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Adaptive and non-adaptive estimation for degenerate diffusion processes*

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Summary We discuss parametric estimation of a degenerate diffusion system from time-discrete observations. The first component of the degenerate diffusion system has a parameter θ_1 in a non-degenerate diffusion coefficient and a parameter θ_2 in the drift term. The second component has a drift term parameterized by θ_3 and no diffusion term. Asymptotic normality is proved in three different situations for an adaptive estimator for θ_3 with some initial estimators for (θ_1, θ_2) , an adaptive one-step estimator for $(\theta_1, \theta_2, \theta_3)$ with some initial estimators for them, and a joint quasi-maximum likelihood estimator for $(\theta_1, \theta_2, \theta_3)$ without any initial estimator. Our estimators incorporate information of the increments of both components. Thanks to this construction, the asymptotic variance of the estimators for θ_1 is smaller than the standard one based only on the first component. The convergence of the estimators for θ_3 is much faster than the other parameters. The resulting asymptotic variance is smaller than that of an estimator only using the increments of the second component.

Keywords and phrases Degenerate diffusion, one-step estimator, quasi-maximum likelihood estimator.

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1 Introduction

In this article, we will discuss parametric estimation for a hypo-elliptic diffusion process. More precisely, given a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with a right-continuous filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$, $\mathbb{R}_+ = [0, \infty)$, suppose that an $\mathbf{F}$ -adapted process $Z_t = (X_t, Y_t)$ satisfies the stochastic differential equation

$$\begin{cases} dX_t &= A(Z_t, \theta_2)dt + B(Z_t, \theta_1)dw_t \\ dY_t &= H(Z_t, \theta_3)dt \end{cases} \quad (1.1)$$

Here $A : \mathbb{R}^{dz} \times \overline{\Theta}_2 \rightarrow \mathbb{R}^{dx}$, $B : \mathbb{R}^{dz} \times \overline{\Theta}_1 \rightarrow \mathbb{R}^{dx} \otimes \mathbb{R}^r$, $H : \mathbb{R}^{dz} \times \overline{\Theta}_3 \rightarrow \mathbb{R}^{dy}$, and $w = (w_t)_{t \in \mathbb{R}_+}$ is an r -dimensional $\mathbf{F}$ -Wiener process. The spaces Θ_i ($i = 1, 2, 3$) are the unknown parameter spaces of the components of $\theta = (\theta_1, \theta_2, \theta_3)$ to be estimated from the data $(Z_{t_j})_{j=0,1,\dots,n}$, where $t_j = jh$, $h = h_n$ satisfying $h \rightarrow 0$, $nh \rightarrow \infty$ and $nh^2 \rightarrow 0$ as $n \rightarrow \infty$.

Estimation theory has been well developed for diffusion processes. Even focusing on parametric estimation for ergodic diffusions, there is huge amount of studies: Kutoyants [22, 24, 23], Prakasa Rao [29, 30], Yoshida [39, 40], Bibby and Sørensen [1], Kessler [20], Kuchler and Sørensen [21], Genon-Catalot et al. [11], Gloter [13, 14, 16], Sakamoto and Yoshida [31], Uchida [35], Uchida and Yoshida [36, 37, 38], Kamatani and Uchida [19], De Gregorio and Iacus [9], Genon-Catalot and Larédo [12], Suzuki and Yoshida [34] among many others. Nakakita and Uchida [28] and Nakakita et al. [27] studied estimation under measurement error; related are Gloter and Jacod [17, 18]. Non parametric estimation for the coefficients of an ergodic diffusion has also been widely studied : Dalalyan and Kutoyants [8], Kutoyants [24], Dalalyan [5], Dalalyan and Reiss [6, 7], Comte and Genon-Catalot [2], Comte et al. [3], Schmisser [33], to name a few. Historically attentions were paid to inference for non-degenerate cases.

Recently there is a growing interest in hypo-elliptic diffusions, that appear in various applied fields. Examples of the hypo-elliptic diffusion include the harmonic oscillator, the Van der Pol oscillator and the FitzHugh-Nagumo neuronal model; see e.g. León and Samson [25]. For parametric estimation of hypo-elliptic diffusions, we refer the reader to Gloter [15] for a discretely observed integrated diffusion process, and Samson and Thieullen [32] for a contrast estimator. Comte et al. [4] gave adaptive estimation under partial observation. Recently, Ditlevsen and Samson [10] studied filtering and inference for hypo-elliptic diffusions from complete and partial observations. When the observations are discrete and complete, they showed asymptotic normality of their estimators under the assumption that the true value of some of parameters are known. Melnykova [26] studied the estimation problem for the model (1.1), comparing contrast functions and least square estimates. The contrast functions we propose in this paper are different from the one in [26].

