a bounded finitely connected domain in \mathbb{R}^2 and $1 < p < \infty$, $k \in \mathbb{N}^*$. The domain Ω is a 2-set with $W^{k,p}(\Omega) = C_p^k(\Omega)$ (with norms' equivalence) if and only if Ω is an (ε, δ) -domain and its boundary $\partial\Omega$ consists of a finite number of points and quasi-circles.

The question about W_p^k -extension domains is equivalent to the question of the continuity of the trace operator $\text{Tr} : W_p^k(\mathbb{R}^n) \rightarrow W_p^k(\Omega)$, the trace operator on the domain Ω . In the next section we introduce the notion of trace to more general Borel sets, and we will use this notion of trace to study boundary conditions.

3 Trace on the boundary and Green formulas

3.1 Framework of d -sets and Markov's local inequality

From [25] p.39, it is also known that all closed d -sets with $d > n - 1$ preserve Markov's local inequality:

Definition 5. (Markov's local inequality) *A closed subset V in \mathbb{R}^n preserves Markov's local inequality if for every fixed $j \in \mathbb{N}^*$, there exists a constant $c = c(V, n, j) > 0$, such that*

$$\max_{V \cap \overline{B_r(x)}} |\nabla P| \leq \frac{c}{r} \max_{V \cap \overline{B_r(x)}} |P|$$

for all polynomials $P \in \mathcal{P}_j$ and all closed balls $\overline{B_r(x)}$, $x \in V$ and $0 < r \leq 1$.

For instance, self-similar sets that are not subsets of any $(n - 1)$ -dimensional subspace of \mathbb{R}^n , the closure of a domain Ω with Lipschitz boundary and also \mathbb{R}^n itself preserve Markov's local inequality (see Refs. [27, 41]). The geometrical characterization of sets preserving Markov's local inequality was initially given in [24] (see Theorem 1.3) and can be simply interpreted as sets which are not too flat anywhere. It can be illustrated by the following theorem of Wingren [42]:

Theorem 2. *A closed subset V in \mathbb{R}^n preserves Markov's local inequality if and only if there exists a constant $c > 0$ such that for every ball $B_r(x)$ centered in $x \in V$ and with the radius $0 < r \leq 1$, there are $n + 1$ affinely independent points $y_i \in V \cap B_r(x)$, $i = 1, \dots, n + 1$, such that the n -dimensional ball inscribed in the convex hull of y_1, y_2, \dots, y_{n+1} , has radius not less than cr .*

Smooth manifolds in \mathbb{R}^n of dimension less than n , as for instance a sphere, are examples of "flat" sets not preserving Markov's local inequality. More precisely, the sets F which do not preserve Markov's inequality [25, Thm. 2, p.38] are exactly the sets satisfying the geometric condition in the following theorem.

Theorem 3. *A closed, non-empty subset F of \mathbb{R}^n preserves Markov's inequality if and only if the following geometric condition does not hold: for every $\varepsilon > 0$ there exists a ball $B_r(x_0)$, $x_0 \in F$, $0 < r \leq 1$, so that $B_r(x_0) \cap F$ is contained in some band of type $\{x \in \mathbb{R}^n \mid (b, x - x_0)_{\mathbb{R}^n} < \varepsilon r\}$, where $b \in \mathbb{R}^n$, $|b| = 1$, and $(b, x - x_0)_{\mathbb{R}^n}$ is the scalar product of b and $x - x_0$.*

Relationships between Markov inequalities and inequalities of Sobolev-Gagliardo-Nirenberg type were studied in [8]. The advantage of d -set boundaries preserving Markov's inequality (thus $0 < d < n$) is that, [40, 2.1], there exists a bounded linear extension operator \hat{E} of the Hölder space $C^{k-1, \alpha-k+1}(\partial\Omega)$ to the Hölder space $C^{k-1, \alpha-k+1}(\mathbb{R}^n)$, where for $k \in \mathbb{N}^*$ $k - 1 < \alpha \leq k$ (see also [25, p. 2]). This allows to show the existence of a linear continuous extension from the Besov space $B_\alpha^{p,p}(\partial\Omega)$ on $\partial\Omega$ to the Sobolev space $W_p^k(\mathbb{R}^n)$ with $\alpha = k - \frac{(n-d)}{p} \geq 1$ and $k \geq 2$ [27]. For the extensions of minimal regularity with $k = 1$, and thus with $\alpha < 1$ (see in addition the definition of the Besov space Def. 3.2 in [20] with the help of the normalized local best approximation in the class of polynomials P_{k-1} of the degree equal to $k - 1$) Markov's inequality is trivially satisfied for $j = 0$ in Definition 5 on all closed sets of \mathbb{R}^n , and hence we do not need to impose it [27, p. 198].

