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RIEMANN-HILBERT CORRESPONDENCE FOR DIFFERENTIAL SYSTEMS OVER RIEMANN SURFACES

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. Let G be a connected reductive affine algebraic group defined over \mathbb{C} and \mathfrak{g} its Lie algebra. We study the monodromy map from the space of \mathfrak{g} -differential systems on a compact connected Riemann surface Σ of genus $g \geq 2$ to the character variety of G -representations of the fundamental group of Σ . If the complex dimension of G is at least three, we show that the monodromy map is an immersion at the generic point.

1. INTRODUCTION

Our aim here is to study the Riemann-Hilbert correspondence for differential systems over compact Riemann surfaces. In order to describe the framework, let us denote by Σ a given compact connected oriented topological surface of genus $g \geq 2$ and by G a connected reductive affine algebraic group defined over \mathbb{C} .

Consider a complex structure X on Σ (it is an element in the Teichmüller space for Σ) and a holomorphic (flat) connection ϕ on the trivial holomorphic principal G -bundle $X \times G$ over X . Recall that ϕ is determined by an element $\delta \in \mathfrak{g} \otimes H^0(X, K_X)$, where \mathfrak{g} is the (complex) Lie algebra of G and K_X is the canonical bundle of X . Fixing a base point $x_0 \in X$, consider the corresponding universal cover $\pi : \tilde{X} \rightarrow X$ of X , and endow the trivial principal G -bundle $\tilde{X} \times G$ over \tilde{X} with the pulled back holomorphic flat connection $\pi^*\phi$.

For any locally defined ϕ -parallel section s of $X \times G$, the pulled back local section π^*s of $\tilde{X} \times G$ extends to a $\pi^*\phi$ -parallel section over entire \tilde{X} . This extension of π^*s produces a holomorphic map $\tilde{X} \rightarrow G$ which is $\pi_1(X, x_0)$ -equivariant with respect to the natural action of $\pi_1(X, x_0)$ on \tilde{X} through deck transformations and the action of $\pi_1(X, x_0)$ on G through a group homomorphism $\pi_1(X, x_0) \rightarrow G$, which is known as the *monodromy* of the flat connection ϕ .

Although the above mentioned monodromy homomorphism depends on the choice of the holomorphic trivialization of the principal G -bundle, the element of the character variety of G -representations

$$\Xi := \text{Hom}(\pi_1(X), G) // G$$

given by it is independent of both the trivialization of the principal G -bundle and the base point x_0 (and also of the choice of s).

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Recall that Ξ is a (singular) complex analytic space of dimension $2((g-1) \cdot \dim [G, G] + g \cdot (\dim G - \dim [G, G]))$; see, for example, [Go], [Si, Proposition 49]. We shall denote the dimension of the commutator group $\dim[G, G]$ by d and $\dim G - \dim [G, G]$ by c (it is the dimension of the center of G). With this notation the complex dimension of Ξ is $2(g-1)d + 2gc$.

Let us adopt the notation of [CDHL] and denote by Syst the space of all pairs (X, ϕ) , where X is an element of the Teichmüller space for Σ and ϕ is a holomorphic connection on the trivial principal G -bundle $X \times G$ over X (recall that a holomorphic connection on a bundle over a Riemann surface is automatically flat). This space of differential systems on Σ is a (singular) complex analytic space of dimension $(g-1)(d+3) + gc$.

Sending a holomorphic connection to its monodromy representation, a holomorphic mapping from Syst to the character variety Ξ is obtained; this map is a restriction to Syst of the Riemann–Hilbert correspondence.

Let us define the nonempty Zariski open subset $\text{Syst}^{\text{irred}}$ of Syst consisting of all pairs (X, ϕ) for which the connection ϕ is *irreducible*, meaning the monodromy homomorphism for ϕ does not factor through any proper parabolic subgroup of G . Then $\text{Syst}^{\text{irred}}$ is a smooth complex manifold (see, for example, Corollary 50 in [Si]). The image of $\text{Syst}^{\text{irred}}$ under the Riemann–Hilbert correspondence lies in the smooth Zariski open subset

$$\Xi^{\text{irred}} \subset \Xi$$

defined by the irreducible homomorphisms $\pi_1(X) \rightarrow G$ (i.e., homomorphisms that do not factor through some proper parabolic subgroup of G). Let

$$\text{Mon} : \text{Syst}^{\text{irred}} \rightarrow \Xi^{\text{irred}} \tag{1.1}$$

be the holomorphic map between complex manifolds given by the restriction of Riemann–Hilbert correspondence to $\text{Syst}^{\text{irred}}$.

The main result proved here is the following (see Theorem 5.3):

Theorem 1.1. *If the complex dimension of G is at least three, the monodromy map*

$$\text{Mon} : \text{Syst}^{\text{irred}} \rightarrow \Xi^{\text{irred}}$$

is an immersion at the generic point.

If $G = \text{SL}(2, \mathbb{C})$, the dimensions of $\text{Syst}^{\text{irred}}$ and Ξ^{irred} are both $6g-6$, and Theorem 1.1 implies that Mon in (1.1) is a local biholomorphism at the generic point (see Corollary 5.5). It should be mentioned that examples constructed in [CDHL] show that for $G = \text{SL}(2, \mathbb{C})$ and Σ of genus $g \geq 3$, the monodromy map Mon is not always a local biholomorphism (over entire $\text{Syst}^{\text{irred}}$).

When $G = \text{SL}(2, \mathbb{C})$ and $g = 2$, the main result of [CDHL] says that the map Mon in (1.1) is a local biholomorphism over entire $\text{Syst}^{\text{irred}}$. An alternative proof of this result of [CDHL] is given in Corollary 5.4. In this context it should be mentioned that our work was greatly influenced by [CDHL].

