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# Partially hyperbolic diffeomorphisms and Lagrangean contact structures

Martin Mion-Mouton

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## Abstract

In this paper, we classify the three-dimensional contact partially hyperbolic diffeomorphisms whose stable, unstable and central distributions are smooth, and whose non-wandering set equals the whole manifold. We prove that up to a finite quotient or a finite power, they are smoothly conjugated either to a time-map of an algebraic contact-Anosov flow, or to an affine partially hyperbolic automorphism of a nil-Heis(3)-manifold. The rigid geometric structure induced by the three dynamical distributions plays a fundamental role in the proof.

## 1 Introduction

In a lot of natural situations, a differentiable dynamical system on a smooth manifold preserves a geometric structure on the tangent bundle coming from invariant distributions. For instance, if it preserves a borelian measure, then Oseledet's theorem provides an almost-everywhere defined splitting of the tangent bundle, given by the rates of expansion or contraction of the tangent vectors by the differentials of the dynamic.

Although invariant geometric structures naturally arise, they are in general highly non-regular (Oseledet's decomposition is for instance only measurable), and this lack of regularity allows a lot of flexibility of the dynamic: former examples can be deformed in order to produce a lot of new ones. In contrast, the smoothness of the dynamical distributions puts a strong restriction on the system, and the known examples with  $C^\infty$  distributions are in general "very symmetric": typically, they arise from compact quotients of Lie groups, with action by affine automorphisms.

It is thus natural to ask to what extent the geometric structure preserved by the dynamic makes the situation rigid, and especially *why*.

Let us give a paradigmatic example of rigidity with the following result of Étienne Ghys concerning three-dimensional Anosov flows (the statement proved by Ghys in [Ghy87] is more precise than the one given below).

**Theorem 1.1** ([Ghy87]). *Let  $(\varphi^t)$  be an Anosov flow of a three-dimensional closed connected manifold. If the stable and unstable distributions of  $(\varphi^t)$  are  $C^\infty$ , then up to a finite covering:*

- *either  $(\varphi^t)$  is smoothly conjugated to the suspension flow of a hyperbolic automorphism of the two-torus,*
- *or  $(\varphi^t)$  is smoothly orbitally equivalent to the geodesic flow of a compact hyperbolic surface.*

We recall that a smooth non-singular flow  $(\varphi^t)$  of a compact manifold  $M$  is *Anosov* if  $D\varphi^t$  preserves two distributions  $E^s$  and  $E^u$  (respectively called the stable and unstable distribution) satisfying  $TM = E^s \oplus E^0 \oplus E^u$ , where  $E^0$  denotes the direction of the flow, and such that:

- $E^s$  is *uniformly contracted* by  $\varphi^t$ , *i.e.* there are constants  $C > 0$  and  $0 < \lambda < 1$  such that for some Riemannian metric on  $M$ , we have for any  $x \in M$  and  $t \in \mathbb{R}^+$ :  $\|D_x \varphi^t|_{E^s}\| \leq C\lambda^t$ ;
- and  $E^u$  is *uniformly expanded* by  $\varphi^t$ , *i.e.* uniformly contracted by  $\varphi^{-t}$ .

Under the smoothness assumption of  $E^s$  and  $E^u$ , Ghys notices that the smooth plane distribution  $E^s \oplus E^u$  can only have two extreme geometrical behaviours: either it integrates into a foliation, or it is a *contact* distribution (*i.e.* it is locally the kernel of a contact one-form). In the first case, former results of Plante and Franks conclude the proof. The work of Ghys in [Ghy87] is therefore entirely devoted to three-dimensional *contact-Anosov flows*, *i.e.* when  $E^s$  and  $E^u$  are smooth, and  $E^s \oplus E^u$  is contact. Under this assumption, the Anosov flow preserves a *rigid geometric structure* that makes the classification possible.

Roughly speaking, a *rigid geometric structure* is a structure with “few automorphisms”. More precisely, they are those smooth geometric structures whose Lie algebra of local Killing fields (vector fields whose flow preserves the structure) is everywhere finite-dimensional.

As d’Ambra and Gromov pointed out in [GD91], it is natural to believe that rigid geometric structures preserved by rich dynamical systems have to be particularly special: “*one does not expect rigid geometry to be accompanied by rich dynamics*” ([GD91, §0.3 p.21]), and it seems thus reasonable to look for classification results in these situations. The general idea is that rich dynamical properties will imply strong restrictions on the rigid geometric structure, inducing in return a rigidity of the dynamical system itself.

Several rigid geometric structures can be preserved by a contact-Anosov flow ( $\varphi^t$ ). First of all, ( $\varphi^t$ ) always preserves a contact one-form  $\alpha$ , and the induced volume form  $\alpha \wedge d\alpha$  is thus also preserved, *i.e.* the contact-Anosov flows are *conservative*. For contact-Anosov flows of any odd dimension, ( $\varphi^t$ ) moreover preserves a natural linear connection on the tangent bundle, initially defined by Kanai in [Kan88]. The invariant Kanai connection allowed for example Benoist, Foulon and Labourie to obtain a classification result for contact-Anosov flows of any odd dimension in [BFL92].

While these invariant rigid geometric structures require the existence of a one-parameter flow of diffeomorphisms, this paper investigates rigid geometric structures preserved by the discrete-time analogs of contact-Anosov flows that are the *contact partially hyperbolic diffeomorphisms*.

## 1.1 Three-dimensional contact partially hyperbolic diffeomorphisms

Let us recall the definition of a partially hyperbolic diffeomorphism.

**Definition 1.2.** A diffeomorphism  $f$  of a compact manifold  $M$  is *partially hyperbolic* if it preserves a splitting  $TM = E^s \oplus E^u \oplus E^c$  of the tangent bundle into three non-trivial continuous distributions, satisfying the following conditions with respect to some Riemannian metric on  $M$ :

- the *stable distribution*  $E^s$  is *uniformly contracted* by  $f$ , *i.e.* for any  $x \in M$  and any unit vector  $v^s \in E^s(x)$ ,  $\|D_x f(v^s)\| < 1$ ;
- the *unstable distribution*  $E^u$  is *uniformly expanded* by  $f$ , *i.e.* uniformly contracted by  $f^{-1}$ ;
- the splitting is dominated, *i.e.* for any  $x \in M$ , and any unit vectors  $v^s \in E^s(x)$ ,  $v^c \in E^c(x)$ , and  $v^u \in E^u(x)$ ,  $\|D_x f(v^s)\| < \|D_x f(v^c)\| < \|D_x f(v^u)\|$  ( $E^c$  is called the *central distribution*).

We refer to [CP15] for a very complete introduction to partially hyperbolic diffeomorphisms. A lot of progress has been made in the recent years concerning their classification in dimension three (see for example the survey [HP18]). Let us quote the recent work of Bonatti and Zhang in [BZ19], who obtain rigidity results for partially hyperbolic diffeomorphisms in the continuous category under specific dynamical conditions.

In this paper, we investigate the particular case when the three distributions  $E^s$ ,  $E^u$ , and  $E^c$  are smooth. Carrasco, Pujals and Rodriguez-Hertz obtain in [CPRH19] a classification result under this smoothness assumption, assuming moreover that the differential of the diffeomorphism is constant when read in the global frame given by three smooth vector fields generating these distributions. The geometric structure  $(E^s, E^u, E^c)$  defined by such a partially hyperbolic diffeomorphism is in general not rigid, and their result is obtained through dynamical arguments.

We will not be interested in this paper in additional assumptions on the partially hyperbolic diffeomorphism, but in geometrical conditions, beside the smoothness of  $E^s$ ,  $E^u$  and  $E^c$ . The generic case for the plane distribution  $E^s \oplus E^u$  is to be contact on a non-empty open subset of the manifold, and we assume in this paper that  $E^s \oplus E^u$  is everywhere contact. The partially hyperbolic diffeomorphisms satisfying these assumptions are the discrete-time analogs of contact-Anosov flows, and will be called *contact partially hyperbolic diffeomorphisms*. Let us recall two well-known families of examples, easily described in an algebraic way.

### 1.1.1 Three-dimensional algebraic contact-Anosov flows

These first examples are the deformations of geodesic flows of hyperbolic surfaces appearing in [Ghy87] (see Theorem 1.1).

With  $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2$  and  $a^t = \exp(tH) \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ , we denote by  $\tilde{A} = \{a^t\}_{t \in \mathbb{R}}$  the one-parameter subgroup of the universal cover  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$  of  $\mathrm{SL}_2(\mathbb{R})$  generated by  $H$ . For any cocompact lattice  $\Gamma_0$  of  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ , the right hyperbolic flow  $(R_{a^t})$  on the left compact quotient  $\Gamma_0 \backslash \widetilde{\mathrm{SL}}_2(\mathbb{R})$  is a finite covering of the geodesic flow of a compact hyperbolic surface (up to a constant rescaling of the time by a factor  $\frac{1}{2}$ ), and is thus Anosov.

Following the terminology of Salein in [Sal99], we will say that a morphism  $u: \Gamma_0 \rightarrow \tilde{A}$  is *admissible* if the graph-group  $\Gamma := \mathrm{gr}(u, \Gamma_0) = \{(\gamma, u(\gamma)) \mid \gamma \in \Gamma_0\}$  acts freely, properly and cocompactly on  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$  by the action  $(g, a) \cdot x = gxa$ . We can then look at the action of the right hyperbolic flow  $(R_{a^t})$  on the new quotient  $\Gamma \backslash \widetilde{\mathrm{SL}}_2(\mathbb{R})$ , defined by  $R_{a^t}(\Gamma \cdot x) = \Gamma \cdot (xa^t)$ . This flow is still Anosov (see for example [Zeg96, Prop. 4.2 p.868] and [Mañ77, Theorem A]), and we will call these flows the *three-dimensional algebraic contact-Anosov flows*.

We will be interested in the sequel with the contact partially hyperbolic diffeomorphisms given by any non-zero time-map of these flows.

### 1.1.2 Partially hyperbolic affine automorphisms of nil-Heis(3)-manifolds

For  $(\lambda, \mu) \in \mathbb{R}^{*2}$ , let us introduce the following automorphism of the Heisenberg group:

$$\varphi_{\lambda, \mu}: \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Heis}(3) \mapsto \begin{pmatrix} 1 & \lambda x & \lambda \mu z \\ 0 & 1 & \mu y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Heis}(3).$$

Let  $\Gamma$  be a cocompact lattice of  $\mathrm{Heis}(3)$ ,  $g \in \mathrm{Heis}(3)$  and  $(\lambda, \mu) \in \mathbb{R}^{*2}$ . If  $g\varphi_{\lambda, \mu}(\Gamma)g^{-1} = \Gamma$ , then the affine automorphism  $L_g \circ \varphi_{\lambda, \mu}$  of  $\mathrm{Heis}(3)$  induces a diffeomorphism of the *nil-Heis(3)-manifold*  $\Gamma \backslash \mathrm{Heis}(3)$ , defined by  $L_g \circ \varphi_{\lambda, \mu}(\Gamma x) = \Gamma(g\varphi_{\lambda, \mu}(x))$ . If we moreover assume that either  $|\lambda| < 1$  and  $|\mu| > 1$ , or the opposite, then  $L_g \circ \varphi_{\lambda, \mu}$  is a contact partially hyperbolic diffeomorphism of  $\Gamma \backslash \mathrm{Heis}(3)$ , that we will call a *partially hyperbolic affine automorphism*.

## 1.2 Principal results

Having in mind these examples, we obtain the following result.

**Theorem A.** *Let  $M$  be a closed, connected and orientable three-dimensional manifold, and  $f$  be a contact partially hyperbolic diffeomorphism of  $M$ , i.e.:*

- *the stable, unstable, and central distributions  $E^s$ ,  $E^u$  and  $E^c$  of  $f$  are smooth,*
- *and  $E^s \oplus E^u$  is a contact distribution.*

*If the non-wandering set  $NW(f)$  equals  $M$ , then we have the following description:*

1. *either some finite power of  $f$  is smoothly conjugated to a non-zero time-map of a three-dimensional algebraic contact-Anosov flow,*
2. *or  $f$  lifts by a covering of order at most 4 to a partially hyperbolic affine automorphism of a nil-Heis(3)-manifold.*

The class of diffeomorphisms that we will consider is actually wider than the partially hyperbolic diffeomorphisms. In fact, our geometrical condition is so rigid that the uniformity of the contraction and the expansion will be obtained as a byproduct, without being assumed.

**Definition 1.3.** A distribution  $E$  of a compact manifold  $M$  is *weakly contracted* by a diffeomorphism  $f$  of  $M$ , if for some Riemannian metric on  $M$ , and for any  $x \in M$ , we have:

$$\text{either } \lim_{n \rightarrow +\infty} \|D_x f^n|_E\| = 0, \text{ or } \lim_{n \rightarrow -\infty} \|D_x f^n|_E\| = 0.$$

We emphasize that the “direction” of weak contraction can *a priori* change from point to point, and that the notion of weak contraction is unchanged when replacing  $f$  by  $f^{-1}$ .

With this definition, the main result of this paper is the following.

**Theorem B.** *Let  $M$  be a closed, connected and orientable three-dimensional manifold, endowed with a smooth splitting  $TM = E^\alpha \oplus E^\beta \oplus E^c$ , such that  $E^\alpha \oplus E^\beta$  is a contact distribution. Let  $f$  be a diffeomorphism of  $M$  that preserves this splitting, and such that:*

- *each of the distributions  $E^\alpha$  and  $E^\beta$  is weakly contracted by  $f$ ,*
- *and  $f$  has a dense orbit.*

*Then the conclusions of Theorem A hold. In particular,  $f$  is a partially hyperbolic diffeomorphism.*

Theorem A stated above for partially hyperbolic diffeomorphisms will directly follow from Theorem B by an argument of Brin. This argument is explained in Paragraph 8.2 at the end of the paper, where a refined version of Theorem A is stated that does not use any domination hypothesis on the central direction (see Corollary 8.2). The rest of the paper is devoted to the proof of Theorem B.

## 1.3 A rigid geometric structure preserved by contact partially hyperbolic diffeomorphisms

Theorem A is a discrete-time analog of Theorem 1.1, and the general strategy of its proof shares a lot with the one of Ghys in [Ghy87]. However, the transition from a continuous dynamic to a discrete one will completely change the situation. First of all, contact partially hyperbolic diffeomorphisms do not anymore preserve a contact one-form, and are thus (*a priori*) not conservative. From a geometrical point of view, the difficulties that appear are analog to the ones of a conformal geometry, with contrast to a metric geometry.

In the discrete-time case, an invariant Kanai connection does not anymore exist, which requires to look for less common rigid geometric structures. A contact plane distribution is far from being rigid: according to Darboux's theorem, they are all locally isomorphic. A single smooth one-dimensional distribution in a contact plane distribution is still not sufficient to make it rigid. But in the case of three-dimensional contact partially hyperbolic diffeomorphisms, the pair  $(E^s, E^u)$  of transverse smooth one-dimensional distributions whose sum is contact happens to be a rigid geometric structure, called a *Lagrangian contact structure*.

The usual machinery of differential geometry being not available, this uncommon structure requires specific tools. The usual invariant linear connection will be replaced by another type of connection called a *Cartan connection*, that does unfortunately not live on the base manifold but on a principal bundle called the *Cartan bundle*. The data of this bundle together with its connection define a *Cartan geometry*, invented by Élie Cartan. This Cartan geometry is implicitly used by Ghys in [Ghy87] under the form of the geometry of second-order ordinary differential equations.

The strength of Cartan geometries is to link the Lagrangian contact structures with the *homogeneous model space*  $\mathbf{X} = \mathrm{PGL}_3(\mathbb{R})/\mathbf{P}_{min}$  of complete flags of  $\mathbb{R}^3$  (where  $\mathbf{P}_{min}$  is the subgroup of upper-triangular matrices). In particular, the *flat* Lagrangian contact structures, *i.e.* the ones whose *curvature* identically vanishes, are locally isomorphic to  $\mathbf{X}$  (see Paragraphs 2.2.2 and 2.3.2). The geometry of  $\mathbf{X}$  will thus play a prominent role in our proof.

Let us quote the work of Barbot in [Bar10], where the geometry of  $\mathbf{X}$  and the dynamic of  $\mathrm{PGL}_3(\mathbb{R})$  are studied with a different approach, the purpose being among others to construct Anosov representations in  $\mathrm{PGL}_3(\mathbb{R})$  and compact quotients of open subsets of  $\mathbf{X}$ .

## 1.4 Organization of the paper

This paper is organised in the following way. Section 2 introduces several notions and results about the geometric structures that will be used in the whole paper. At the end of the paper in Paragraph 8.2, we prove Theorem A from Theorem B, and the rest of the paper is devoted to the proof of Theorem B. In Section 3, we begin this proof by showing that the triplet  $\mathcal{S} = (E^\alpha, E^\beta, E^c)$  is *quasi-homogeneous*, *i.e.* locally homogeneous in restriction to a dense open subset  $\Omega$  of  $M$ , and that its isotropy on  $\Omega$  is non-trivial. This implies that the Lagrangian contact structure  $(E^\alpha, E^\beta)$  is *flat*, *i.e.* described by a  $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$ -structure on  $M$ . In Section 4, we refine this description, proving that  $\mathcal{S}|_\Omega$  is locally isomorphic to one of two possible homogeneous models  $(Y_t, \mathcal{S}_t)$  or  $(Y_a, \mathcal{S}_a)$ , defined in Paragraphs 4.2.2 and 4.2.3. This relies on a technical classification of the underlying infinitesimal model, done in Section 5. A critical step of the proof of Theorem B is to show in Section 6 that the open dense subset  $\Omega$  is actually equal to  $M$ , implying that  $M$  has a  $(\mathcal{H}, Y)$ -structure, with two possible models  $(H_t, Y_t)$  or  $(H_a, Y_a)$ . We prove in Section 7 that this  $(\mathcal{H}, Y)$ -structure is complete, implying that  $M$  is a compact quotient  $\Gamma \backslash Y$  of one of these two models, with  $\Gamma$  a discrete subgroup of  $\mathcal{H} = \mathrm{Aut}(Y)$ . This description allows us to conclude the proof of Theorem B in Paragraph 8.1.

## 1.5 Conventions and notations

From now on, every differential geometric object will be supposed to be smooth (*i.e.*  $C^\infty$ ) if nothing is precised, and the manifolds will be supposed to be boundaryless. We denote by  $\pi_M: \tilde{M} \rightarrow M$  the universal cover of a manifold  $M$ . The flow of a vector field  $X \in \mathfrak{X}(M)$  is denoted by  $(\varphi_X^t)$ . The Lie algebra of a Lie group  $G$  is denoted by  $\mathfrak{g}$ , and for any  $v \in \mathfrak{g}$ , we denote by  $\tilde{v}$  the left-invariant vector field of  $G$  generated by  $v$ . If  $\Theta: G \times M \rightarrow M$  is a smooth group action (on the left or the right) of  $G$  on  $M$ , then the orbital map of the action at  $x \in M$  is denoted by  $\theta_x = \Theta(\cdot, x)$ , and we denote by  $L_g = \Theta(g, \cdot)$  the translation by  $g \in G$  if the action is on the left (respectively by  $R_g$  if the action is on the right). For any  $v \in \mathfrak{g}$  we denote by  $v^\dagger \in \mathfrak{X}(M)$  the *fundamental vector field of the action generated by  $v$* , defined by

$v^\dagger(x) = D_e \theta_x(v)$ . Finally, for any subset  $Q$  of  $\mathbb{R}^{n+1}$  we denote by  $[Q]$  the projection in  $\mathbb{RP}^n$  of the linear subspace of  $\mathbb{R}^{n+1}$  generated by  $Q$ .

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## 2 Three-dimensional Lagrangean contact structures

The rigid geometric structures that will be studied in the rest of this paper are the following.

**Definition 2.1.** A *Lagrangean contact structure*  $\mathcal{L}$  on a three-dimensional manifold  $M$  is a pair  $\mathcal{L} = (E^\alpha, E^\beta)$  of transverse one-dimensional smooth distributions, such that  $E^\alpha \oplus E^\beta$  is a contact distribution. An *enhanced Lagrangean contact structure*  $\mathcal{S}$  on  $M$  is a triplet  $\mathcal{S} = (E^\alpha, E^\beta, E^c)$  of one-dimensional smooth distributions such that  $TM = E^\alpha \oplus E^\beta \oplus E^c$ , and  $E^\alpha \oplus E^\beta$  is a contact distribution.

A (local) isomorphism between two Lagrangean contact structures is a (local) diffeomorphism that preserves individually the distributions  $\alpha$  and  $\beta$ , and the (local) isomorphisms of enhanced Lagrangean contact structures preserve in addition the central distribution  $E^c$ .

We first define what will be for us the most important example of three-dimensional Lagrangean contact structure.

### 2.1 Homogeneous model space

We will call *projective line* the projection in  $\mathbb{RP}^2$  of a plane of  $\mathbb{R}^3$ , and we denote by  $\mathbb{RP}_*^2$  the set of projective lines of  $\mathbb{RP}^2$  (called the *dual projective plane*). A *pointed projective line* is a pair  $(m, D)$  with  $D \in \mathbb{RP}_*^2$  and  $m \in D$ , and we denote by

$$\mathbf{X} = \left\{ (m, D) \mid D \in \mathbb{RP}_*^2, m \in D \right\} \subset \mathbb{RP}^2 \times \mathbb{RP}_*^2$$

the *space of pointed projective lines*. In other words,  $\mathbf{X}$  is the space of complete flags of  $\mathbb{R}^3$ . We will denote in the whole paper by

$$\mathbf{G} = \mathrm{PGL}_3(\mathbb{R})$$

the group of projective transformations of  $\mathbb{RP}^2$ . As the projective action of  $\mathbf{G}$  on  $\mathbb{RP}^2$  and  $\mathbb{RP}_*^2$  preserves the incidence relation  $m \in D$ , it induces a natural diagonal action of  $\mathbf{G}$  on  $\mathbf{X} \subset \mathbb{RP}^2 \times \mathbb{RP}_*^2$ . The action of  $\mathbf{G}$  on  $\mathbf{X}$  is *transitive*, and the stabilizer in  $\mathbf{G}$  of the base-point  $o = ([e_1], [e_1, e_2])$  of  $\mathbf{X}$  is the subgroup

$$\mathrm{Stab}_{\mathbf{G}}(o) = \mathbf{P}_{min} = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\} < \mathbf{G}$$

of upper-triangular matrices. From now on, we will identify  $\mathbf{X}$  and  $\mathbf{G}/\mathbf{P}_{min}$  by the orbital map  $\theta_o: \mathbf{G}/\mathbf{P}_{min} \rightarrow \mathbf{X}$  at  $o$ . The space  $\mathbf{X}$  is a  $\mathbb{RP}^1$ -bundle over  $\mathbb{RP}^2$  and  $\mathbb{RP}_*^2$  through the projections

$$\pi_\alpha: (m, D) \in \mathbf{X} \mapsto m \in \mathbb{RP}^2 \text{ and } \pi_\beta: (m, D) \in \mathbf{X} \mapsto D \in \mathbb{RP}_*^2. \quad (2.1)$$

For  $x = (m, D) \in \mathbf{X}$ , we will denote by  $\mathcal{C}^\alpha(x) = \mathcal{C}^\alpha(m)$  (resp.  $\mathcal{C}^\beta(x) = \mathcal{C}^\beta(D)$ ) the fiber of  $x$  with respect to  $\pi_\alpha$  (resp.  $\pi_\beta$ ) and call it the  $\alpha$ -circle (resp.  $\beta$ -circle) of  $x$ . We denote by

$$\mathcal{E}^\alpha = \mathrm{Ker}(D\pi_\alpha) \text{ and } \mathcal{E}^\beta = \mathrm{Ker}(D\pi_\beta)$$

the one-dimensional distributions of  $\mathbf{X}$  respectively tangent to the foliations by  $\alpha$  and  $\beta$ -circles. The sum  $\mathcal{E}^\alpha \oplus \mathcal{E}^\beta$  is contact and we will call  $\mathcal{L}_\mathbf{X} = (\mathcal{E}^\alpha, \mathcal{E}^\beta)$  the *standard Lagrangean contact structure* of  $\mathbf{X}$ .

**Lemma 2.2.** *The group of automorphisms of the standard Lagrangean contact structure  $\mathcal{L}_\mathbf{X}$  is equal to  $\mathbf{G}$ . In particular, the structure  $(\mathbf{X}, \mathcal{L}_\mathbf{X})$  is homogeneous.*

*Proof.* First of all, the action of  $\mathbf{G}$  preserves the foliations of  $\mathbf{X}$  by  $\alpha$  and  $\beta$ -circles, *i.e.* preserves  $\mathcal{L}_\mathbf{X}$ . Conversely, if  $f$  is a diffeomorphism of  $\mathbf{X}$  preserving  $\mathcal{L}_\mathbf{X}$ , the fact that  $f$  preserves the foliation by  $\alpha$ -circles shows that  $f$  can be pushed down by  $\pi_\alpha$  to a diffeomorphism  $\bar{f}$  of  $\mathbb{RP}^2$ . Since  $f$  moreover preserves the foliation by  $\beta$ -circles,  $\bar{f}$  maps any projective line to a projective line. This implies that  $\bar{f}$  is a projective transformation according to a classical result of projective geometry (proved for example in [Sam89, Theorem 7 p.32]), *i.e.* that  $f$  is induced by the action of an element of  $\mathbf{G}$ .  $\square$

## 2.2 Lagrangean-contact structures as Cartan geometries

We now introduce the Cartan geometries modelled on the homogeneous space  $\mathbf{X} = \mathbf{G}/\mathbf{P}_{min}$ , and make the link with Lagrangean contact structures. This notion will be our principal technical tool to deal with Lagrangean contact structures. We refer the reader to [Sha97] or [ČS09] for further details about Cartan geometries in a more general context.

### 2.2.1 Cartan geometries modelled on $\mathbf{G}/\mathbf{P}_{min}$

**Definition 2.3.** A *Cartan geometry*  $\mathcal{C} = (\hat{M}, \omega)$  modelled on  $\mathbf{G}/\mathbf{P}_{min}$  on a three-dimensional manifold  $M$  is the data of a  $\mathbf{P}_{min}$ -principal bundle over  $M$  denoted by  $\pi: \hat{M} \rightarrow M$  and called the *Cartan bundle*, together with a  $\mathfrak{sl}_3$ -valued one-form  $\omega: T\hat{M} \rightarrow \mathfrak{sl}_3$  on  $\hat{M}$  called the *Cartan connection*, that satisfies the three following properties:

1.  $\omega$  defines a *parallelism* of  $\hat{M}$ , *i.e.* for any  $\hat{x} \in \hat{M}$ ,  $\omega_{\hat{x}}$  is a linear isomorphism from  $T_{\hat{x}}\hat{M}$  to  $\mathfrak{sl}_3$ ,
2.  $\omega$  reproduces the fundamental vector fields of the right action of  $\mathbf{P}_{min}$ , *i.e.* for any  $v \in \mathfrak{sl}_3$  and  $\hat{x} \in \hat{M}$  we have:  $v^\dagger(\hat{x}) = \frac{d}{dt}|_{t=0} \hat{x} \cdot e^{tv} = \omega_{\hat{x}}^{-1}(v)$ ,
3. and  $\omega$  is  $\mathbf{P}_{min}$ -equivariant, *i.e.* for any  $p \in \mathbf{P}_{min}$  and  $\hat{x} \in \hat{M}$  we have:  $R_p^* \omega = \text{Ad}(p)^{-1} \circ \omega$  (where  $\text{Ad}(p)$  stands for the adjoint action of  $p$  on  $\mathfrak{sl}_3$ ).

A (local) automorphism  $f$  of the Cartan geometry  $\mathcal{C}$  between two open sets  $U$  and  $V$  of  $M$  is a (local) diffeomorphism from  $U$  to  $V$  that lifts to a  $\mathbf{P}_{min}$ -equivariant (local) diffeomorphism  $\hat{f}$  between  $\pi^{-1}(U)$  and  $\pi^{-1}(V)$ , such that  $\hat{f}$  preserves the Cartan connexion  $\omega$  (*i.e.*  $\hat{f}^* \omega = \omega$ ).

*Example 2.4.* The homogeneous model space  $\mathbf{X}$  is endowed with the *Cartan geometry of the model*  $\mathcal{C}_\mathbf{X} = (\mathbf{G}, \omega_\mathbf{G})$ , given by the canonical  $\mathbf{P}_{min}$ -bundle  $\pi_\mathbf{G}: \mathbf{G} \rightarrow \mathbf{G}/\mathbf{P}_{min} = \mathbf{X}$  over  $\mathbf{X}$ , together with the *Maurer-Cartan form*  $\omega_\mathbf{G}: T\mathbf{G} \rightarrow \mathfrak{sl}_3$  (defined by  $\omega_\mathbf{G}(\tilde{v}) \equiv v$  on the left-invariant vector fields of  $\mathbf{G}$ ).

We fix for the rest of this subsection a Cartan geometry  $(M, \mathcal{C}) = (M, \hat{M}, \omega)$  modelled on  $\mathbf{G}/\mathbf{P}_{min}$ .

### 2.2.2 Curvature of a Cartan geometry

The following definition replaces the curvature of a Riemannian metric in the case of Cartan geometries.



**Definition 2.5.** The *curvature form* of  $\mathcal{C}$  is the  $\mathfrak{sl}_3$ -valued two-form  $\Omega$  of  $\hat{M}$  defined by the following relation for two vector fields  $X$  and  $Y$  on  $\hat{M}$ :

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]. \quad (2.2)$$

Thanks to the Cartan connection, the curvature form  $\Omega$  is equivalent to a *curvature map*  $K: \hat{M} \rightarrow \text{End}(\Lambda^2 \mathfrak{sl}_3, \mathfrak{sl}_3)$  on  $\hat{M}$  (that we will often simply call the *curvature* of  $\mathcal{C}$ ), having values in the vector space of  $\mathfrak{sl}_3$ -valued alternated bilinear maps on  $\mathfrak{sl}_3$ , and defined by the following relation for  $\hat{x} \in \hat{M}$  and  $v, w \in \mathfrak{sl}_3$ :

$$K_{\hat{x}}(v, w) = \Omega(\omega_{\hat{x}}^{-1}(v), \omega_{\hat{x}}^{-1}(w)). \quad (2.3)$$

We will say that the Cartan geometry  $\mathcal{C}$  (or the Cartan connection  $\omega$ ) is *torsion-free* if  $K_{\hat{x}}(v, w) \in \mathfrak{p}_{min}$  for any  $\hat{x} \in \hat{M}$  and  $v, w \in \mathfrak{sl}_3$ .