In this paper, we will present several estimation schemes. Since we assume discrete-time observations of $Z = (Z_t)_{t \in \mathbb{R}_+}$, quasi-likelihood estimation for θ_1 and θ_2 is known; only difference from the standard diffusion case is the existence of the covariate $Y = (Y_t)_{t \in \mathbb{R}_+}$ in the equation of $X = (X_t)_{t \in \mathbb{R}_+}$ but it causes no theoretical difficulty. We will give an exposition for construction of those standard estimators in Sections 7 and 8 for selfcontainedness. Thus, our first approach in Section 4 is toward estimation of θ_3 with initial estimators for θ_1 and θ_2 . The idea for construction of the quasi-likelihood function in the elliptic case was based on the local Gaussian approximation of the transition density. Then it is natural to approximate the distribution of

the increments of Y by that of the principal Gaussian variable in the expansion of the increment. However, this method causes deficiency, as we will observe there; see Remark 4.6 on p.18. We present a more efficient method by incorporating an additional Gaussian part from X . The rate of convergence attained by the estimator for θ_3 is $n^{-1/2}h^{1/2}$ and it is much faster than the rate $(nh)^{-1/2}$ for θ_2 and $n^{-1/2}$ for θ_1 . Section 5 treats some adaptive estimators using suitable initial estimators for $(\theta_1, \theta_2, \theta_3)$, and shows joint asymptotic normality. Then it should be remarked that the asymptotic variance of our estimator $\hat{\theta}_1$ for θ_1 has improved that of the ordinary volatility parameter estimator, e.g. $\hat{\theta}_1^0$ recalled in Section 7, that would be asymptotically optimal if the system consisted only of X . In Section 6, we consider a non-adaptive joint quasi-maximum likelihood estimator. This method does not require initial estimators. From computational point of view, adaptive methods often have merits by reducing dimension of parameters, but the non-adaptive method is still theoretically interesting. Section 2 collects the assumptions under which we will work. Section 3 offers several basic estimates to the increments of Z . To investigate efficiency of the presented estimators, we need the LAN property of the exact likelihood function of the hypo-elliptic diffusion. We will discuss this problem elsewhere.

2 Assumptions

We assume that Θ_i ($i = 1, 2, 3$) are bounded open domain in $\mathbb{R}^{p_i}$, respectively, and $\Theta = \prod_{i=1}^3 \Theta_i$ has a good boundary so that Sobolev's embedding inequality holds, that is, there exists a positive constant C_Θ such that

$$\sup_{\theta \in \Theta} |f_i(\theta_i)| \leq C_\Theta \sum_{k=0}^1 \|\partial_\theta^k f\|_{L^p(\Theta)} \quad (2.1)$$

for all $f \in C^1(\Theta)$ and $p > \sum_{i=1}^3 p_i$. If Θ has a Lipschitz boundary, then this condition is satisfied. Obviously, the embedding inequality (2.1) is valid for functions depending only on a part of components of θ .

In this paper, we will propose an estimator for θ and show its consistency and asymptotic normality.

Given a finite-dimensional real vector space $\mathbf{E}$, denote by $C_p^{a,b}(\mathbb{R}^{dz} \times \Theta_i; \mathbf{E})$ the set of functions $f : \mathbb{R}^{dz} \times \Theta_i \rightarrow \mathbf{E}$ such that f is continuously differentiable a times in $z \in \mathbb{R}^{dz}$ and b times in $\theta_i \in \Theta$ in any order and f and all such derivatives are continuously extended to $\mathbb{R}^{dz} \times \bar{\Theta}_i$, moreover, they are of at most polynomial growth in $z \in \mathbb{R}^{dz}$ uniformly in $\theta \in \Theta$. Let $C = BB^*$, $\star$ denoting the matrix transpose. We suppose that the process $(Z_t)_{t \in \mathbb{R}_+}$ that generates the data satisfies the stochastic differential equation (1.1) for a true value $\theta^* = (\theta_1^*, \theta_2^*, \theta_3^*) \in \Theta_1 \times \Theta_2 \times \Theta_3$.