Just as for the authors of [CDHL] our main motivation came from a question of E. Ghys for $G = \text{SL}(2, \mathbb{C})$ relating the monodromy of $\mathfrak{sl}(2, \mathbb{C})$ -differential systems to the existence of holomorphic curves of genus $g > 1$ lying in compact quotients of $\text{SL}(2, \mathbb{C})$ by lattices Γ . Such compact quotients of $\text{SL}(2, \mathbb{C})$ are non-Kähler manifolds which admit no complex

hypersurfaces [HM]. It is known that elliptic curves do exist in some of those manifolds, but the existence of holomorphic curves of genus $g > 1$ is still an open question. E. Ghys realized that constructing a $\mathfrak{sl}(2, \mathbb{C})$ -differential system on a Riemann surface X with monodromy lying inside a cocompact lattice $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ would provide a nontrivial holomorphic map from X into the quotient of $\mathrm{SL}(2, \mathbb{C})/\Gamma$ (in fact the two problems are equivalent). While the question asked by Ghys is still open, the above Theorem 1.1 extends the results of [CDHL] and provides an enhancement of the understanding of the monodromy of the differential systems.

The strategy of the proof of Theorem 1.1 and the organization of the paper are as follow. We consider the Hilbert-Riemann correspondence (given by the monodromy map) defined on the space of triples (X, E_G, ϕ) , with X an element of the Teichmüller space for Σ , E_G a holomorphic principal G -bundle over X and ϕ a holomorphic connection on E_G . In Section 2 we define a 2-term complex \mathcal{C}_\bullet over X whose first hypercohomology gives the infinitesimal deformations of (X, E_G, ϕ) (Theorem 2.1 (2)). Moreover the kernel of the differential of the monodromy map coincides with the image of the space of deformations of the complex structure $H^1(X, TX)$ through the homomorphism β induced on 1-hypercohomologies of \mathcal{C}_\bullet (Theorem 2.1 (4)). In Section 3 we fix E_G to be the holomorphically trivial principal G -bundle over X and ϕ an irreducible holomorphic connection on it. We show that the tangent space of $\mathrm{Syst}^{\mathrm{irred}}$ at (X, ϕ) , which is naturally embedded in $\mathbb{H}^1(X, \mathcal{C}_\bullet)$, is transverse to the kernel of the monodromy map if ϕ satisfies a geometric criteria described in Proposition 3.1. In section 4 we consider the special case of $G = \mathrm{SL}(2, \mathbb{C})$ and we prove that the criteria in Proposition 3.1 is satisfied at any point $(X, \phi) \in \mathrm{Syst}^{\mathrm{irred}}$ for surfaces Σ of genus two (see Proposition 4.1); the same holds for the generic point in $\mathrm{Syst}^{\mathrm{irred}}$ for surfaces Σ of genus three (see Lemma 4.2). The main result (Theorem 1.1) is obtained in Section 5 where Lemma 5.1 proves that the transversality criteria is satisfied at the generic point in $\mathrm{Syst}^{\mathrm{irred}}$. More precisely, the proof of Lemma 5.1 shows that the transversality criteria (in Proposition 3.1) is implied by the fact that for a non hyper-elliptic Riemann surface X and a generic three dimensional subspace $W \subset H^0(X, K_X)$, the natural homomorphism

$$\Theta_W : H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2)$$

is surjective. This is precisely the statement of Theorem 1.1 in [Gi, p. 221], where the proof is attributed to R. Lazarsfeld.

2. INFINITESIMAL DEFORMATIONS OF BUNDLES AND CONNECTIONS

In this section we introduce several infinitesimal deformation spaces and natural morphisms between them.

The holomorphic tangent bundle of a complex manifold Y will be denoted by TY .

Let X be a compact connected Riemann surface. The holomorphic cotangent bundle of X will be denoted by K_X . Let G be a connected reductive affine algebraic group defined over \mathbb{C} . The Lie algebra of G will be denoted by \mathfrak{g} .

Take a holomorphic principal G -bundle over X

$$p : E_G \longrightarrow X. \tag{2.1}$$

So E_G is equipped with a holomorphic action of G on the right which is both free and transitive, and furthermore, $E_G/G = X$. Consider the holomorphic right action of G on the holomorphic tangent bundle TE_G given by the action of G on E_G . The quotient

$$\text{At}(E_G) := (TE_G)/G$$

is a holomorphic vector bundle over $E_G/G = X$; it is called the *Atiyah bundle* for E_G . The differential

$$dp : TE_G \longrightarrow p^*TX$$

of the projection p in (2.1) is G -equivariant for the trivial action of G on the fibers of p^*TX . The action of G on E_G produces a holomorphic homomorphism from the trivial holomorphic bundle

$$E_G \times \mathfrak{g} \longrightarrow \text{kernel}(dp)$$

which is an isomorphism. Therefore, we have a short exact sequence of holomorphic vector bundles on E_G

$$0 \longrightarrow \text{kernel}(dp) = E_G \times \mathfrak{g} \longrightarrow \text{At}(E_G) \xrightarrow{dp} p^*TX \longrightarrow 0 \quad (2.2)$$

in which all the homomorphisms are G -equivariant. The quotient $\text{kernel}(dp)/G$ is the adjoint vector bundle $\text{ad}(E_G) = E_G(\mathfrak{g})$, which is the holomorphic vector bundle over X associated to E_G for the adjoint action of G on \mathfrak{g} . Taking quotient of the bundles in (2.2), by the actions of G , the following short exact sequence of holomorphic vector bundles on X is obtained:

$$0 \longrightarrow \text{ad}(E_G) \xrightarrow{\iota} \text{At}(E_G) \xrightarrow{d'p} TX \longrightarrow 0 \quad (2.3)$$

[At]; it is known as the Atiyah exact sequence.