Before coming to details of the mentioned results, let us generalize the notion of the trace:

Definition 6. *For an arbitrary open set Ω of \mathbb{R}^n , the trace operator Tr is defined [25] for $u \in L_{loc}^1(\Omega)$ by*

$$\text{Tr}u(x) = \lim_{r \rightarrow 0} \frac{1}{\lambda(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) dy,$$

where λ denotes the Lebesgue measure on \mathbb{R}^n . The trace operator Tr is considered for all $x \in \overline{\Omega}$ for which the limit exists.

Using this definition of the trace one can prove a trace theorem on closed d -sets [25] Ch.VII and [41] Proposition 4, in which we think it should be made more precise that the closed set F should preserve Markov's local inequality not necessarily for all $k \in \mathbb{N}^*$, but at least up to $k - 1$ with $k \in \mathbb{N}^*$, the fixed regularity of the Sobolev space of which we take the trace on F :

Theorem 4. *Let F be a closed d -set preserving Markov's local inequality at least up to $k - 1$ for a fixed $k \in \mathbb{N}^*$. If*

$$0 < d < n, \quad 1 < p < \infty, \quad \text{and} \quad \alpha = k - \frac{(n-d)}{p} > 0,$$

then the trace operator $\text{Tr} : W_p^k(\mathbb{R}^n) \rightarrow B_\alpha^{p,p}(F)$ is bounded linear surjection with a bounded right inverse $E : B_\alpha^{p,p}(F) \rightarrow W_p^k(\mathbb{R}^n)$, i.e. $\text{Tr} \circ E = \text{Id}$ on $B_\alpha^{p,p}(F)$.

The definition of the Besov space $B_\alpha^{p,p}(F)$ on a closed d -set F can be found, for instance, in Ref. [25] p. 135 and Ref. [41]. See also Triebel for equivalent definitions [39].

Thanks to [24, Thm. 1.2, p. 145] we notice that, if the Markov inequality is preserved by a set F for $j = 1$, then it is preserved by F for all $j \in \mathbb{N}^*$, $j \geq 1$. Thus the added condition in Theorem 4 for these sets, especially for d -sets with $d > n - 1$, is useless. Nevertheless, for the case of W_p^1 with $1 < p < \infty$ the theorem explicitly holds for all $n - 1$ -sets, also not satisfying the Markov inequality with $j = 1$. Note that for $d = n - 1$, as it also mentioned in [6], one has $\alpha = \frac{1}{2}$ and $B_{\frac{1}{2}}^{2,2}(F) = H^{\frac{1}{2}}(F)$ as usual in the case of the classical results [31, 33] for Lipschitz boundaries $\partial\Omega = F$. Since $\alpha = \frac{1}{2} < 1$, as noticed previously the geometrical condition for the boundary to preserve Markov's inequality does not occur.

Moreover, considering only $W^{1,2}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid \nabla u \in L^2(\mathbb{R}^n)\}$ we deduce the following from Theorem 4

Theorem 5. *Let F be a closed d -set,*

$$0 \leq n - 2 < d < n, \quad \text{and} \quad \alpha = 1 - \frac{(n-d)}{2} > 0.$$

Then the trace operator $\text{Tr} : W^{1,2}(\mathbb{R}^n) \rightarrow B_\alpha^{2,2}(F)$ is bounded linear surjection with a bounded right inverse $E : B_\alpha^{2,2}(F) \rightarrow W^{1,2}(\mathbb{R}^n)$, i.e. $\text{Tr} \circ E = \text{Id}$ on $B_\alpha^{2,2}(F)$.

3.2 General framework of closed subsets of \mathbb{R}^n

It is possible to consider more general measures than d -dimensional measures which can describe by their supports a boundary of a domain [11, 22, 23].

We follow [22, Section 1] and say that a Borel measure μ on \mathbb{R}^n with support $\text{supp } \mu = F$ satisfies the D_s -condition for an exponent $0 < s \leq n$ if there is a constant $c_s > 0$ such that

$$\mu(B_{kr}(x)) \leq c_s k^s \mu(B_r(x)), \quad x \in F, \quad r > 0, \quad k \geq 1, \quad 0 < kr \leq 1. \quad (4)$$

Here as previously $B_r(x) \subset \mathbb{R}^n$ denotes an open ball centered at x and of radius r . We say that μ satisfies the L_d -condition for an exponent $0 \leq d \leq n$ if for some constant $c > 0$ we have

$$\mu(B_{kr}(x)) \geq c_d k^d \mu(B_r(x)), \quad x \in F, \quad r > 0, \quad k \geq 1, \quad 0 < kr \leq 1. \quad (5)$$

We also introduce so called the normalization condition

$$c_1 \leq \mu(B_1(x)) \leq c_2, \quad x \in F, \quad (6)$$

where $c_1 > 0$ and $c_2 > 0$ are constants independent of x .