If  $v$  or  $w$  is tangent to the fiber of the principal bundle  $\hat{M}$ , then the curvature form satisfies  $\Omega(v, w) = 0$  (this is proved in [Sha97, Chapter 5 Corollary 3.10]). Since  $\omega$  maps the tangent space of the fibers of  $\hat{M}$  to  $\mathfrak{p}_{min}$  (because the fundamental vector fields are  $\omega$ -invariant), this implies that the curvature  $K(v, w)$  vanishes whenever  $v$  or  $w$  is in  $\mathfrak{p}_{min}$ . As a consequence at any point  $\hat{x} \in \hat{M}$ ,  $K_{\hat{x}}$  induces a  $\mathfrak{sl}_3$ -valued alternated bilinear map on  $\mathfrak{sl}_3/\mathfrak{p}_{min}$ , and we will identify in the sequel  $K$  with the induced map

$$K: \hat{M} \rightarrow \text{End}(\Lambda^2(\mathfrak{sl}_3/\mathfrak{p}_{min}), \mathfrak{sl}_3). \quad (2.4)$$

The adjoint action of  $\mathbf{P}_{min}$  induces a linear left action on  $\text{End}(\Lambda^2(\mathfrak{sl}_3/\mathfrak{p}_{min}), \mathfrak{sl}_3)$  defined for  $p \in \mathbf{P}_{min}$  and  $K \in \text{End}(\Lambda^2(\mathfrak{sl}_3/\mathfrak{p}_{min}), \mathfrak{sl}_3)$  by

$$p \cdot K: u \wedge v \mapsto \text{Ad}(p) \cdot (K(\overline{\text{Ad}}(p)^{-1} \cdot u, \overline{\text{Ad}}(p)^{-1} \cdot v)). \quad (2.5)$$

Using the linear right action of  $\mathbf{P}_{min}$  on  $\text{End}(\Lambda^2(\mathfrak{sl}_3/\mathfrak{p}_{min}), \mathfrak{sl}_3)$  defined by  $K \cdot p := p^{-1} \cdot K$ ,  $K$  is  $\mathbf{P}_{min}$ -equivariant (this is proved in [Sha97, Chapter 5 Lemma 3.23]). Moreover,  $K$  is preserved by any local automorphism  $f$  of the Cartan geometry (*i.e.*  $K \circ \hat{f} = K$  for any such local automorphism).

### 2.2.3 Lagrangean-contact structure induced by a Cartan geometry

At any point  $x \in M$  and for any  $\hat{x} \in \pi^{-1}(x)$ , we denote by  $i_{\hat{x}}: T_x M \rightarrow \mathfrak{sl}_3/\mathfrak{p}_{min}$  the unique isomorphism satisfying

$$i_{\hat{x}} \circ D_{\hat{x}} \pi = \overline{\omega}_{\hat{x}}, \quad (2.6)$$

where  $\overline{\omega}$  denotes the projection of  $\omega$  on  $\mathfrak{sl}_3/\mathfrak{p}_{min}$ . Since the adjoint action of  $\mathbf{P}_{min}$  preserves  $\mathfrak{p}_{min}$ , it induces a representation  $\overline{\text{Ad}}: \mathbf{P}_{min} \rightarrow \text{GL}(\mathfrak{sl}_3/\mathfrak{p}_{min})$  on  $\mathfrak{sl}_3/\mathfrak{p}_{min}$ , and the  $\mathbf{P}_{min}$ -equivariance of  $\omega$  implies the following relation for any  $p \in \mathbf{P}_{min}$ :

$$i_{\hat{x} \cdot p} = \overline{\text{Ad}}(p)^{-1} \circ i_{\hat{x}}. \quad (2.7)$$

This relation shows that any  $\overline{\text{Ad}}(\mathbf{P}_{min})$ -invariant object on  $\mathfrak{sl}_3/\mathfrak{p}_{min}$  gives rise, through the isomorphisms  $i_{\hat{x}}$ , to a well-defined object on the tangent bundle of  $M$ . Let us apply this idea to define a Lagrangean contact structure on  $M$  associated to the Cartan geometry  $\mathcal{C}$ . We introduce

$$e_{\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, e_{\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } e_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2.8)$$

defining a basis  $(\bar{e}_{\alpha}, \bar{e}_{\beta}, \bar{e}_0)$  of  $\mathfrak{sl}_3/\mathfrak{p}_{min}$ , in which the matrix of the adjoint action of

$$p = \begin{pmatrix} a & x & z \\ 0 & a^{-1}b^{-1} & y \\ 0 & 0 & b \end{pmatrix} \in \mathbf{P}_{min}$$

is equal to:

$$\text{Mat}_{(\bar{e}_\alpha, \bar{e}_\beta, \bar{e}_0)}(\overline{\text{Ad}}(p)) = \begin{pmatrix} a^{-2}b^{-1} & 0 & a^{-1}y \\ 0 & ab^2 & -b^2x \\ 0 & 0 & a^{-1}b \end{pmatrix}. \quad (2.9)$$

In particular, the adjoint action of  $\mathbf{P}_{\min}$  individually preserves the lines  $\mathbb{R}\bar{e}_\alpha$  and  $\mathbb{R}\bar{e}_\beta$  of  $\mathfrak{sl}_3/\mathfrak{p}_{\min}$ . Together with the relation (2.7), this shows that for  $x \in M$ , the lines  $i_{\hat{x}}^{-1}(\mathbb{R}\bar{e}_\alpha)$  and  $i_{\hat{x}}^{-1}(\mathbb{R}\bar{e}_\beta)$  of  $T_x M$  do not depend on the lift  $\hat{x}$  of  $x$ . The Cartan geometry  $\mathcal{C}$  induces thus two one-dimensional distributions  $E_{\mathcal{C}}^\alpha(x) = i_{\hat{x}}^{-1}(\mathbb{R}\bar{e}_\alpha)$  and  $E_{\mathcal{C}}^\beta(x) = i_{\hat{x}}^{-1}(\mathbb{R}\bar{e}_\beta)$  on  $M$ .

**Lemma 2.6.** *Any torsion-free Cartan geometry  $(M, \mathcal{C})$  modelled on  $\mathbf{G}/\mathbf{P}_{\min}$  induces a Lagrangean contact structure  $(E_{\mathcal{C}}^\alpha, E_{\mathcal{C}}^\beta)$  on the three-dimensional base manifold  $M$ .*

*Sketch of proof.* For  $x \in M$ , considering a local section of the Cartan bundle over  $x$ , we can push down by  $\pi$  the  $\omega$ -constant vector fields  $\bar{e}_\alpha$  and  $\bar{e}_\beta$  of  $\hat{M}$  (characterized by  $\omega(\bar{e}_\epsilon) \equiv e_\epsilon$ ) to local vector fields  $X_\alpha$  and  $X_\beta$  of  $M$  defined on a neighbourhood of  $x$ , that respectively generate the distributions  $E_{\mathcal{C}}^\alpha$  and  $E_{\mathcal{C}}^\beta$ . If the curvature of  $\mathcal{C}$  has values in  $\mathfrak{p}_{\min}$ , then the identity  $\omega([\bar{e}_\alpha, \bar{e}_\beta]) = [e_\alpha, e_\beta] - K(e_\alpha, e_\beta)$  (deduced from Cartan's formula for the differential of a one-form) implies that  $[X_\alpha, X_\beta] \notin \text{Vect}(X_\alpha, X_\beta)$  in the neighbourhood of  $x$ . This shows that  $E_{\mathcal{C}}^\alpha \oplus E_{\mathcal{C}}^\beta$  is contact, and concludes the proof of the lemma.  $\square$

*Remark 2.7.* In the case of the homogeneous model space, the Lagrangean contact structure  $(E_{\mathcal{C}_X}^\alpha, E_{\mathcal{C}_X}^\beta)$  induced by  $\mathcal{C}_X$  is the standard Lagrangean contact structure  $\mathcal{L}_X$ .

## 2.3 Normal Cartan geometry of a Lagrangean contact structure

Actually, any three-dimensional Lagrangean contact structure is induced by a torsion-free Cartan geometry modelled on  $\mathbf{G}/\mathbf{P}_{\min}$ . This equivalence between three-dimensional Lagrangean contact structures and Cartan geometries modelled on  $\mathbf{G}/\mathbf{P}_{\min}$  was discovered by Élie Cartan, who developed this notion and after whom these geometries are named.

### 2.3.1 Equivalence problem for Lagrangean contact structures

A given three-dimensional Lagrangean-contact structure is induced by several Cartan connections, but to obtain an equivalence between both formulations, we have to choose a particular one. This choice will be done through a *normalisation condition* on the curvature. Using the basis  $(\bar{e}_\alpha \wedge \bar{e}_0, \bar{e}_\beta \wedge \bar{e}_0, \bar{e}_\alpha \wedge \bar{e}_\beta)$  of  $\Lambda^2(\mathfrak{sl}_3/\mathfrak{p}_{\min})$ , we define the following four-dimensional subspace of  $\text{End}(\Lambda^2(\mathfrak{sl}_3/\mathfrak{p}_{\min}), \mathfrak{sl}_3)$ :

$$W_K = \left\{ K: \bar{e}_\alpha \wedge \bar{e}_0 \mapsto \begin{pmatrix} 0 & 0 & K_\alpha^\alpha \\ 0 & 0 & K_\alpha^\beta \\ 0 & 0 & 0 \end{pmatrix}, \bar{e}_\beta \wedge \bar{e}_0 \mapsto \begin{pmatrix} 0 & K_\beta & K_\beta^\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{e}_\alpha \wedge \bar{e}_\beta \mapsto 0 \mid (K_\alpha, K_\alpha^\alpha, K_\beta, K_\beta^\beta) \in \mathbb{R}^4 \right\}. \quad (2.10)$$

The linear action of  $\mathbf{P}_{\min}$  preserves  $W_K$ , that will be called the *space of normal curvatures*. Theorem 2.8 below is proved in [DK16, Theorem 3 p.14], where the normalisation condition is explicitly calculated through Cartan's *method of equivalence* (see also [ČS09, Theorem 3.1.14 p.271 and Paragraph 4.2.3] that makes the link with general parabolic Cartan geometries).

**Theorem 2.8** (É. Cartan, [DK16], [ČS09]). *For any Lagrangean contact structure  $\mathcal{L}$  on a three-dimensional manifold  $M$ , there exists a torsion-free Cartan geometry modelled on  $\mathbf{G}/\mathbf{P}_{\min}$  inducing  $\mathcal{L}$  on  $M$ , and whose curvature map has values in the space  $W_K$  of normal curvatures. Such a Cartan geometry is unique (up to action of principal bundle automorphisms covering the identity on  $M$ ), and will be called the normal Cartan geometry of  $\mathcal{L}$ .*

Furthermore, if  $(M_1, \mathcal{L}_1)$  and  $(M_2, \mathcal{L}_2)$  are two three-dimensional Lagrangean contact structures, and  $\mathcal{C}_1, \mathcal{C}_2$  are the associated normal Cartan geometries, then the (local) isomorphisms

between  $\mathcal{L}_1$  and  $\mathcal{L}_2$  and the (local) isomorphism between  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are the same (this a direct consequence of the quasi-unicity of the normal Cartan geometry). The curvature map  $K: \hat{M} \rightarrow W_K$  of the normal Cartan geometry of a three-dimensional Lagrangean contact structure  $\mathcal{L}$  will simply be called *the curvature of  $\mathcal{L}$* .

### 2.3.2 Flat Lagrangean contact structures

The homogeneous model space  $(\mathbf{X}, \mathcal{L}_{\mathbf{X}})$  verifies the following analog of Liouville's theorem.

**Theorem 2.9.** *For any connected open subsets  $U$  and  $V$  of the homogeneous model space  $\mathbf{X}$ , and any diffeomorphism  $f$  from  $U$  to  $V$  that preserves its standard Lagrangean contact structure  $\mathcal{L}_{\mathbf{X}}$ , there exists  $g \in \mathbf{G} = \text{Aut}(\mathbf{X}, \mathcal{L}_{\mathbf{X}})$  such that  $f$  is the restriction to  $U$  of the translation by  $g$ .*

*Proof.* The Maurer-Cartan form satisfies for any tangent vectors  $v$  and  $w$  the structural equation  $d\omega_{\mathbf{G}}(v, w) + [\omega_{\mathbf{G}}(v), \omega_{\mathbf{G}}(w)] = 0$  (see [Sha97, §3.3 p.108]), implying that the curvature of the Cartan connection  $\omega_{\mathbf{G}}$  is zero. The Cartan geometry of the model satisfies thus the normalisation condition of Theorem 2.8, *i.e.* is the normal Cartan geometry of  $\mathcal{L}_{\mathbf{X}}$  (see Remark 2.7). According to Theorem 2.8, any local isomorphism of  $\mathcal{L}_{\mathbf{X}}$  between two connected open subset  $U$  and  $V$  of  $\mathbf{X}$  lifts therefore to a local isomorphism of the Cartan geometry  $\mathcal{C}_{\mathbf{X}}$  between  $\pi_{\mathbf{G}}^{-1}(U)$  and  $\pi_{\mathbf{G}}^{-1}(V)$ . Such an automorphism is the left translation by an element of  $\mathbf{G}$  according to [Sha97, Chapter 5 Theorem 5.2], which concludes the proof.  $\square$

A three-dimensional Lagrangean contact structure  $(M, \mathcal{L})$  is *flat* if its curvature vanishes identically. According to the proof of Theorem 2.9, the model space is flat, and since this property is local, any Lagrangean contact structure locally isomorphic to  $(\mathbf{X}, \mathcal{L}_{\mathbf{X}})$  is flat.

The power of Cartan geometries lies in the converse of this statement: any flat three-dimensional Lagrangean contact structure  $\mathcal{L}$  is locally isomorphic to the homogeneous model space (see [Sha97, Theorem 5.1 and Theorem 5.2 p. 212]). There exists in this case an atlas of charts from  $M$  to  $\mathbf{X}$  consisting of local isomorphisms of Lagrangean contact structures from  $\mathcal{L}$  to  $\mathcal{L}_{\mathbf{X}}$ , and whose transition maps are restrictions of left translations by elements of  $\mathbf{G}$  (according to Theorem 2.9). A maximal atlas satisfying these conditions is called a  $(\mathbf{G}, \mathbf{X})$ -structure on  $M$ . Any  $(\mathbf{G}, \mathbf{X})$ -structure conversely induces on  $M$  a Lagrangean contact structure  $\mathcal{L}$  locally isomorphic to  $\mathcal{L}_{\mathbf{X}}$ , whose charts are local isomorphisms from  $\mathcal{L}$  to  $\mathcal{L}_{\mathbf{X}}$ .

**Theorem 2.10.** *Any flat three-dimensional Lagrangean contact structure  $(M, \mathcal{L})$  is induced by a  $(\mathbf{G}, \mathbf{X})$ -structure on  $M$ .*

Denoting by  $\pi: \tilde{M} \rightarrow M$  the universal cover of  $M$ , we recall that any  $(\mathbf{G}, \mathbf{X})$ -structure on  $M$  is described by a local diffeomorphism  $\delta: \tilde{M} \rightarrow \mathbf{X}$  called the *developping map*, that is equivariant for a morphism  $\rho: \pi_1(M) \rightarrow \mathbf{G}$  called the *holonomy morphism* (see for example [Thu97, §3.4 p.139-141]). Moreover for any  $g \in \mathbf{G}$ , the pair  $(g \circ \delta, g\rho g^{-1})$  of developping map and holonomy morphism describes the same  $(\mathbf{G}, \mathbf{X})$ -structure. The Lagrangean contact structure  $\mathcal{L}$  induced by a  $(\mathbf{G}, \mathbf{X})$ -structure is characterized by:  $\delta^* \mathcal{L}_{\mathbf{X}} = \pi_M^* \mathcal{L}$ .

### 2.3.3 Harmonic curvature

For  $K \in W_K$  an element of the space of normal curvatures defined by

$$K: \bar{e}_\alpha \wedge \bar{e}_0 \mapsto \begin{pmatrix} 0 & 0 & K^\alpha \\ 0 & 0 & K_\alpha \\ 0 & 0 & 0 \end{pmatrix}, \bar{e}_\beta \wedge \bar{e}_0 \mapsto \begin{pmatrix} 0 & K_\beta & K^\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{e}_\alpha \wedge \bar{e}_\beta \mapsto 0,$$

and

$$p = \begin{pmatrix} a & x & z \\ 0 & a^{-1}b^{-1} & y \\ 0 & 0 & b \end{pmatrix} \in \mathbf{P}_{min},$$

the adjoint action (2.9) of  $\mathbf{P}_{min}$  given in Paragraph 2.3 enables to compute the components  $\cdot_\alpha$  and  $\cdot_\beta$  of  $p \cdot K \in W_K$ :

$$(p \cdot K)_\alpha = a^5 b K_\alpha \text{ and } (p \cdot K)_\beta = a^{-1} b^{-5} K_\beta. \quad (2.11)$$

These expressions show in particular that the two-dimensional subspace  $W_H = \{K \in W_K \mid K_\alpha = K_\beta = 0\}$  of  $W_K$  is preserved by the linear action of  $\mathbf{P}_{min}$ .

**Proposition 2.11.** *If the curvature map of a three-dimensional Lagrangean contact structure  $\mathcal{L}$  has values in the subspace  $W_H$  (i.e. if  $K_\alpha$  and  $K_\beta$  identically vanish), then  $\mathcal{L}$  is flat.*

We will use in the proof of this result the *graduation* of  $\mathfrak{sl}_3$  given by the following block-decomposition:

$$\mathfrak{sl}_3 = \begin{pmatrix} (\mathfrak{sl}_3)_0 & (\mathfrak{sl}_3)_1 & (\mathfrak{sl}_3)_2 \\ (\mathfrak{sl}_3)_{-1} & (\mathfrak{sl}_3)_0 & (\mathfrak{sl}_3)_1 \\ (\mathfrak{sl}_3)_{-2} & (\mathfrak{sl}_3)_{-1} & (\mathfrak{sl}_3)_0 \end{pmatrix}. \quad (2.12)$$

The graduation property simply means that the components of the splitting  $\mathfrak{sl}_3 = \bigoplus_{i=-2}^2 (\mathfrak{sl}_3)_i$  satisfy  $[(\mathfrak{sl}_3)_i, (\mathfrak{sl}_3)_j] \subset (\mathfrak{sl}_3)_{i+j}$  for any  $i$  and  $j$  (where  $(\mathfrak{sl}_3)_i = \{0\}$  for any  $|i| > 2$ ). This graduation of  $\mathfrak{sl}_3$  gives rise to a filtration defined by  $\mathfrak{sl}_3^i = \bigoplus_{j \geq i} (\mathfrak{sl}_3)_j$ , with respect to which  $\mathfrak{sl}_3$  is a filtered Lie algebra, i.e.  $[\mathfrak{sl}_3^i, \mathfrak{sl}_3^j] \subset \mathfrak{sl}_3^{i+j}$  (with  $\mathfrak{sl}_3^i = \mathfrak{sl}_3$  for  $i \leq -2$  and  $\mathfrak{sl}_3^i = \{0\}$  for  $i > 2$ ).

*Proof of Proposition 2.11.* Let  $(M, \hat{M}, \omega)$  be the normal Cartan geometry of  $\mathcal{L}$ , and let us introduce the following basis of  $\mathfrak{sl}_3$ :  $e_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $e_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ ,  $e_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ ,  $e^\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $e^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $e^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ , that we denote by  $\mathcal{B}$ . The graduation of  $\mathfrak{sl}_3$  implies  $[e^0, (\mathfrak{sl}_3)_1 \oplus (\mathfrak{sl}_3)_2] = \{0\}$ , and we have the following Lie brackets relations between the elements of  $\mathcal{B}$ :  $[e^0, e_0] = e_1 + e_2$ ,  $[e^0, e_\alpha] = e^\beta$ ,  $[e^0, e_\beta] = -e^\alpha$ ,  $[e^0, e_1] = [e^0, e_2] = -e^0$ . We denote the coordinate of the Cartan connection  $\omega$  with respect to an element  $e$  of the basis  $\mathcal{B}$  as a real-valued one-form  $\omega_e$  on  $\hat{M}$ . We thus have  $\omega = \sum_{e \in \mathcal{B}} \omega_e e$ , and in the same way, the curvature form  $\Omega$  of  $\omega$  will be denoted as  $\Omega = \sum_{e \in \mathcal{B}} \Omega_e e$ , where the  $\Omega_e$ 's are real-valued two-forms on  $\hat{M}$ . According to the normalisation condition (2.10), if  $K_\alpha = K_\beta = 0$  identically, then the only non-zero two-form  $\Omega_e$  is  $\Omega^0 = K^\alpha \omega_\alpha \wedge \omega_0 + K^\beta \omega_\beta \wedge \omega_0$ . The curvature form satisfies moreover the Bianchi identity  $d\Omega = [\Omega, \omega]$ , where  $[\Omega, \omega] = L \circ (\Omega \wedge \omega)$  with  $L: v \otimes w \in \mathfrak{sl}_3 \otimes \mathfrak{sl}_3 \mapsto [v, w] \in \mathfrak{sl}_3$  (see [Sha97, Chapter 1.5 p.61 and Chapter 5 Lemma 3.30]). Projecting the Bianchi identity to  $\mathbb{R}e^\beta$  and  $\mathbb{R}e^\alpha$ , we obtain the following equalities:

$$0 = -K^\alpha \omega_\alpha \wedge \omega_0 \wedge \omega_\beta, 0 = K^\beta \omega_\beta \wedge \omega_0 \wedge \omega_\alpha.$$

Since  $(\omega_\alpha, \omega_0, \omega_\beta)$  is at each point  $\hat{x} \in \hat{M}$  a basis of the dual space  $(\omega_{\hat{x}}^{-1}((\mathfrak{sl}_3)_{-2} \oplus (\mathfrak{sl}_3)_{-1}))^*$ , the three-form  $\omega_\alpha \wedge \omega_0 \wedge \omega_\beta$  does not vanish, and the above equalities imply therefore  $K^\alpha = K^\beta = 0$  identically, i.e.  $K = 0$  as announced.  $\square$

*Remark 2.12.* The components  $K_\alpha$  and  $K_\beta$  of the curvature actually encode the *harmonic curvature* of a normal Cartan geometry modelled on  $\mathbf{G}/\mathbf{P}_{min}$ , which is known to be the only obstruction to the flatness for parabolic Cartan geometries. With this point of view, the above Proposition 2.11 is the translation in the specific case of Lagrangean contact structures of a general phenomena arising for any parabolic Cartan geometry (see for example [ČS09, Theorem 3.1.12]).

### 2.3.4 Normal generalized Cartan geometry of an enhanced Lagrangean contact structure

Let  $\mathcal{S} = (E^\alpha, E^\beta, E^c)$  be an enhanced Lagrangean contact structure on a three-dimensional manifold  $M$ , and  $\mathcal{C} = (\hat{M}, \omega)$  be the normal Cartan geometry of the underlying Lagrangean

contact structure  $(E^\alpha, E^\beta)$ . Using the isomorphisms  $i_{\hat{x}}$  defined in (2.6), the transverse distribution  $E^c$  is encoded by the map

$$\varphi: \hat{x} \in \hat{M} \mapsto i_{\hat{x}}(E^c(\pi(\hat{x}))) \in \mathbb{V},$$

having values in the open subset

$$\mathbb{V} = \{L \in \mathbf{P}(\mathfrak{sl}_3/\mathfrak{p}_{min}) \mid L \notin \text{Vect}(\bar{e}_\alpha, \bar{e}_\beta)\}$$

of the projective space  $\mathbf{P}(\mathfrak{sl}_3/\mathfrak{p}_{min})$ . Endowing  $\mathbb{V}$  with the right  $\mathbf{P}_{min}$ -action defined by  $L \cdot p = \overline{\text{Ad}}(p)^{-1}(L)$ ,  $\varphi$  is  $\mathbf{P}_{min}$ -equivariant. Conversely, any  $\mathbf{P}_{min}$ -equivariant application  $\varphi: \hat{M} \rightarrow \mathbb{V}$  defines on  $M$  a distribution  $E^c(\pi(\hat{x})) = i_{\hat{x}}^{-1}(\varphi(\hat{x}))$  which is transverse to the contact plane  $E^\alpha \oplus E^\beta$  of  $\mathcal{L}$ .

**Definition 2.13.** We will call  $(\mathcal{C}, \varphi) = (\hat{M}, \omega, \varphi)$  the *normal generalized Cartan geometry* of the enhanced Lagrangean contact structure  $\mathcal{S}$ .

## 2.4 Killing fields of (enhanced) Lagrangean contact structures

### 2.4.1 Some classical properties of Killing fields

A (local) *Killing field* of a Lagrangean contact structure  $(M, \mathcal{L})$  is a (local) vector field  $X$  of  $M$  whose flow preserves  $\mathcal{L}$ . We will denote by  $\mathfrak{Kill}(U, \mathcal{L})$  the subalgebra of Killing fields of  $\mathcal{L}$  defined on an open subset  $U \subset M$ , and by  $\mathfrak{kill}_{\mathcal{L}}^{loc}(x)$  the Lie algebra of germs of Killing fields of  $\mathcal{L}$  defined on a neighbourhood of  $x$ . The Killing fields of an enhanced Lagrangean contact structure  $\mathcal{S}$  are defined in the same way.

The following statement summarizes some important properties of Killing fields, coming from their description through Cartan geometries and well-known in this context. The results are stated for Lagrangean contact structures, but are true as well for enhanced Lagrangean contact structures.

**Lemma 2.14.** *Let  $M$  be a three-dimensional connected manifold endowed with a Lagrangean contact structure  $\mathcal{L}$ , and  $\mathcal{C} = (\hat{M}, \omega)$  be the normal Cartan geometry of  $\mathcal{L}$ .*

1. *If  $\hat{f}$  is a  $\mathbf{P}_{min}$ -equivariant diffeomorphism of  $\hat{M}$  that covers  $\text{id}_M$  and preserves  $\omega$ , then  $\hat{f} = \text{id}_{\hat{M}}$ . If  $\hat{X}$  is a  $\mathbf{P}_{min}$ -invariant vector field on  $\hat{M}$  whose flow preserves  $\omega$  and whose projection on  $M$  vanishes, then  $\hat{X} = 0$ . As a consequence, the lift of a local automorphism  $f$  (respectively Killing field  $X$ ) of  $\mathcal{L}$  to a  $\mathbf{P}_{min}$ -equivariant diffeomorphism  $\hat{f}$  of  $\hat{M}$  that preserves  $\omega$  (resp. to a  $\mathbf{P}_{min}$ -invariant vector field  $\hat{X}$  on  $\hat{M}$  whose flow preserves  $\omega$ ), is unique.*
2. *If the lift  $\hat{X}$  of a Killing field  $X$  of  $\mathcal{L}$  vanishes at some point  $\hat{x} \in \hat{M}$ , then  $X = 0$ . In other words, the linear map  $X \in \mathfrak{Kill}(M, \mathcal{L}) \mapsto \omega_{\hat{x}}(\hat{X}_{\hat{x}}) \in \mathfrak{sl}_3$  is injective.*
3. *The Lie algebra morphism  $X \in \mathfrak{Kill}(M, \mathcal{L}) \mapsto [X]_x \in \mathfrak{kill}_{\mathcal{L}}^{loc}(x)$  sending a Killing field of  $\mathcal{L}$  to its germ at a point  $x \in M$  is injective.*

*Sketch of proof.* 1. The first assertion is a direct consequence of [ČS09, Proposition 1.5.3] for Cartan geometries modelled on  $\mathbf{G}/\mathbf{P}_{min}$ , and implies the second one.

2. Let us assume that a local automorphism  $\hat{f}$  of  $\mathcal{C}$  fixes a point  $\hat{x} \in \hat{M}$ . Then, since  $\hat{f}$  preserves the parallelism defined by  $\omega$ , a classical argument implies that  $\hat{f}$  is trivial on the connected component of  $\hat{x}$ . This remark easily implies the assertion about Killing fields.

3. According to [BFM09, Lemma 7.1], a local automorphism that is trivial in the neighbourhood of  $x$  is trivial on the connected component of its domain of definition that contains  $x$ . This result easily implies the statement concerning Killing fields.  $\square$

*Remark 2.15.* The third statement of the previous lemma shows in particular that for any connected open neighbourhood  $U$  of  $x \in M$ , the dimension of  $\mathfrak{Kill}(U, \mathcal{L})$  is bounded from above by  $\dim \mathfrak{sl}_3 = 8$ . If we consider a decreasing sequence of connected open neighbourhoods  $U_k$  of  $x$  such that  $\cap_k U_k = \{x\}$ ,  $(\dim \mathfrak{Kill}(U_k, \mathcal{L}))$  is thus a bounded increasing sequence, and is therefore constant for  $k$  large enough. This proves the existence of a connected open neighbourhood  $U$  of  $x$  such that

$$X \in \mathfrak{Kill}(U, \mathcal{L}) \mapsto [X]_x \in \mathfrak{kill}_{\mathcal{L}}^{loc}(x)$$

is a Lie algebra isomorphism.

The following Lemma is the translation of Theorem 2.9 for Killing fields of  $(\mathbf{X}, \mathcal{L}_{\mathbf{X}})$ .

**Lemma 2.16.** *1. At any point  $x \in \mathbf{X}$ , the Lie algebra of local Killing fields of  $\mathcal{L}_{\mathbf{X}}$  at  $x$  is identified with  $\mathfrak{sl}_3$  through the fundamental vector fields of the action of  $\mathbf{G}$ . In other words, the application  $v \in \mathfrak{sl}_3 \mapsto [v^\dagger]_x \in \mathfrak{kill}_{\mathcal{L}_{\mathbf{X}}}^{loc}(x)$  sending  $v \in \mathfrak{sl}_3$  to the germ of  $v^\dagger$  at  $x$ , is an anti-isomorphism of Lie algebras.*

*2. Any local Killing field of  $(\mathbf{X}, \mathcal{L}_{\mathbf{X}})$  defined on a connected neighbourhood of a point  $x \in \mathbf{X}$  is the restriction of a global Killing field defined on  $\mathbf{X}$ . In other words,  $X \in \mathfrak{Kill}(\mathbf{X}, \mathcal{L}_{\mathbf{X}}) \mapsto [X]_x \in \mathfrak{kill}_{\mathcal{L}_{\mathbf{X}}}^{loc}(x)$  is a Lie algebra isomorphism.*

*Proof.* 1. If  $v^\dagger$  is trivial in the neighbourhood of  $x$ , then for any  $t \in \mathbb{R}$ ,  $e^{tv}$  acts trivially on an open neighbourhood of  $x$ . But the action of  $\mathbf{G}$  on  $\mathbf{X}$  is *analytic*: if  $g$  and  $h$  in  $\mathbf{G}$  have the same action on some non-empty open subset of  $\mathbf{X}$ , then  $g = h$  (because the linear subspace generated by the pre-image in  $\mathbb{R}^3$  of a non-empty open subset of  $\mathbb{RP}^2$  is equal to  $\mathbb{R}^3$ ). Therefore,  $e^{tv} = \text{id}$  for any  $t \in \mathbb{R}$ , i.e.  $v = 0$ . The application  $v \mapsto [v^\dagger]_x$  is thus injective, and as  $\dim \mathfrak{kill}_{\mathcal{L}_{\mathbf{X}}}^{loc}(x) \leq \dim \mathfrak{sl}_3$  according to the third assertion of Lemma 2.14, it is an isomorphism. Finally,  $v \mapsto v^\dagger$  is known to be an anti-morphism of Lie algebras.