A holomorphic connection on E_G is a holomorphic homomorphism of vector bundles

$$\phi : TX \longrightarrow \text{At}(E_G)$$

such that

$$(d'p) \circ \phi = \text{Id}_{TX}, \quad (2.4)$$

where $d'p$ is the projection in (2.3) (see [At]). A holomorphic connection on a bundle over X is automatically flat, because $\Omega_X^{2,0} = 0$. A holomorphic connection ϕ on E_G gives a holomorphic decomposition $\text{At}(E_G) = TX \oplus \text{ad}(E_G)$. This decomposition produces a holomorphic homomorphism

$$\phi' : \text{At}(E_G) \longrightarrow \text{ad}(E_G) \quad (2.5)$$

such that $\phi' \circ \iota = \text{Id}_{\text{ad}(E_G)}$, where ι is the homomorphism in (2.3).

Take a holomorphic connection

$$\phi : TX \longrightarrow \text{At}(E_G) \quad (2.6)$$

on E_G . Since $\text{At}(E_G) = (TE_G)/G$, this homomorphism ϕ produces a G -equivariant holomorphic homomorphism of vector bundles

$$\widehat{\phi} := p^*\phi : p^*TX \longrightarrow TE_G \quad (2.7)$$

over E_G . Take any analytic open subset $U \subset X$. Let s be a holomorphic section of $\text{At}(E_G)|_U$ over U . Since $\text{At}(E_G) = (TE_G)/G$, we have

$$\widehat{s} := p^*s \in H^0(p^{-1}(U), TE_G)^G \subset H^0(p^{-1}(U), TE_G).$$

For any holomorphic vector field $t \in H^0(U, TU)$, consider the Lie bracket

$$[\widehat{\phi}(p^*t), \widehat{s}] \in H^0(p^{-1}(U), TE_G),$$

where $\widehat{\phi}$ is the homomorphism in (2.7). This vector field $[\widehat{\phi}(p^*t), \widehat{s}]$ on $p^{-1}(U)$ is G -invariant, because both \widehat{s} and $\widehat{\phi}(p^*t)$ are so. Therefore, $[\widehat{\phi}(p^*t), \widehat{s}]$ produces a holomorphic section of $\text{At}(E_G)$ over U ; this section of $\text{At}(E_G)|_U$ will be denoted by $A(t, s)$. Let

$$\phi'(A(t, s)) \in H^0(U, \text{ad}(E_G)) \quad (2.8)$$

be the section of $\text{ad}(E_G)|_U$, where ϕ' is the projection in (2.5).

Now, for any holomorphic function f defined on U , we have

$$[\widehat{\phi}(p^*(f \cdot t)), \widehat{s}] = (f \circ p) \cdot [\widehat{\phi}(p^*t), \widehat{s}] - \widehat{s}(f \circ p) \cdot \widehat{\phi}(p^*t).$$

Since $\phi'(\widehat{\phi}(p^*t)) = 0$, where ϕ' is constructed in (2.5), from this it follows immediately that

$$\phi'(A(f \cdot t, s)) = f \cdot \phi'(A(t, s));$$

$\phi'(A(t, s))$ is defined in (2.8). Let

$$\Phi : \text{At}(E_G) \longrightarrow \text{ad}(E_G) \otimes K_X \quad (2.9)$$

be the homomorphism of sheaves defined by the equation

$$\langle \Phi(s), t \rangle = \phi'(A(t, s)) \in H^0(U, \text{ad}(E_G)),$$

where s and t are holomorphic sections, over U , of $\text{At}(E_G)$ and TX respectively, while $\langle - - \rangle$ is the contraction of K_X by T_X . It should be clarified that Φ is not \mathcal{O}_X -linear.

The composition

$$\Phi \circ \phi : TX \longrightarrow \text{ad}(E_G) \otimes K_X$$

coincides with the curvature of the connection ϕ . Since ϕ is flat, we have

$$\Phi \circ \phi = 0. \quad (2.10)$$

Let \mathcal{C}_\bullet be the 2-term complex

$$\mathcal{C}_\bullet : \mathcal{C}_0 := \text{At}(E_G) \xrightarrow{\Phi} \mathcal{C}_1 := \text{ad}(E_G) \otimes K_X,$$

where \mathcal{C}_i is at the i -th position and Φ is the homomorphism constructed in (2.9). Using (2.10) we have the following commutative diagram of homomorphisms of complexes of sheaves on X :

$$\begin{array}{ccc} & 0 & 0 \\ & \downarrow & \downarrow \\ & TX & \longrightarrow 0 \\ & \downarrow \phi & \downarrow \\ \mathcal{C}_\bullet : & \mathcal{C}_0 & \xrightarrow{\Phi} \mathcal{C}_1 \\ & \downarrow = & \downarrow \\ & \text{At}(E_G) & \longrightarrow 0 \\ & \downarrow & \downarrow \\ & 0 & 0 \end{array} \quad (2.11)$$

It should be clarified that this is not a complex of complexes of sheaves — the composition map does not vanish. Let

$$H^1(X, TX) \xrightarrow{\beta} \mathbb{H}^1(X, \mathcal{C}_\bullet) \xrightarrow{\gamma} H^1(X, \text{At}(E_G)) \quad (2.12)$$

be the homomorphisms of (hyper)cohomologies associated to the homomorphisms (2.11), where \mathbb{H}^i denotes the i -th hypercohomology. It should be clarified that $\gamma \circ \beta$ need not vanish.