Combining (4) and (6) one can find a constant $c > 0$ such that

$$\mu(B_r(x)) \geq c r^s, \quad x \in F, \quad 0 < r \leq 1, \quad (7)$$

what implies $\dim_H F \leq s$, where $\dim_H F$ denotes the Hausdorff dimension of F . Similarly (5) and (6) yield a constant $c' > 0$ such that

$$\mu(B_r(x)) \leq c' r^d, \quad x \in F, \quad 0 < r \leq 1, \quad (8)$$

hence $\dim_H F \geq d$. Moreover, (4) implies the doubling condition

$$\mu(B_{2r}(x)) \leq c \mu(B_r(x)), \quad x \in F, \quad 0 < r \leq 1/2,$$

where $c > 0$ is a suitable constant, [22, Section 1].

If a Borel measure μ with support F satisfies (7) and (8) with $s = d$ for some $0 < d \leq n$, then, according to Definition 3, μ is called a d -measure and F is called a d -set. Obviously, if we have (4), (5) and (6) and $d = s$ then μ is a d -measure and F a d -set. Otherwise, we consider measures, which by (7) and (8) satisfy for some constants $c > 0$ and $c' > 0$

$$c r^s \leq \mu(B_r(x)) \leq c' r^d, \quad x \in F, \quad 0 < r \leq 1. \quad (9)$$

For this general measure μ supported on a closed subset $F \subset \mathbb{R}^n$ it is possible thanks to [22] to define the corresponding Lebesgue spaces $L^p(F, \mu)$ and Besov spaces $B_\beta^{p,p}(F, \mu)$ on closed subsets $F \subset \mathbb{R}^n$ in such a way that we have the following theorem

Theorem 6. *Let $0 \leq d \leq n$, $d \leq s \leq n$, $s > 0$, $1 \leq p \leq +\infty$,*

$$\frac{n-d}{p} < \beta < 1 + \frac{n-s}{p}, \quad (10)$$

and let $F \subset \mathbb{R}^n$ be a closed set which is the support of a Borel measure μ satisfying (4), (5) and (6).

Then, considering the Besov space $B_\beta^{p,p}(F, \mu)$ on F , defined as the space of μ -classes of real-valued functions f on F such that the norm

$$\|f\|_{B_\beta^{p,p}(F, \mu)} := \|f\|_{L^p(F, \mu)} + \left(\sum_{\nu=0}^{\infty} 2^{\nu(\beta - \frac{n}{p})} \int \int_{|x-y| < 2^{-\nu}} \frac{|f(x) - f(y)|^p}{\mu(B(x, 2^{-\nu}))\mu(B(y, 2^{-\nu}))} \mu(dy)\mu(dx) \right)^{1/p}$$

is finite, the following statements hold:

(i) Tr_F is a continuous linear operator from $W_p^\beta(\mathbb{R}^n)$ onto $B_\beta^{p,p}(F)$, and

$$\|\text{Tr}_F f\|_{B_\beta^{p,p}(F)} \leq c_\beta \|f\|_{W_p^\beta(\mathbb{R}^n)}, \quad f \in W_p^\beta(\mathbb{R}^n), \quad (11)$$

with a constant $c_\beta > 0$ depending only on β , s , d , n , c_s , c_d , c_1 and c_2 .

(ii) There is a continuous linear extension operator $E_F : B_\beta^{p,p}(F) \rightarrow W_p^\beta(\mathbb{R}^n)$ such that $\text{Tr}_F(E_F f) = f$ for $f \in B_\beta^{p,p}(F)$.

Theorem 6 is a particular case of [22, Theorem 1].

The spaces $B_\beta^{p,p}(F, \mu)$ are Banach spaces, while $B_\beta^{2,2}(F, \mu)$ are Hilbert spaces, and their corresponding scalar product is denoted by $\langle \cdot, \cdot \rangle_{B_\beta^{2,2}(F, \mu)}$.

A priori the definition of $B_\beta^{p,p}(F, \mu)$ depends on both F and μ . However, it was shown in [22, Section 3.5] that for two different measures μ_1 and μ_2 satisfying hypotheses of Theorem 6 and with common support F , if $f \in B_\beta^{p,p}(F, \mu_2)$, then f can be altered on a set with μ_2 -measure zero, in such a way that f becomes a function in $B_\beta^{p,p}(F, \mu_1)$. In other words, also by Theorem 6, the spaces $B_\beta^{2,2}(F, \mu_1)$ and $B_\beta^{p,p}(F, \mu_2)$ are equivalent. Thus, we simplify the notations and instead of $B_\beta^{p,p}(F, \mu)$ simply write $B_\beta^{p,p}(F)$.