2. Any local Killing field at  $x$  is the restriction of  $v^\dagger$  for some  $v \in \mathfrak{sl}_3$  according to the first assertion, and extends therefore to a Killing field defined on  $\mathbf{X}$ .  $\square$

#### 2.4.2 Total curvature map of an enhanced Lagrangean contact structure

Let  $(\mathcal{C}, \varphi) = (\hat{M}, \omega, \varphi)$  be the normal Cartan geometry of a three-dimensional enhanced Lagrangean contact structure  $(M, \mathcal{S})$ . With  $K: \hat{M} \rightarrow W_K$  the curvature map of  $\mathcal{C}$ , we define the curvature map

$$\mathcal{K} := (K, \varphi): \hat{M} \rightarrow W_{\mathcal{K}} := W_K \times \mathbb{V},$$

of the enhanced Lagrangean contact structure  $(M, \mathcal{S})$ , which is  $\mathbf{P}_{min}$ -equivariant for the right diagonal action of  $\mathbf{P}_{min}$  on  $W_{\mathcal{K}}$ .

If  $W$  is any manifold endowed with a right action of  $\mathbf{P}_{min}$ , we define  $B(W) := \{(w, l) \mid w \in W, l \in \text{End}(\mathfrak{sl}_3, T_w W)\}$  (this is a vector bundle over  $W$ ), that we endow with the right  $\mathbf{P}_{min}$ -action  $(w, l) \cdot p = (w \cdot p, D_w R_p \circ l \circ \text{Ad}(p))$ . For any smooth  $\mathbf{P}_{min}$ -equivariant map  $\psi: \hat{M} \rightarrow W$ , we define a  $\mathbf{P}_{min}$ -equivariant map  $D^1\psi: \hat{M} \rightarrow B(W)$  encoding the differential of  $\psi$  as follows:  $D^1\psi(\hat{x}) = (\psi(\hat{x}), D_{\hat{x}}\psi \circ \omega_{\hat{x}}^{-1})$ . We then define inductively  $B^{k+1}(W) = B(B^k(W))$  and  $D^{k+1}\psi = D(D^k\psi): \hat{M} \rightarrow B^{k+1}(W)$  for any  $k \in \mathbb{N}$  (with  $B^0(W) = W$  and  $D^0\psi = \psi$ ).

Denoting  $m = \dim \mathfrak{sl}_3 = 8$ , we define  $W_{\mathcal{K}^{tot}} := B^m(W_{\mathcal{K}})$ , and the *total curvature*

$$\mathcal{K}^{tot} := D^m \mathcal{K}: \hat{M} \rightarrow W_{\mathcal{K}^{tot}}$$

of the enhanced Lagrangean contact structure  $\mathcal{S}$ . The total curvature  $\mathcal{K}^{tot}$  is  $\mathbf{P}_{min}$ -equivariant and preserved by local automorphisms of  $\mathcal{S}$  (i.e. for any such local automorphism  $f$  we have  $\mathcal{K}^{tot} \circ \hat{f} = \mathcal{K}^{tot}$ ). We also define for  $k \in \mathbb{N}^*$  the space of Killing generators of order  $k$  by  $\text{Kill}^k(\hat{x}) = \omega_{\hat{x}}(\text{Ker}(D_{\hat{x}} D^{k-1} \mathcal{K})) \subset \mathfrak{sl}_3$ , and the space of *Killing generators of total order* by  $\text{Kill}^{tot}(\hat{x}) = \text{Kill}^{m+1}(\hat{x}) = \omega_{\hat{x}}(\text{Ker}(D_{\hat{x}} \mathcal{K}^{tot})) \subset \mathfrak{sl}_3$ .

### 2.4.3 Gromov's theory

The *integrability locus*  $\hat{M}^{int}$  of  $\hat{M}$  is the set of points  $\hat{x} \in \hat{M}$  such that for any  $v \in \text{Kill}^{tot}(\hat{x})$ , there exists a local Killing field  $X$  of  $\mathcal{S}$  defined around  $\pi(\hat{x})$  and such that  $\omega_{\hat{x}}(\hat{X}_{\hat{x}}) = v$ . It is easy to check that  $\hat{M}^{int}$  is a  $\mathbf{P}_{min}$ -equivariant set, and we define the *integrability locus* of  $M$  as  $M^{int} = \pi(\hat{M}^{int})$ .

**Theorem 2.17** (Integrability theorem). *Let  $(M, \mathcal{S})$  be a three-dimensional enhanced Lagrangean contact structure of total curvature  $\mathcal{K}^{tot}$ , and  $\hat{M}$  be its normal Cartan bundle. Then the integrability locus  $\hat{M}^{int}$  of  $\hat{M}$  is equal to the set of points  $\hat{x} \in \hat{M}$  where the rank of  $D_{\hat{x}}\mathcal{K}^{tot}$  is locally constant. In particular,  $\hat{M}^{int}$  is open and dense, and so is the integrability locus  $M^{int}$  of  $M$ .*

Gromov investigates in [Gro88] the integration of “jets” of Killing fields for very general rigid geometric structures, and proves results related to the above Theorem. In the case of three-dimensional enhanced Lagrangean contact structures, the equivalence with normal generalized Cartan geometries allows to avoid the notion of jets of Killing fields, replaced by the one of Killing generators of total order. In this setting, Theorem 2.17 is a consequence of [Pec16, Theorem 4.19]. We use here a modification of the statement of Pecastaing proved by Frances in [Fra16, Theorem 2.2]. The proof of Frances statement for *generalized* Cartan geometries is straightforward by following the lines of the proof he does for Cartan geometries, and using [Pec16, Lemma 4.20 and Lemma 4.9].

## 3 Quasi homogeneity and flatness

From now on and until the end of this paper, we are under the hypotheses of Theorem B and we adopt its notations. The manifold  $M$  is thus three-dimensional, closed, connected and orientable,  $\mathcal{S} = (E^\alpha, E^\beta, E^c)$  is an enhanced Lagrangean contact structure on  $M$ , and we will denote by  $\mathcal{L} = (E^\alpha, E^\beta)$  the underlying Lagrangean contact structure. Finally,  $f$  is an automorphism of  $(M, \mathcal{S})$  such that:

- each of the distributions  $E^\alpha$  and  $E^\beta$  is weakly contracted by  $f$  (see Definition 1.3),
- and  $f$  has a dense orbit.

In particular, the non-wandering set  $NW(f) = NW(f^{-1})$  equals  $M$ , and the set  $\text{Rec}(f)$  (respectively  $\text{Rec}(f^{-1})$ ) of recurrent points of  $f$  (respectively  $f^{-1}$ ) is thus a dense  $G_\delta$ -subset of  $M$ . In particular,  $\text{Rec}(f) \cap \text{Rec}(f^{-1})$  is dense in  $M$  as well.

### 3.1 Quasi homogeneity of the enhanced Lagrangean contact structure

At a point  $x \in M$ , we introduce the subalgebra

$$\mathfrak{is}_S^{loc}(x) = \left\{ X \in \mathfrak{kill}_S^{loc}(x) \mid X(x) = 0 \right\} \quad (3.1)$$

of local Killing fields vanishing at  $x$ , that we call the *isotropy subalgebra* of  $\mathcal{S}$  at  $x$ .

**Definition 3.1.** The  $\text{Kill}^{loc}$ -orbit (for  $\mathcal{S}$ , respectively  $\mathcal{L}$ ) of a point  $x \in M$  is the set of points that can be reached from  $x$  by flowing along finitely many local Killing fields of  $\mathcal{S}$  (respectively  $\mathcal{L}$ ). An enhanced Lagrangean contact structure  $(M, \mathcal{S})$  (resp. a Lagrangean contact structure  $(M, \mathcal{L})$ ) is *locally homogeneous* if any connected component of  $M$  is a  $\text{Kill}^{loc}$ -orbit (for  $\mathcal{S}$ , resp.  $\mathcal{L}$ ).

The first claim of the following Proposition is a consequence of Gromov’s “open-dense orbit theorem”, and the second one is the translation in the context of enhanced Lagrangean contact structures of a work done by Frances in [Fra16, Proposition 5.1] for pseudo-Riemannian structures.

**Proposition 3.2.** *There exists an open and dense subset  $\Omega$  of  $M$ , such that the enhanced Lagrangean contact structure  $\mathcal{S}$  is locally homogeneous in restriction to  $\Omega$ . Moreover for any  $x \in \Omega$ , the isotropy subalgebra  $\mathfrak{is}_S^{\text{loc}}(x)$  is non-trivial.*

*Proof.* Since  $\mathcal{S}$  has an automorphism  $f$  with a dense orbit, Gromov's dense orbit theorem directly implies the first claim (see [Gro88, Corollary 3.3.A], and [Pec16, Theorem 4.13] for a proof in the case of generalized Cartan geometries). We fix from now on a connected component  $O$  of  $\Omega$ . By local homogeneity of  $\mathcal{S}|_O$ , it suffices to prove the second claim for one point of  $O$ . Since the integrability locus  $M^{\text{int}}$  is open and dense (see Theorem 2.17), and  $\text{Rec}(f) \cap \text{Rec}(f^{-1})$  is dense in  $M$ , there exists a point  $x \in O \cap M^{\text{int}} \cap \text{Rec}(f) \cap \text{Rec}(f^{-1})$ . We show now that  $\mathfrak{is}_S^{\text{loc}}(x)$  is non-zero.

Let us denote by  $(\hat{M}, \omega, \varphi)$  the normal generalized Cartan geometry of  $\mathcal{S}$  (see Definition 2.13), and let us choose  $\hat{x} \in \pi^{-1}(x)$ . Possibly replacing  $f$  by  $f^{-1}$ , we have  $\lim_{n \rightarrow +\infty} \|D_x f^n|_{E^\alpha}\| = 0$ , and by hypothesis on  $x$ , there exists a strictly increasing sequence  $(n_k)$  of integers such that  $f^{n_k}(x)$  converges to  $x$ , implying the existence of a sequence  $p_k \in \mathbf{P}_{\min}$  such that  $\hat{f}^{n_k}(\hat{x}) \cdot p_k^{-1}$  converges to  $\hat{x}$ . We claim that the sequence  $(\hat{f}^{n_k}(\hat{x}))$  has to leave every compact subset of  $\hat{M}$ , implying that  $(p_k)$  also leaves every compact subset of  $\mathbf{P}_{\min}$ . In fact, if not, some subsequence  $(\hat{f}^{n'_k}(\hat{x}))$  converges in  $\hat{M}$ , implying that  $(\hat{f}^{n'_k})$  converges in  $\text{Diff}^\infty(\hat{M})$  for the  $\mathcal{C}^\infty$ -topology, because  $\hat{f}$  preserves the parallelism defined by  $\omega$  (see [Kob95, Theorem I.3.2]). Therefore,  $(\hat{f}^{n'_k})$  also converges for the  $\mathcal{C}^\infty$ -topology, which contradicts  $\lim_{k \rightarrow +\infty} \|D_x \hat{f}^{n'_k}|_{E^\alpha}\| = 0$ .

The sequel of the proof of [Fra16, Proposition 5.1] will enable us to conclude, using the total curvature  $\mathcal{K}^{\text{tot}}: \hat{M} \rightarrow W_{\mathcal{K}^{\text{tot}}}$  of  $\mathcal{S}$  (see Paragraph 2.4.3). By  $\mathbf{P}_{\min}$ -equivariance of the total curvature and its invariance by automorphisms, we have  $p_k \cdot \mathcal{K}^{\text{tot}}(\hat{x}) = \mathcal{K}^{\text{tot}}(\hat{f}^{n_k}(\hat{x}) \cdot p_k^{-1})$ , and  $p_k \cdot \mathcal{K}^{\text{tot}}(\hat{x})$  converges thus to  $\mathcal{K}^{\text{tot}}(\hat{x})$ . The manifold  $W_{\mathcal{K}^{\text{tot}}}$  has a canonical structure of algebraic variety for which the action of  $\mathbf{P}_{\min}$  is algebraic (because its action on the space  $W_K$  of normal curvatures and on the algebraic variety  $\mathbb{V} \subset \mathbf{P}(\mathfrak{sl}_3/\mathfrak{p}_{\min})$  are algebraic, see [Pec16, Remark 4.16] for more details). Therefore, the orbits of the action of  $\mathbf{P}_{\min}$  on  $W_{\mathcal{K}^{\text{tot}}}$  are locally closed, and are thus imbedded submanifolds. In particular, there exists a sequence  $\varepsilon_k \in \mathbf{P}_{\min}$  converging to the identity and such that  $p_k \cdot \mathcal{K}^{\text{tot}}(\hat{x}) = \varepsilon_k \cdot \mathcal{K}^{\text{tot}}(\hat{x})$ , i.e. such that  $\varepsilon_k^{-1} p_k \in \text{Stab}_{\mathbf{P}_{\min}}(\mathcal{K}^{\text{tot}}(\hat{x}))$ . Since  $\varepsilon_k^{-1} p_k$  leaves every compact subset of  $\mathbf{P}_{\min}$ ,  $\text{Stab}_{\mathbf{P}_{\min}}(\mathcal{K}^{\text{tot}}(\hat{x}))$  is non-compact. But  $\text{Stab}_{\mathbf{P}_{\min}}(\mathcal{K}^{\text{tot}}(\hat{x}))$  is an algebraic subgroup of  $\mathbf{P}_{\min}$  and has thus a finite number of connected components, which implies that its identity component is also non-compact.

There exists thus a non-zero vector  $v \in \mathfrak{p}_{\min}$  in the Lie algebra of  $\text{Stab}_{\mathbf{P}_{\min}}(\mathcal{K}^{\text{tot}}(\hat{x}))$ . For any  $t \in \mathbb{R}$  we have by hypothesis  $\mathcal{K}^{\text{tot}}(\hat{x} \cdot \exp(tv)) = \mathcal{K}^{\text{tot}}(\hat{x}) \cdot \exp(tv) = \mathcal{K}^{\text{tot}}(\hat{x})$ , and deriving this equality at  $t = 0$  we obtain  $D_{\hat{x}} \mathcal{K}^{\text{tot}}(\omega_{\hat{x}}^{-1}(v)) = 0$ , i.e.  $v \in \omega_{\hat{x}}(\text{Ker}(D_{\hat{x}} \mathcal{K}^{\text{tot}})) = \text{Kill}^{\text{tot}}(\hat{x})$ . Since  $\hat{x}$  is in the integrability locus  $\hat{M}^{\text{int}}$ , there exists a local Killing field  $X \in \mathfrak{kill}_S^{\text{loc}}(x)$  such that  $\omega_{\hat{x}}(\hat{X}_{\hat{x}}) = v \neq 0$ . This shows that  $X \neq 0$  and  $X(x) = 0$ , i.e. that  $X \in \mathfrak{is}_S^{\text{loc}}(x) \setminus \{0\}$ , and concludes the proof of the proposition.  $\square$

### 3.2 Flatness of the Lagrangean contact structure

In particular, the underlying Lagrangean contact structure  $\mathcal{L} = (E^\alpha, E^\beta)$  is also locally homogeneous with non-zero isotropy in restriction to the open and dense subset  $\Omega$ . The following result due to Tresse in [Tre96] (see also [KT17, §4.5.2]) implies that  $\mathcal{L}|_\Omega$  is flat.

**Theorem 3.3** (Tresse [Tre96]). *Any three-dimensional locally homogeneous connected Lagrangean contact structure with non-zero isotropy is flat.*

By density of  $\Omega$  and continuity of the curvature, the Lagrangean contact structure  $(M, \mathcal{L})$  is therefore flat, and according to Paragraph 2.3.2, we obtain the following corollary.

**Corollary 3.4.** *The Lagrangean contact structure  $\mathcal{L}$  is described by a  $(\mathbf{G}, \mathbf{X})$ -structure on  $M$ .*



The rest of this paragraph is devoted to prove Tresse's Theorem 3.3. We consider a locally homogeneous Lagrangean contact structure  $\mathcal{L}$  with non-zero isotropy defined on a three-dimensional connected manifold  $M$ . We denote by  $\mathcal{C} = (\hat{M}, \omega)$  the normal Cartan geometry of  $\mathcal{L}$ , and by  $K: \hat{M} \rightarrow W_K$  its curvature map. Choosing  $x \in M$  and  $\hat{x} \in \pi^{-1}(x)$ , it suffices to prove that  $K(\hat{x}) = 0$  by local homogeneity of  $\mathcal{C}$ . We will denote by

$$\mathfrak{h} = \mathfrak{kill}_{\mathcal{L}}^{loc}(x) \text{ and } \mathfrak{i} = \mathfrak{is}_{\mathcal{L}}^{loc}(x)$$

the algebra of local Killing fields of  $\mathcal{L}$  at  $x$  and its isotropy subalgebra. As  $\mathcal{L}$  is locally homogeneous,  $\text{ev}_x(\mathfrak{h}) := \{X(x) \mid X \in \mathfrak{h}\} = T_x M$ , and in particular  $\dim \mathfrak{h} - \dim \mathfrak{i} = 3$ . The following result gives us a sufficient condition for the vanishing of the curvature.

**Lemma 3.5.** *Let  $f$  be a local automorphism of a locally homogeneous three-dimensional Lagrangean contact structure  $(M, \mathcal{L})$  fixing a point  $x \in M$ , let  $\hat{x} \in \pi^{-1}(x)$  be a lift of  $x$  in the normal Cartan bundle of  $\mathcal{L}$ , and let  $p \in \mathbf{P}_{min}$  be the holonomy of  $\hat{f}$  at  $\hat{x}$ , characterized by  $\hat{f}(\hat{x}) = \hat{x} \cdot p^{-1}$ . If  $p = \exp(v)$  with*

$$v = \begin{pmatrix} a & * & * \\ 0 & -a - b & * \\ 0 & 0 & b \end{pmatrix} \in \mathfrak{p}_{min} \text{ such that } b \neq -5a \text{ and } a \neq -5b, \quad (3.2)$$

then  $\mathcal{L}$  is flat.

*Proof.* Since the curvature  $K$  is preserved by  $\hat{f}$  and  $\mathbf{P}_{min}$ -equivariant (see Paragraph 2.3), we obtain  $p \cdot K(\hat{x}) = K(\hat{x} \cdot p^{-1}) = K(\hat{f}(\hat{x})) = K(\hat{x})$ , where the holonomy  $p$  is of the form

$$p = \begin{pmatrix} \lambda & * & * \\ 0 & \lambda^{-1}\mu^{-1} & * \\ 0 & 0 & \mu \end{pmatrix}$$

with  $\mu \neq \lambda^{-5}$  and  $\lambda \neq \mu^{-5}$  by hypothesis. According to the expression of the components  $(p.K)_\alpha$  and  $(p.K)_\beta$  of the curvature given in (2.11), we have  $\lambda^5 \mu K(\hat{x})_\alpha = K(\hat{x})_\alpha$  and  $\lambda \mu^5 K(\hat{x})_\beta = K(\hat{x})_\beta$ , implying  $K(\hat{x})_\alpha = K(\hat{x})_\beta = 0$ . The structure being locally homogeneous and the subspace  $W_H = \{K \in W_K \mid K_\alpha = K_\beta = 0\}$  being  $\mathbf{P}_{min}$ -invariant,  $K$  has values in  $W_H$  on a neighbourhood of  $\hat{x}$ , and therefore  $K = 0$  on this neighbourhood according to Proposition 2.11. By local homogeneity,  $\mathcal{L}$  is flat.  $\square$

We introduce the Cartan subalgebra  $\mathfrak{a} \simeq \mathbb{R}^2$  of diagonal matrices of  $\mathfrak{p}_{min}$ , and the projection  $p: \mathfrak{p}_{min} \rightarrow \mathfrak{a}$  on  $\mathfrak{a}$  parallel to  $\mathfrak{heis}(3)$ , which is a Lie algebra morphism. The following linear map will play an important role in the proof:

$$\phi: X \in \mathfrak{i} \mapsto p(\omega_{\hat{x}}(\hat{X}_{\hat{x}})) \in \mathfrak{a}.$$

**Fact 3.6.** *If there exists  $X \in \mathfrak{i}$  such that  $\phi(X)$  satisfies the hypotheses (3.2) of Lemma 3.5, then  $\mathcal{L}$  is flat.*

*Proof.* Denoting by  $p(t)$  the element of  $\mathbf{P}_{min}$  such that  $\varphi_{\hat{X}}^t(\hat{x}) = \hat{x} \cdot p(t)$ ,  $\{p(t)\}_{t \in \mathbb{R}}$  is a one-parameter subgroup. There exists thus  $w \in \mathfrak{p}_{min}$  such that  $p(t) = \exp(tw)$ , and deriving the relation  $\varphi_{\hat{X}}^t(\hat{x}) = \hat{x} \cdot \exp(tw)$  at  $t = 0$  we obtain  $w = \omega_{\hat{x}}(\hat{X}_{\hat{x}})$  (because  $\omega$  reproduces the fundamental vector fields of the action of  $\mathbf{P}_{min}$ ). There exists thus an automorphism  $\varphi = \varphi_X^{-1}$  of  $(M, \mathcal{L})$  fixing  $x$  and such that  $\hat{\varphi}(\hat{x}) = \hat{x} \cdot \exp(\omega_{\hat{x}}(\hat{X}_{\hat{x}}))^{-1}$ . Since  $\phi(X) = p(\omega_{\hat{x}}(\hat{X}_{\hat{x}}))$  satisfies the conditions (3.2),  $\omega_{\hat{x}}(\hat{X}_{\hat{x}})$  also does, and Lemma 3.5 implies that  $\mathcal{L}$  is flat.  $\square$

**Fact.** *If  $\text{Ker}(\phi) \neq \{0\}$  then  $\mathcal{L}$  is flat.*

*Proof.* There exists then  $X \in \mathfrak{i}$  such that  $v := \omega_{\hat{x}}(\hat{X}_{\hat{x}}) \in \mathfrak{heis}(3) = (\mathfrak{sl}_3)^1$ , i.e.

$$v = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$$

We first assume that  $(a, b) \neq (0, 0)$ . For an element of the form  $w = \begin{pmatrix} 0 & 0 & 0 \\ a' & 0 & 0 \\ 0 & b' & 0 \end{pmatrix}$  in  $\mathfrak{sl}_3$ , we have  $[v, w] = \begin{pmatrix} aa' & * & 0 \\ 0 & bb' - aa' & * \\ 0 & 0 & -bb' \end{pmatrix}$ , and since  $a \neq 0$  or  $b \neq 0$ , there exists such an element  $w \in \mathfrak{sl}_3$  satisfying  $[v, w] = \begin{pmatrix} 1 & * & 0 \\ 0 & -1 & * \\ 0 & 0 & 0 \end{pmatrix}$  or  $[v, w] = \begin{pmatrix} 0 & * & 0 \\ 0 & 1 & * \\ 0 & 0 & -1 \end{pmatrix}$ . Since  $\mathcal{L}$  is locally homogeneous, there exists a Killing field  $Y \in \mathfrak{h}$  such that  $Y_x = D_{\hat{x}}\pi(\omega_{\hat{x}}^{-1}(w))$ , implying  $\omega_{\hat{x}}(\hat{Y}_{\hat{x}}) = w + w_0$  with  $w_0 \in \mathfrak{p}_{min} = (\mathfrak{sl}_3)^0$ . We now use the relation

$$\omega([\hat{X}, \hat{Y}]) = -[\omega(\hat{X}), \omega(\hat{Y})] + K(\omega(\hat{X}), \omega(\hat{Y})), \quad (3.3)$$

verified for any Killing fields of the Cartan geometry  $\mathcal{C}$ , that will be proved at the end of this demonstration. This relation implies  $\omega_{\hat{x}}([\hat{X}, \hat{Y}]_{\hat{x}}) = -[v, w] + [v, w_0] + K(v, w + w_0) \in \mathfrak{p}_{min}$ , where  $[v, w_0] \in (\mathfrak{sl}_3)^1$  according to the filtration property of  $\mathfrak{sl}_3$ , and  $K(v, w + w_0) = 0$  because  $v \in \mathfrak{p}_{min}$  (see Paragraph 2.2.2). In particular  $[X, Y] \in \mathfrak{i}$ , and  $\phi([X, Y])$  is equal to one of the diagonal matrices  $[1, -1, 0]$  or  $[0, 1, -1]$ , that both satisfy the conditions (3.2). This shows that  $\mathcal{L}$  is flat according to Fact 3.6.

If  $a = b = 0$ , we can find an element  $w \in \mathfrak{sl}_3$  such that  $[v, w] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ , and by the same argument as before we find  $Y \in \mathfrak{h}$  such that  $[X, Y] \in \mathfrak{i}$  and  $\phi([X, Y]) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ . Since this element of  $\mathfrak{a}$  satisfies the conditions (3.2),  $\mathcal{L}$  is flat according to Fact 3.6.

We now prove the relation (3.3) for two Killing fields  $X$  and  $Y$  of the Cartan geometry  $\mathcal{C}$ . Since the flow of  $X$  preserves  $\omega$ , the Lie derivative  $L_X\omega$  vanishes identically, and applying Cartan's formula  $L_X = d \circ \iota_X + \iota_X \circ d$  to  $Y$ , we obtain  $Y \cdot \omega(X) + d\omega(X, Y) = 0$ . Cartan's formula  $d\omega(X, Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X, Y])$  implies then  $X \cdot \omega(Y) = \omega([X, Y])$ , and since  $L_Y\omega = 0$  as well, we also have  $-Y \cdot \omega(X) = \omega([X, Y])$ . Equation (3.3) then follows from the definition of the curvature.  $\square$

**Fact.** *If  $\phi(\mathfrak{i}) = \mathfrak{a}$  then  $\mathcal{L}$  is flat.*

*Proof.* There exists in this case a Killing field  $X \in \mathfrak{i}$  such that  $\phi(X) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ , which satisfies the conditions (3.2). This implies that  $\mathcal{L}$  is flat according to Fact 3.6.  $\square$

It remains to handle the case when  $\phi$  is injective, and  $\phi(\mathfrak{i})$  is one-dimensional. There exists then  $V \in \mathfrak{i}$  such that  $\mathfrak{i} = \mathbb{R}V$ , and we can moreover assume without loss of generality that  $v := \omega_{\hat{x}}(\hat{V}_{\hat{x}}) \in \mathfrak{p}_{min}$  does not verify the conditions (3.2) (because if it does, then  $\mathcal{L}$  is flat according to Fact 3.6). In other words, denoting the components of  $v$  in  $\mathfrak{a}$  by

$$\phi(V) = p(v) = \begin{pmatrix} a & 0 & 0 \\ 0 & -a - b & 0 \\ 0 & 0 & b \end{pmatrix} \in \mathfrak{a},$$

with  $(a, b) \in \mathbb{R}^2$ , we assume that:

$$\text{either } a = -5b \neq 0, \text{ or } b = -5a \neq 0. \quad (3.4)$$

Since  $v \in \mathfrak{p}_{min}$ , the curvature part of the relation (3.3) vanishes, and for any  $X \in \mathfrak{h}$  we have:

$$\omega_{\hat{x}}(\widehat{[V, X]}_{\hat{x}}) = -[v, \omega_{\hat{x}}(\hat{X}_{\hat{x}})]. \quad (3.5)$$

The linear map

$$\varphi: X \in \mathfrak{h} \mapsto \omega_{\hat{x}}(\hat{X}_{\hat{x}}) \in \mathfrak{sl}_3$$

is injective according to Lemma 2.14, and since  $\text{ev}_x(\mathfrak{h}) = T_x M$  by local homogeneity of  $\mathcal{L}$ ,  $\varphi$  induces an isomorphism  $\bar{\varphi}$  between  $\mathfrak{h}/\mathfrak{i}$  and  $\mathfrak{sl}_3/\mathfrak{p}_{\min}$ . Using the notations (2.8) of Paragraph 2.2.3 for the basis  $(\bar{e}_\alpha, \bar{e}_\beta, \bar{e}_0)$  of  $\mathfrak{sl}_3/\mathfrak{p}_{\min}$ , there exists  $X, Y$ , and  $Z$  in  $\mathfrak{h}$  such that  $\varphi(X) \in e_\alpha + \mathfrak{p}_{\min}$ ,  $\varphi(Y) \in e_\beta + \mathfrak{p}_{\min}$ , and  $\varphi(Z) \in e_0 + \mathfrak{p}_{\min}$ . According to (3.5),  $\bar{\varphi}$  intertwines the adjoint action of  $V$  on  $\mathfrak{h}/\mathfrak{i}$  and the adjoint action of  $-v$  on  $\mathfrak{sl}_3/\mathfrak{p}_{\min}$ , implying:

$$\text{Mat}_{(\bar{X}, \bar{Y}, \bar{Z})}(\overline{\text{ad}}(V)) = \text{Mat}_{(\bar{e}_\alpha, \bar{e}_\beta, \bar{e}_0)}(\overline{\text{ad}}(-v)) = \begin{pmatrix} -a - 2b & 0 & * \\ 0 & 2a + b & * \\ 0 & 0 & a - b \end{pmatrix}. \quad (3.6)$$

We will denote by  $A = -a - 2b$  and  $B = 2a + b$  the eigenvalues of  $\overline{\text{ad}}(V)$  with respect to  $\bar{X}$  and  $\bar{Y}$ . Our hypotheses (3.4) on  $a$  and  $b$  imply  $A \neq 0$  and  $B \neq 0$ , allowing us to choose  $X$  and  $Y$  in  $\mathfrak{h}$  satisfying

$$[V, X] = AX \text{ and } [V, Y] = BY.$$

In fact, if  $X \in \mathfrak{h}$  satisfies  $\bar{\varphi}(X + \mathfrak{i}) = \bar{e}_\alpha$ , there exists  $\lambda \in \mathbb{R}$  such that  $[V, X] = AX + \lambda V$  according to (3.6) (recall that  $\mathfrak{i} = \mathbb{R}V$ ), and  $X' := X + \frac{\lambda}{A}V$  satisfies then  $[V, X'] = AX'$ . We deal with the case of  $Y$  by the same computations.