The following known theorem will be used (see [Ch2], [Do], [In], [BHH]).

Theorem 2.1.

- (1) *The infinitesimal deformations of the pair (X, E_G) are parametrized by the elements of the cohomology $H^1(X, \text{At}(E_G))$.*
- (2) *The infinitesimal deformations of the triple (X, E_G, ϕ) are parametrized by the elements of the hypercohomology $\mathbb{H}^1(X, \mathcal{C}_\bullet)$.*
- (3) *The forgetful map from the infinitesimal deformations of the triple (X, E_G, ϕ) to the infinitesimal deformations of the pair (X, E_G) , that forgets the connection ϕ , is the homomorphism γ in (2.12).*
- (4) *The infinitesimal isomonodromy map, from the infinitesimal deformations of X to the infinitesimal deformations of the triple (X, E_G, ϕ) , coincides with the homomorphism β in (2.12).*

For the proof of Theorem 2.1 the reader is referred to [Ch2, p. 1413, Proposition 4.3] (for Theorem 2.1(1)), [Ch2, p. 1415, Proposition 4.4] (for Theorem 2.1(2)) and [Ch2, p. 1417, Proposition 5.1] (for Theorem 2.1(4)); see also [Ch1].

The Atiyah exact sequence in (2.3) produces a long exact sequence of cohomologies

$$H^1(X, \text{ad}(E_G)) \xrightarrow{\iota_*} H^1(X, \text{At}(E_G)) \xrightarrow{(d'p)_*} H^1(X, TX) \longrightarrow 0.$$

We note that the infinitesimal deformations of E_G (keeping the Riemann surface X fixed) are parametrized by $H^1(X, \text{ad}(E_G))$ (see [Do]), and the above homomorphism ι_* coincides with the natural homomorphism of infinitesimal deformations. The above projection $(d'p)_*$ is the forgetful map that sends the infinitesimal deformations of the pair (X, E_G) to the infinitesimal deformations of X that forgets principal G -bundle.

3. INFINITESIMAL DEFORMATIONS OF CONNECTIONS ON THE TRIVIAL BUNDLE

In this section we define the tangent space to $(X, \phi) \in \text{Syst}^{\text{irred}}$ as embedded in $\mathbb{H}^1(X, \mathcal{C}_\bullet)$, the space of infinitesimal deformations of triples (X, E_G, ϕ) (see below) and we prove a criteria for transversality to the kernel of the monodromy map (Proposition 3.1).

Let Y be a compact connected Riemann surface, and let ψ be a holomorphic connection on the holomorphically trivial principal G -bundle $Y \times G \longrightarrow Y$ over Y . Let

$$\mathcal{T}(Y, \psi) \quad (3.1)$$

denote the infinitesimal deformations of the pair (Y, ψ) (keeping the underlying principal bundle to be the trivial principal G -bundle on the moving Riemann surface).

Henceforth, we assume that $\text{genus}(X) = g \geq 2$.

Now take E_G in (2.1) to be the holomorphically trivial principal G -bundle $X \times G$ on X . As in (2.6), take a holomorphic connection ϕ on $E_G = X \times G$. Since the infinitesimal deformations of the triple (X, E_G, ϕ) are parametrized by $\mathbb{H}^1(X, \mathcal{C}_\bullet)$ (see Theorem 2.1(2)), we have a natural homomorphism

$$\mathcal{T}(X, \phi) \longrightarrow \mathbb{H}^1(X, \mathcal{C}_\bullet),$$

where $\mathcal{T}(X, \phi)$ is defined in (3.1). Let

$$\mathcal{S}(X, \phi) \subseteq \mathbb{H}^1(X, \mathcal{C}_\bullet) \tag{3.2}$$

be the image of this homomorphism from $\mathcal{T}(X, \phi)$.

The trivial holomorphic principal G -bundle $E_G = X \times G$ has a unique holomorphic connection whose monodromy is the trivial representation. Once we fix an isomorphism of E_G with $X \times G$, the trivial holomorphic connection on $X \times G$ induces a holomorphic connection on E_G using the chosen isomorphism. However, this induced connection on E_G does not depend on the choice of the trivialization of E_G ; this unique connection on E_G will be called the trivial connection. The monodromy of the trivial connection is evidently trivial.

Using the trivial connection on E_G , we have a canonical holomorphic decomposition

$$\text{At}(E_G) = \text{ad}(E_G) \oplus TX. \tag{3.3}$$

Using (3.3), the holomorphic connections on E_G are identified with holomorphic homomorphisms from TX to $\text{ad}(E_G)$. More precisely, to any holomorphic homomorphism

$$\rho : TX \longrightarrow \text{ad}(E_G) \tag{3.4}$$

we assign the corresponding homomorphism

$$\widehat{\rho} : TX \longrightarrow \text{ad}(E_G) \oplus TX = \text{At}(E_G), \quad v \longmapsto (\rho(v), v) \tag{3.5}$$

clearly $\widehat{\rho}$ satisfies the equation in (2.4). Note that the decomposition in (3.3) is used in the construction of $\widehat{\rho}$ in (3.5).

Proposition 3.1. *Let ϕ be a holomorphic connection on $E_G = X \times G$ such that the homomorphism ρ in (3.4) corresponding to ϕ satisfies the following condition: the homomorphism of first cohomologies corresponding to ρ , namely*

$$\rho_* : H^1(X, TX) \longrightarrow H^1(X, \text{ad}(E_G)),$$

is injective. Then

$$\mathbb{H}^1(X, \mathcal{C}_\bullet) \supset \mathcal{S}(X, \phi) \cap \beta(H^1(X, TX)) = 0,$$

where $\mathcal{S}(X, \phi)$ is the subspace constructed in (3.2) and β is the homomorphism in (2.12).