Let us notice [22] that this time if F is a d -set with $0 < d \leq n$ as defined in Subsection 3.1, then $\mu = m_d$ satisfies (4), (5) and (6) and hence it is possible to apply Theorem 6. The restriction on β in Theorem 6 becomes $0 < \alpha < 1$ with $\alpha = \beta - \frac{n-d}{p}$. Consequently, from one hand, the space $B_\beta^{p,p}(F)$ is equivalent to the Besov space $B_\alpha^{p,p}(F)$ with $0 < \alpha < 1$ from Subsection 3.1 (see [25]), which, from the other hand, by our previous remark for $\alpha < 1$, explains why we don't need to impose that F preserves the local Markov inequality. Thus in the framework of d -sets this theorem coincides with Theorem 4 for $\alpha < 1$.

Remark 1. *If we apply Theorem 6 for $W^{1,2}(\mathbb{R}^n)$ we obtain the image of the trace equal to the Hilbert space $B_1^{2,2}(F)$ with the restrictions*

$$n \geq s \geq d > n - 2 \geq 0.$$

3.3 Integration by parts and the Green formula

Let us generalize the Green formula formulated for d -sets in [5] (initially proposed by Lancia [28, Thm. 4.15] for a von Koch curve, see also [29] for an other specific d -set boundary case) and the integration by parts from Appendix A Theorem A.3 [32] (see also the proof of formula (4.11) of Theorem 4.5 in [11]).

Proposition 1. (Green formula) *Let Ω be a domain in \mathbb{R}^n ($n \geq 2$) with a closed boundary $\partial\Omega$ which is the support of a Borel measure μ satisfying the conditions of Theorem 6 with $n \geq s \geq d > n - 2 \geq 0$. Then*

1. *the Green formula holds for all u and v from $W^{1,2}(\Omega)$ with $\Delta u \in L^2(\Omega)$,*

$$\int_{\Omega} v \Delta u dx + \int_{\Omega} \nabla v \cdot \nabla u dx = \langle \frac{\partial u}{\partial n}, \text{Tr} v \rangle_{(B_1^{2,2}(\partial\Omega))', B_1^{2,2}(\partial\Omega)}, \quad (12)$$

where $(B_1^{2,2}(\partial\Omega))'$ is the dual space of $B_1^{2,2}(\partial\Omega)$.

2. *In addition the usual integration by parts holds for all u and v from $W^{1,2}(\Omega)$ in the following weak sense*

$$\langle uv_i, v \rangle_{(B_1^{2,2}(\partial\Omega))', B_1^{2,2}(\partial\Omega)} := \int_{\Omega} \frac{\partial u}{\partial x_i} v dx + \int_{\Omega} u \frac{\partial v}{\partial x_i} dx \quad i = 1, \dots, n, \quad (13)$$

where by uv_i is denoted the linear continuous functional on $B_1^{2,2}(\partial\Omega)$.

The statement of proposition follows, thanks to Theorem 6, from the surjectivity of the linear continuous trace operator $\text{Tr}_{\partial\Omega} : W^{1,2}(\Omega) \rightarrow B_1^{2,2}(\partial\Omega)$. To prove (12) and (13) it sufficient to follow [28] and [11] respectively. Thus the proof is omitted.

4 Sobolev admissible domains and the generalization of the Rellich-Kondrachov theorem

Thanks to Theorems 1 and 6 we can generalize now the notion of admissible domains introduced in [5] in the framework of d -sets:

Definition 7. (Sobolev admissible domain) *Let $1 < p < \infty$ and $k \in \mathbb{N}^*$ be fixed. A domain $\Omega \subset \mathbb{R}^n$ is called a $(W^{k,p,-})$ Sobolev admissible domain if it is an n -set, such that $W^{k,p}(\Omega) = C_p^k(\Omega)$ as sets with equivalent norms (hence, Ω is a $W^{k,p}$ -extension domain), with a closed boundary $\partial\Omega$ which is the support of a Borel measure μ satisfying the conditions of Theorem 6.*

We summarize several useful results on the trace and extension operators (see [37] for more general results for the case $p > n$) for the trace and the extension operators:

Theorem 7. *Let Ω be a Sobolev admissible domain in \mathbb{R}^n , $1 < p < \infty$, $k \in \mathbb{N}^*$ be fixed and β defined in (10). Then the following trace operators (see Definition 6)*

1. $Tr : W^{\beta,p}(\mathbb{R}^n) \rightarrow B_{\beta}^{p,p}(\partial\Omega)$,
2. $Tr_{\Omega} : W^{k,p}(\mathbb{R}^n) \rightarrow W^{k,p}(\Omega)$,
3. $Tr_{\partial\Omega} : W^{1,p}(\Omega) \rightarrow B_1^{p,p}(\partial\Omega)$

are linear continuous and surjective with linear bounded right inverse, i.e. extension, operators $E : B_{\beta}^{p,p}(\partial\Omega) \rightarrow W^{\beta,p}(\mathbb{R}^n)$, $E_{\Omega} : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$, $E_{\partial\Omega} : B_1^{p,p}(\partial\Omega) \rightarrow W^{1,p}(\Omega)$.