The Jacobi identity yields  $[V, [X, Y]] = (A + B)[X, Y]$ , implying in particular that  $[X, Y] \notin \text{Vect}(X, Y, V)$  since  $A + B$  is distinct from  $A$ ,  $B$  and  $0$ . A second application of the same identity gives  $[V, [X, [X, Y]]] = (2A + B)[X, [X, Y]]$  and  $[V, [Y, [X, Y]]] = (A + 2B)[Y, [X, Y]]$ . Furthermore, if  $[X, [X, Y]] \neq 0$ , then  $2A + B$  is an eigenvalue of  $\text{ad}(V) \in \text{End}(\mathfrak{h})$ , and is thus equal to one of the eigenvalues  $A$ ,  $B$ , or  $A + B$  (since  $\dim \mathfrak{h} = 4$  and  $2A + B \neq 0$ ). But the equalities  $2A + B = A + B$  or  $2A + B = B$  would contradict  $A \neq 0$ , and the equality  $2A + B = A$  would likewise contradict our hypotheses (3.4) on  $a$  and  $b$ . Consequently,  $[X, [X, Y]] = 0$ , and for the same reasons  $[Y, [X, Y]] = 0$ .

This shows that  $\mathcal{E} := \text{Vect}(X, Y, [X, Y])$  is a subalgebra of  $\mathfrak{h}$  isomorphic to  $\mathfrak{heis}(3)$ . There is a connected open neighbourhood  $U$  of  $x$  such that the injective linear map  $X \in \mathfrak{Kill}(U, \mathcal{L}) \mapsto [X]_x \in \mathfrak{Kill}_{\mathcal{L}}^{\text{loc}}(x)$  is an isomorphism (see Remark 2.15), and there is thus an injective Lie algebra morphism  $\iota: \mathfrak{heis}(3) \rightarrow \mathfrak{Kill}(U, \mathcal{L})$  of image  $\mathcal{E}$ . According to the work of Palais in [Pal57], chapter II Theorem XI and its corollary, there exists a (unique) local action of  $\text{Heis}(3)$  on  $U$  that integrates this infinitesimal action, *i.e.* such that  $X^\dagger = \iota(X)|_U$  for any  $X \in \mathfrak{heis}(3)$ . In particular, the local action of  $\text{Heis}(3)$  on  $U$  preserves  $\mathcal{L}$ , and since  $\iota(\mathfrak{heis}(3)) \cap \mathfrak{i} = \{0\}$ , this action is moreover simply transitive at  $x$ . The Lagrangean contact structure  $\mathcal{L}$  is thus locally isomorphic to a left-invariant Lagrangean contact structure on  $\text{Heis}(3)$ . The following lemma implies then that  $\mathcal{L}$  is flat, finishing the proof of Theorem 3.3.

**Lemma 3.7.** *Any left-invariant Lagrangean contact structure on  $\text{Heis}(3)$  is flat.*

*Proof.* The left-invariant Lagrangean contact structure  $\mathcal{M}_0 = (\mathbb{R}\tilde{X}, \mathbb{R}\tilde{Y})$  of  $\text{Heis}(3)$  generated by  $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$  is flat. In fact, we will see in Paragraph 4.2.3 that  $(\text{Heis}(3), \mathcal{M}_0)$  is isomorphic to an open subset of the homogeneous model space  $(\mathbf{X}, \mathcal{L}_{\mathbf{X}})$ . Considering a left-invariant Lagrangean contact structure  $\mathcal{M}$  on  $\text{Heis}(3)$ , it suffices thus to find an isomorphism of Lagrangean contact structures from  $\mathcal{M}_0$  to  $\mathcal{M}$  to prove our claim.

There exists  $v, w \in \mathfrak{heis}(3)$  such that  $\mathcal{M} = (\mathbb{R}\tilde{v}, \mathbb{R}\tilde{w})$ , and since  $\mathbb{R}\tilde{v} \oplus \mathbb{R}\tilde{w}$  is a contact distribution,  $[v, w] \notin \text{Vect}(v, w)$ . Denoting  $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $v = aX + bY + cZ$ , and  $w = a'X + b'Y + c'Z$ , we have  $[v, w] = (ab' - ba')Z$ , which implies  $ab' - ba' \neq 0$ . The Lie algebra automorphism  $\varphi$  of  $\mathfrak{heis}(3)$  whose matrix in the basis  $(X, Y, Z)$  is  $\begin{pmatrix} a & a' & 0 \\ b & b' & 0 \\ c & c' & ab' - ba' \end{pmatrix}$  sends  $(X, Y)$  to  $(v, w)$ , and since  $\text{Heis}(3)$  is simply-connected, there exists a Lie group automorphism  $\phi$  of  $\text{Heis}(3)$  whose differential at identity is  $\varphi$ . As  $\phi$  is an automorphism,  $D_e\phi(X, Y) = (v, w)$  implies  $\phi^*\tilde{v} = \tilde{X}$  and  $\phi^*\tilde{w} = \tilde{Y}$ , *i.e.*  $\phi$  is an isomorphism of Lagrangean contact structures from  $\mathcal{M}_0$  to  $\mathcal{M}$ .  $\square$

## 4 Local model of the enhanced Lagrangean contact structure

In the previous section, we proved that the Lagrangean contact structure  $\mathcal{L}$  is locally isomorphic to the homogeneous model space  $(\mathbf{X}, \mathcal{L}_{\mathbf{X}})$ , and thus described by a  $(\mathbf{G}, \mathbf{X})$ -structure on  $M$ . The classical strategy is then to reduce the possibilities for the images of the developing map  $\delta: \tilde{M} \rightarrow M$  and of the holonomy morphism  $\rho: \pi_1(M) \rightarrow \mathbf{G}$  of this structure.

In the case studied by Ghys in [Ghy87] of an Anosov flow preserving the structure, the holonomy group  $\rho(\pi_1(M)) \subset \mathbf{G}$  is centralized by a one-parameter subgroup of  $\mathbf{G}$ , which reduces dramatically the possibilities for  $\rho(\pi_1(M))$ . But in the case of a discrete-time dynamic, we do not have any relevant algebraic restriction of this kind on  $\rho(\pi_1(M))$ .

For this reason, we have to look not only at the local homogeneity of  $\mathcal{L}$  on  $\Omega$ , but at the local homogeneity of the whole *enhanced* Lagrangean contact structure  $\mathcal{S} = (E^\alpha, E^\beta, E^c)$  on this open dense subset. In this section, we will show that in restriction to  $\Omega$ ,  $\mathcal{S}$  is locally isomorphic to an *infinitesimal homogeneous model*, that preserves a distribution transverse to the contact plane.

### 4.1 Two algebraic models

We first describe the infinitesimal models in an algebraic way.

#### 4.1.1 Left-invariant structure on $\mathrm{SL}_2(\mathbb{R})$

We will use the following basis for the Lie algebra  $\mathfrak{sl}_2$  of  $\mathrm{SL}_2(\mathbb{R})$ :

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.1)$$

The Lie bracket relation  $[E, F] = H$  in  $\mathfrak{sl}_2$  shows that the left-invariant distributions generated by  $E$ ,  $F$  and  $H$  induce a left-invariant enhanced Lagrangean contact structure  $\mathcal{S}_{\mathrm{SL}_2(\mathbb{R})} = (\mathbb{R}\tilde{E}, \mathbb{R}\tilde{F}, \mathbb{R}\tilde{H})$  on  $\mathrm{SL}_2(\mathbb{R})$ . Moreover, the right action of the one-parameter subgroup  $A$  generated by  $H$  also preserves  $\mathcal{S}_{\mathrm{SL}_2(\mathbb{R})}$ . We endow the universal cover  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$  of  $\mathrm{SL}_2(\mathbb{R})$  with the pullback of  $\mathcal{S}_{\mathrm{SL}_2(\mathbb{R})}$ , so that the right action of the one-parameter subgroup  $\tilde{A}$  of  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$  generated by  $H$  preserves  $\mathcal{S}_{\widetilde{\mathrm{SL}}_2(\mathbb{R})}$ .

For any cocompact lattice  $\Gamma_0$  of  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$  and any admissible morphism  $u: \Gamma_0 \rightarrow \tilde{A}$  (see Paragraph 1.1.1 of the introduction), the left-invariant enhanced Lagrangean contact structure of  $\widetilde{\mathrm{SL}}_2(\mathbb{R})$  is preserved by the graph-group  $\Gamma = \mathrm{gr}(u, \Gamma_0) \subset \widetilde{\mathrm{SL}}_2(\mathbb{R}) \times \tilde{A}$ , and  $\Gamma \backslash \widetilde{\mathrm{SL}}_2(\mathbb{R})$  is endowed with the induced enhanced Lagrangean contact structure. The stable and unstable distributions of the algebraic contact-Anosov flow  $(R_{a^t})$  on  $\Gamma \backslash \widetilde{\mathrm{SL}}_2(\mathbb{R})$  are given by this enhanced Lagrangean contact structure.

#### 4.1.2 Left-invariant structure on $\mathrm{Heis}(3)$

We will use the following basis for the Lie algebra  $\mathfrak{heis}(3)$  of  $\mathrm{Heis}(3)$ :

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie bracket relation  $[X, Y] = Z$  in  $\mathfrak{heis}(3)$  shows that the left-invariant distributions  $(\mathbb{R}\tilde{X}, \mathbb{R}\tilde{Y}, \mathbb{R}\tilde{Z})$  induce a left-invariant enhanced Lagrangean contact structure  $\mathcal{S}_{\mathrm{Heis}(3)}$  on  $\mathrm{Heis}(3)$ . The subgroup

$$\mathcal{A} = \left\{ \varphi_{\lambda, \mu} \mid (\lambda, \mu) \in \mathbb{R}^{*2} \right\}$$

of automorphisms introduced in Paragraph 1.1.2 is exactly the subgroup of automorphisms of  $\mathrm{Heis}(3)$  that preserve  $\mathcal{S}_{\mathrm{Heis}(3)}$ .

Any cocompact lattice  $\Gamma$  of  $\mathrm{Heis}(3)$  preserves  $\mathcal{S}_{\mathrm{Heis}(3)}$ , and the quotient  $\Gamma \backslash \mathrm{Heis}(3)$  is endowed with the induced enhanced Lagrangean contact structure. The stable, unstable and central

distributions of any partially hyperbolic affine automorphism of  $\Gamma \backslash \text{Heis}(3)$  are given by this enhanced Lagrangean contact structure.

## 4.2 Two homogeneous open subsets of $\mathbf{X}$

The left-invariant enhanced Lagrangean contact structures of  $\text{SL}_2(\mathbb{R})$  and  $\text{Heis}(3)$  can be geometrically imbedded in  $\mathbf{X}$  as homogeneous open subsets, that will be the local models of the enhanced Lagrangean contact structure  $\mathcal{S}$  in restriction to  $\Omega$ .

### 4.2.1 Some specific surfaces of $\mathbf{X}$ , and one affine chart

For  $D$  a projective line of  $\mathbb{RP}^2$ , we define the  $\beta - \alpha$  surface

$$\mathcal{S}_{\beta, \alpha}(D) = \pi_{\alpha}^{-1}(D) = \cup_{y \in \mathcal{C}^{\beta}(D)} \mathcal{C}^{\alpha}(y),$$

and for  $m \in \mathbb{RP}^2$ , the analog  $\alpha - \beta$  surface

$$\mathcal{S}_{\alpha, \beta}(m) = \pi_{\beta}^{-1}(\{L \in \mathbb{RP}_*^2 \mid m \in L\}) = \cup_{y \in \mathcal{C}^{\alpha}(m)} \mathcal{C}^{\beta}(y).$$

The open subset

$$\Omega_a := \mathbf{X} \setminus \mathcal{S}_{\beta, \alpha}([e_1, e_2])$$

of  $\mathbf{X}$ , composed by pointed projective lines  $(m, D)$  for which  $m \notin [e_1, e_2]$ , will be identified with the set  $\mathbf{X}_a$  of pointed affine lines of  $\mathbb{R}^2$  as follows:

$$\phi_a: (m, D) \in \Omega_a \mapsto (m \cap P, D \cap P) \in \mathbf{X}_a, \quad (4.2)$$

where  $\text{Vect}(e_1, e_2) + (0, 0, 1)$  is identified with  $\mathbb{R}^2$  by translation. The diffeomorphism  $\phi_a$  is moreover equivariant for the canonical identification

$$\begin{bmatrix} A & X \\ 0 & 1 \end{bmatrix} \in \text{Stab}_{\mathbf{G}}(\Omega_a) \mapsto A + X \in \text{Aff}(\mathbb{R}^2) \quad (4.3)$$

of  $\text{Stab}_{\mathbf{G}}(\Omega_a)$  with the group of affine transformations of  $\mathbb{R}^2$ .

### 4.2.2 The open subset $Y_{\mathbf{t}}$

We will embed  $\text{SL}_2(\mathbb{R})$  in  $\mathbf{G}$  as follows:

$$\iota: g \in \text{SL}_2(\mathbb{R}) \mapsto \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \in \mathbf{G}.$$

The resulting copy  $S_0$  of  $\text{SL}_2(\mathbb{R})$  acts simply transitively at  $o_{\mathbf{t}} = ([1, 0, 1], [(1, 0, 1), e_2]) = \phi_a^{-1}(e_1 + \mathbb{R}e_2) \in \Omega_a$ , and its orbit  $Y_{\mathbf{t}} = S_0 \cdot o_{\mathbf{t}}$  can be described as

$$Y_{\mathbf{t}} = \Omega_a \setminus \mathcal{S}_{\alpha, \beta}[e_3] = \phi_a^{-1} \left( \left\{ m + L \mid m \in \mathbb{R}^2 \setminus \{(0, 0)\}, L \in \mathbb{RP}^1 \setminus \{\mathbb{R}m\} \right\} \right).$$

The left-invariant structure of  $\text{SL}_2(\mathbb{R})$  induces on  $Y_{\mathbf{t}}$  a  $S_0$ -invariant enhanced Lagrangean contact structure

$$\mathcal{S}_{\mathbf{t}} = (\theta_{o_{\mathbf{t}}} \circ \iota)_* \mathcal{S}_{\text{SL}_2(\mathbb{R})}, \quad (4.4)$$

which is *compatible with*  $\mathcal{L}_{\mathbf{X}}$  in the sense that its  $\alpha$  and  $\beta$ -distributions coincide with the ones of  $\mathcal{L}_{\mathbf{X}}$ , and whose central distribution is entirely described by its value at  $o_{\mathbf{t}}$ :

$$\mathcal{E}_{\mathbf{t}}^c(o_{\mathbf{t}}) = \mathbb{R}H_0^{\dagger}(o_{\mathbf{t}}), \text{ where } H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.5)$$

Let us denote by  $A^\pm$  the subgroup of  $\mathrm{SL}_2(\mathbb{R})$  composed by diagonal matrices. The right action of  $A^\pm$  preserves  $\mathcal{S}_{\mathrm{SL}_2(\mathbb{R})}$ , and the direct product  $\mathrm{SL}_2(\mathbb{R}) \times A^\pm$  acts on  $\mathrm{SL}_2(\mathbb{R})$  by  $(g, a) \cdot h = gha$ . The isomorphism from  $\mathrm{SL}_2(\mathbb{R})$  to  $(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$  given by the orbital map at  $o_{\mathbf{t}}$  is equivariant for:

$$\left(g, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}) \times A^\pm \mapsto \lambda g \in \mathrm{GL}_2(\mathbb{R}).$$

In particular,

$$H_{\mathbf{t}} := \begin{bmatrix} \mathrm{GL}_2(\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}$$

is contained in the automorphism group of  $(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$ .

#### 4.2.3 The open subset $Y_{\mathbf{a}}$

The action of  $\mathrm{Heis}(3)$  is simply transitive at  $o_{\mathbf{a}} = ([e_3], [e_3, e_2]) = \phi_a^{-1}((0, 0) + \mathbb{R}e_2) \in \Omega_a$ , and its orbit  $Y_{\mathbf{a}} = \mathrm{Heis}(3) \cdot o_{\mathbf{a}}$  can be described as

$$Y_{\mathbf{a}} = \Omega_a \setminus S_{\alpha, \beta}[e_1] = \phi_a^{-1} \left( \left\{ m + L \mid m \in \mathbb{R}^2, L \in \mathbb{R}\mathbf{P}^1 \setminus \{\mathbb{R}e_1\} \right\} \right).$$

We endow  $Y_{\mathbf{a}}$  with the  $\mathrm{Heis}(3)$ -invariant enhanced Lagrangean contact structure

$$\mathcal{S}_{\mathbf{a}} = (\theta_{o_{\mathbf{a}}} |_{\mathrm{Heis}(3)})_* \mathcal{S}_{\mathrm{Heis}(3)} \quad (4.6)$$

which is compatible with  $\mathcal{L}_{\mathbf{X}}$ , and whose central distribution is entirely determined by

$$\mathcal{E}_{\mathbf{a}}^c(o_{\mathbf{a}}) = \mathbb{R}Z^\dagger(o_{\mathbf{a}}). \quad (4.7)$$

Let us recall that  $\mathcal{A}$  is the subgroup of automorphisms of  $\mathrm{Heis}(3)$  that moreover preserve  $\mathcal{S}_{\mathrm{Heis}(3)}$  (see Paragraph 4.1.2). The group of affine automorphisms  $L_g \circ \varphi$  of  $\mathrm{Heis}(3)$ , where  $g \in \mathrm{Heis}(3)$  and  $\varphi \in \mathcal{A}$ , will be seen as a semi-direct subgroup  $\mathrm{Heis}(3) \rtimes \mathcal{A}$ . With this notation, the isomorphism from  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$  to  $\mathrm{Heis}(3)$  given by the orbital map at  $o_{\mathbf{a}}$  is equivariant for:

$$\begin{bmatrix} \lambda & x & z \\ 0 & \lambda^{-1}\mu^{-1} & y \\ 0 & 0 & \mu \end{bmatrix} \in \mathbf{P}_{min} \mapsto \left( \begin{pmatrix} 1 & \lambda\mu x & \mu^{-1}z \\ 0 & 1 & \mu^{-1}y \\ 0 & 0 & 1 \end{pmatrix}, \varphi_{\lambda^2\mu, \lambda^{-1}\mu^{-2}} \right) \in \mathrm{Heis}(3) \rtimes \mathcal{A}. \quad (4.8)$$

In particular,  $H_{\mathbf{a}} := \mathbf{P}_{min}$  is contained in the automorphism group of  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$ .

### 4.3 From the infinitesimal model to the local model

We take back the notations of Theorem B. We recall that  $\pi_M: \tilde{M} \rightarrow M$  denotes the universal cover of  $M$  and that  $\Omega$  is a dense and open subset of  $M$  where  $\mathcal{S}$  is locally homogeneous (see Proposition 3.2). We will denote  $\tilde{\mathcal{S}} = \pi_M^* \mathcal{S} = (\tilde{E}^\alpha, \tilde{E}^\beta, \tilde{E}^c)$ ,  $\tilde{\mathcal{L}} = \pi_M^* \mathcal{L}$ ,  $\tilde{\Omega} = \pi_M^{-1}(\Omega)$ , and  $\delta: \tilde{M} \rightarrow \mathbf{X}$  a developping map of the  $(\mathbf{G}, \mathbf{X})$ -structure of  $M$  describing the Lagrangean contact structure  $\mathcal{L}$  (see Corollary 3.4 and Paragraph 2.3.2). We finally choose for this whole section a connected component  $O$  of  $\tilde{\Omega}$ , *i.e.* an open  $\mathrm{Kill}^{loc}$ -orbit of  $\tilde{\mathcal{S}}$ .

Our goal in this section is to describe the local model of  $\tilde{\mathcal{S}}$  in restriction to  $O$ .

#### 4.3.1 Infinitesimal model

At any point of  $\mathbf{X}$ , we will identify the Lie algebra of local Killing fields of  $\mathcal{L}_{\mathbf{X}}$  with  $\mathfrak{sl}_3$  through the fundamental vector fields of the action of  $\mathbf{G}$  (see Lemma 2.16). Since the developping map  $\delta$  is a local isomorphism from  $\tilde{\mathcal{L}}$  to  $\mathcal{L}_{\mathbf{X}}$ , it induces at each point  $x \in \tilde{M}$  an isomorphism

$$\delta^*: v \in \mathfrak{sl}_3 = \mathfrak{kill}_{\mathcal{L}_{\mathbf{X}}}^{loc}(\delta(x)) \mapsto \delta^* v \in \mathfrak{kill}_{\tilde{\mathcal{L}}}^{loc}(x), \quad (4.9)$$

of Lie algebras, whose inverse will be denoted by  $\delta_*: \mathfrak{kill}_{\tilde{\mathcal{L}}}^{loc}(x) \rightarrow \mathfrak{sl}_3$ . For  $X \in \mathfrak{kill}_{\tilde{\mathcal{L}}}^{loc}(x)$  and  $t \in \mathbb{R}$  for which  $\varphi_X^t(x)$  exists, denoting  $v = \delta_*[X]_x \in \mathfrak{sl}_3$ , we have

$$\delta(\varphi_X^t(x)) = e^{tv} \cdot \delta(x). \quad (4.10)$$

**Lemma 4.1.** *There exists a subalgebra  $\mathfrak{h}$  of  $\mathfrak{sl}_3$  such that:*

$$\mathfrak{Kill}(O, \tilde{\mathcal{S}}|_O) = (\delta^* \mathfrak{h})|_O = \{(\delta^* v)|_O \mid v \in \mathfrak{h}\}.$$

*Moreover, any local Killing field of  $\tilde{\mathcal{S}}$  on  $O$  extends to the whole  $\text{Kill}^{\text{loc}}$ -orbit  $O$ .*

*Proof.* It suffices to show that the subalgebra  $\mathfrak{h}(x) = \delta_* \mathfrak{Kill}_{\tilde{\mathcal{S}}}^{\text{loc}}(x)$  is locally constant on  $O$ . This will in fact imply by connexity of  $O$  that  $\mathfrak{h}(x)$  is constant equal to some Lie subalgebra  $\mathfrak{h}$  on  $O$ , and then  $(\delta^* \mathfrak{h})|_O \subset \mathfrak{Kill}(O, \tilde{\mathcal{S}}|_O)$ . But for  $x \in O$ ,  $\dim \mathfrak{h} = \dim \mathfrak{Kill}_{\tilde{\mathcal{S}}}^{\text{loc}}(x) \geq \mathfrak{Kill}(O, \tilde{\mathcal{S}}|_O)$  (see Lemma 2.14), and this inclusion is thus an equality.

For any  $x \in O$  there exists an open connected neighbourhood  $U$  of  $x$  such that any local Killing field of  $\tilde{\mathcal{S}}$  at  $x$  extends to a Killing field defined on  $U$  (see Remark 2.15), and for any  $y \in U$  we thus have  $\mathfrak{h}(x) \subset \mathfrak{h}(y)$ . But  $\mathfrak{h}(x)$  and  $\mathfrak{h}(y)$  have the same dimension since  $x$  and  $y$  are in the same  $\text{Kill}^{\text{loc}}$ -orbit of  $\tilde{\mathcal{S}}$ , and this inclusion is thus an equality. This shows that  $\mathfrak{h}(x)$  is locally constant and finishes the proof.  $\square$

We denote from now on by  $H$  the connected Lie subgroup of  $\mathbf{G}$  of subalgebra  $\mathfrak{h}$ . It is not necessarily closed in  $\mathbf{G}$ , but the action of  $H$  on  $\mathbf{X}$  is smooth for the structure of immersed submanifold of  $H$ .

**Lemma 4.2.** *All the points of  $\delta(O)$  are in the same orbit  $Y$  under the action of  $H$ . In particular,  $Y$  is open.*

*Proof.* We consider  $x$  and  $y$  in  $O$ , and we want to find  $h \in H$  such that  $\delta(y) = h \cdot \delta(x)$ . By hypothesis, as  $x$  and  $y$  are in the same  $\text{Kill}^{\text{loc}}$ -orbit of  $\tilde{\mathcal{S}}$ , there exists a finite number of points  $x_1 = x, \dots, x_n = y$  such that for any  $i \leq n-1$  there exists a local Killing field  $X_i$  of  $\tilde{\mathcal{S}}$  satisfying  $x_{i+1} = \varphi_{X_i}^1(x_i)$ . According to Lemma 4.1, there exists for each  $i$  an element  $v_i \in \mathfrak{h}$  such that  $X_i = \delta^* v_i$ , and we have  $\delta(x_{i+1}) = e^{v_i} \delta(x_i)$  according to the equation (4.10), implying  $\delta(y) = e^{v_{n-1}} \dots e^{v_1} x_0 \in H \cdot \delta(x)$ .  $\square$

We choose from now on a point  $x \in O$ , we denote  $x_0 = \delta(x) \in Y$ , and we consider the isotropy subalgebra

$$\mathfrak{i} = \mathfrak{stab}_{\mathfrak{h}}(x_0) := \{v \in \mathfrak{h} \mid v(x_0) = 0\} \quad (4.11)$$

of  $\mathfrak{h}$  at  $x_0$ , characterized by  $\delta^* \mathfrak{i} = \mathfrak{is}_{\tilde{\mathcal{S}}}^{\text{loc}}(x)$ . Since the orbit  $Y$  of  $x_0$  under  $H$  is open,  $\dim \mathfrak{h} - \dim \mathfrak{i} = 3$ , and  $\mathfrak{i}$  is non-trivial according to Proposition 3.2. We also denote  $\mathcal{E}^c(x_0) = D_x \delta(\tilde{E}^c(x))$ , and  $\mathfrak{h}/\mathfrak{i} = D^\alpha \oplus D^\beta \oplus D^c$  the splitting sent to  $T_{x_0} Y = (\mathcal{E}^\alpha \oplus \mathcal{E}^\beta \oplus \mathcal{E}^c)(x_0)$  by the isomorphism  $\overline{D_e \theta_{x_0}}$  induced by the orbital map at  $x_0$ .

**Lemma 4.3.** *1. The adjoint representation  $\overline{\text{ad}}: \mathfrak{i} \rightarrow \text{End}(\mathfrak{h}/\mathfrak{i})$  preserves the line  $D^c$  in  $\mathfrak{h}/\mathfrak{i}$ , i.e. for any  $v \in \mathfrak{i}$  we have  $\overline{\text{ad}}(v)(D^c) \subset D^c$ .*

*2. There exists in the neighbourhood of  $x_0$  an unique  $H$ -invariant germ of a smooth one-dimensional distribution  $\mathcal{E}^c$  that extends  $\mathcal{E}^c(x_0)$  on a neighbourhood of  $x_0$ , and this distribution is everywhere transverse to  $\mathcal{E}^\alpha \oplus \mathcal{E}^\beta$ .*

*3. The developing map is an isomorphism between the enhanced Lagrangean contact structures  $\tilde{\mathcal{S}}$  and  $\mathcal{S}_Y := (\mathcal{E}^\alpha, \mathcal{E}^\beta, \mathcal{E}^c)$ , from a neighbourhood of  $x$  to a neighbourhood of  $x_0$ .*

*4.  $\mathfrak{h} = \mathfrak{Kill}_{\mathcal{S}_Y}^{\text{loc}}(x_0)$  and  $\mathfrak{i} = \mathfrak{is}_{\mathcal{S}_Y}^{\text{loc}}(x_0)$ .*

*5. If  $I = \text{Stab}_H(x_0)$  is a connected subgroup of  $H$ , then there exists an unique  $H$ -invariant smooth one-dimensional distribution  $\mathcal{E}^c$  that extends  $\mathcal{E}^c(x_0)$  on the whole open orbit  $Y$ , and  $\mathcal{E}^c$  is transverse to  $\mathcal{E}^\alpha \oplus \mathcal{E}^\beta$ . Furthermore,  $\delta|_O$  is a local isomorphism from  $(O, \tilde{\mathcal{S}}|_O)$  to  $(Y, \mathcal{S}_Y)$ .*

*Proof.* 1. For  $v \in \mathfrak{i}$ , denoting  $X = \delta^*v \in \mathfrak{is}_{\tilde{\mathcal{S}}}^{loc}(x)$ , equation (4.10) implies  $\mathcal{E}^c(x_0) = D_{x_0}e^{tv}(\mathcal{E}^c(x_0))$  for any  $t \in \mathbb{R}$ , and thus  $D^c = \overline{\text{Ad}}(e^{tv}) \cdot D^c$ . Deriving this equality at  $t = 0$ , we obtain  $\overline{\text{ad}}(v) \cdot D^c \subset D^c$ .

2. The isotropy subgroup  $I = \text{Stab}_H(x_0)$  and its identity component  $I^0$  are closed in  $H$  for its topology of immersed submanifold, and the orbital map at  $x_0$  induces a local diffeomorphism  $\bar{\theta}_{x_0}: H/I^0 \rightarrow Y$ , equivariant for the action of  $H$ . We saw previously that  $\overline{\text{Ad}}(\exp(\mathfrak{i}))$  preserves  $D^c$ , implying that the subgroup  $\{i \in I^0 \mid \overline{\text{Ad}}(i) \cdot D^c = D^c\}$  is equal to  $I^0$  by connexity, *i.e.* that  $I^0$  preserves  $D^c$ . Therefore,  $H/I^0$  supports a unique  $H$ -invariant smooth one-dimensional distribution extending  $D^c$ , that can be pushed by  $\bar{\theta}_{x_0}$  to a  $H$ -invariant distribution extending  $\mathcal{E}^c(x_0)$  on a neighbourhood of  $x_0$ . Conversely, the pullback of any  $H$ -invariant distribution extending  $\mathcal{E}^c(x_0)$  on a neighbourhood of  $x_0$  is  $H$ -invariant on  $H/I^0$ , which proves the unicity of the germ of  $\mathcal{E}^c$ . Since  $\mathcal{E}^c$  is preserved by  $H$ , it must moreover remain transverse to  $\mathcal{E}^\alpha \oplus \mathcal{E}^\beta$ .

3. For  $y$  sufficiently close to  $x$ , there exists  $X \in \mathfrak{Kill}(O, \tilde{\mathcal{S}}|_O)$  such that  $y = \varphi_X^1(x)$ . Denoting  $y_0 = \delta(y)$  and  $v \in \mathfrak{h}$  such that  $\delta^*v = X$ , we have  $D_{y_0}e^{-v} \circ D_y\delta(\tilde{E}^c(y)) = D_x\delta \circ D_y\varphi_X^{-1}(\tilde{E}^c(y)) = \mathcal{E}^c(x_0)$ , implying  $D_y\delta(\tilde{E}^c(y)) = \mathcal{E}^c(y_0)$  by  $H$ -invariance of  $\mathcal{E}^c$ .