Proof. Let

$$q : \text{At}(E_G) = \text{ad}(E_G) \oplus TX \longrightarrow TX$$

be the projection constructed using the decomposition in (3.3). Note that q coincides with the projection $d'p$ in (2.3); indeed, this follows from the construction of the decomposition in (3.3). Let

$$q_* : H^1(X, \text{At}(E_G)) = H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX) \longrightarrow H^1(X, TX) \tag{3.6}$$

be the homomorphism of first cohomologies induced by q . From the equation

$$q \circ \phi = \text{Id}_{TX}$$

(see (2.4)) it can be deduced that

$$q_* \circ \gamma \circ \beta = \text{Id}_{H^1(X, TX)}, \quad (3.7)$$

where β, γ are the homomorphisms in (2.12) and q_* is constructed in (3.6). To see this, just note that $\gamma \circ \beta$ coincides with the homomorphism of cohomologies induced by ϕ .

Consider the two subspaces

$$\gamma(\mathcal{S}(X, \phi)), \gamma(\beta(H^1(X, TX))) \subset H^1(X, \text{At}(E_G)) \quad (3.8)$$

of $H^1(X, \text{At}(E_G))$, where γ is the homomorphism in (2.12). From (3.7) it follows immediately that the homomorphism $\gamma \circ \beta$ is injective.

Consequently, to prove the proposition it suffices to show that

$$\gamma(\mathcal{S}(X, \phi)) \cap (\gamma(\beta(H^1(X, TX)))) = 0, \quad (3.9)$$

where $\gamma(\mathcal{S}(X, \phi))$ and $\gamma(\beta(H^1(X, TX)))$ are the subspaces in (3.8).

The isomonodromic deformation of the trivial connection on $E_G \rightarrow X$ is evidently the trivial connection on the trivial principal G -bundle over the moving Riemann surface. Since the decomposition in (3.3) is given by the trivial connection on E_G , from Theorem 2.1(4) and Theorem 2.1(3) it follows that the subspace

$$\gamma(\mathcal{S}(X, \phi)) \subset H^1(X, \text{At}(E_G))$$

coincides with the natural subspace

$$\begin{aligned} H^1(X, TX) &\subset H^1(X, \text{ad}(E_G)) \oplus H^1(X, TX) \\ &= H^1(X, \text{ad}(E_G) \oplus TX) = H^1(X, \text{At}(E_G)) \end{aligned}$$

(inclusion into the direct summand) corresponding to the decomposition in (3.3) (given by the trivial connection). Therefore, using the construction of the homomorphism $\widehat{\rho}$ from ρ (see (3.5); note that $\phi = \widehat{\rho}$ by the definition of ρ given in the statement of the proposition) it follows that the given condition that

$$\rho_* : H^1(X, TX) \longrightarrow H^1(X, \text{ad}(E_G))$$

is injective implies that (3.9) holds. As noted before, (3.9) completes the proof of the proposition. \square

4. SOME EXAMPLES WITH $G = \text{SL}(2, \mathbb{C})$

This section focuses on the case $G = \text{SL}(2, \mathbb{C})$. In this case we prove that the criteria in Proposition 3.1 is satisfied at any point $(X, \phi) \in \text{Syst}^{\text{irred}}$ for $g = 2$ (see Proposition 4.1) and at the generic point in $\text{Syst}^{\text{irred}}$ for $g = 3$ (see Lemma 4.2).

Proposition 4.1. *Let X be a compact connected Riemann surface of genus two. Set $G = \text{SL}(2, \mathbb{C})$. Let ϕ be an irreducible holomorphic connection on the trivial holomorphic principal G -bundle $E_G = X \times G \rightarrow X$. Then the homomorphism ρ as in (3.4) corresponding to ϕ satisfies the following condition: the homomorphism of cohomologies corresponding to ρ , namely*

$$\rho_* : H^1(X, TX) \longrightarrow H^1(X, \text{ad}(E_G)),$$

is injective.

Proof. Fix a holomorphic trivialization of E_G . Then the adjoint vector bundle $\text{ad}(E_G)$ is the trivial holomorphic vector bundle $X \times \mathfrak{sl}(2, \mathbb{C})$ over X , where $\mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra consisting of 2×2 complex matrices of trace zero. So ρ as in (3.4) corresponding to ϕ

$$\rho : TX \longrightarrow \text{ad}(E_G) = X \times \mathfrak{sl}(2, \mathbb{C})$$

sends any $v \in T_x X$ to

$$\left(x, \begin{pmatrix} \omega_1(x)(v) & \omega_2(x)(v) \\ \omega_3(x)(v) & -\omega_1(x)(v) \end{pmatrix} \right) \in X \times \mathfrak{sl}(2, \mathbb{C}),$$

where $\omega_j \in H^0(X, K_X)$, $1 \leq j \leq 3$.

Let $V \subset H^0(X, K_X)$ be the linear span of $\{\omega_j\}_{j=1}^3$. The given condition that the connection ϕ is irreducible implies that $\dim V > 1$. Indeed, if we assume by contradiction that $\dim V \leq 1$, then

$$\rho = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & -\omega_1 \end{pmatrix} = B \otimes \omega,$$

where $\omega \in H^0(X, K_X)$ and $B \in \mathfrak{sl}(2, \mathbb{C})$ is a fixed element. Since the standard action on \mathbb{C}^2 of B is reducible, the connection ϕ is reducible: a contradiction. So we have $\dim V > 1$ and, since $\dim H^0(X, K_X) = 2$, we conclude that $V = H^0(X, K_X)$.