Proof. It is a corollary of results given in Sections 2 and 3. Indeed, if Ω is Sobolev admissible, then by Theorem 6, the trace operator $Tr : W_p^{\beta}(\mathbb{R}^n) \rightarrow B_{\beta}^{p,p}(\partial\Omega) \subset L^p(\partial\Omega)$ is linear continuous and surjective with linear bounded right inverse $E : B_{\beta}^{p,p}(\partial\Omega) \rightarrow W_p^{\beta}(\mathbb{R}^n)$ (point 1). We notice that since $d \geq n - 1$, the interval of suitable β , given by $]\frac{n-d}{p}, 1 + \frac{n-s}{p}[$, includes as a subset $]\frac{1}{p}, 1[$ for all $p > 1$. The upper regularity limit for β can be estimated by $1 < 1 + \frac{n-s}{p} \leq 1 + \frac{1}{p} < 2$.

On the other hand, by [17], Ω is a $W^{k,p}$ -extension domain and $Tr_{\Omega} : W^{k,p}(\mathbb{R}^n) \rightarrow W^{k,p}(\Omega)$ and $E_{\Omega} : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$ are linear continuous (point 2). Hence, the embeddings for $k = 1$

$$B_1^{p,p}(\partial\Omega) \rightarrow W^{1,p}(\mathbb{R}^n) \rightarrow W^{1,p}(\Omega) \quad \text{and} \quad W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n) \rightarrow B_1^{p,p}(\partial\Omega)$$

are linear continuous (point 3). □

By updating the class of admissible domains, all results of [5] still hold in this new class. For instance it is also possible to consider Sobolev admissible truncated domains for which two disjoint boundaries satisfy Theorem 6. Without any particular motivation here for the truncated domain, let us just formulate the compactness of the embedding $W^{1,2}(\Omega)$ to $L^2(\Omega)$ for the Sobolev admissible domains:

Proposition 2. *Let Ω be a bounded Sobolev admissible domain for $p = 2$ and $k = 1$. Then the Sobolev space $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$:*

$$W^{1,2}(\Omega) \subset\subset L^2(\Omega).$$

The proof follows with small modifications the proof of Proposition 2 in [5] and thus is omitted.

However, we would like to recall main compactness results of [5] putting them in the new framework of Sobolev admissible domains with not necessarily a d -set boundary.

Remark 2. *To have a compact embedding it is important that the domain Ω be a W_p^k -extension domain. The boundness or unboundness of Ω is not important to have $W_p^k(\Omega) \subset\subset W_p^{\ell}(\Omega)$ with $k > \ell \geq 1$ ($1 < p < \infty$). But the boundness of Ω is important for the compact embedding in $L^q(\Omega)$.*

As a direct corollary we have the following generalization of the classical Rellich-Kondrachov theorem (see for instance Adams [2] p.144 Theorem 6.2):

Theorem 8. (Compact Sobolev embeddings for n -sets, [5]) *Let $\Omega \subset \mathbb{R}^n$ be an n -set with $W^{k,p}(\Omega) = C_p^k(\Omega)$, $1 < p < \infty$, $k, \ell \in \mathbb{N}^*$. Then the following embeddings are compact:*

1. $W_p^{k+\ell}(\Omega) \subset\subset W_q^{\ell}(\Omega)$,
2. $W_p^k(\Omega) \subset\subset L_{loc}^q(\Omega)$, or $W_p^k(\Omega) \subset\subset L^q(\Omega)$ if Ω is bounded,

with $q \in [1, +\infty[$ if $kp = n$, $q \in [1, +\infty]$ if $kp > n$, and with $q \in [1, \frac{pn}{n-kp}[$ if $kp < n$.

5 Compactness of the trace

We generalize the classical Rellich-Kondrachov theorem for fractals in the following way:

Theorem 9. (Compact Besov embeddings) *Let $F \subset \mathbb{R}^n$ be a compact set satisfying conditions of Theorem 6 with $\beta = 1$. Assume $1 \leq q \leq p < \infty$.*

Then

1. *the linear trace operator $\text{Tr} : W^{1,p}(\mathbb{R}^n) \rightarrow L^q(F)$ is compact,*
2. *the continuous embedding $B_1^{p,p}(F) \subset L^q(F)$ is compact.*

Proof. Indeed, for $\beta = 1$, thanks to Theorem 6, the extension $E_F : B_1^{p,p}(F) \rightarrow W_p^1(\mathbb{R}^n)$ is continuous. Thus, if the trace operator $\text{Tr} : W^{1,p}(\mathbb{R}^n) \rightarrow L^q(F)$ is compact, by the composition of the bounded and compact operators, the embedding $B_1^{p,p}(F) \subset L^q(F)$ is compact too.