4. This is a direct consequence of the equalities  $\delta^*\mathfrak{h} = \mathfrak{kill}_{\tilde{\mathcal{S}}}^{loc}(x)$  and  $\delta^*\mathfrak{i} = \mathfrak{is}_{\tilde{\mathcal{S}}}^{loc}(x)$ , and of the fact that  $\delta$  is a local isomorphism from  $\tilde{\mathcal{S}}$  to  $\mathcal{S}_Y$  at  $x$ .

5. Concerning the first assertion, the orbital map at  $x_0$  induces a  $H$ -equivariant diffeomorphism from  $H/I$  to  $Y$ , and we saw in the proof of the second assertion that  $H/I^0 = H/I$  supports a unique  $H$ -invariant distribution extending  $D^c$  on  $H/I^0$ , which stays transverse to the contact plane.

The set  $\mathcal{E}$  of points  $y \in O$  such that  $\delta$  is a local isomorphism in the neighbourhood of  $y$  is open and non-empty, and we only have to prove that  $\mathcal{E}$  is closed to conclude by connexity of  $O$ . Let  $z \in O$  be an adherent point of  $\mathcal{E}$ , and let us denote  $z_0 = \delta(z)$ . There exists a point  $y \in \mathcal{E}$  sufficiently close to  $z$  such that, for some Killing field  $X$  of  $\tilde{\mathcal{S}}$ ,  $z = \varphi_X^1(y)$ . Denoting  $v \in \mathfrak{h}$  such that  $X = \delta^*v$ , we have  $D_{z_0}e^{-v} \circ D_z\delta(\tilde{E}^c(z)) = D_y\delta \circ D_z\varphi_X^{-1}(\tilde{E}^c(z)) = \mathcal{E}^c(y_0)$ , implying  $D_z\delta(\tilde{E}^c(z)) = \mathcal{E}^c(z_0)$  by  $H$ -invariance of  $\mathcal{E}^c$ . By local homogeneity of  $\mathcal{S}|_O$ , we can reach all the points of some neighbourhood  $U$  of  $z$  in  $O$  by a Killing field, and the same computation as before shows that  $\delta|_U$  is a local isomorphism, *i.e.* that  $z \in \mathcal{E}$ , which concludes the proof.  $\square$

#### 4.3.2 Local model of an open $\text{Kill}^{loc}$ -orbit

We will call

$$\kappa: (m, D) \in \mathbf{X} \mapsto (D^\perp, m^\perp) \in \mathbf{X} \quad (4.12)$$

the *flip diffeomorphism* of the homogeneous model space. This involution switches the distributions  $\mathcal{E}^\alpha$  and  $\mathcal{E}^\beta$  of the standard Lagrangean contact structure, and is moreover equivariant for the Lie group morphism  $\kappa_{\mathbf{G}}: g \mapsto {}^t g^{-1}$  of  $\mathbf{G}$ .

Consequently, interverting the distributions  $E^\alpha$  and  $E^\beta$  of the Lagrangean contact structure of  $M$  is equivalent to composing the developping map of the  $(\mathbf{G}, \mathbf{X})$ -structure with  $\kappa$ . At the level of the subalgebra  $\mathfrak{h}$  introduced in the previous paragraph, it is equivalent to apply the Lie algebra morphism  $\kappa_{\mathfrak{sl}_3} := D_e \kappa_{\mathbf{G}}: A \mapsto -{}^t A$ .

Denoting

$$(4.13) \quad \begin{cases} \mathfrak{h}_{\mathfrak{t}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -\text{tr}(A) \end{pmatrix} \mid A \in \mathfrak{gl}_2 \right\}, \\ \mathfrak{h}_{\mathfrak{a}} = \mathfrak{p}_{\min}, \end{cases}$$

we will prove in the next section that:

**Proposition 4.4.** *Up to conjugacy in  $\mathbf{G}$  or image by  $\kappa_{\mathfrak{sl}_3} = -{}^t \cdot$ ,  $\mathfrak{h}$  is equal to  $\mathfrak{h}_{\mathfrak{t}}$  or  $\mathfrak{h}_{\mathfrak{a}}$ .*

To deduce a local information about  $\tilde{\mathcal{S}}|_O$  from this infinitesimal classification, it only remains to look at the action of the connected Lie subgroups  $H_{\mathfrak{t}}^0 := \text{GL}_2^+(\mathbb{R})$  and  $H_{\mathfrak{a}}^0 := \mathbf{P}_{\min}^+$  of  $\mathbf{G}$  of respective Lie algebras  $\mathfrak{h}_{\mathfrak{t}}$  and  $\mathfrak{h}_{\mathfrak{a}}$ .



**Proposition 4.5.** 1.  $Y_{\mathbf{t}}$  (respectively  $Y_{\mathbf{a}}$ ) is the only open orbit of  $H_{\mathbf{t}}^0$  (resp. of  $H_{\mathbf{a}}^0$ ) on  $\mathbf{X}$ .

2.  $\mathcal{S}_{\mathbf{t}}$  (respectively  $\mathcal{S}_{\mathbf{a}}$ ) is the only  $H_{\mathbf{t}}^0$ -invariant (resp.  $H_{\mathbf{a}}^0$ -invariant) enhanced Lagrangean contact structure of  $Y_{\mathbf{t}}$  (resp.  $Y_{\mathbf{a}}$ ) that is compatible with  $\mathcal{L}_{\mathbf{X}}$ .

*Proof.* We refer to Paragraphs 4.2.2 and 4.2.3 for the description of  $(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$  and  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$ .

1. Both of these groups are contained in  $\text{Stab}_{\mathbf{G}}([e_1, e_2]) = \left\{ \begin{bmatrix} A & X \\ 0 & 1 \end{bmatrix} \mid A \in \text{GL}_2(\mathbb{R}), X \in \mathbb{R}^2 \right\}$ , that preserves the surface  $S_{\beta, \alpha}[e_1, e_2]$ , and whose only open orbit is thus  $\Omega_a = \mathbf{X} \setminus S_{\beta, \alpha}[e_1, e_2]$ . Any open orbit of one these groups is therefore contained in  $\Omega_a$ . Since  $H_{\mathbf{t}}^0$  preserves the surface  $S_{\alpha, \beta}[e_3]$ , any open orbit of  $H_{\mathbf{t}}^0$  is contained in  $Y_{\mathbf{t}} = \mathbf{X} \setminus (S_{\beta, \alpha}[e_1, e_2] \cup S_{\alpha, \beta}[e_3]) = H_{\mathbf{t}}^0 \cdot o_{\mathbf{t}}$ . In the same way, since  $H_{\mathbf{a}}^0$  preserves  $S_{\alpha, \beta}[e_1]$ , any open orbit of  $H_{\mathbf{a}}^0$  is contained in  $Y_{\mathbf{a}} = \mathbf{X} \setminus (S_{\beta, \alpha}[e_1, e_2] \cup S_{\alpha, \beta}[e_1]) = H_{\mathbf{a}}^0 \cdot o_{\mathbf{a}}$ .

2. We start with  $Y_{\mathbf{t}}$ , and we denote

$$\mathfrak{i}_{\mathbf{t}} = \text{Lie}(\text{Stab}_{H_{\mathbf{t}}^0}(o_{\mathbf{t}})) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & -2a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in \mathbb{R} \right\},$$

and

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The standard Lagrangean contact structure of  $\mathbf{X}$  satisfies  $\mathbb{R}E^{\dagger}(o_{\mathbf{t}}) = \mathcal{E}^{\alpha}(o_{\mathbf{t}})$  and  $\mathbb{R}F^{\dagger}(o_{\mathbf{t}}) = \mathcal{E}^{\beta}(o_{\mathbf{t}})$ , and for  $a \in \mathbb{R}$ , the adjoint action of the diagonal element  $[a, -2a, a]$  of  $\mathfrak{i}_{\mathbf{t}}$  has the following diagonal matrix in the basis  $(\bar{E}, \bar{F}, \bar{H})$  of  $\mathfrak{h}/\mathfrak{i}$ :

$$\text{Mat}_{(\bar{E}, \bar{F}, \bar{H})}(\overline{\text{ad}}([a, -2a, a])) = [3a, -3a, 0].$$

Any line  $D^c$  of  $\mathfrak{h}_{\mathbf{t}}/\mathfrak{i}_{\mathbf{t}}$  that is transverse to  $\text{Vect}(\bar{E}, \bar{F})$  has projective coordinates  $[x, y, 1]$  in the basis  $(\bar{E}, \bar{F}, \bar{H})$  for some  $(x, y) \in \mathbb{R}^2$ , and  $\overline{\text{ad}}([a, -2a, a])(D^c)$  is generated by the vector of coordinates  $(3ax, -3ay, 0)$ . The only transverse line stabilized by  $\overline{\text{ad}}(\mathfrak{i}_{\mathbf{t}})$  is therefore  $\mathbb{R}\bar{H}$ , and  $\mathcal{E}_{\mathbf{t}}^c$  is the only  $H_{\mathbf{t}}^0$ -invariant distribution of  $Y_{\mathbf{t}}$  transverse to  $\mathcal{L}_{\mathbf{X}}$ .

Let us denote

$$\mathfrak{i}_{\mathbf{a}} = \text{Lie}(\text{Stab}_{H_{\mathbf{a}}^0}(o_{\mathbf{a}})) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & -a-b & 0 \\ 0 & 0 & b \end{pmatrix} \mid (a, b) \in \mathbb{R}^2 \right\}.$$

and

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.14)$$

We have  $\mathbb{R}X^{\dagger}(o_{\mathbf{a}}) = \mathcal{E}^{\alpha}(o_{\mathbf{a}})$ ,  $\mathbb{R}Y^{\dagger}(o_{\mathbf{a}}) = \mathcal{E}^{\beta}(o_{\mathbf{a}})$ , and for  $(a, b) \in \mathbb{R}^2$ , the adjoint action of the diagonal element  $[a, -a-b, b]$  of  $\mathfrak{i}_{\mathbf{a}}$  has the following diagonal matrix in the basis  $(\bar{X}, \bar{Y}, \bar{Z})$  of  $\mathfrak{p}_{\min}/\mathfrak{i}_{\mathbf{a}}$ :

$$\text{Mat}_{(\bar{X}, \bar{Y}, \bar{Z})}(\overline{\text{ad}}([a, -a-b, b])) = [2a+b, -a-2b, a-b]. \quad (4.15)$$

Any line  $D^c$  of  $\mathfrak{p}_{\min}/\mathfrak{i}_{\mathbf{a}}$  that is transverse to  $\text{Vect}(\bar{X}, \bar{Y})$  has projective coordinates of the form  $[x, y, 1]$  in the basis  $(\bar{X}, \bar{Y}, \bar{Z})$  for some  $(x, y) \in \mathbb{R}^2$ , and  $\overline{\text{ad}}([a, -a-b, b])(D^c)$  is generated by the vector of coordinates  $((2a+b)x, (-a-2b)y, a-b)$ . The only transverse line stabilized by  $\overline{\text{ad}}(\mathfrak{i}_{\mathbf{a}})$  is therefore  $\mathbb{R}\bar{Z}$ , and  $\mathcal{E}_{\mathbf{a}}^c$  is the only  $H_{\mathbf{a}}^0$ -invariant distribution of  $Y_{\mathbf{a}}$  transverse to  $\mathcal{L}_{\mathbf{X}}$ .  $\square$

We can finally describe the local geometry of  $O$ .

**Corollary 4.6.** Up to inversion of the distributions  $E^{\alpha}$  and  $E^{\beta}$ , the restriction  $\delta|_O$  of the developping map to  $O$  is a local isomorphism from  $(O, \tilde{\mathcal{S}}|_O)$  to  $(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$ , or to  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$ .

*Proof.* Inverting the distributions  $E^{\alpha}$  and  $E^{\beta}$  is equivalent to apply  $\kappa_{\mathfrak{s}1_3}$  to  $\mathfrak{h}$ , and the conjugation of  $\mathfrak{h}$  by  $g \in \mathbf{G}$  is equivalent to replace the developping map  $\delta$  by  $g \circ \delta$  (that describes the same  $(\mathbf{G}, \mathbf{X})$ -structure on  $M$ ). According to Proposition 4.4, we can thus assume that  $\mathfrak{h}$  is equal to  $\mathfrak{h}_{\mathbf{t}}$  or  $\mathfrak{h}_{\mathbf{a}}$ , and the open orbit  $Y$  is therefore equal to  $Y_{\mathbf{t}}$  (respectively  $Y_{\mathbf{a}}$ ) according to Proposition 4.5. Since the isotropy subgroups  $\text{Stab}_{H_{\mathbf{t}}^0}(o_{\mathbf{t}})$  and  $\text{Stab}_{H_{\mathbf{a}}^0}(o_{\mathbf{a}})$  are connected, there exists a  $H_{\mathbf{t}}^0$ -invariant (resp.  $H_{\mathbf{a}}^0$ -invariant) enhanced Lagrangean contact structure  $\mathcal{S}_Y$  on  $Y$  that is compatible with  $\mathcal{L}_{\mathbf{X}}$  and such that  $\delta|_O$  is a local isomorphism from  $(O, \tilde{\mathcal{S}}|_O)$  to  $(Y, \mathcal{S}_Y)$  (see Lemma 4.3). According to Proposition 4.5,  $\mathcal{S}_Y$  is equal to  $\mathcal{S}_{\mathbf{t}}$  (resp.  $\mathcal{S}_{\mathbf{a}}$ ).  $\square$

## 5 Classification of the infinitesimal model

The goal of this section is to prove Proposition 4.4. Let us recall that the Lie subalgebras  $\mathfrak{i} \subset \mathfrak{h}$  of  $\mathfrak{sl}_3$  are characterized by  $(\delta^*\mathfrak{h})|_O = \mathfrak{Kill}(O, \tilde{\mathcal{S}}|_O)$  and  $[\delta^*\mathfrak{i}]_x = \mathfrak{is}_{\tilde{\mathcal{S}}}^{loc}(x)$  (see Lemma 4.1 and (4.11)).

### 5.1 Algebraic reduction

We first prove some purely algebraic restrictions on  $\mathfrak{h}$ .

**Lemma 5.1.** *The dimension of  $\mathfrak{h}$  is either 4 or 5.*

*Proof.* Possibly translating the developping map by an element of  $\mathbf{G}$ , we can assume that  $x_0 = o = ([e_1], [e_1, e_2]) \in \mathbf{X}$ , and since the adjoint action of  $\mathbf{P}_{min}$  on the lines of  $\mathfrak{sl}_3/\mathfrak{p}_{min}$  transverse to  $\text{Vect}(\bar{e}_\alpha, \bar{e}_\beta)$  is transitive (see equation (2.9) in Paragraph 2.2.3), we can moreover assume that  $D^c = \overline{D_e \theta_o}(\mathbb{R}\bar{e}_0)$  with  $e_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ . As a consequence,  $\mathfrak{i} = \mathfrak{h} \cap \mathfrak{p}_{min}$  is contained in

$$\mathfrak{o} = \left\{ v \in \mathfrak{p}_{min} \mid \overline{\text{ad}}(v)(\mathbb{R}\bar{e}_0) \subset \mathbb{R}\bar{e}_0 \right\} = \left\{ \begin{pmatrix} a & 0 & z \\ 0 & -a-b & 0 \\ 0 & 0 & b \end{pmatrix} \mid (a, b, z) \in \mathbb{R}^3 \right\}. \quad (5.1)$$

Denoting  $e^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{o}$ , we now prove that  $\mathfrak{i} \cap \mathbb{R}e^0 = \{0\}$ , which implies  $\dim \mathfrak{i} \leq 2$  and concludes the proof of the Lemma since  $\mathfrak{i}$  is non-zero and  $\dim \mathfrak{h} - \dim \mathfrak{i} = 3$ .

Let us assume by contradiction that  $e^0 \in \mathfrak{i}$ . As  $\mathfrak{h} + \mathfrak{p}_{min} = \mathfrak{sl}_3$  (because the orbit of  $o$  under  $H$  is open), there exists  $v \in \mathfrak{h}$  and  $w \in \mathfrak{p}_{min}$  such that  $e_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = v + w$ . We have then  $[v, e^0] \in \mathfrak{h}$ , and since  $w \in \mathfrak{p}_{min}$ ,  $[w, e^0] \in \mathbb{R}e^0 \subset \mathfrak{i}$ . This implies that  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = [e_\beta, e^0] = [v, e^0] + [w, e^0] \in \mathfrak{h} \cap \mathfrak{p}_{min} = \mathfrak{i} \subset \mathfrak{o}$ , which contradicts the description of  $\mathfrak{o}$  in (5.1) and concludes the proof of the lemma.  $\square$

Let

$$\mathfrak{h} = \mathfrak{s}_\varphi \ltimes \mathfrak{r}, \quad (5.2)$$

be the Levi decomposition of  $\mathfrak{h}$ , where  $\mathfrak{s}$  is a semi-simple subalgebra of  $\mathfrak{h}$  (or is trivial if  $\mathfrak{h}$  is solvable),  $\mathfrak{r}$  is the solvable radical of  $\mathfrak{h}$  (it is an ideal of  $\mathfrak{h}$ ), and  $\varphi$  is the restriction of the adjoint representation  $\text{ad}: \mathfrak{h} \rightarrow \text{Der } \mathfrak{h}$  to  $\mathfrak{s}$  ( $\varphi: \mathfrak{s} \rightarrow \text{Der } \mathfrak{r}$  describes the bracket in  $\mathfrak{h}$  by  $[v, w] = \varphi(v)(w)$  for  $v \in \mathfrak{s}$  and  $w \in \mathfrak{r}$ ).

A proper semi-simple subalgebra of  $\mathfrak{sl}_3$  of dimension less than 5 is three-dimensional, and is thus isomorphic to  $\mathfrak{sl}_2$  or to  $\mathfrak{so}(3)$ . Moreover, up to conjugacy in  $\text{SL}_3(\mathbb{R})$ , the only embedding of  $\mathfrak{so}(3)$  in  $\mathfrak{sl}_3$  is the inclusion, and the only embeddings of  $\mathfrak{sl}_2$  in  $\mathfrak{sl}_3$  are:

$$\mathfrak{s}_0 := \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{sl}_2 \right\} \text{ and } \mathfrak{so}(1, 2). \quad (5.3)$$

If  $\mathfrak{h}$  is not solvable,  $\mathfrak{s}$  is thus equal to  $\mathfrak{s}_0$ ,  $\mathfrak{so}(1, 2)$  or  $\mathfrak{so}(3)$  up to conjugacy in  $\text{SL}_3(\mathbb{R})$ . The centralizers of these subalgebras in  $\mathfrak{sl}_3$  are:

$$\begin{cases} C_{\mathfrak{sl}_3}(\mathfrak{so}(1, 2)) = C_{\mathfrak{sl}_3}(\mathfrak{so}(3)) = \{0\}, \\ C_{\mathfrak{sl}_3}(\mathfrak{s}_0) = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -2x \end{pmatrix} \mid x \in \mathbb{R} \right\}. \end{cases} \quad (5.4)$$

**Lemma 5.2.** *Up to conjugacy in  $\text{SL}_3(\mathbb{R})$  or image by  $\kappa_{\mathfrak{sl}_3} = -^t \cdot$ , we have the following results.*

1. *If  $\mathfrak{h}$  is not solvable, then:*

(a)  *$\mathfrak{s}$  is equal to  $\mathfrak{s}_0$ ,*

(b) and  $\mathfrak{h}$  is equal to  $\mathfrak{h}_{\mathfrak{t}}$  or to

$$\mathfrak{h}_1 = \mathbb{R}^2 \rtimes \mathfrak{sl}_2 = \left\{ \begin{pmatrix} A & X \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{sl}_2, X \in \mathbb{R}^2 \right\}. \quad (5.5)$$

2. If  $\mathfrak{h}$  is solvable, then either  $\mathfrak{h}$  is contained in  $\mathfrak{h}_{\mathfrak{a}} = \mathfrak{p}_{\min}$ , or  $\mathfrak{h}$  is equal to

$$\mathfrak{h}_2 = \mathbb{R}^2 \rtimes \mathfrak{sim}(\mathbb{R}^2) = \left\{ \begin{pmatrix} A & X \\ 0 & -\text{tr } A \end{pmatrix} \mid A \in \mathfrak{sim}(\mathbb{R}^2), X \in \mathbb{R}^2 \right\}, \quad (5.6)$$

$$\text{where } \mathfrak{sim}(\mathbb{R}^2) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid (a, b) \in \mathbb{R}^2 \right\}.$$

*Proof.* 1.a) Let us assume by contradiction that  $\mathfrak{s}$  is conjugated to  $\mathfrak{so}(1, 2)$  or  $\mathfrak{so}(3)$ , implying that  $C_{\mathfrak{sl}_3}(\mathfrak{s}) = \{0\}$  according to (5.4). Since  $\mathfrak{s}$  is simple, if the Lie algebra morphism  $\varphi$  is not injective then it is trivial, implying  $\mathfrak{r} \subset C_{\mathfrak{sl}_3}(\mathfrak{s}) = \{0\}$ , and therefore  $\dim \mathfrak{h} = \dim \mathfrak{s} = 3$  which contradicts Lemma 5.1. Our hypothesis on  $\mathfrak{s}$  implies therefore that  $\varphi$  is injective, and in particular that  $\dim \text{Der } \mathfrak{r} \geq \dim \mathfrak{s} = 3$ .

Since  $\dim \mathfrak{s} = 3$ , the solvable radical  $\mathfrak{r}$  is of dimension 1 or 2 according to Lemma 5.1, and is thus isomorphic to  $\mathbb{R}$ ,  $\mathfrak{aff}(\mathbb{R})$ , or  $\mathbb{R}^2$ . But if  $\mathfrak{r}$  is isomorphic to  $\mathbb{R}$  or  $\mathfrak{aff}(\mathbb{R})$ , then  $\text{Der } \mathfrak{r}$  is of dimension 1 or 2 which contradicts the injectivity of  $\varphi$ , and  $\mathfrak{r}$  is thus isomorphic to  $\mathbb{R}^2$ . Since  $\mathfrak{so}(3)$  has no non-zero two-dimensional representation, this implies that  $\mathfrak{s}$  is conjugated to  $\mathfrak{so}(1, 2)$ . The connected Lie subgroup  $H$  of  $\text{SL}_3(\mathbb{R})$  of Lie algebra  $\mathfrak{h}$  contains then  $\text{SO}^0(1, 2)$ , and its adjoint action induces thus by restriction a two-dimensional representation  $\phi$  of  $\text{SO}^0(1, 2)$  on  $\mathfrak{r}$  (because  $\mathfrak{r}$  is an ideal of  $\mathfrak{h}$ ). Since  $\text{SO}^0(1, 2)$  is isomorphic to  $\text{PSL}_2(\mathbb{R})$ ,  $\phi$  is trivial, implying that  $\varphi$  is trivial as well. This contradiction concludes the proof of the first claim.

1.b) Let us assume by contradiction that  $\mathfrak{r}$  is isomorphic to  $\mathfrak{aff}(\mathbb{R})$ . Then  $\text{Der } \mathfrak{r}$  is two-dimensional and  $\varphi$  is thus non-injective, *i.e.* trivial by simplicity of  $\mathfrak{s}_0$ . But  $\mathfrak{r}$  is then contained in the centralizer of  $\mathfrak{s}_0$  which is one-dimensional according to (5.4), contradicting  $\mathfrak{r} \simeq \mathfrak{aff}(\mathbb{R})$ . Therefore,  $\mathfrak{r}$  is isomorphic to  $\mathbb{R}^2$  or  $\mathbb{R}$ .

We first assume that  $\mathfrak{r}$  is isomorphic to  $\mathbb{R}^2$ , implying that  $\varphi$  is injective (otherwise  $\mathfrak{r} \subset C_{\mathfrak{sl}_3}(\mathfrak{s}_0)$  which is one-dimensional). We use the linear mapping  $\text{ev}_{e_3}|_{\mathfrak{r}}: M \in \mathfrak{r} \mapsto M(e_3) \in \mathbb{R}^3$  and discuss according to the dimension of its image  $\mathfrak{r}(e_3)$ . Let us emphasize that  $\mathfrak{r}$  is normalized by the connected Lie subgroup  $S_0$  of  $\text{SL}_3(\mathbb{R})$  of Lie algebra  $\mathfrak{s}_0$ , and that  $\mathfrak{r}(e_3)$  is thus preserved by  $S_0$ . If  $\mathfrak{r}(e_3)$  is a plane then  $\mathfrak{r}(e_3) = \text{Vect}(e_1, e_2)$ , because it is the only plane of  $\mathbb{R}^3$  preserved by  $S_0$ . Moreover,  $\text{ev}_{e_3}|_{\mathfrak{r}}$  is then injective. There exists  $v \in \mathfrak{r}$  such that  $\text{ev}_{e_3}(v) = e_1$ , and with  $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2$  and  $u = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{s}_0$  we have  $\text{ev}_{e_3}([u, v]) = e_1 = \text{ev}_{e_3}(v)$ . This implies  $[u, v] = v$  by injectivity of  $\text{ev}_{e_3}|_{\mathfrak{r}}$ , and finally  $v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & x & 0 \end{pmatrix}$  for some  $x \in \mathbb{R}$ . The same reasoning with  $w \in \mathfrak{r}$  such that  $\text{ev}_{e_3}(w) = e_2$  and  $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{sl}_2$ , implies that  $w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ y & 0 & 0 \end{pmatrix}$  for some  $y \in \mathbb{R}$ . Since  $\mathfrak{r}$  is abelian we have  $[v, w] = 0$ , which implies  $x = y = 0$  and proves that  $\mathfrak{r} = \begin{pmatrix} 0 & \mathbb{R}^2 \\ 0 & 0 \end{pmatrix}$ , *i.e.* that  $\mathfrak{h} = \mathbb{R}^2 \rtimes \mathfrak{sl}_2$ . If  $\mathfrak{r}(e_3) = \{0\}$ , then  $p: \begin{pmatrix} B & 0 \\ X & 0 \end{pmatrix} \in \mathfrak{r}' \mapsto X \in \mathbb{R}^2$  is injective (recall that  $\mathfrak{r} \cap \mathfrak{s}_0 = \{0\}$ ), implying  $p(\mathfrak{r}) = \mathbb{R}^2$ . Therefore  $\dim(\kappa_{\mathfrak{sl}_3}(\mathfrak{r}))(e_3) = 2$  which brings us back to the first case, and  $\kappa_{\mathfrak{sl}_3}(\mathfrak{h}) = \mathbb{R}^2 \rtimes \mathfrak{sl}_2$ . Finally,  $\dim \mathfrak{r}(e_3) = 1$  is impossible. Otherwise,  $\mathfrak{r}' := \ker \text{ev}_{e_3}|_{\mathfrak{r}}$  is one-dimensional, and since  $p|_{\mathfrak{r}'}$  is injective,  $p(\mathfrak{r}')$  is a line of  $\mathbb{R}^2$ . But for  $w \in \mathfrak{r}'$  and  $v = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{s}_0$  we have  $p([v, w]) = -p(w)A$ , *i.e.*  $p(\mathfrak{r}')$  is preserved by  $\mathfrak{sl}_2$  and cannot be a line.

We now assume that  $\mathfrak{r}$  is isomorphic to  $\mathbb{R}$ . Then  $\varphi$  is non-injective and thus trivial, implying  $\mathfrak{r} \subset C_{\mathfrak{sl}_3}(\mathfrak{s}_0)$ . This inclusion is an equality by equality of dimensions, proving  $\mathfrak{h} = \mathfrak{h}_{\mathfrak{t}}$ .

2. Since  $\mathfrak{h}$  is solvable, it preserves a complex line in  $\mathbb{C}^3$  according to Levi's theorem. More precisely, either  $\mathfrak{h}$  preserves a real line, or it preserves a plane on which it acts by similarities. The second case implies  $\mathfrak{h} \subset \mathbb{R}^2 \rtimes \mathfrak{sim}(\mathbb{R}^2) = \mathfrak{h}_2$  up to conjugacy in  $\text{SL}_3(\mathbb{R})$ . In the first case we can assume that  $\mathfrak{h}$  preserves  $\mathbb{R}e_1$ , and if the representation  $\begin{pmatrix} * & * \\ 0 & A \end{pmatrix} \in \mathfrak{h} \mapsto A \in \mathfrak{gl}_2$  also preserves a real line, then  $\mathfrak{h} \subset \mathfrak{p}_{\min} = \mathfrak{h}_{\mathfrak{a}}$  up to conjugacy. If not, then  $\kappa_{\mathfrak{sl}_3}(\mathfrak{h}) \subset \left\{ \begin{pmatrix} -\text{tr } A & 0 \\ X & A \end{pmatrix} \mid A \in \mathfrak{sim}(\mathbb{R}^2), X \in \mathbb{R}^2 \right\}$ , according to the same remark than before. This last subalgebra being conjugated to  $\mathbb{R}^2 \rtimes \mathfrak{sim}(\mathbb{R}^2) = \mathfrak{h}_2$ , this concludes the proof of the lemma.  $\square$

## 5.2 Two further properties of the infinitesimal model

In order to eliminate the subalgebras  $\mathfrak{h}_1$  and  $\mathfrak{h}_2$  appearing in Lemma 5.2, we prove two additional properties of the infinitesimal model  $(\mathfrak{h}, \mathfrak{i})$ .