Let

$$\Theta : H^1(X, TX) \otimes H^0(X, K_X) \longrightarrow H^1(X, TX \otimes K_X) = H^1(X, \mathcal{O}_X) \quad (4.1)$$

be the natural homomorphism. For any \mathbb{C} -linear map

$$h : \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathbb{C}, \quad (4.2)$$

let

$$\tilde{h}_* : H^1(X, TX) \longrightarrow H^1(X, \mathcal{O}_X) \quad (4.3)$$

be the homomorphism induced by the composition

$$TX \xrightarrow{\rho} \text{ad}(E_G) = \mathcal{O}_X \times_{\mathbb{C}} \mathfrak{sl}(2, \mathbb{C}) \xrightarrow{\text{Id} \otimes h} \mathcal{O}_X.$$

For any

$$\mu \in \text{kernel}(\rho_*) \subset H^1(X, TX),$$

we evidently have $\tilde{h}_*(\mu) = 0$. Now, since $V = H^0(X, K_X)$, it follows that

$$\Theta(\mu, \omega) = 0 \quad (4.4)$$

for all $\omega \in H^0(X, K_X)$ and $\mu \in \text{kernel}(\rho_*)$, where Θ is the homomorphism in (4.1).

To complete the proof of the proposition we need to show that there is no nonzero cohomology class $\mu \in H^1(X, TX)$ that satisfies (4.4) for all $\omega \in H^0(X, K_X)$.

Using Serre duality, it suffices to prove that the tensor product homomorphism

$$\Theta' : H^0(X, K_X) \otimes H^0(X, K_X) \longrightarrow H^0(X, K_X^2) \quad (4.5)$$

is surjective; note that Θ' is given by the dual of Θ .

It is known that for a genus two Riemann surface X , the homomorphism Θ' in (4.5) is indeed surjective. To be somewhat self-contained, we give the outline of an argument for it. Consider the short exact sequence of sheaves on $X \times X$

$$0 \longrightarrow (p_1^* K_X) \otimes (p_2^* K_X) \otimes \mathcal{O}_{X \times X}(-\Delta) \longrightarrow (p_1^* K_X) \otimes (p_2^* K_X) \longrightarrow i_* K_X^2 \longrightarrow 0, \quad (4.6)$$

where p_j is the projection of $X \times X$ to the j -th factor for $j = 1, 2$, and i is the inclusion map of the diagonal $\Delta \subset X \times X$. Using the short exact sequence

$$0 \longrightarrow (p_2^*K_X) \otimes \mathcal{O}_{X \times X}(-\Delta) \longrightarrow p_2^*K_X \longrightarrow i_*K_X \longrightarrow 0$$

we have

$$p_{1*}((p_2^*K_X) \otimes \mathcal{O}_{X \times X}(-\Delta)) = TX.$$

Therefore, the projection formula gives that

$$p_{1*}((p_1^*K_X) \otimes (p_2^*K_X) \otimes \mathcal{O}_{X \times X}(-\Delta)) = \mathcal{O}_X.$$

Hence we have

$$H^0(X \times X, (p_1^*K_X) \otimes (p_2^*K_X) \otimes \mathcal{O}_{X \times X}(-\Delta)) = H^0(X, \mathcal{O}_X). \quad (4.7)$$

Consider the long exact sequence of cohomologies for the short exact sequence of sheaves in (4.6)

$$\begin{aligned} 0 &\longrightarrow H^0(X \times X, (p_1^*K_X) \otimes (p_2^*K_X) \otimes \mathcal{O}_{X \times X}(-\Delta)) & (4.8) \\ &\longrightarrow H^0(X \times X, (p_1^*K_X) \otimes (p_2^*K_X)) = H^0(X, K_X)^{\otimes 2} \xrightarrow{\Theta'} H^0(X, K_X^2). \end{aligned}$$

Since $\dim H^0(X, K_X)^{\otimes 2} = 4 = \dim H^0(X, K_X^2) + 1$, from (4.7) and (4.8) it follows that Θ' is surjective. \square

Lemma 4.2. *Let X be a compact connected Riemann surface of genus three which is not hyperelliptic. Set $G = \mathrm{SL}(2, \mathbb{C})$. Then there is a nonempty Zariski open subset \mathcal{U} of the space of all holomorphic connections on the trivial holomorphic principal G -bundle $E_G = X \times G \longrightarrow X$ such that for any $\phi \in \mathcal{U}$, the homomorphism ρ in (3.4) corresponding to ϕ satisfies the following condition: the homomorphism of cohomologies corresponding to ρ , namely*

$$\rho_* : H^1(X, TX) \longrightarrow H^1(X, \mathrm{ad}(E_G)),$$

is injective.

Proof. As in the proof of Proposition 4.1, fixing a holomorphic trivialization of E_G , identifies $\mathrm{ad}(E_G)$ with the trivial holomorphic vector bundle $X \times \mathfrak{sl}(2, \mathbb{C})$ over X . Let $\{\omega_j\}_{j=1}^3$ be a basis of $H^0(X, K_X)$ (recall that $H^0(X, K_X)$ has dimension three). Define the homomorphism ρ

$$\rho : TX \longrightarrow \mathrm{ad}(E_G) = X \times \mathfrak{sl}(2, \mathbb{C})$$

that sends any $v \in T_x X$ to

$$\left(x, \begin{pmatrix} \omega_1(x)(v) & \omega_2(x)(v) \\ \omega_3(x)(v) & -\omega_1(x)(v) \end{pmatrix} \right) \in X \times \mathfrak{sl}(2, \mathbb{C}).$$

As in (4.1), let

$$\Theta : H^1(X, TX) \otimes H^0(X, K_X) \longrightarrow H^1(X, \mathcal{O}_X)$$

be the natural homomorphism. For any h as in (4.2), the homomorphism \tilde{h}_* as in (4.3) vanishes. Therefore, from the above construction of ρ it follows immediately that every

$$\mu \in \mathrm{kernel}(\rho_*) \subset H^1(X, TX),$$

satisfies the equation

$$\Theta(\mu, \omega) = 0 \quad (4.9)$$

for all $\omega \in H^0(X, K_X)$.