Hence, let us prove the first statement of the theorem. Taking, for instance, a nontrivial ball $B_r \subset \mathbb{R}^n$, or, more generally, any extension domain, in the way that $F \subset \overline{B_r}$, we notice that, as the operator $W^{1,p}(\mathbb{R}^n) \rightarrow W^{1,p}(B_r)$ is continuous, the question can be reduced to the compactness of $\text{Tr}_{B_r,F} : W^{1,p}(B_r) \rightarrow L^q(F)$.

Using the idea of the proof of [4, Proposition 8.1], we take a bounded sequence $(u_n)_{n \in \mathbb{N}}$ of $W^{1,p}(B_r)$:

$$\exists \hat{M} > 0 : \quad \forall k \in \mathbb{N} \quad \|u_{n_k}\|_{W^{1,p}(B_r)} \leq \hat{M}.$$

As B_r is a bounded extension domain, then by point 2 of Theorem 8 (here, as ∂B_r is C^∞ , even by the classical Rellich-Kondrachov theorem) the inclusion $W^{1,p}(B_r) \rightarrow L^p(B_r)$ is compact. Taking $1 \leq q \leq p$, we have the usual continuous embedding $L^p(B_r) \subset L^q(B_r)$, holding actually for all bounded sets.

Therefore, there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}} \subset (u_n)_{n \in \mathbb{N}}$ with an element $u \in W^{1,p}(B_r)$ such that

$$u_{n_k} \rightharpoonup u \text{ in } W^{1,p}(B_r), \quad u_{n_k} \rightarrow u \text{ in } L^p(B_r) \text{ for } k \rightarrow +\infty.$$

Moreover, still by the boudness of $(u_{n_k})_{k \in \mathbb{N}}$ in $W^{1,p}(B_r)$, there exists a constant $M > 0$ such that

$$\forall k \in \mathbb{N} \quad \|\nabla u_{n_k}\|_{L^p(B_r)}^p \leq M.$$

Since $B_1^{p,p}(F) \subset L^p(F) \subset L^q(F)$, combining the previously viewed continuous embeddings and Theorem 6, we obtain that the linear operator $\text{Tr}_{B_r,F} : W^{1,p}(B_r) \rightarrow L^q(F)$ is bounded.

Consequently, the sequence $(\text{Tr}_{B_r,F} u_{n_k})_{k \in \mathbb{N}}$ is bounded in $L^q(F)$ with $1 \leq q \leq p$ and thus in addition (passing if necessarily again to a subsequence, denoted again by $(\text{Tr}_{B_r,F} u_{n_k})_{k \in \mathbb{N}}$) $\text{Tr}_{B_r,F} u_{n_k} \rightharpoonup w$ in $L^q(F)$ for an element $w \in L^q(F)$ (we will see in the following that $w = \text{Tr}_{B_r,F} u$). Without loss of generality let us assume that $u = 0$ and hence let us prove that $\text{Tr}_{B_r,F} u_{n_k} \rightarrow 0$ in $L^q(F)$.

We firstly notice that for all $\sigma \in]0, 1]$ there exists a constant $c = c(\sigma, \hat{M}) > 0$ it holds

$$\|\text{Tr}_{B_r,F} u_{n_k}\|_{L^q(F)}^p \leq \sigma \|\nabla u_{n_k}\|_{L^p(B_r)}^p + c(\sigma, \hat{M}) \|u_{n_k}\|_{L^p(B_r)}^p. \quad (14)$$

Actually, by the boudness of $(u_{n_k})_{k \in \mathbb{N}}$ in $W^{1,p}(\Omega)$, for all $\delta \in]0, 1]$ there exists a constant $c(\delta, \hat{M}) > 0$ such that for all $k \in \mathbb{N}$

$$\|\nabla u_{n_k}\|_{L^p(B_r)}^p + \|u_{n_k}\|_{L^p(B_r)}^p \leq \delta \|\nabla u_{n_k}\|_{L^p(B_r)}^p + c(\delta, \hat{M}) \|u_{n_k}\|_{L^p(B_r)}^p.$$