**Lemma 5.3.** *Let  $\mathfrak{l}$  be a subalgebra of  $\mathfrak{sl}_3$  containing  $\mathfrak{h}$ ,  $\mathfrak{j} = \text{stab}_{\mathfrak{l}}(x_0)$  be its isotropy at  $x_0$ , and  $D^c$  be the line of  $\mathfrak{l}/\mathfrak{j}$  sent to  $\mathcal{E}^c(x_0)$  by the orbital map at  $x_0$ . If  $\overline{\text{ad}}(\mathfrak{j})(D^c) \subset D^c$ , then  $\mathfrak{l} = \mathfrak{h}$ .*

*Proof.* Let us denote by  $L$  the connected Lie subgroup of  $\mathbf{G}$  of Lie algebra  $\mathfrak{l}$ , and by  $J^0$  the identity component of  $J = \text{Stab}_L(x_0)$ . As  $\overline{\text{ad}}(\mathfrak{j})$  preserves  $D^c$ ,  $\overline{\text{Ad}}(\exp(\mathfrak{j}))$  preserves  $D^c$ , and the subgroup of elements  $j \in J^0$  such that  $\overline{\text{Ad}}(j)$  preserves  $D^c$  is thus equal to  $J^0$  by connexity. The construction made in the second assertion of Lemma 4.3 is thus valid for  $L/J^0$ , and proves the existence of an unique  $L$ -invariant enhanced Lagrangean contact structure  $\mathcal{S}'_Y$  extending  $(\mathcal{E}^\alpha(x_0), \mathcal{E}^\beta(x_0), \mathcal{E}^c(x_0))$  in the neighbourhood of  $x_0$ . As  $\mathfrak{h} \subset \mathfrak{l}$ , we have  $H \subset L$ , and  $\mathcal{S}'_Y$  is thus  $H$ -invariant, implying  $\mathcal{S}'_Y = \mathcal{S}_Y$  by the unicity of such a structure (see Lemma 4.3). Therefore  $\mathfrak{l} \subset \text{fill}_{\mathcal{S}'_Y}^{\text{loc}}(x_0) = \mathfrak{h}$ , which concludes the proof.  $\square$

**Lemma 5.4.** *Let us assume that  $\mathfrak{i}$  is one-dimensional, and let  $v$  be a non-zero element of  $\mathfrak{i}$ . Then the eigenvalues of  $\overline{\text{ad}}(v) \in \text{End}(\mathfrak{h}/\mathfrak{i})$  with respect to the eigenlines  $D^\alpha$  and  $D^\beta$  are non-zero.*

*Proof.* We already know that  $\overline{\text{ad}}(\mathfrak{i})$  is diagonalizable with eigenlines  $D^\alpha$ ,  $D^\beta$ , and  $D^c$  (see Lemma 4.3). The proof is the same for the eigenvalues of both eigenlines  $D^\alpha$  and  $D^\beta$ , and we only do it for  $D^\alpha$ . By density of  $\text{Rec}(f) \cap \text{Rec}(f^{-1})$  in  $M$  (see the introduction of Section 3), there exists  $x \in O$  such that  $\bar{x} = \pi_M(x) \in \text{Rec}(f) \cap \text{Rec}(f^{-1})$ , and possibly replacing  $f$  by  $f^{-1}$ , we have  $\lim_{n \rightarrow +\infty} \|D_{\bar{x}} f^n|_{E^\alpha(\bar{x})}\|_M = 0$  for a given Riemannian metric that we fix on  $M$ .

By hypothesis on  $\bar{x}$ , there exists a sequence  $(\gamma_k)$  in  $\pi_1(M)$  and a strictly increasing sequence  $(n_k)$  of integers such that  $\gamma_k \tilde{f}^{n_k}(x)$  converges to  $x$ . We can moreover assume up to extraction that  $x_k \in O$  for any  $k$ , implying that  $\gamma_k \tilde{f}^{n_k}$  preserves  $O$ . Endowing  $\tilde{M}$  with the pullback  $\tilde{\mu}_M$  of the Riemannian metric of  $M$ , we have  $\lim_{k \rightarrow +\infty} \|D_x(\gamma_k \tilde{f}^{n_k})|_{\tilde{E}^\alpha(x)}\|_{\tilde{\mu}_M} = 0$  (since  $\pi_1(M)$  acts by isometries).

Theorem 2.9 implies the existence of a unique sequence  $(g_k)$  in  $\mathbf{G}$  satisfying

$$\delta \circ \gamma_k \tilde{f}^{n_k} = g_k \circ \delta \text{ on a neighbourhood of } x. \quad (5.7)$$

Denoting  $x_0 = \delta(x)$ ,  $g_k \cdot x_0 = \delta \circ \gamma_k \tilde{f}^{n_k}(x) \in Y = H \cdot x_0$  converges to  $x_0$ . There exists thus a sequence  $h_k \in H$  converging to the identity in  $\mathbf{G}$  and such that  $h_k \cdot x_0 = g_k \cdot x_0$ . Since  $\delta$  is a local isomorphism from  $\tilde{\mathcal{S}}|_O$  to  $\mathcal{S}_Y$  on a neighbourhood of  $x$ , the equation (5.7) defining  $g_k$  shows that  $g_k$  preserves  $\mathcal{S}_Y$  on a neighbourhood of  $x_0$ . By  $H$ -invariance of  $\mathcal{S}_Y$ ,  $i_k = h_k^{-1} g_k$  also preserves  $\mathcal{S}_Y$ , and  $i_k$  is thus contained in the closed subgroup

$$I := \{i \in \text{Stab}_{\mathbf{G}}(x_0) \mid i \text{ preserves } \mathcal{S}_Y \text{ on a neighbourhood of } x_0\}$$

of  $\mathbf{G}$ . The Lie algebra of  $I$  is equal to  $\mathfrak{i}$  because  $\text{is}_{\mathcal{S}_Y}^{\text{loc}}(x_0) = \mathfrak{i}$  (see Lemma 4.3).

**Fact.**  $I = \{i \in \text{Stab}_{\mathbf{G}}(x_0) \mid \text{Ad}(i) \cdot \mathfrak{h} = \mathfrak{h} \text{ and } \overline{\text{Ad}}(i) \cdot D^c = D^c\}$ . In particular  $I$  is algebraic and has a finite number of connected components.

*Proof.* For  $i \in I$  and  $v \in \mathfrak{h}$ , the relation  $D_{x_0} i \circ D_e \theta_{x_0} = D_e \theta_{x_0} \circ \text{Ad}(i)$  implies  $i^{-1*} v = \text{Ad}(i) \cdot v$ . Since  $i$  is a local automorphism of  $\mathcal{S}_Y$  and  $v$  a Killing field of  $\mathcal{S}_Y$ ,  $\text{Ad}(i) \cdot v$  is also a Killing field of  $\mathcal{S}_Y$ , implying  $\text{Ad}(i) \cdot v \in \mathfrak{h}$  since  $\text{fill}_{\mathcal{S}_Y}^{\text{loc}}(x_0) = \mathfrak{h}$  (see Lemma 4.3). Moreover,  $D_{x_0} i(\mathcal{E}_{x_0}^c) = \mathcal{E}_{x_0}^c$  implies  $\overline{\text{Ad}}(i) \cdot D^c = D^c$ .

Let us conversely assume that  $i \in \text{Stab}_{\mathbf{G}}(x_0)$  satisfies  $\text{Ad}(i) \cdot \mathfrak{h} = \mathfrak{h}$  and  $\overline{\text{Ad}}(i) \cdot D^c = D^c$ . We consider  $v \in \mathfrak{h}$  sufficiently close to 0, such that with  $h = e^v \in H$  and  $y = h \cdot x_0 \in Y$ ,  $\mathcal{S}_Y$  is defined at  $y$ . Since  $\overline{\text{Ad}}(i) \cdot D^c = D^c$ ,  $D_{x_0} i(\mathcal{E}^c(x_0)) = \mathcal{E}^c(x_0)$ , and  $h' := i h i^{-1} = e^{\text{Ad}(i) \cdot v} \in H$  because  $\text{Ad}(i) \cdot \mathfrak{h} = \mathfrak{h}$ . By  $H$ -invariance of  $\mathcal{E}^c$ , we obtain  $D_y i(\mathcal{E}^c(y)) = D_{x_0} h' \circ D_{x_0} i(\mathcal{E}^c(x_0)) = \mathcal{E}^c(i \cdot y)$ , proving that  $i \in I$ .  $\square$

We can thus assume up to extraction that  $(i_k)$  lies in a given connected component of  $I$ , and there exists then  $g \in I$  such that  $j_k = gi_k$  is contained in the identity component  $I^0$ . We endow  $\mathbf{X}$  with a Riemannian metric  $\mu_{\mathbf{X}}$ , and denote by  $\tilde{\mu}_{\mathbf{X}} = \delta^* \mu_{\mathbf{X}}$  its pullback on  $\tilde{M}$ . Since  $(\gamma_k \tilde{f}^{n_k}(x))$  is relatively compact in  $\tilde{M}$ , the metrics  $\tilde{\mu}_M$  and  $\tilde{\mu}_{\mathbf{X}}$  are equivalent in restriction to  $(\gamma_k \tilde{f}^{n_k}(x))$ , and the limit stated above for  $\tilde{\mu}_M$  is thus valid for  $\tilde{\mu}_{\mathbf{X}}$ , implying that  $\lim \left\| D_{x_0} g_k|_{\mathcal{E}^\alpha(x_0)} \right\|_{\mu_{\mathbf{X}}} = 0$ . Since  $j_k = gh_k^{-1}g_k$  with  $(gh_k^{-1})$  relatively compact in  $\mathbf{G}$ , we also have  $\lim \left\| D_{x_0} j_k|_{\mathcal{E}^\alpha(x_0)} \right\|_{\mu_{\mathbf{X}}} = 0$ .

The identity component  $I^0$  being connected and one-dimensional, there exists a non-zero  $v \in \mathfrak{i}$  and a sequence  $t_k \in \mathbb{R}$  such that  $i_k = \exp(t_k v)$ , implying that  $D_{x_0} j_k$  is conjugated by the orbital map to  $\exp(t_k \overline{\text{ad}}(v))$ , and thus  $\lim \left\| \exp(t_k \overline{\text{ad}}(v))|_{D^\alpha} \right\| = 0$ . Denoting by  $\lambda_\alpha$  the eigenvalue of  $\overline{\text{ad}}(v)$  with respect to  $D^\alpha$ ,  $\exp(t_k \overline{\text{ad}}(v))|_{D^\alpha} = \exp(\lambda_\alpha t_k) \text{id}_{D^\alpha}$  implies then  $\lambda_\alpha \neq 0$ .  $\square$

### 5.3 End of the classification

We are now able to conclude the classification.

**Lemma 5.5.**  $\mathfrak{h}_1 = \mathbb{R}^2 \rtimes \mathfrak{sl}_2$  does not satisfy the geometrical conditions of Lemma 4.3.

*Proof.* The only open orbit of the connected Lie subgroup  $H_1$  of  $\mathbf{G}$  of Lie algebra  $\mathfrak{h}_1$  is the open subset  $\Omega_a$  defined in Paragraph 4.2.1. If  $H_1 \cdot x_0$  is open for some point  $x_0 \in \mathbf{X}$ , we can thus assume that  $x_0 = ([e_3], [e_3, e_1]) \in \Omega_a$  up to conjugacy in  $H_1$ , implying that  $\mathfrak{i}_1 = \text{Lie}(\text{Stab}_{H_1}(x_0)) = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \mid a, b \in \mathbb{R}^2 \right\}$ . Denoting  $v_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  and  $v_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathfrak{h}_1$ , we have  $\mathbb{R}v_\alpha^\dagger(x_0) = \mathcal{E}^\alpha(x_0)$  and  $\mathbb{R}v_\beta^\dagger(x_0) = \mathcal{E}^\beta(x_0)$ , and defining  $v_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{i}_1$ , the matrix of  $\overline{\text{ad}}(i)$  in the basis  $(\bar{v}_\alpha, \bar{v}_\beta, \bar{v}_c)$  of  $\mathfrak{h}_1/\mathfrak{i}_1$  is:

$$\text{Mat}_{(\bar{v}_\alpha, \bar{v}_\beta, \bar{v}_c)} \overline{\text{ad}}(i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Any line of  $\mathfrak{h}_1/\mathfrak{i}_1$  that is transverse to  $\text{Vect}(\bar{v}_\alpha, \bar{v}_\beta)$  has projective coordinates  $[a, b, 1]$  in the basis  $(\bar{v}_\alpha, \bar{v}_\beta, \bar{v}_c)$  for some  $(a, b) \in \mathbb{R}^2$ , and  $\overline{\text{ad}}(i)(D^c)$  has thus coordinates  $[0, 1, 0]$ . This proves that  $\overline{\text{ad}}(i)(D^c) \not\subset D^c$ , i.e. that  $\mathfrak{h}_1$  does not satisfy the geometrical conditions of Lemma 4.3.  $\square$

**Lemma 5.6.** If  $\mathfrak{h}$  is a four-dimensional subalgebra of  $\mathfrak{h}_a = \mathfrak{p}_{\min}$ , or is equal to  $\mathfrak{h}_2 = \mathbb{R}^2 \rtimes \mathfrak{sim}(\mathbb{R}^2)$ , then  $\mathfrak{h}$  does not respect both the geometrical conditions of Lemma 4.3 and the dynamical condition of Lemma 5.4.

*Proof.* We first assume that  $\mathfrak{h}$  is a four-dimensional subalgebra of  $\mathfrak{p}_{\min}$ . Therefore  $H \subset \mathbf{P}_{\min}$ , and if  $H \cdot x_0$  is open then  $x_0 \in Y_a$  according to Proposition 4.5. We can thus assume up to conjugacy in  $H$  that  $x_0 = o_a = ([e_3], [e_3, e_2]) \in Y_a$ , implying:

$$\mathfrak{i} = \mathfrak{stab}_{\mathfrak{h}}(o_a) \subset \mathfrak{i}_a = \mathfrak{stab}_{\mathfrak{p}_{\min}}(o_a) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & -a-b & 0 \\ 0 & 0 & b \end{pmatrix} \mid (a, b) \in \mathbb{R}^2 \right\}.$$

Let  $D^c \subset \mathfrak{h}/\mathfrak{i}$  be a line preserved by  $\overline{\text{ad}}(i)$ , and such that  $D_e \bar{\theta}_{o_a}(D^c)$  is transverse to  $(\mathcal{E}^\alpha \oplus \mathcal{E}^\beta)(o_a)$ . Since  $\mathfrak{h}$  is a proper subalgebra of  $\mathfrak{p}_{\min}$ , Lemma 5.3 implies that  $\overline{\text{ad}}(i_a)(D^c) \not\subset D^c$ , and thus that  $\mathfrak{stab}_{\mathfrak{i}_a}(D^c) := \left\{ v \in \mathfrak{i}_a \mid \overline{\text{ad}}(v)(D^c) \subset D^c \right\}$  is equal to  $\mathfrak{i}$ . Consequently,  $\dim \mathfrak{stab}_{\mathfrak{i}_a}(D^c) = 1$ . Any line  $D^c$  of  $\mathfrak{p}_{\min}/\mathfrak{i}_a$  which is transverse to the contact plane has projective coordinates  $[x, y, 1]$  in the basis  $(\bar{X}, \bar{Y}, \bar{Z})$  of  $\mathfrak{p}_{\min}/\mathfrak{i}_a$ , for some  $(x, y) \in \mathbb{R}^2$  (see Proposition 4.6), and according to equation (4.15):

- if  $x = y = 0$ , i.e.  $D^c = \mathbb{R}\bar{Z}$ , then  $\mathfrak{stab}_{\mathfrak{i}_a}(\mathbb{R}\bar{Z}) = \mathfrak{i}_a$ ;
- if  $x = 0$  and  $y \neq 0$ , i.e.  $D^c = D_Y^c(t) := \mathbb{R}(\bar{Z} + t\bar{Y})$  for some  $t \in \mathbb{R}$ , then  $\mathfrak{stab}_{\mathfrak{i}_a}(D_Y^c(t))$  is equal to the line  $\mathfrak{i}_Y$  generated by the diagonal matrix  $[1, 1, -2] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ ;

- if  $x \neq 0$  and  $y = 0$ , i.e.  $D^c = D_X^c(t) := \mathbb{R}(\bar{Z} + t\bar{X})$  for some  $t \in \mathbb{R}$ , then  $\mathfrak{stab}_{\mathbf{i}_a}(D_X^c(t))$  is equal to the line  $\mathbf{i}_X$  generated by the diagonal matrix  $[-2, 1, 1] = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ;
- if  $x \neq 0$  and  $y \neq 0$ , then  $\mathfrak{stab}_{\mathbf{i}_a}(D^c) = \{0\}$ .

The only transverse lines of  $\mathfrak{p}_{min}/\mathbf{i}_a$  having a one-dimensional stabilizer in  $\mathbf{i}_a$  are thus  $D_X^c(t)$  and  $D_Y^c(t)$ , and  $\mathbf{i}$  is therefore equal to  $\mathbf{i}_X$  or  $\mathbf{i}_Y$ . But  $\text{Mat}_{(\bar{X}, \bar{Y}, \bar{Z})} \overline{\text{ad}}([1, 1, -2]) = [0, 3, 3]$  and  $\text{Mat}_{(\bar{X}, \bar{Y}, \bar{Z})} \overline{\text{ad}}([-2, 1, 1]) = [-3, 0, -3]$  according to (4.15), i.e. the elements of  $\mathbf{i}_X$  and  $\mathbf{i}_Y$  have zero eigenvalue with respect to either the  $\alpha$  or the  $\beta$ -direction. This shows that  $\mathfrak{h}$  does not satisfy the dynamical condition of Lemma 5.4, and concludes the proof in the case  $\mathfrak{h} \subsetneq \mathfrak{p}_{min}$ .

In the same way, if  $\mathfrak{h} = \mathfrak{h}_2$ , then we can assume that  $x_0 = o_a \in \Omega_a$  up to conjugacy in  $H_2 = \mathbb{R}^2 \rtimes \text{Sim}(\mathbb{R}^2)$ , implying  $\mathbf{i}_2 = \mathfrak{stab}_{\mathfrak{h}_2}(o_a) = \mathbf{i}_Y$  defined above. But we saw that the elements of  $\mathbf{i}_Y$  have zero eigenvalue with respect to the  $\alpha$ -direction, which shows that  $\mathfrak{h}_2$  does not satisfy the dynamical condition of Lemma 5.4.  $\square$

Proposition 4.4 directly follows from Lemmas 5.2, 5.5 and 5.6.

## 6 Global structure

From the local model that we determined for the enhanced Lagrangean contact structure  $\mathcal{S}$ , we will now deduce a global information.

### 6.1 Local homogeneity of the enhanced Lagrangean contact structure

So far, we only have informations about  $\mathcal{S}$  on a dense and open subset  $\Omega$  of  $M$  (see Proposition 3.2), and the first step to obtain a global information is to prove the following result.

**Proposition 6.1.** *The open dense subset  $\Omega$  equals  $M$ , i.e.  $\mathcal{S}$  is locally homogeneous on  $M$ .*

We will denote in this paragraph by  $(\mathcal{C}, \varphi) = (\hat{M}, \omega, \varphi)$  the normal generalized Cartan geometry of the enhanced Lagrangean contact structure  $\tilde{\mathcal{S}} = \pi_M^* \mathcal{S}$  of  $\tilde{M}$ , and by  $\mathcal{K}^{tot}: \hat{M} \rightarrow W_{\mathcal{K}^{tot}}$  its total curvature (see Paragraphs 2.3.4 and 2.4.2). We recall that  $\tilde{\Omega} = \pi_M^{-1}(\Omega) \subset \tilde{M}$ , and that the projection of the Cartan bundle is denoted by  $\pi: \hat{M} \rightarrow \tilde{M}$ .

We also recall that the local homogeneity of  $\tilde{\mathcal{S}}|_{\tilde{\Omega}}$  means that the connected components of  $\tilde{\Omega}$  are exactly its  $\text{Kill}^{loc}$ -orbits (see Definition 3.1). Since the rank of  $D\mathcal{K}^{tot}$  is invariant by the right action of  $\mathbf{P}_{min}$  and by the flow of Killing fields, this shows that  $\text{rk}(D\mathcal{K}^{tot})$  is constant over any connected component of  $\tilde{\Omega}$ .

We choose for this whole paragraph a connected component  $O$  of  $\tilde{\Omega}$  (i.e. an open  $\text{Kill}^{loc}$ -orbit of  $\tilde{\mathcal{S}}$ ) such that  $\text{rk}(D_{\hat{x}}\mathcal{K}^{tot})$  for  $\hat{x} \in \pi^{-1}(O)$  is maximal among  $\text{rk}(D_{\hat{x}}\mathcal{K}^{tot})$  for  $\hat{x} \in \pi^{-1}(\tilde{\Omega})$ . We will denote by  $(Y, \mathcal{S}_Y)$  the local model of  $\tilde{\mathcal{S}}|_O$ , equal to  $(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$  or  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$  and such that  $\delta|_O: (O, \tilde{\mathcal{S}}|_O) \rightarrow (Y, \mathcal{S}_Y)$  is a local isomorphism (see Corollary 4.6). We still denote by  $\mathfrak{h}$  the subalgebra of Killing fields of  $\mathcal{S}_Y$ , respectively equal to  $\mathfrak{h}_{\mathbf{t}}$  or  $\mathfrak{h}_{\mathbf{a}}$  (see Proposition 4.4), and by  $H$  the corresponding Lie connected subgroup

$$H_{\mathbf{t}}^0 = \begin{bmatrix} \text{GL}_2^+(\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix} \text{ or } H_{\mathbf{a}}^0 = \mathbf{P}_{min}^+,$$

of  $\mathbf{G}$  of Lie algebra  $\mathfrak{h}$ , preserving  $\mathcal{S}_Y$ .

We recall that  $\delta: \tilde{M} \rightarrow \mathbf{X}$  denotes the developping map of the  $(\mathbf{G}, \mathbf{X})$ -structure of  $M$  describing the flat Lagrangean contact structure  $\mathcal{L}$  (see Corollary 3.4).

**Lemma 6.2.** *The boundary of  $O$  is mapped to  $\mathbf{X} \setminus Y$  by the developping map:  $\delta(\partial O) \subset \mathbf{X} \setminus Y$ .*

*Proof.* Let us assume by contradiction that there exists  $x \in \partial O$  such that  $x_0 = \delta(x) \in Y$ . The pullback  $\tilde{\mathfrak{h}} := \delta^* \mathfrak{h} = \{\delta^* v \mid v \in \mathfrak{h}\}$  is a subalgebra of vector fields of  $\tilde{M}$ , such that  $\mathfrak{Kill}(O, \tilde{\mathcal{S}}|_O) = \tilde{\mathfrak{h}}|_O$  according to Lemma 4.1. As  $x_0 \in Y$ , there exists an open and convex neighbourhood  $W_0$  of 0 in  $\mathfrak{h}$  such that  $V = \exp(W_0) \cdot x_0 \subset Y$  is an open neighbourhood of  $x_0$ . Denoting  $W = \delta^* W_0 \subset \tilde{\mathfrak{h}}$ ,  $U = \{\varphi_X^1(x) \mid X \in W\}$  is thus an open neighbourhood of  $x$ , and possibly shrinking  $W_0$ , we can moreover assume that  $\delta|_U$  is a diffeomorphism from  $U$  to  $V$ . As  $x \in \partial O$ , there exists  $y \in U \cap O$ , and  $X \in W$  such that  $x = \varphi_X^1(y)$ , implying that  $\varphi_X^t(y) \in U$  for any  $t \in [0; 1]$ , and thus  $\delta(\varphi_X^t(y)) \in V \subset Y$ . Denoting  $t_0 = \inf \{t \in [0; 1] \mid \varphi_X^t(y) \in \partial O\}$ ,  $t_0 > 0$  because  $O$  is open, and  $\varphi_X^{t_0}(y) \in \partial O$  because  $\partial O$  is closed. Replacing  $x$  by  $\varphi_X^{t_0}(y)$  and  $X$  by  $\frac{X}{t_0} \in W$ , we finally have  $y \in O$ ,  $x = \varphi_X^1(y) \in \partial O$ , and for any  $t \in [0; 1[$ ,  $\varphi_X^t(y) \in O$ , with  $X|_O \in \text{Kill}(O, \tilde{\mathcal{S}}|_O)$ .

Choosing  $\hat{y} \in \pi^{-1}(y)$ , the invariance of  $D^1 \mathcal{K}^{tot}$  by local automorphisms and the fact that  $\varphi_X^t$  is a local automorphism of  $(\mathcal{C}, \varphi)$  on the neighbourhood of  $y$  for any  $t \in [0; 1[$  implies  $D^1 \mathcal{K}^{tot}(\hat{\varphi}_X^t(\hat{y})) = D^1 \mathcal{K}^{tot}(\hat{y})$  for any  $t \in [0; 1[$ . Denoting  $\hat{x} = \hat{\varphi}_X^1(\hat{y})$ , we obtain  $D^1 \mathcal{K}^{tot}(\hat{x}) = D^1 \mathcal{K}^{tot}(\hat{y})$  by continuity, *i.e.*  $\mathcal{K}^{tot}(\hat{x}) = \mathcal{K}^{tot}(\hat{y})$  and  $D_{\hat{x}} \mathcal{K}^{tot} \circ \omega_{\hat{x}}^{-1} = D_{\hat{y}} \mathcal{K}^{tot} \circ \omega_{\hat{y}}^{-1}$  (see Paragraph 2.4.2 for the definition of  $D^1 \mathcal{K}^{tot}$ ).

This implies  $\hat{x} \in \hat{M}^{int}$ . In fact since the rank of  $D\mathcal{K}^{tot}$  can only increase locally, there is an open neighbourhood  $\mathcal{U}$  of  $\hat{x}$  where the rank of  $D\mathcal{K}^{tot}$  is greater than  $\text{rk}(D_{\hat{x}} \mathcal{K}^{tot})$ . Let us assume by contradiction that the open subset of  $\mathcal{U}$  where  $\text{rk}(D_{\hat{x}'} \mathcal{K}^{tot}) > \text{rk}(D_{\hat{x}} \mathcal{K}^{tot})$  is non-empty. Then by density of  $\pi^{-1}(\tilde{\Omega})$ , there exists  $\hat{z} \in \pi^{-1}(\tilde{\Omega})$  such that  $\text{rk}(D_{\hat{z}} \mathcal{K}^{tot}) > \text{rk}(D_{\hat{x}} \mathcal{K}^{tot})$ . But  $\text{rk}(D_{\hat{x}} \mathcal{K}^{tot}) = \text{rk}(D_{\hat{y}} \mathcal{K}^{tot})$  because  $D^1 \mathcal{K}^{tot}(\hat{x}) = D^1 \mathcal{K}^{tot}(\hat{y})$ , and thus  $\text{rk}(D_{\hat{z}} \mathcal{K}^{tot}) > \text{rk}(D_{\hat{y}} \mathcal{K}^{tot})$  with  $\hat{y} \in \pi^{-1}(O)$ , which contradicts our hypothesis of maximality of  $\text{rk}(D\mathcal{K}^{tot})$  on  $O$ . Therefore  $\text{rk}(D\mathcal{K}^{tot})$  is constant on the open neighbourhood  $\mathcal{U}$  of  $\hat{x}$ , proving that  $\hat{x} \in \hat{M}^{int}$  according to Theorem 2.17.

For any  $X \in \mathfrak{Kill}_S^{loc}(y)$ ,  $\hat{X}_{\hat{y}} \in \text{Ker}(D_{\hat{y}} \mathcal{K}^{tot})$ , which implies  $\omega_{\hat{y}}^{-1}(\mathfrak{p}_{min}) + \text{Ker}(D_{\hat{y}} \mathcal{K}^{tot}) = T_{\hat{y}} \hat{M}$  since the  $\text{Kill}^{loc}$ -orbit  $O$  of  $y$  is open, and therefore  $\omega_{\hat{x}}^{-1}(\mathfrak{p}_{min}) + \text{Ker}(D_{\hat{x}} \mathcal{K}^{tot}) = T_{\hat{x}} \hat{M}$  (because  $D_{\hat{x}} \mathcal{K}^{tot} \circ \omega_{\hat{x}}^{-1} = D_{\hat{y}} \mathcal{K}^{tot} \circ \omega_{\hat{y}}^{-1}$ ). Since  $\hat{x} \in \hat{M}^{int}$ , this equality implies  $\{X_x \mid X \in \mathfrak{Kill}_S^{loc}(x)\} = T_x \tilde{M}$ , *i.e.* that the  $\text{Kill}^{loc}$ -orbit of  $x$  is open. But  $x \in \partial O$ , and the  $\text{Kill}^{loc}$ -orbit of  $x$  intersects thus  $O$ , *i.e.*  $x \in O$  since  $O$  is also a  $\text{Kill}^{loc}$ -orbit, which contradicts our initial hypothesis. This contradiction concludes the proof of the lemma.  $\square$

Lemma 6.2 allows us to reduce the study of the central direction  $\tilde{E}^c$  on the boundary of  $O$ , to the study of the central direction  $\mathcal{E}^c$  on the boundary of  $Y$ . We first do some geometrical remarks about the open subsets  $Y_{\mathbf{a}}$  and  $Y_{\mathbf{t}}$  of  $\mathbf{X}$ , defined in Paragraphs 4.2.2 and 4.2.3.

Let us recall that, denoting  $D_{\infty} = [e_1, e_2]$ ,  $m_{\mathbf{t}} = [e_3]$  and  $m_{\mathbf{a}} = [e_1]$ , we have:

$$Y_{\mathbf{t}} = \mathbf{X} \setminus (S_{\beta, \alpha}(D_{\infty}) \cup S_{\alpha, \beta}(m_{\mathbf{t}})) \text{ and } Y_{\mathbf{a}} = \mathbf{X} \setminus (S_{\beta, \alpha}(D_{\infty}) \cup S_{\beta, \alpha}(m_{\mathbf{a}})).$$

In particular, for  $\varepsilon = \mathbf{a}$  and  $\mathbf{t}$  we have  $\mathbf{X} \setminus Y_{\varepsilon} = \partial Y_{\varepsilon} = S_{\beta, \alpha}(D_{\infty}) \cup S_{\alpha, \beta}(m_{\varepsilon})$ .

We define in both cases

$$\mathcal{G} := \left\{ x \in \partial Y \mid \mathcal{C}^{\alpha}(x) \not\subset \partial Y \text{ or } \mathcal{C}^{\beta}(x) \not\subset \partial Y \right\}.$$

It is easy to check that for  $\varepsilon = \mathbf{a}$  and  $\mathbf{t}$ , we have:

$$\mathcal{G}_{\varepsilon} = \partial Y_{\varepsilon} \setminus \{\mathcal{C}^{\beta}(D_{\infty}) \cup \mathcal{C}^{\alpha}(m_{\varepsilon}) \cup (S_{\beta, \alpha}(D_{\infty}) \cap S_{\alpha, \beta}(m_{\varepsilon}))\},$$

and that for any  $x \in \mathcal{G}$ , if  $\mathcal{C}^{\varepsilon}(x) \not\subset \partial Y$  for  $\varepsilon = \alpha$  or  $\beta$ , then  $\mathcal{C}^{\varepsilon}(x) \setminus \{x\} \subset Y$ .