We will show that there is no nonzero cohomology class $\mu \in H^1(X, TX)$ that satisfies (4.9) for all $\omega \in H^0(X, K_X)$.

Using Serre duality, it suffices to prove that the tensor product homomorphism

$$\Theta' : H^0(X, K_X) \otimes H^0(X, K_X) \longrightarrow H^0(X, K_X^2)$$

is surjective. Now, Max Noether's theorem says that the homomorphism Θ' is surjective because X is not hyperelliptic [ACGH, p. 117].

The condition on a homomorphism $\rho : TX \longrightarrow \text{ad}(E_G)$ that

$$\rho_* : H^1(X, TX) \longrightarrow H^1(X, \text{ad}(E_G))$$

is injective, is Zariski open (in the space of all holomorphic homomorphisms). This completes the proof of the lemma. \square

5. HOLOMORPHIC CONNECTIONS ON THE TRIVIAL BUNDLE

In this section we prove the main result of the article (Theorem 5.3) and deduce several consequences (Corollary 5.4 and Corollary 5.5). Corollary 5.4 is the main result in [CDHL]. Corollary 5.5 answers positively a question of B. Deroin.

As before, G is a connected reductive affine algebraic group defined over \mathbb{C} . In this section we further assume that $\dim G \geq 3$. As before, the Lie algebra of G will be denoted by \mathfrak{g} .

As before, X is a compact connected Riemann surface of genus g , with $g \geq 2$. Given an element

$$\delta \in \mathfrak{g} \otimes H^0(X, K_X),$$

we have an \mathcal{O}_X -linear homomorphism

$$M(\delta) : TX \longrightarrow \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g} \tag{5.1}$$

that sends any $v \in T_x X$ to the contraction $\langle \delta(x), v \rangle \in \mathfrak{g}$. Let

$$M(\delta)_* : H^1(X, TX) \longrightarrow H^1(X, \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}) = H^1(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathfrak{g} \tag{5.2}$$

be the homomorphism of first cohomologies induced by the homomorphism $M(\delta)$ in (5.1).

Notice that the homomorphism $M(\delta)$ in (5.1) is similar to ρ in (3.4).

Lemma 5.1. *Let X be a compact connected Riemann surface of genus $g \geq 2$ such that one of the following two holds:*

- (1) X is non-hyperelliptic;
- (2) $g = 2$.

Then there is a nonempty Zariski open subset $\mathcal{U} \subset \mathfrak{g} \otimes H^0(X, K_X)$ such that for every $\delta \in \mathcal{U}$, the homomorphism $M(\delta)_$ constructed in (5.2) is injective.*

Proof. First assume that X is non-hyperelliptic. Under this assumption, Theorem 1.1 of [Gi, p. 221] says that for a generic three dimensional subspace $W \subset H^0(X, K_X)$, the natural homomorphism

$$\Theta_W : H^0(X, K_X) \otimes W \longrightarrow H^0(X, K_X^2)$$

is surjective; in [Gi], the proof of this theorem is attributed to R. Lazarsfeld (see the sentence in [Gi] just after Theorem 1.1). The dual homomorphism for it

$$\Theta_W^* : H^1(X, TX) \longrightarrow H^1(X, \mathcal{O}_X) \otimes W^*$$

(obtained using Serre duality) is injective if Θ_W is surjective.

Take any W as above such that Θ_W^* is injective. Set

$$\delta \in \mathfrak{g} \otimes H^0(X, K_X)$$

to be such that the image of the homomorphism $\mathfrak{g}^* \longrightarrow H^0(X, K_X)$ corresponding to δ contains W ; note that the given condition that $\dim G \geq 3$ ensures that such a δ exists. Then from the injectivity of Θ_W^* it follows immediately that the homomorphism $M(\delta)_*$ constructed in (5.2) is injective.

Since the condition on $\delta \in \mathfrak{g} \otimes H^0(X, K_X)$ that $M(\delta)_*$ is injective is Zariski open (in $\mathfrak{g} \otimes H^0(X, K_X)$), the proof of the lemma is complete under the assumption that X is non-hyperelliptic.

Next assume that $g = 2$. This case is actually covered in the proof of Proposition 4.1. More precisely, the proof of Proposition 4.1 shows that as long as δ is not of the form $B \otimes \omega$, where $B \in \mathfrak{g}$ and $\omega \in H^0(X, K_X)$ are fixed elements, the homomorphism $M(\delta)_*$ is injective. \square

Corollary 5.2. *Let X be a compact connected Riemann surface of genus $g \geq 2$ such that one of the following two holds:*

- (1) X is non-hyperelliptic;
- (2) $g = 2$.

Then for the generic holomorphic connection ϕ on the holomorphically trivial principal G -bundle $E_G = X \times G \longrightarrow X$,

$$\mathbb{H}^1(X, \mathcal{C}_\bullet) \supset \mathcal{S}(X, \phi) \cap \beta(H^1(X, TX)) = 0,$$

where $\mathcal{S}(X, \phi)$ is the subspace constructed in (3.2) and β is the homomorphism in (2.12).