By the continuity of $\text{Tr}_{B_r,F} : W^{1,p}(B_r) \rightarrow L^q(F)$, there exists a constant $C_{tr} > 0$ (depending on all constants from Theorem 6) such that for all $k \in \mathbb{N}$

$$\|\text{Tr}_{B_r,F} u_{n_k}\|_{L^q(F)}^p \leq C_{tr} \delta \|\nabla u_{n_k}\|_{L^p(B_r)}^p + C_{tr} c(\delta, \hat{M}) \|u_{n_k}\|_{L^p(B_r)}^p.$$

Thus we denote $\sigma = C_{tr}\delta > 0$ which can be arbitrary small thanks to the arbitrariness on $]0, 1]$ of δ and obtain (14). Let us define now $\varepsilon = \sigma M > 0$ for $M > 0$, the boundness constant of ∇u_{n_k} in $L^p(B_r)$. Then for all $k \in \mathbb{N}$ it holds

$$\|\mathrm{Tr}_{B_r, F} u_{n_k}\|_{L^q(F)}^p \leq \varepsilon + c \|u_{n_k}\|_{L^p(B_r)}^p$$

with a constant $c > 0$ independent on k . Since $\|u_{n_k}\|_{L^p(B_r)} \rightarrow 0$ for $k \rightarrow +\infty$, we find that for all $\varepsilon > 0$

$$\limsup_{k \rightarrow +\infty} \|\mathrm{Tr}_{B_r, F} u_{n_k}\|_{L^q(F)}^p \leq \varepsilon,$$

which directly ensures that $\mathrm{Tr}_{B_r, F} u_{n_k} \rightarrow 0$ in $L^q(F)$. This finishes the proof of point 2. \square

Remark 3. *The statement of point 2 [5, Theorem 2.13] for compact d -set preserving Markov's local inequality should be updated by the general result on the compactness of Besov embeddings on d -sets from [39, Proposition 20.5, Theorem 20.6], which actually contains the case $q \in [1, p]$ of Theorem 9 independently on the value of the dimensions d and n . A complement result about compactness of the embeddings $W_p^1(\Omega) \rightarrow L^q(F)$ for an extension domain Ω and $F \subset \overline{\Omega}$, defined as a support of a measure satisfying only the upper estimate in the d -set condition with $0 \leq n - p \leq d \leq n$, is given in [7]: for $p = n \geq 2$ with $q \in [1, +\infty[$, for $p > n$ and $p = n = 1$ with $q = \infty$, and finally, for $1 < p < n$ and $s = \frac{dp}{n-p} > p$ for $q \in [1, s]$. We also notice that working in the framework of Theorem 6, if $p = +\infty$, then β is necessarily strictly less than 1.*

In particular, the compactness of the trace operator implies the following equivalence of the norms on $W^{1,p}(\Omega)$:

Proposition 3. *Let Ω be a Sobolev admissible domain in \mathbb{R}^n with a compact boundary $\partial\Omega$ and $1 < p < \infty$, $k \in \mathbb{N}^*$. Then*

1. $W^{k,p}(\Omega) \subset\subset L_{loc}^p(\Omega)$, $k \in \mathbb{N}^*$;
2. $\mathrm{Tr} : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ is compact;
3. If in addition the measure μ is Borel regular then the image $\mathrm{Im}(\mathrm{Tr}) = B_1^{p,p}(\partial\Omega)$ is dense in $L^p(\partial\Omega)$.
4. $\|u\|_{W^{1,p}(\Omega)}$ is equivalent to $\|u\|_{\mathrm{Tr}} = \left(\int_{\Omega} |\nabla u|^p dx + \int_{\partial\Omega} |\mathrm{Tr} u|^p d\mu\right)^{\frac{1}{p}}$.

Proof. Let us prove point 3. To prove all other points it is sufficient to follow the proof of Proposition 3 in [5].

If $\partial\Omega$ is endowed with a Borel regular measure μ , then the space $\{v|_{\partial\Omega} : v \in \mathcal{D}(\mathbb{R}^n)\}$, which is dense in $C(\partial\Omega)$ by the Stone-Weierstrass theorem for the uniform norm, is also dense in $L^p(\partial\Omega)$ (see Theorem 2.11 in [14]). Hence, $B_k^{p,p}(\partial\Omega)$ is dense in $L^p(\partial\Omega)$. \square

The Poincaré's inequality stays also true on a bounded Sobolev admissible domain [12]:

Theorem 10. *(Poincaré's inequality) Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be a bounded connected Sobolev admissible domain. For all $u \in W_0^{1,p}(\Omega)$ with $1 \leq p < +\infty$, there exists $C > 0$ depending only on Ω , p and n such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Therefore the semi-norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$, defined by $\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)}$, is a norm which is equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$ on $W_0^{1,p}(\Omega)$.