We have  $S_{\beta, \alpha}(D_{\infty}) \cap S_{\alpha, \beta}(m_{\mathbf{a}}) = \mathcal{C}^{\beta}(D_{\infty}) \cup \mathcal{C}^{\alpha}(m_{\mathbf{a}})$ , and  $S_{\beta, \alpha}(D_{\infty}) \cap S_{\alpha, \beta}(m_{\mathbf{t}})$  is equal to the chain defined by  $(m_{\mathbf{t}}, D_{\infty})$ , denoted by  $\mathcal{C}(m_{\mathbf{t}}, D_{\infty})$  and defined as follows:

$$\mathcal{C}(m_{\mathbf{t}}, D_{\infty}) := \{(m, [m, m_{\mathbf{t}}]) \mid m \in D_{\infty}\}.$$

Finally, we will use the following description of the respective orbits of  $H$  on  $\mathcal{G}$ :

1. the orbits of  $H_{\mathbf{t}}^0$  on  $\mathcal{G}_{\mathbf{t}}$  are  $\mathcal{G}_{\mathbf{t}}^1 = S_{\alpha,\beta}(m_{\mathbf{t}}) \setminus (\mathcal{C}^\alpha(m_{\mathbf{t}}) \cup \mathcal{C}(m_{\mathbf{t}}, D_\infty))$  where  $\mathcal{C}^\alpha(x) \setminus \{x\} \subset Y_{\mathbf{t}}$ , and  $\mathcal{G}_{\mathbf{t}}^2 = S_{\beta,\alpha}(D_\infty) \setminus (\mathcal{C}^\beta(D_\infty) \cup \mathcal{C}(m_{\mathbf{t}}, D_\infty))$  where  $\mathcal{C}^\beta(x) \setminus \{x\} \subset Y_{\mathbf{t}}$ ;
2. the orbits of  $H_{\mathbf{a}}^0$  on  $\mathcal{G}_{\mathbf{a}}$  are  $\mathcal{G}_{\mathbf{a}}^1 = S_{\alpha,\beta}(m_{\mathbf{a}}) \setminus (\mathcal{C}^\alpha(m_{\mathbf{a}}) \cup \mathcal{C}^\beta(D_\infty))$  where  $\mathcal{C}^\alpha(x) \setminus \{x\} \subset Y_{\mathbf{a}}$ , and  $\mathcal{G}_{\mathbf{a}}^2 = S_{\beta,\alpha}(D_\infty) \setminus (\mathcal{C}^\alpha(m_{\mathbf{a}}) \cup \mathcal{C}^\beta(D_\infty))$  where  $\mathcal{C}^\beta(x) \setminus \{x\} \subset Y_{\mathbf{a}}$ .

We now prove that the central direction  $\mathcal{E}^c$  degenerates along the  $\alpha$  and  $\beta$ -circles when converging to a point of  $\mathcal{G}$ .

**Lemma 6.3.** *Let  $\gamma: [0; 1] \rightarrow \mathbf{X}$  be a smooth path such that  $\gamma([0; 1]) \subset Y$ ,  $x = \gamma(0) \in \mathcal{G}$ , and  $\gamma([0; 1])$  is entirely contained in  $\mathcal{C}^\alpha(x)$ , or entirely contained in  $\mathcal{C}^\beta(x)$ . Then  $\mathcal{E}^c(\gamma(t))$  converges at  $t = 0$  to a line contained in  $(\mathcal{E}^\alpha \oplus \mathcal{E}^\beta)(x)$ .*

*Proof.* Since the action of  $H$  on  $Y$  preserves  $\mathcal{E}^c$ , it will be sufficient to prove this result for one point of each of the two orbits of  $H$  on  $\mathcal{G}$  described above, in each of the two cases  $Y_{\mathbf{t}}$  or  $Y_{\mathbf{a}}$ . Moreover, we saw that in each case, either  $\mathcal{C}^\alpha(x) \setminus \{x\} \subset Y$  and  $\mathcal{C}^\beta(x) \subset \partial Y$ , or the contrary. We thus have only one possibility to consider for  $\gamma$  in each of these four cases, either that  $\gamma([0; 1]) \subset \mathcal{C}^\alpha(x)$ , or that  $\gamma([0; 1]) \subset \mathcal{C}^\beta(x)$ . To clarify our strategy, let  $x$  be a point of  $\mathcal{G}_\mu^i$  for  $\mu = \mathbf{t}$  or  $\mathbf{a}$  and  $i = 1$  or  $2$ , and let us consider the following data:

- a one-parameter subgroup  $\{g^t\}_{t \in \mathbb{R}}$  of  $\mathbf{G}$  such that, denoting  $x(t) = g^t \cdot x$ , we have  $\{x(t) \mid t \in \mathbb{R}\} = \mathcal{C}^\varepsilon(x) \setminus \{y\}$ , with  $y \in \mathcal{C}^\varepsilon(x) \cap Y$ , and  $\varepsilon = \alpha$  or  $\beta$  according to the case considered,
- a one-parameter subgroup  $\{h^t\}_{t \in \mathbb{R}}$  of  $H$  such that  $g^t \cdot x = x(t) = h^{t^{-1}} \cdot y$  for any  $t \in \mathbb{R}^*$ ,
- $A$  in  $\mathfrak{sl}_3$  such that  $D_e \theta_y(\mathbb{R}A) = \mathcal{E}^c(y)$ , where  $\theta_y: \mathbf{G} \rightarrow \mathbf{X}$  is the orbital map at  $y$ ,
- and  $g_0 \in \mathbf{G}$  such that  $g_0 \cdot x = o$  where  $o = ([e_1], [e_1, e_2])$  is the usual base-point of  $\mathbf{X}$ .

Then for any  $t \in \mathbb{R}^*$  we have

$$D_{x(t)}(g_0 g^{-t})(\mathcal{E}^c(x(t))) = D_e \theta_o(\mathbb{R} \text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A).$$

Denoting by  $p: \mathfrak{sl}_3 \rightarrow \mathfrak{sl}_3/\mathfrak{p}_{\min}$  the canonical projection, let us assume that  $p(\mathbb{R} \text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A)$  converges at  $t = 0$  to a line contained in  $\text{Vect}(\bar{e}_\alpha, \bar{e}_\beta)$ . Then  $D_e \theta_o(\mathbb{R} \text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A) \subset T_o \mathbf{X}$  converges to a line  $L \subset (\mathcal{E}^\alpha \oplus \mathcal{E}^\beta)(o)$ , and since  $g^t g_0^{-1}$  converges to  $g_0^{-1}$  at  $t = 0$ , we deduce by continuity that  $\mathcal{E}^c(x(t))$  converges at  $t = 0$  to  $D_o g_0^{-1}(L)$ , contained in  $(\mathcal{E}^\alpha \oplus \mathcal{E}^\beta)(x)$  because  $g_0^{-1}$  preserves  $\mathcal{E}^\alpha \oplus \mathcal{E}^\beta$ .

In conclusion, we only have to find, in each of the four cases  $\mu = \mathbf{t}$  or  $\mathbf{a}$  and  $i = 1$  or  $2$ , a point  $x \in \mathcal{G}_\mu^i$ , together with  $g^t$ ,  $h^t$ ,  $A$ , and  $g_0$  satisfying the above conditions, and to prove that  $p(\mathbb{R} \text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A)$  converges at  $t = 0$  to a line contained in  $\text{Vect}(\bar{e}_\alpha, \bar{e}_\beta)$ .

We begin with  $Y_{\mathbf{t}}$ , for which we choose for both orbits  $\mathcal{G}_{\mathbf{t}}^1$  and  $\mathcal{G}_{\mathbf{t}}^2$  the point  $y := o_{\mathbf{t}} = ([1, 0, 1], [(1, 0, 1), e_2]) \in Y_{\mathbf{t}}$ . Let us recall that in this case,  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  satisfies  $\mathcal{E}^c(o_{\mathbf{t}}) = D_e \theta_{o_{\mathbf{t}}}(\mathbb{R}A)$  (see Paragraph 4.2.2).

- For  $\mathcal{G}_{\mathbf{t}}^1$ , choosing  $x = ([1, 0, 1], [(1, 0, 1), e_1]) = ([1, 0, 1], [e_1, e_3])$ ,  $g_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ , and the one-parameter subgroups  $g^t = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}$  of  $\mathbf{G}$  and  $h^t = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  of  $H_{\mathbf{t}}^0$  such that  $g^t \cdot x = h^{t^{-1}} \cdot o_{\mathbf{t}} \in \mathcal{C}^\alpha(x)$ , we obtain:

$$\text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A = \begin{pmatrix} 1 & -2 & -2t^{-1} \\ 1 & -2 & -2t^{-1} \\ -t & t & 1 \end{pmatrix},$$

and thus  $p(\mathbb{R} \text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A)$  converges at  $t = 0$  to  $\mathbb{R}\bar{e}_\beta$ .



- For  $\mathcal{G}_{\mathbf{t}}^2$ , choosing  $x = ([e_2], [e_2, (1, 0, 1)])$ ,  $g_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ , and the one-parameter subgroups  $g^t = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  of  $\mathbf{G}$  and  $h^t = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  of  $H_{\mathbf{t}}^0$  such that  $g^t \cdot x = h^{t^{-1}} \cdot o_{\mathbf{t}} \in \mathcal{C}^\beta(x)$ , we obtain

$$\text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A = \begin{pmatrix} 1 & 2t^{-1} & 0 \\ 0 & -1 & 0 \\ t & 1 & 0 \end{pmatrix},$$

and thus  $p(\mathbb{R} \text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A)$  converges at  $t = 0$  to  $\mathbb{R}\bar{e}_\alpha$ .

We now consider the case of  $Y_{\mathbf{a}}$ , for which we choose for both orbits  $\mathcal{G}_{\mathbf{a}}^1$  and  $\mathcal{G}_{\mathbf{a}}^2$  the point  $y := o_{\mathbf{a}} = ([e_3], [e_3, e_2]) \in Y_{\mathbf{a}}$ , and we recall that in this case  $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  satisfies the above condition  $\mathcal{E}^c(o_{\mathbf{a}}) = D_e \theta_{o_{\mathbf{a}}}(\mathbb{R}A)$  (see Paragraph 4.2.3).

- For  $\mathcal{G}_{\mathbf{a}}^1$ , choosing  $x = ([e_3], [e_3, e_1])$ ,  $g_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ , and the one-parameter subgroups  $g^t = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  of  $\mathbf{G}$  and  $h^t = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  of  $H_{\mathbf{a}}^0$  such that  $g^t \cdot x = h^{t^{-1}} \cdot o_{\mathbf{a}} \in \mathcal{C}^\alpha(x)$ , we obtain:

$$\text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -t & 0 & 0 \end{pmatrix},$$

and thus  $p(\mathbb{R} \text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A)$  converges at  $t = 0$  to  $\mathbb{R}\bar{e}_\beta$ .

- For  $\mathcal{G}_{\mathbf{a}}^2$ , choosing  $x = ([e_2], [e_2, e_3])$ ,  $g_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ , and the one-parameter subgroups  $g^t = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  of  $\mathbf{G}$  and  $h^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$  of  $H_{\mathbf{a}}^0$  such that  $g^t \cdot x = h^{t^{-1}} \cdot o_{\mathbf{a}} \in \mathcal{C}^\beta(x)$ , we obtain

$$\text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t & 1 & 0 \end{pmatrix},$$

and thus  $p(\mathbb{R} \text{Ad}(g_0 g^{-t} h^{t^{-1}}) \cdot A)$  converges at  $t = 0$  to  $\mathbb{R}\bar{e}_\alpha$ .

According to the discussion above, this concludes the proof of the lemma.  $\square$

We are now able to prove Proposition 6.1.

*Proof of Proposition 6.1.* Let us assume by contradiction that  $\Omega \neq M$ . We choose a connected component  $O$  of  $\tilde{\Omega}$  such that the rank of  $D_{\hat{x}} \mathcal{K}^{tot}$  for  $\hat{x} \in \pi^{-1}(O)$  is maximal among the ranks of  $D_{\hat{x}} \mathcal{K}^{tot}$  for  $\hat{x} \in \pi^{-1}(\tilde{\Omega})$ . Since  $\emptyset \neq O \neq \tilde{M}$  there exists  $x \in \partial O$ , and since  $\tilde{E}^\alpha \oplus \tilde{E}^\beta$  is contact, [Sus73, Theorem 4.1] implies the existence of a piecewise smooth path  $\gamma: [0; 1] \rightarrow \tilde{M}$  constituted of a finite concatenation of segments of  $\alpha$  and  $\beta$ -leaves, joining  $x = \gamma(1)$  to a point  $y = \gamma(0) \in O$ . Denoting  $t_0 = \inf \{t \in [0; 1] \mid \gamma(t) \in \partial O\}$ ,  $t_0 > 0$  and  $\gamma(t_0) \in \partial O$ . Replacing  $x$  by  $\gamma(t_0)$ , keeping only the last smooth arc of  $\gamma$ , replacing  $y$  by the origin of this arc, and choosing a parametrization of this arc by  $[0; 1]$ , we finally obtain a smooth path  $\gamma: [0; 1] \rightarrow \tilde{M}$  such that  $\gamma([0; 1]) \subset O$ ,  $x = \gamma(1) \in \partial O$ , and  $\gamma([0; 1])$  is entirely contained in a same  $\alpha$  or  $\beta$ -leaf. The proof being the same in the two cases, we assume that  $\gamma([0; 1]) \subset \tilde{\mathcal{F}}^\alpha(x)$  to fix the ideas. Denoting  $x_0 = \delta(x)$ ,  $x_0 \in \mathbf{X} \setminus Y$  according to Lemma 6.2, and  $\delta(\gamma([0; 1])) \subset Y$  because  $\delta(O) \subset Y$  (see Lemma 4.2). Finally  $\delta(\gamma([0; 1]))$  is an open interval of  $C^\alpha(x_0)$  contained in  $Y$ , and  $x_0 \in \mathbf{X} \setminus Y$ , i.e.  $x_0 \in \mathcal{G}$ . Denoting  $\gamma_0(t) = \delta(\gamma(t))$ , Lemma 6.3 implies therefore that  $\mathcal{E}^c(\gamma_0(t))$  converges to a line  $D_0^c \subset (\mathcal{E}^\alpha \oplus \mathcal{E}^\beta)(x_0)$  at  $t = 1$ . As  $\delta|_O$  is a local isomorphism between  $\tilde{\mathcal{S}}|_O$  and  $\mathcal{S}_Y$ , we have  $\tilde{E}^c(\gamma(t)) = (D_{\gamma(t)} \delta)^{-1}(\mathcal{E}^c(\gamma_0(t)))$  for any  $t \in [0; 1]$ , implying  $\tilde{E}^c(x) = (D_x \delta)^{-1}(D_0^c)$  by continuity. Since  $\delta$  is a local isomorphism between the Lagrangean contact structures  $\tilde{\mathcal{L}}$  and  $\mathcal{L}_{\mathbf{X}}$ , this implies that  $\tilde{E}^c(x) \subset (\tilde{E}^\alpha \oplus \tilde{E}^\beta)(x)$ , which contradicts the definition of the transverse distribution  $\tilde{E}^c$ . This contradiction concludes the proof of the proposition.  $\square$

## 6.2 Reduction of the holonomy group

Let us describe the global and local automorphisms of  $(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$  and  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$ .

**Proposition 6.4.** 1.  $\text{Aut}(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}}) = H_{\mathbf{t}} = \begin{bmatrix} \text{GL}_2(\mathbb{R}) & 0 \\ 0 & 1 \end{bmatrix}$  and  $\text{Aut}(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}}) = H_{\mathbf{a}} = \mathbf{P}_{\min}$ .

2. Let  $(Y, \mathcal{S}_Y)$  be one of the two models  $(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$  or  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$ . Then any local isomorphism of  $\mathcal{S}_Y$  between two connected open subsets of  $Y$  is the restriction of the action of a global automorphism of  $\text{Aut}(Y, \mathcal{S}_Y)$ .

*Proof.* 1. The inclusions  $H_{\mathbf{t}} \subset \text{Aut}(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$  and  $H_{\mathbf{a}} \subset \text{Aut}(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$  were explained in Paragraphs 4.2.2 and 4.2.3. Since the automorphism groups are contained in the respective stabilizers of  $Y_{\mathbf{t}}$  and  $Y_{\mathbf{a}}$ , the equalities follow because  $H_{\mathbf{t}} = \text{Stab}_{\mathbf{G}}(Y_{\mathbf{t}})$  and  $H_{\mathbf{a}} = \text{Stab}_{\mathbf{G}}(Y_{\mathbf{a}})$ .

2. Let us emphasize that in both cases,  $\text{Aut}(Y, \mathcal{S}_Y)$  is the normalizer of  $\mathfrak{h}$  in  $\mathbf{G}$ . Let  $\varphi$  be a local automorphism of  $\mathcal{S}_Y$  between two connected open subsets  $U$  and  $V$  of  $Y$ . For any  $v \in \mathfrak{h}$ , since  $v|_V$  is a Killing field of  $\mathcal{S}_Y$ ,  $\varphi^*(v|_V)$  is a Killing field of  $\mathcal{S}_Y$ , and therefore  $\varphi^*(v|_V) = w|_U$  for some  $w \in \mathfrak{h}$ . But  $\varphi$  is in particular a local automorphism of the Lagrangean contact structure  $\mathcal{L}_{\mathbf{X}}$  of  $\mathbf{X}$ , and is thus the restriction to an open subset  $U \subset Y$  of the left translation by an element  $g \in \mathbf{G}$ , according to Theorem 2.9. Therefore  $w|_U = \varphi_*(v|_V) = (\text{Ad}(g) \cdot v)|_U$ , implying that  $\text{Ad}(g) \cdot v = w \in \mathfrak{h}$  since the action of  $\mathbf{G}$  on  $\mathbf{X}$  is analytic (see Lemma 2.16). Consequently,  $g \in \text{Nor}_{\mathbf{G}}(\mathfrak{h}) = \text{Aut}(Y, \mathcal{S}_Y)$ .  $\square$

We recall that  $\rho: \pi_1(M) \rightarrow \mathbf{G}$  denotes the holonomy morphism associated to the developping map  $\delta: \tilde{M} \rightarrow \mathbf{X}$  of the  $(\mathbf{G}, \mathbf{X})$ -structure of  $M$  (see Corollary 3.4 and Paragraph 2.3.2).

**Proposition 6.5.** The holonomy group  $\rho(\pi_1(M))$  is contained in  $\text{Aut}(Y, \mathcal{S}_Y)$ . Consequently,  $M$  has either a  $(H_{\mathbf{t}}, Y_{\mathbf{t}})$ -structure or a  $(H_{\mathbf{a}}, Y_{\mathbf{a}})$ -structure, and its developping map is a local isomorphism of enhanced Lagrangean contact structures from  $\tilde{\mathcal{S}}$  to  $\mathcal{S}_{\mathbf{t}}$  (respectively  $\mathcal{S}_{\mathbf{a}}$ ).

*Proof.* According to Proposition 6.1,  $\mathcal{S}$  is locally homogeneous, and we thus deduce from Corollary 4.6 that, up to inversion of the distributions  $E^{\alpha}$  and  $E^{\beta}$ , the developping map of the  $(\mathbf{G}, \mathbf{X})$ -structure of  $M$  is a local isomorphism from  $(\tilde{M}, \tilde{\mathcal{S}})$  to one of the two models  $(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$  or  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$ . According to Proposition 6.4, the holonomy morphism has moreover values in the corresponding automorphism group  $H_{\mathbf{t}}$  (respectively  $H_{\mathbf{a}}$ ) described in the same result, *i.e.*  $\mathcal{S}$  is described by a  $(H_{\mathbf{t}}, Y_{\mathbf{t}})$ -structure (resp.  $(H_{\mathbf{a}}, Y_{\mathbf{a}})$ -structure) on  $M$ . Concerning the inversion of  $E^{\alpha}$  and  $E^{\beta}$ , it is easy to construct for both models  $(Y_{\mathbf{t}}, \mathcal{S}_{\mathbf{t}})$  and  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$ , a diffeomorphism of  $Y$  inverting the distributions  $\mathcal{E}^{\alpha}$  and  $\mathcal{E}^{\beta}$  and fixing the transverse distribution  $\mathcal{E}_{\mathcal{Y}}^{\mathcal{C}}$ . In other words, for these both models, the structures  $(\mathcal{E}^{\alpha}, \mathcal{E}^{\beta}, \mathcal{E}_{\mathcal{Y}}^{\mathcal{C}})$  and  $(\mathcal{E}^{\beta}, \mathcal{E}^{\alpha}, \mathcal{E}_{\mathcal{Y}}^{\mathcal{C}})$  are isomorphic, so that *a posteriori*, the order of the distributions  $E^{\alpha}$  and  $E^{\beta}$  in the statement of Proposition 6.5 does not matter.  $\square$

## 7 Completeness of the structure

The goal of this section is to prove that:

**Proposition 7.1.** The developping map  $\delta$  is a covering map from  $\tilde{M}$  to  $Y$ .

It is a known fact that a local diffeomorphism satisfying the path-lifting property is a covering map (the reader can for example look for a proof in [DC76, §5.6, Proposition 6 p. 383]). According to the following statement, it will be sufficient to check the path-lifting property in the  $\alpha$ ,  $\beta$  and central directions, to prove that  $\delta$  is a covering map.

**Lemma 7.2.** Let  $h: N \rightarrow B$  be a local diffeomorphism between two smooth three-dimensional manifolds,  $B$  being connected. We assume that there is a smooth splitting  $E_1 \oplus E_2 \oplus E_3 = \text{TB}$

of the tangent bundle of  $B$  into three one-dimensional smooth distributions, such that for any  $i \in \{1, 2, 3\}$ ,  $x \in \text{Im}(h)$ , and  $\tilde{x} \in h^{-1}(x)$ , any path tangent to  $E_i$  and starting from  $x$  entirely lifts through  $h$  to a path starting from  $\tilde{x}$ . Then  $h$  is a covering map from  $N$  to  $B$  (and in particular,  $h$  is surjective).

*Proof.* Since  $h$  is a local diffeomorphism, it suffices to prove that our weaker hypothesis implies the lift of any path. By compactness, it is moreover sufficient to locally lift the paths in  $B$ , around any point. We choose  $x \in B$  and a sufficiently small open neighbourhood  $U$  of  $x$ , such that there are three smooth vector fields  $X$ ,  $Y$  and  $Z$  generating  $E_1$ ,  $E_2$  and  $E_3$  on  $U$ , and  $\varepsilon > 0$  such that  $(t, u, v) \in ]-\varepsilon; \varepsilon[^3 \mapsto \phi(t, u, v) := \varphi_X^t \circ \varphi_Y^u \circ \varphi_Z^v(x) \in U$  is well-defined, and is a diffeomorphism (this exists according to Inverse mapping theorem). Let us choose  $\tilde{x} \in h^{-1}(x)$ . Then, denoting by  $\tilde{X} = h^*X$ ,  $\tilde{Y} = h^*Y$  and  $\tilde{Z} = h^*Z$  the pullbacks, the path-lifting property in the directions  $E_1$ ,  $E_2$  and  $E_3$ , and from any point, implies that  $\tilde{\phi}(t, u, v) := \varphi_{\tilde{X}}^t \circ \varphi_{\tilde{Y}}^u \circ \varphi_{\tilde{Z}}^v(\tilde{x})$  is well-defined on  $]-\varepsilon; \varepsilon[^3$ . If  $\gamma: [0; 1] \rightarrow U$  is a continuous path starting from  $x$  and contained in  $U$ , there are three continuous maps  $t$ ,  $u$  and  $v$  from  $[0; 1]$  to  $]-\varepsilon; \varepsilon[$  such that  $\gamma(s) = \phi(t(s), u(s), v(s))$ . Since  $h \circ \tilde{\phi} = \phi$  by construction,  $\tilde{\gamma}(s) := \tilde{\phi}(t(s), u(s), v(s))$  is a lift of  $\gamma$  starting from  $\tilde{x}$ , which concludes the proof.  $\square$

*Remark 7.3.* Proving that the paths in  $\delta(\tilde{M})$  in the  $\alpha$ -direction (respectively  $\beta$  or central direction) lift to  $\tilde{M}$  is equivalent to prove that for any  $x \in \delta(\tilde{M})$  and  $\tilde{x} \in \delta^{-1}(x)$ , we have:

$$\delta(\tilde{\mathcal{F}}^\alpha(\tilde{x})) = \mathcal{C}^\alpha(x) \cap \delta(\tilde{M}),$$

(respectively the same equality for  $\beta$ -leaves and  $\beta$ -circles, or for central leaves).

We first prove that the image of any  $\alpha$  (respectively  $\beta$ ) leaf in  $\tilde{M}$  miss exactly one point in the associated  $\alpha$ -circle (respectively  $\beta$ -circle) of  $Y$ . We recall that  $\partial Y = \mathbf{X} \setminus Y$ , as explained before Lemma 6.3.

**Lemma 7.4.** *For any  $\tilde{x} \in \tilde{M}$ , denoting  $x = \delta(\tilde{x}) \in Y$ , there exists  $x^* \in \mathcal{C}^\beta(x) \cap \partial Y$  such that  $\delta(\tilde{\mathcal{F}}^\beta(\tilde{x})) = \mathcal{C}^\beta(x) \setminus \{x^*\} = \mathcal{C}^\beta(x) \cap Y$ . The same happens for  $\alpha$ -leaves and their associated  $\alpha$ -circles.*

*Proof.* We will only write the proof for  $\beta$ -leaves and  $\beta$ -circles as in the statement, the case of the  $\alpha$ -direction being the same. Denoting  $\bar{x} = \pi_M(\tilde{x}) \in M$ , and possibly replacing  $f$  by  $f^{-1}$ , we have  $\lim_{n \rightarrow +\infty} \|D_{\bar{x}} f^n|_{E^\alpha(\bar{x})}\|_M = 0$  for some Riemannian metric that we fix on  $M$ .

The description of the open subsets  $Y_t$  and  $Y_a$  in Paragraphs 4.2.2 and 4.2.3 easily shows that in these both cases, the intersection of any  $\beta$ -circle (respectively  $\alpha$ -circle) with  $Y$  miss exactly one point of the circle. In other words, the intersection  $\mathcal{C}^\beta(x) \cap \partial Y$  is a single point  $\{x^*\}$ , and as a consequence  $\delta(\tilde{\mathcal{F}}^\beta(\tilde{x})) \subset \mathcal{C}^\beta(x) \setminus \{x^*\} = \mathcal{C}^\beta(x) \cap Y$ . To finish the proof of the lemma, we have to prove that  $\delta(\tilde{\mathcal{F}}^\beta(\tilde{x}))$  cannot miss more than one point of  $\mathcal{C}^\beta(x)$ . To achieve this, we assume by contradiction the following:

$$\text{there exists } x^- \neq x^+ \in \mathcal{C}^\beta(x) \setminus \{x, x^*\} \text{ such that } \delta(\tilde{\mathcal{F}}^\beta(\tilde{x})) = ]x^-; x^+[ \subsetneq \mathcal{C}^\beta(x) \setminus \{x^*\}, \quad (7.1)$$

where  $]x^-; x^+[$  is the connected component of  $\mathcal{C}^\beta(x) \setminus \{x^-, x^+\}$  that contains  $x$ .

Since  $M$  is compact, there exists a strictly increasing sequence  $(n_k)$  of positive integers such that  $f^{n_k}(\bar{x})$  converges to a point  $\bar{x}_\infty \in M$ , and there exists then a sequence  $\gamma_k \in \pi_1(M)$  such that  $\tilde{x}_k := \gamma_k \cdot \tilde{f}^{n_k}(\tilde{x})$  converges to a point  $\tilde{x}_\infty \in \pi^{-1}(\bar{x}_\infty)$ . Since  $\gamma_k \cdot \tilde{f}^{n_k}$  is an automorphism of the Lagrangean contact structure  $\tilde{\mathcal{L}}$  and  $\delta$  a local isomorphism from  $\tilde{\mathcal{L}}$  to  $\mathcal{L}_X$ , Theorem 2.9 implies the existence of a unique sequence  $g_k \in \mathbf{G}$  satisfying:

$$\delta(\gamma_k \cdot \tilde{f}^{n_k}(\tilde{x})) = g_k \cdot \delta(\tilde{x}).$$

We denote  $x_k = \delta(\tilde{x}_k) = g_k(x) \in Y$ , that converges to  $x_\infty := \delta(\tilde{x}_\infty)$ . Denoting  $x_k^- = g_k(x^-)$  and  $x_k^+ = g_k(x^+)$ ,  $x_k$ ,  $x_k^-$  and  $x_k^+$  are three distincts points of  $\mathcal{C}^\beta(x_k)$  for any  $k$ . By compactness of  $\mathbf{X}$ , we can assume up to extraction that  $x_k^-$  and  $x_k^+$  respectively converge to points  $x_\infty^-$  and  $x_\infty^+$  of  $\mathcal{C}^\beta(x_\infty)$ , and the hypothesis (7.1) allows us to obtain the following crucial statement.