Proof. This follows from the combination of Proposition 3.1 and Lemma 5.1. \square

For any holomorphic connection ϕ on a holomorphic principal G -bundle E_G (it need not be trivial) on X , consider the monodromy representation for ϕ . Let $\mathbb{L}(\phi)$ be the \mathbb{C} -local system on X for the flat connection on $\text{ad}(E_G)$ induced by ϕ . The infinitesimal deformations of the monodromy representation are parametrized by the elements of $H^1(X, \mathbb{L}(\phi))$ [Go]. The differential of the monodromy map is a homomorphism

$$\mathcal{H}(\phi) : \mathbb{H}^1(X, \mathcal{C}_\bullet) \longrightarrow H^1(X, \mathbb{L}(\phi)) \tag{5.3}$$

(see Theorem 2.1(2)).

Theorem 5.3. *Let X be a compact connected Riemann surface of genus $g \geq 2$ such that one of the following two holds:*

- (1) X is non-hyperelliptic;
- (2) $g = 2$.

Then for the generic holomorphic connection ϕ on the holomorphically trivial principal G -bundle $E_G = X \times G \rightarrow X$, the restriction of the homomorphism $\mathcal{H}(\phi)$ in (5.3) to the subspace $\mathcal{S}(X, \phi) \subset \mathbb{H}^1(X, \mathcal{C}_\bullet)$ in (3.2) is injective.

Proof. From Theorem 2.1(4) we know that the kernel of the homomorphism $\mathcal{H}(\phi)$ is the image of the homomorphism β in (2.12). In view of this, the theorem is an immediate consequence of Corollary 5.2. \square

The following result was first proved in [CDHL]:

Corollary 5.4. *Let X be a compact connected Riemann surface of genus two. Set $G = \mathrm{SL}(2, \mathbb{C})$. Let ϕ be an irreducible holomorphic connection on the trivial holomorphic principal G -bundle $E_G = X \times G \rightarrow X$. Then the restriction of the homomorphism $\mathcal{H}(\phi)$ in (5.3) to the subspace $\mathcal{S}(X, \phi)$ in (3.2) is an isomorphism between $\mathcal{S}(X, \phi)$ and $H^1(X, \mathbb{L}(\phi))$.*

Proof. Since $\mathrm{kernel}(\mathcal{H}(\phi)) = \beta(H^1(X, TX))$ (see Theorem 2.1(4)), it follows from the combination of Proposition 4.1 and Proposition 3.1 that the restriction of $\mathcal{H}(\phi)$ to $\mathcal{S}(X, \phi)$ is injective. Since the complex dimensions agree:

$$\dim \mathcal{S}(X, \phi) = 6 = \dim H^1(X, \mathbb{L}(\phi)),$$

injectivity implies isomorphism. \square

For Riemann surfaces of higher genus we have the following:

Corollary 5.5. *Let X be a compact connected Riemann surface of genus $g \geq 3$ which is not hyperelliptic. Set $G = \mathrm{SL}(2, \mathbb{C})$.*

Then for the generic holomorphic connection ϕ on the holomorphically trivial principal G -bundle $E_G = X \times G \rightarrow X$, the restriction of the homomorphism $\mathcal{H}(\phi)$ in (5.3) to the subspace $\mathcal{S}(X, \phi) \subset \mathbb{H}^1(X, \mathcal{C}_\bullet)$ is an isomorphism between $\mathcal{S}(X, \phi)$ and $H^1(X, \mathbb{L}(\phi))$.

Proof. This follows directly from the injectivity statement obtained in Theorem 5.3. Indeed, here the complex dimensions agree:

$$\dim \mathcal{S}(X, \phi) = 6g - 6 = \dim H^1(X, \mathbb{L}(\phi));$$

consequently, injectivity implies isomorphism. \square

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REFERENCES

- [ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves*. Vol. I, Grundlehren der Mathematischen Wissenschaften, 267. Springer-Verlag, New York, 1985.
- [At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.
- [BHH] I. Biswas, V. Heu and J. Hurtubise, Isomonodromic deformations of logarithmic connections and stability, *Math. Ann.* **366** (2016), 121–140.
- [CDHL] G. Calsamiglia, B. Deroin, V. Heu and F. Loray, The Riemann-Hilbert mapping for \mathfrak{sl}_2 systems over genus two curves, *Bull. Soc. Math. France* **147** (2019), 159–195.
- [Ch1] T. Chen, The associated map of the nonabelian Gauss–Manin connection, Thesis (Ph.D.)–University of Pennsylvania (2012), <https://www.math.upenn.edu/grad/dissertations/ChenThesis.pdf>.
- [Ch2] T. Chen, The associated map of the nonabelian Gauss–Manin connection, *Cent. Eur. Jour. Math.* **10** (2012), 1407–1421.
- [Do] I. F. Donin, Construction of a versal family of deformations for holomorphic bundles over a compact complex space, *Math. USSR Sb.* **23** (1974), 405–416.
- [Gi] D. Gieseker, A lattice version of the KP equation, *Acta Math.* **168** (1992), 219–248.
- [Go] W. M. Goldman, The symplectic nature of fundamental groups of surfaces, *Adv. in Math.* **54** (1984), 200–225.
- [HM] A. H. Huckleberry and G. A. Margulis, Invariant Analytic hypersurfaces, *Invent. Math.* **71** (1983), 235–240.
- [In] M. Inaba, Moduli of parabolic connections on a curve and Riemann-Hilbert correspondence, *Jour. Alg. Geom.* **22** (2013), 407–480.
- [Si] A. S. Sikora, Character varieties, *Trans. Amer. Math. Soc.* **364** (2012), 173–208.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

Email address: `indranil@math.tifr.res.in`

UNIVERSITÉ CÔTE D’AZUR, CNRS, LJAD, FRANCE

Email address: `dumitres@unice.fr`