Moreover for all $u \in W^{1,p}(\Omega)$ there exists $C > 0$ depending only on Ω , p and n such that

$$\left\| u - \frac{1}{\lambda(\Omega)} \int_{\Omega} u \, d\lambda \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Proof. The result for $u \in W_0^{1,p}(\Omega)$ comes from the boundness of Ω . The result for $u \in W^{1,p}(\Omega)$ comes from the compactness of the embedding $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ from Theorem 8 and following for instance the proof from [14] (see section 5.8.1 Theorem 1). \square

Thus the results of [5] on the Dirichlet-to-Neumann operator can be also updated in the framework of the Sobolev admissible domains of Definition 7. For instance we have

Theorem 11. *Let Ω be a bounded Sobolev admissible domain in \mathbb{R}^n ($n \geq 2$) for $p = 2$ and $k = 1$. Then the Poincaré-Steklov operator*

$$A : B_1^{2,2}(\partial\Omega) \rightarrow (B_1^{2,2}(\partial\Omega))'$$

mapping $u|_{\partial\Omega}$ to $\partial_\nu u|_{\partial\Omega}$ is a linear bounded self adjoint operator with $\ker A \neq 0$.

6 Application to the Poisson boundary valued and spectral problems

In this section we show the application of the theory of functional spaces developed in the previous sections on the example of the Poisson equation with Robin boundary conditions which we can weakly solve on the Sobolev admissible domains.

Let Ω be a $W^{1,2}$ -Sobolev admissible domain with a compact boundary $\partial\Omega$ and $f \in L^2(\Omega)$. For $a > 0$ we define $W^{1,2}(\Omega)$ endowed with the equivalent by Proposition 3 norm

$$\|u\|_{\text{Tr}}^2 = \int_{\Omega} |\nabla u|^2 dx + a \int_{\partial\Omega} |Tr_{\partial\Omega} u|^2 d\mu. \quad (15)$$

Then $u \in W^{1,2}(\Omega)$ is called a weak solution of the Poisson problem (1) if for all $v \in W^{1,2}(\Omega)$

$$(u, v)_{\text{Tr}} = \int_{\Omega} \nabla u \nabla v dx + a \int_{\partial\Omega} Tr_{\partial\Omega} u Tr_{\partial\Omega} v d\mu = \int_{\Omega} f v dx.$$

Thus, the Riesz representation theorem gives us the well-posedness result:

Theorem 12. *Let Ω be a $W^{1,2}$ -Sobolev admissible domain with a compact boundary $\partial\Omega$. Then for all $f \in L^2(\Omega)$ and $a > 0$ there exists a unique weak solution $u \in W^{1,2}(\Omega)$ of the Poisson problem (1) and it holds the stability estimate*

$$\|u\|_{\text{Tr}} \leq C \|f\|_{L^2(\Omega)}.$$

In the same time with the additional assumption that Ω is bounded, ensuring the compactness of the embedding $i_{L^2(\Omega)} : W^{1,2}(\Omega) \rightarrow L^2(\Omega)$ by Proposition 3, we also have the compactness of the operator $B : f \in L^2(\Omega) \mapsto B(f) = u \in W^{1,2}(\Omega)$ mapping a source term f to the weak solution of the Poisson problem (1) (see for instance Theorem 3.6 [5]). The compactness of the embedding $i_{L^2(\Omega)}$ allows also to apply the spectral Hilbert-Schmidt theorem for a self-adjoint compact operator on a Hilbert space to obtain the usual properties of the spectral problem for the $-\Delta$ on the Sobolev admissible domains:

Theorem 13. *Let Ω be a bounded $W^{1,2}$ -Sobolev admissible domain. The weak eigenvalue problem*

$$\forall v \in W^{1,2}(\Omega) \quad (u, v)_{\text{Tr}} = \lambda(u, v)_{L^2(\Omega)}$$

has a countable number of strictly positive eigenvalues of finite multiplicity, which is possible to numerate in the non-decreasing way:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \quad \lambda_j \rightarrow +\infty \quad j \rightarrow +\infty.$$

In addition the corresponding eigenfunctions forms an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $W^{1,2}(\Omega)$.

Proof. It is sufficient to notice that the eigenvalue problem is equivalent to the spectrum problem $Tu = \frac{1}{\lambda}u$ for the operator $T = A \circ i_{L^2(\Omega)} : W^{1,2}(\Omega) \rightarrow W^{1,2}(\Omega)$ which is linear compact and self-adjoint on the Hilbert space $W^{1,2}(\Omega)$. Here A is the linear bounded operator (existing by the Riesz representation theorem) which maps $v \in L^2(\Omega)$ to $Av \in W^{1,2}(\Omega)$ such that

$$\forall \phi \in W^{1,2}(\Omega) \quad (v, \phi)_{L^2(\Omega)} = (Av, \phi)_{\text{Tr}}.$$

□

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