**Fact 7.5.**  $x_\infty \neq x_\infty^-$ , and  $x_\infty \neq x_\infty^+$ .

*Proof.* Let us assume by contradiction that  $x_\infty^- = x_\infty$ . Considering a neighbourhood  $U$  of  $\tilde{x}_\infty$  such that  $\delta|_U$  is injective, we can choose  $\tilde{y}_\infty \in (\tilde{\mathcal{F}}^\beta(\tilde{x}_\infty) \cap U) \setminus \{\tilde{x}_\infty\}$ . There exists a sequence  $\tilde{y}_k \in \tilde{\mathcal{F}}^\beta(\tilde{x}_k)$  converging to  $\tilde{y}_\infty$ , and possibly changing  $\tilde{y}_\infty$ , we can moreover assume that  $\delta(\tilde{y}_k) \in ]x_k^-; x_k[$ , implying that  $\delta(\tilde{y}_\infty) \in [x_\infty^-; x_\infty]$  by continuity. But  $[x_\infty^-; x_\infty] = \{x_\infty\}$  since  $x_\infty^- = x_\infty$ , and therefore  $\delta(\tilde{y}_\infty) = x_\infty = \delta(\tilde{x}_\infty)$ , implying  $\tilde{y}_\infty = \tilde{x}_\infty$  by injectivity of  $\delta|_U$ , which contradicts our hypothesis on  $\tilde{y}_\infty$ . This contradiction concludes the proof of  $x_\infty \neq x_\infty^-$ , and  $x_\infty \neq x_\infty^+$  is proved in the same way.  $\square$

The subgroup  $\text{SO}(3)$  of  $\mathbf{G}$  acts transitively on  $\mathbf{X}$ , and we can thus choose  $\phi \in \text{SO}(3)$  and a sequence  $(\phi_k)$  in  $\text{SO}(3)$ , satisfying  $\phi(x) = o$  and  $\phi_k(x_k) = o$  for any  $k$  (we recall that  $o = ([e_1], [e_1, e_2])$ ). Since  $\text{Stab}_{\text{SO}(3)}(\mathcal{C}^\beta(o)) = \begin{bmatrix} \text{SO}(2) & 0 \\ 0 & 1 \end{bmatrix}$  acts transitively on  $\mathcal{C}^\beta(o)$ , we can moreover assume that  $\phi(x^+) = o^+$  and  $\phi_k(x_k^+) = o^+$ , where  $o^+ = ([e_2], [e_1, e_2]) \in \mathcal{C}^\beta(o)$ . For any  $k$ ,  $\phi_k \circ g_k \circ \phi^{-1}$  is an element of  $\text{Stab}_{\mathbf{G}}(o) \cap \text{Stab}_{\mathbf{G}}(o^+)$ , *i.e.* is of the form  $\begin{bmatrix} 1 & 0 & x \\ 0 & \lambda_k & y \\ 0 & 0 & \mu_k \end{bmatrix}$ . Since  $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix}$  acts trivially in restriction to  $\mathcal{C}^\beta(o)$ ,  $A_k := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_k & 0 \\ 0 & 0 & 1 \end{bmatrix}$  satisfies thus:

$$g_k|_{\mathcal{C}^\beta(x)} = \phi_k^{-1} \circ A_k \circ \phi|_{\mathcal{C}^\beta(x)}.$$

The following commutative diagram summarizes the situation.

$$\begin{array}{ccccccc} \mathcal{C}^\beta(o) & \xleftarrow{\phi} & \mathcal{C}^\beta(x) & \xleftarrow{\delta} & \tilde{\mathcal{F}}^\beta(\tilde{x}) & \xrightarrow{\pi_M} & \mathcal{F}^\beta(\bar{x}) \\ \downarrow A_k & & \downarrow g_k & & \downarrow \gamma_k \tilde{f}^{n_k} & & \downarrow f^{n_k} \\ \mathcal{C}^\beta(o) & \xleftarrow{\phi_k} & \mathcal{C}^\beta(x_k) & \xleftarrow{\delta} & \tilde{\mathcal{F}}^\beta(\tilde{x}_k) & \xrightarrow{\pi_M} & \mathcal{F}^\beta(f^{n_k}(\bar{x})) \end{array} \quad (7.2)$$

The action of  $A_k \in \mathbf{G}$  on  $\mathcal{C}^\beta(o)$  is conjugated to the action of the projective transformations  $\begin{bmatrix} 1 & 0 \\ 0 & \lambda_k \end{bmatrix} \in \text{PGL}_2(\mathbb{R})$  on  $\mathbb{RP}^1$ , *i.e.* to the action of the homotheties of ratio  $\lambda_k$  on  $\mathbb{R} \cup \{\infty\}$ . By this conjugation,  $o$  corresponds to 0,  $o^+$  to  $\infty$ , and  $o^- := \phi(x^-) \in \mathcal{C}^\beta(o) \setminus \{o, o^+\}$  corresponds to a non-zero point of  $\mathbb{R}$ . Fact 7.5 implies that  $A_k(o^-) = \phi_k(x_k^-) \in \mathcal{C}^\beta(o)$  stays bounded away from  $o$  (since  $\phi_k \in \text{SO}(3)$ ), and therefore that  $\lambda_k$  is bounded away from 0.

On the other hand, endowing  $\tilde{M}$  with the pullback of the Riemannian metric of  $M$ , the diagram (7.2) implies  $\lim_{k \rightarrow +\infty} \|D_{\tilde{x}}(\gamma_k \tilde{f}^{n_k})|_{\tilde{E}^\beta(\tilde{x})}\|_{\tilde{M}} = 0$  (since  $\pi_1(M)$  acts by isometries). Fixing any Riemannian metric on  $\mathbf{X}$ , as  $(\tilde{x}_k)$  is relatively compact we also have  $\lim \|D_x g_k|_{\mathcal{E}^\beta(x)}\|_{\mathbf{X}} = 0$ , and since  $(\phi_k)$  and  $(x_k)$  are relatively compact as well, we finally obtain  $\lim \|D_o A_k|_{\mathcal{E}^\beta(o)}\|_{\mathbf{X}} = 0$ .

This contradicts the fact that  $\lambda_k$  is bounded away from 0, and this contradiction concludes the proof of the lemma.  $\square$

Lemma 7.4 allows us to easily infer the path-lifting property in the  $\alpha$  and  $\beta$ -directions.

**Corollary 7.6.** 1. For any  $x \in \delta(\tilde{M})$ ,  $\mathcal{C}^\alpha(x) \cap \delta(\tilde{M}) = \mathcal{C}^\alpha(x) \cap Y$  and  $\mathcal{C}^\beta(x) \cap \delta(\tilde{M}) = \mathcal{C}^\beta(x) \cap Y$ .

2. The paths in  $\delta(\tilde{M})$  in the  $\alpha$  and  $\beta$ -directions lift to  $\tilde{M}$  from any point.

*Proof.* We only write the proof of the statements for the  $\alpha$ -direction, the case of the  $\beta$ -direction being proved in the same way.

1. For any  $\tilde{x} \in \tilde{M}$ , denoting  $\delta(\tilde{x}) = x$ , we know that  $\partial Y \cap \mathcal{C}^\alpha(x)$  is equal to a single point  $\{x^*\}$  that satisfies  $\mathcal{C}^\alpha(x) \setminus \{x^*\} = \mathcal{C}^\alpha(x) \cap Y$ . Furthermore,  $\delta(\tilde{\mathcal{F}}^\alpha(\tilde{x})) = \mathcal{C}^\alpha(x) \setminus \{x^*\} = \mathcal{C}^\alpha(x) \cap Y$  according to Lemma 7.4. Since  $\mathcal{C}^\alpha(x) \cap \delta(\tilde{M}) \subset \cup_{\tilde{x} \in \delta^{-1}(x)} \delta(\tilde{\mathcal{F}}^\alpha(\tilde{x})) = \mathcal{C}^\alpha(x) \cap Y$ , we finally

obtain  $\mathcal{C}^\alpha(x) \cap \delta(\tilde{M}) = \mathcal{C}^\alpha(x) \cap Y$ .

2. The first claim together with Lemma 7.4 imply  $\delta(\tilde{\mathcal{F}}(\tilde{x})) = \mathcal{C}^\alpha(x) \cap \delta(\tilde{M})$ , for any  $x \in \delta(\tilde{M})$  and  $\tilde{x} \in \delta^{-1}(x)$ . According to Remark 7.3, this proves that any path starting from  $x$  in the  $\alpha$ -direction lifts to  $\tilde{M}$  from  $\tilde{x}$ .  $\square$

The accessibility property of Lagrangean contact structures allows us to deduce that:

**Corollary 7.7.** *The developping map is surjective:  $\delta(\tilde{M}) = Y$ .*

*Proof.* Let  $x$  be a point of  $\delta(\tilde{M})$ , and  $y$  be any point in  $Y$ . Restricting the standard Lagrangean contact structure of  $\mathbf{X}$  to the connected open subset  $Y$ , [Sus73, Theorem 4.1] implies the existence of a finite number  $x = x_1, \dots, x_n = y$  of points of  $Y$  such that for any  $i = 1, \dots, n-1$ ,  $x_{i+1} \in \mathcal{C}^\alpha(x_i) \cap Y$  or  $x_{i+1} \in \mathcal{C}^\beta(x_i) \cap Y$ . Applying the first statement of Corollary 7.6, we deduce by a direct finite recurrence that for any  $i$ ,  $x_i \in \delta(\tilde{M})$ , implying  $y \in \delta(\tilde{M})$ .  $\square$

We finally prove that the central paths also lift, by a specific method for each model.

**Lemma 7.8.** *In the case of  $Y_{\mathbf{t}}$ , any central path starting at any point  $x \in Y_{\mathbf{t}}$  lifts in  $\tilde{M}$  from any point  $\tilde{x} \in \delta^{-1}(x)$ .*

*Proof.* Denoting  $Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \in \mathfrak{h}_{\mathbf{t}}$ , the transverse distribution on  $Y_{\mathbf{t}}$  is generated by  $Z$  at  $\mathfrak{o}_{\mathbf{t}}$  (see Paragraph 4.2.2). But  $Z$  is  $H_{\mathbf{t}}$ -invariant since  $Z$  is central in  $\mathfrak{h}_{\mathbf{t}}$ , and  $\mathcal{E}_{\mathbf{t}}^c$  is therefore generated by  $Z$  on  $Y_{\mathbf{t}}$ , i.e. the central leaves are the integral curves of  $Z$  on  $Y_{\mathbf{t}}$ . Since  $\rho(\pi_1(M)) \subset H_{\mathbf{t}}$  according to Proposition 6.5,  $Z$  is preserved by  $\rho(\pi_1(M))$ , and  $\tilde{Z} := \delta^*Z$  is thus  $\pi_1(M)$ -invariant. This allows us to push it down by  $\pi_M$  to a vector field  $\bar{Z}$  of  $M$  generating  $E^c$ , and since  $M$  is compact,  $\bar{Z}$  is a complete vector field. Since  $\pi_M$  is a covering map,  $\tilde{Z} = \pi_M^* \bar{Z}$  is also complete, which allows us to lift any central path in  $Y_{\mathbf{t}}$  to  $\tilde{M}$ .  $\square$

**Lemma 7.9.** *In the case of  $Y_{\mathbf{a}}$ , any central path starting at any point  $x \in Y_{\mathbf{a}}$  lifts in  $\tilde{M}$  from any point  $\tilde{x} \in \delta^{-1}(x)$ .*

*Proof.* Let us first emphasize that the argument used in the previous lemma for the case of  $Y_{\mathbf{t}}$  does not work here, because the center of  $\mathfrak{h}_{\mathbf{a}}$  is trivial.

We identify  $Y_{\mathbf{a}}$  with  $\mathbb{R}^3$  through  $(x, y, z) \in \mathbb{R}^3 \mapsto ([x, y, 1], [(x, y, 1), (z, 1, 0)]) \in Y_{\mathbf{a}}$ , and we define three vector fields of  $Y_{\mathbf{a}}$  as follows in these global coordinates:

$$X^\alpha(x, y, z) = e_3, X^\beta(x, y, z) = (z, 1, 0), \text{ and } X^c(x, y, z) = e_1.$$

These vector fields are complete and generate the enhanced Lagrangean contact structure  $\mathcal{S}_{\mathbf{a}} = (\mathcal{E}^\alpha, \mathcal{E}^\beta, \mathcal{E}_{\mathbf{a}}^c)$  on  $Y_{\mathbf{a}}$  (see Paragraph 4.2.3). Since the paths tangent to the  $\alpha$  and  $\beta$ -distributions entirely lift to  $\tilde{M}$  according to Corollary 7.6,  $\tilde{X}^\alpha := \delta^*X^\alpha$  and  $\tilde{X}^\beta := \delta^*X^\beta$  are complete as well. We can furthermore realize the flow of the central vector field  $X^c$  by  $\alpha - \beta$  curves through the following equalities:

$$\begin{cases} \varphi_{X^\beta}^{-t} \circ \varphi_{X^\alpha}^{-t} \circ \varphi_{X^\beta}^t \circ \varphi_{X^\alpha}^t(x) = x + t^2 e_1 = \varphi_{X^c}^{t^2}(x), \\ \varphi_{X^\beta}^t \circ \varphi_{X^\alpha}^{-t} \circ \varphi_{X^\beta}^{-t} \circ \varphi_{X^\alpha}^t(x) = x - t^2 e_1 = \varphi_{X^c}^{-t^2}(x). \end{cases}$$

The same equalities are true for  $\tilde{X}^\alpha$ ,  $\tilde{X}^\beta$  and  $\tilde{X}^c = \delta^*X^c$ , and since the flows of  $\tilde{X}^\alpha$  and  $\tilde{X}^\beta$  are defined for all times, these equalities show that  $\tilde{X}^c$  is complete. This allows us to lift any central path of  $Y_{\mathbf{a}}$  from any point of  $\tilde{M}$ , and concludes the proof of the lemma.  $\square$

*End of the proof of Proposition 7.1.* According to Corollary 7.6 and to Lemmas 7.8 and 7.9, the local diffeomorphism  $\delta$  satisfies the path-lifting property on  $Y$  in the  $\alpha$ ,  $\beta$ , and central directions, and is thus a covering map from  $\tilde{M}$  to  $Y$  according to Lemma 7.2.  $\square$

## 8 Conclusion

### 8.1 End of the proof of Theorem B

#### 8.1.1 Case of $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$

We first assume that  $(M, \mathcal{S})$  is locally isomorphic to  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$ . We identify  $Y_{\mathbf{a}}$  with  $\text{Heis}(3)$  and the action of  $H_{\mathbf{a}}$  on  $Y_{\mathbf{a}}$  to the action of  $\text{Heis}(3) \rtimes \mathcal{A}$  on  $\text{Heis}(3)$  as explained in Paragraph 4.2.3. The developing map being a covering according to Proposition 7.1, it is a diffeomorphism  $\delta: \tilde{M} \rightarrow \text{Heis}(3)$  by simple connexity of  $\text{Heis}(3)$ . Moreover,  $\delta$  conjugates the action of  $\pi_1(M)$  on  $\tilde{M}$  to the one of  $\Gamma := \rho(\pi_1(M)) \subset \text{Heis}(3) \rtimes \mathcal{A}$  on  $\text{Heis}(3)$  (see Proposition 6.5). We can thus assume that  $M$  is a quotient  $\Gamma \backslash \text{Heis}(3)$ , with  $\Gamma$  a discrete subgroup of  $\text{Heis}(3) \rtimes \mathcal{A}$  acting freely, properly and cocompactly on  $\text{Heis}(3)$ , and that  $f$  is an element of  $\text{Heis}(3) \rtimes \mathcal{A}$  such that  $f\Gamma f^{-1} = \Gamma$ .

Denoting  $[x, y, z] = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ , the identification  $[x, y, z] \in \text{Heis}(3) \mapsto (x, y, z) \in \mathbb{R}^3$  of  $\text{Heis}(3)$  with  $\mathbb{R}^3$  is equivariant for the following injective morphism from  $\text{Heis}(3) \rtimes \mathcal{A}$  to the affine transformations of  $\mathbb{R}^3$ :

$$\Theta: ([x, y, z], \varphi_{\lambda, \mu}) \in \text{Heis}(3) \rtimes \mathcal{A} \mapsto \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & \mu x & \lambda \mu \end{pmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{Aff}(\mathbb{R}^3).$$

$M$  is thus diffeomorphic to the quotient  $\Lambda \backslash \mathbb{R}^3$ , where  $\Lambda := \Theta(\Gamma)$  is a discrete subgroup of affine transformations of  $\mathbb{R}^3$  contained in  $S := \Theta(\text{Heis}(3) \rtimes \mathcal{A})$ , acting freely, properly and cocompactly on  $\mathbb{R}^3$ . Since  $S$  is solvable (because  $\text{Heis}(3) \rtimes \mathcal{A} \simeq \mathbf{P}_{\min}$  is), the work of Fried and Goldmann in [FG83] (more precisely Theorem 1.4, Corollary 1.5 and Paragraphs 3 and 4 of their paper) implies the existence of a *crystallographic hull*  $C$  for  $\Lambda$ . This group  $C$  is a closed subgroup of  $S$  containing  $\Lambda$ , and whose identity component  $C^0$  satisfies the following assumptions:  $\Lambda \cap C^0$  has finite index in  $\Lambda$  and is cocompact in  $C^0$ ,  $C^0$  acts simply transitively on  $\mathbb{R}^3$ , and  $C^0$  is isomorphic to  $\mathbb{R}^3$ ,  $\text{Heis}(3)$ , or  $\text{Sol}$ . One can check that  $S$  does not contain any subgroup isomorphic to  $\mathbb{R}^3$ , that the subgroups of  $S$  isomorphic to  $\text{Sol}$  do not act simply transitively on  $\mathbb{R}^3$ , and that  $\Theta(\text{Heis}(3))$  is the only subgroup of  $S$  isomorphic to  $\text{Heis}(3)$ . Finally,  $C^0$  is equal to  $\Theta(\text{Heis}(3))$ , and therefore,  $\Lambda \cap \Theta(\text{Heis}(3))$  has finite index in  $\Lambda$  and is cocompact in  $\Theta(\text{Heis}(3))$ . As a consequence,  $\Gamma_0 := \Gamma \cap \text{Heis}(3)$  has finite index in  $\Gamma$  and is a cocompact lattice of  $\text{Heis}(3)$ .

The kernel of  $p: (g, \varphi) \in \text{Heis}(3) \rtimes \mathcal{A} \mapsto \varphi \in \mathcal{A}$  is equal to  $\text{Heis}(3)$ , and  $\Gamma/\Gamma_0$  is thus isomorphic to  $p(\Gamma) \subset \mathcal{A}$ . But  $\mathcal{A}$  is isomorphic to  $(\mathbb{R}^*)^2$ , and a finite subgroup of  $\mathcal{A}$  is thus contained in the subgroup  $\{\varphi_{\pm 1, \pm 1}\}$  of cardinal 4, implying that the index of  $\Gamma_0$  in  $\Gamma$  is at most 4. Since  $f = (g, \varphi) \in \text{Nor}_{\text{Heis}(3) \rtimes \mathcal{A}}(\Gamma)$ , we have  $g\varphi(\Gamma_0)g^{-1} = \Gamma_0$ , and the affine automorphism  $x \mapsto g\varphi(x)$  of  $\text{Heis}(3)$  induces therefore a diffeomorphism  $\check{f}$  of  $\check{M} := \Gamma_0 \backslash \text{Heis}(3)$ . The canonical projection  $\check{\pi}: \check{M} = \Gamma_0 \backslash \text{Heis}(3) \rightarrow M = \Gamma \backslash \text{Heis}(3)$  is a covering of finite order equal to  $[\Gamma_0 : \Gamma]$ , and we have  $\check{\pi} \circ \check{f} = f \circ \check{\pi}$ .

We denote  $\varphi = \varphi_{\lambda, \mu}$ . To conclude that  $\check{f}$  is a partially hyperbolic affine automorphism of  $\text{Heis}(3)$ , it only remains to show that  $\lambda < 1$  and  $\mu > 1$ , or the contrary. Let us assume by contradiction that  $\lambda < 1$  and  $\mu < 1$ . Choosing a left-invariant volume form  $\nu$  on  $\text{Heis}(3)$ , we have  $((D_e \varphi)^* \nu)_e = \lambda^2 \mu^2 \nu_e$ , and  $\nu$  induces a volume form  $\bar{\nu}$  on  $\check{M} = \Gamma_0 \backslash \text{Heis}(3)$  such that  $\check{f}^* \bar{\nu} = \lambda^2 \mu^2 \bar{\nu}$  (because  $L_g$  preserves  $\nu$ ). Since  $\check{f}$  is a diffeomorphism of the compact manifold  $\check{M}$ , we must have  $\int_{\check{M}} \bar{\nu} = \int_{\check{M}} \check{f}^* \bar{\nu} = \lambda^2 \mu^2 \int_{\check{M}} \bar{\nu}$ , which is a contradiction because  $\int_{\check{M}} \bar{\nu} \neq 0$  and  $\lambda^2 \mu^2 < 1$ . The same argument shows that we cannot have  $\lambda > 1$  and  $\mu > 1$  neither, which concludes the proof of Theorem B in the case of the local model  $(Y_{\mathbf{a}}, \mathcal{S}_{\mathbf{a}})$ .

### 8.1.2 Case of $(Y_t, S_t)$

We now assume that  $\mathcal{S}$  is locally isomorphic to  $(Y_t, S_t)$ . According to Proposition 7.1,  $\delta: \tilde{M} \rightarrow Y_t$  is a covering map. Identifying  $Y_t$  with  $\mathrm{SL}_2(\mathbb{R})$  as explained in Paragraph 4.2.2,  $\delta$  lifts thus to a diffeomorphism  $\tilde{\delta}: \tilde{M} \rightarrow \tilde{\mathrm{SL}}_2(\mathbb{R})$ . According to Proposition 6.5,  $\rho(\pi_1(M)) \subset H_t \equiv \mathrm{GL}_2(\mathbb{R})$ . But  $M$  is orientable, and  $\pi_1(M)$  preserves thus the induced orientation of  $\tilde{M}$ , implying that  $\rho(\pi_1(M))$  is contained in  $\mathrm{GL}_2^+(\mathbb{R})$ . We saw in Paragraph 4.2.2 that the identification of  $Y_t$  with  $\mathrm{SL}_2(\mathbb{R})$  conjugates the action of  $\mathrm{GL}_2^+(\mathbb{R})$  on  $Y_t$  to the action of  $\mathrm{SL}_2(\mathbb{R}) \times A$  on  $\mathrm{SL}_2(\mathbb{R})$ . This shows that  $M$  is a quotient  $\tilde{\Gamma} \backslash \tilde{\mathrm{SL}}_2(\mathbb{R})$ , with  $\tilde{\Gamma}$  a discrete subgroup of  $\tilde{\mathrm{SL}}_2(\mathbb{R}) \times \tilde{A}$  acting freely, properly and cocompactly on  $\tilde{\mathrm{SL}}_2(\mathbb{R})$ . Possibly replacing  $f$  by  $f^2$ , we can moreover assume that  $f$  preserves the orientation of  $M$ , and Theorem 6.4 implies then that  $f = L_g \circ R_{a^t}$  with  $(g, a^t) \in \mathrm{Nor}_{\tilde{\mathrm{SL}}_2(\mathbb{R}) \times \tilde{A}}(\tilde{\Gamma})$ .

Denoting by  $r_1: \tilde{\mathrm{SL}}_2(\mathbb{R}) \times \tilde{A} \rightarrow \tilde{\mathrm{SL}}_2(\mathbb{R})$  the projection on the first factor, and  $\tilde{\Gamma}_0 := r_1(\tilde{\Gamma}) \subset \tilde{\mathrm{SL}}_2(\mathbb{R})$ , we now prove that

**Fact 8.1.**  $\tilde{\Gamma}_0$  is a cocompact lattice of  $\tilde{\mathrm{SL}}_2(\mathbb{R})$ , and  $\tilde{\Gamma}$  is the graph-group  $\mathrm{gr}(\tilde{u}, \tilde{\Gamma}_0)$  of a morphism  $\tilde{u}: \tilde{\Gamma} \rightarrow \tilde{A}$ .

*Proof.* Choosing a generator  $z$  of the center  $\tilde{\mathcal{Z}}$  of  $\tilde{\mathrm{SL}}_2(\mathbb{R})$ , the finiteness of the level proved by Salein in [Sal99, Theorem 3.3.2.3] implies the existence of a non-zero integer  $k \in \mathbb{N}^*$  such that  $\tilde{\Gamma} \cap (\tilde{\mathcal{Z}} \times \{e\}) = \langle (z^k, e) \rangle$ . We will denote by  $\langle g \rangle$  the group generated by an element  $g$ , and we introduce the group  $\mathrm{PSL}_2^{(k)}(\mathbb{R}) := \tilde{\mathrm{SL}}_2(\mathbb{R}) / \langle z^k \rangle$  and denote by  $p_k: \tilde{\mathrm{SL}}_2(\mathbb{R}) \rightarrow \mathrm{PSL}_2^{(k)}(\mathbb{R})$  its universal cover. Then, denoting  $A_k = p_k(\tilde{A})$  and  $\Gamma_k := (p_k \times p_k)(\tilde{\Gamma}) \subset \mathrm{PSL}_2^{(k)}(\mathbb{R}) \times A_k$ ,  $p_k$  induces a diffeomorphism  $\tilde{\Gamma} \backslash \tilde{\mathrm{SL}}_2(\mathbb{R}) \rightarrow \Gamma_k \backslash \mathrm{PSL}_2^{(k)}(\mathbb{R})$  (because  $\mathrm{Ker} p_k = \langle z^k \rangle$  and  $(z^k, e) \in \tilde{\Gamma}$ ), implying in particular that  $\Gamma_k$  acts freely, properly and cocompactly on  $\mathrm{PSL}_2^{(k)}(\mathbb{R})$ .

We can now apply the work of Kulkarni-Raymond in [KR85] to  $\Gamma_k$ . Let us denote by  $\pi: \tilde{\mathrm{SL}}_2(\mathbb{R}) \rightarrow \mathrm{PSL}_2(\mathbb{R})$  the universal cover morphism of  $\mathrm{PSL}_2(\mathbb{R})$  (of kernel  $\tilde{\mathcal{Z}}$ ), and by  $\pi_k: \mathrm{PSL}_2^{(k)}(\mathbb{R}) \rightarrow \mathrm{PSL}_2(\mathbb{R})$  the induced  $k$ -fold covering by  $\mathrm{PSL}_2^{(k)}(\mathbb{R})$ . Then, with  $\Gamma = (\pi \times \pi)(\tilde{\Gamma})$  and  $\Gamma_0 = r_1(\Gamma) \subset \mathrm{PSL}_2(\mathbb{R})$  the projection on the first factor, the form of Kulkarni-Raymond's results proved by Tholoan in [Tho14, Lemma 4.3.1] implies that  $\Gamma_0$  is a cocompact lattice of  $\mathrm{PSL}_2(\mathbb{R})$ , and that  $\pi_k \circ r_1|_{\Gamma_k}$  is injective.

The first assertion ensures that  $\tilde{\Gamma}_0$  is discrete in  $\tilde{\mathrm{SL}}_2(\mathbb{R})$ . The second one implies that  $\Gamma = \mathrm{gr}(u, \Gamma_0)$  is the graph-group of a morphism  $u: \Gamma_0 \rightarrow A = \pi(\tilde{A})$ . Since  $r_1|_{\tilde{\Gamma}}$  is also injective, this implies that  $\tilde{\Gamma}$  is the graph of a morphism  $\tilde{u}: \tilde{\Gamma}_0 \rightarrow \tilde{A}$ , trivial on  $\tilde{\Gamma}_0 \cap \tilde{\mathcal{Z}}$ .

Since  $\tilde{\mathcal{Z}} \cap \tilde{\Gamma}_0 = \langle z^k \rangle$  is finite, the projection  $\tilde{\Gamma}_0 \backslash \tilde{\mathrm{SL}}_2(\mathbb{R}) \rightarrow \Gamma_0 \backslash \mathrm{PSL}_2(\mathbb{R})$  has finite fibers, implying that  $\tilde{\Gamma}$  is a cocompact lattice since  $\Gamma_0 \backslash \mathrm{PSL}_2(\mathbb{R})$  is compact.  $\square$

The projection  $\Gamma_0 = \pi(\tilde{\Gamma}_0)$  is a cocompact lattice of  $\mathrm{PSL}_2(\mathbb{R})$  according to the proof of Fact 8.1, and  $\Gamma_0 \backslash \mathrm{Nor}_{\mathrm{PSL}_2(\mathbb{R})}(\Gamma_0)$  is thus finite. Therefore,  $\tilde{\Gamma}_0 \backslash \mathrm{Nor}_{\tilde{\mathrm{SL}}_2(\mathbb{R})}(\tilde{\Gamma}_0)$  is finite as well since the projection  $\tilde{\Gamma}_0 \backslash \mathrm{Nor}_{\tilde{\mathrm{SL}}_2(\mathbb{R})}(\tilde{\Gamma}_0) \rightarrow \Gamma_0 \backslash \mathrm{Nor}_{\mathrm{PSL}_2(\mathbb{R})}(\Gamma_0)$  has finite fibers ( $\tilde{\mathcal{Z}} \cap \tilde{\Gamma}_0 = \langle z^k \rangle$  is finite according to the finiteness of the level).

Recall that  $f = L_g \circ R_a$ , where  $(g, a^t) \in \mathrm{Nor}_{\tilde{\mathrm{SL}}_2(\mathbb{R}) \times \tilde{A}}(\tilde{\Gamma})$ . Therefore  $g \in \mathrm{Nor}_{\tilde{\mathrm{SL}}_2(\mathbb{R})}(\tilde{\Gamma}_0)$ , and since  $\tilde{\Gamma}_0 \backslash \mathrm{Nor}_{\tilde{\mathrm{SL}}_2(\mathbb{R})}(\tilde{\Gamma}_0)$  is finite, there exists  $n \in \mathbb{N}^*$  such that  $\gamma := g^n \in \tilde{\Gamma}_0$ . Denoting  $a := a^n \tilde{u}(\gamma)^{-1}$ , we have  $f^n = L_\gamma \circ R_{a^n} = R_a \circ (L_\gamma \circ R_{\tilde{u}(\gamma)})$ . But  $L_\gamma \circ R_{\tilde{u}(\gamma)}$  acts trivially on the quotient  $\tilde{\Gamma} \backslash \tilde{\mathrm{SL}}_2(\mathbb{R})$ , and therefore  $f = R_a$  is a non-zero time-map of the algebraic contact-Anosov flow  $(R_{a^t})$  on  $\tilde{\Gamma} \backslash \tilde{\mathrm{SL}}_2(\mathbb{R})$ . This concludes the proof of Theorem B in the case where  $\mathcal{S}$  is locally isomorphic to  $(Y_t, S_t)$ .

This paragraph concludes the proof of Theorem B.

## 8.2 Proof of Theorem A

Theorem B implies the following refined version of Theorem A stated in the introduction.

**Corollary 8.2.** *Let  $M$  be a closed, connected and orientable three-dimensional manifold, endowed with a smooth splitting  $TM = E^\alpha \oplus E^\beta \oplus E^c$ , such that  $E^\alpha \oplus E^\beta$  is a contact distribution. Let  $f$  be a diffeomorphism of  $M$  that preserves this splitting, and such that:*

- *each of the distributions  $E^\alpha$  and  $E^\beta$  is either uniformly contracted, or uniformly expanded by  $f$ ,*
- *and  $NW(f) = M$ .*

*Then the conclusions of Theorem A hold. In particular,  $f$  is a partially hyperbolic diffeomorphism.*

Let us emphasize that no domination is required on the central direction, and that the two remaining directions can *a priori* be both contracted, or both expanded.

*Proof of Corollary 8.2.* Since  $E^\alpha \oplus E^\beta$  is contact and  $M$  connected, any two points of  $M$  are linked by the concatenation of a finite number of paths, tangent either to  $E^\alpha$  or to  $E^\beta$  (this is for example a consequence of the work of Sussmann in [Sus73, Theorem 4.1]). In other words, the pair  $(\mathcal{F}^\alpha, \mathcal{F}^\beta)$  of foliations associated to  $(E^\alpha, E^\beta)$  is topologically transitive in the terminology of Brin in [Bri75]. Our hypothesis of uniform contraction or expansion of the distributions  $E^\alpha$  and  $E^\beta$  directly implies that  $\mathcal{F}^\alpha$  and  $\mathcal{F}^\beta$  are uniformly contracted or expanded in the sense of [Bri75]. Since  $NW(f) = M$  by hypothesis, [Bri75, Theorem 1.1] implies that  $f$  is topologically transitive. In fact, Brin states this result assuming that one of the distributions is contracted, and the other one expanded, but it is easy to see that his proof does not use this assumption, and that the same proof works if both distributions are expanded, or both contracted.

We are now under the hypotheses of Theorem B, and its conclusions hold.  $\square$

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