[A1] (i) $A \in C_p^{i_A, j_A}(\mathbb{R}^{dz} \times \Theta_2; \mathbb{R}^{dx})$ and $B \in C_p^{i_B, j_B}(\mathbb{R}^{dz} \times \Theta_1; \mathbb{R}^{dx} \otimes \mathbb{R}^r)$.

(ii) $H \in C_p^{i_H, j_H}(\mathbb{R}^{dz} \times \Theta_3; \mathbb{R}^{dy})$.

We will denote F_x for $\partial_x F$, F_y for $\partial_y F$, and F_i for $\partial_{\theta_i} F$.

[A2] (i) $\sup_{t \in \mathbb{R}_+} \|Z_t\|_p < \infty$ for every $p > 1$.

(ii) There exists a probability measure ν on $\mathbb{R}^{dz}$ such that

$$\frac{1}{T} \int_0^T f(Z_t) dt \xrightarrow{p} \int f(z) \nu(dz) \quad (T \rightarrow \infty)$$

for any bounded continuous function $f : \mathbb{R}^{dz} \rightarrow \mathbb{R}$.

(iii) The function $\theta_1 \mapsto C(Z_t, \theta_1)^{-1}$ is continuous on $\overline{\Theta}_1$ a.s., and

$$\sup_{\theta_1 \in \Theta_1} \sup_{t \in \mathbb{R}_+} \|\det C(Z_t, \theta_1)^{-1}\|_p < \infty$$

for every $p > 1$.

(iv) For the $\mathbb{R}^{dy} \otimes \mathbb{R}^{dy}$ valued function $V(z, \theta_1, \theta_3) = H_x(z, \theta_3)C(z, \theta_1)H_x(z, \theta_3)^*$, the function $(\theta_1, \theta_3) \mapsto V(Z_t, \theta_1, \theta_3)^{-1}$ is continuous on $\overline{\Theta}_1 \times \overline{\Theta}_3$ a.s., and

$$\sup_{(\theta_1, \theta_3) \in \Theta_1 \times \Theta_3} \sup_{t \in \mathbb{R}_+} \|\det V(Z_t, \theta_1, \theta_3)^{-1}\|_p < \infty$$

for every $p > 1$.

Remark 2.1. (a) It follows from [A2] that the convergence in [A2] (ii) holds for any continuous function f of at most polynomial growth.

(b) We implicitly assume the existence of $C(Z_T, \theta_1)^{-1}$ and $V(Z_t, \theta_1, \theta_3)^{-1}$ in (iii) and (iv) of [A2].

(c) Fatou's lemma implies

$$\int |z|^p \nu(dz) + \sup_{\theta_1 \in \overline{\Theta}_1} \int (\det C(z, \theta_1))^{-p} \nu(dz) + \sup_{(\theta_1, \theta_3) \in \overline{\Theta}_1 \times \overline{\Theta}_3} \int (\det V(z, \theta_1, \theta_3))^{-p} \nu(dz) < \infty$$

for any $p > 0$.

Let

$$\mathbb{Y}^{(1)}(\theta_1) = -\frac{1}{2} \int \left\{ \text{Tr}(C(z, \theta_1)^{-1} C(z, \theta_1^*)) - \mathbf{d}_X + \log \frac{\det C(z, \theta_1)}{\det C(z, \theta_1^*)} \right\} \nu(dz).$$

Since $|\log x| \leq x + x^{-1}$ for $x > 0$, $\mathbb{Y}^{(1)}(\theta_1)$ is a continuous function on $\overline{\Theta}_1$ well defined under [A1] and [A2]. Let

$$\begin{aligned} \mathbb{Y}^{(J,1)}(\theta_1) &= -\frac{1}{2} \int \left\{ \text{Tr}(C(z, \theta_1)^{-1} C(z, \theta_1^*)) + \text{Tr}(V(z, \theta_1, \theta_3^*)^{-1} V(z, \theta_1^*, \theta_3^*)) - \mathbf{d}_Z \right. \\ &\quad \left. + \log \frac{\det C(z, \theta_1) \det V(\theta_1, \theta_3^*)}{\det C(z, \theta_1^*) \det V(\theta_1^*, \theta_3^*)} \right\} \nu(dz) \end{aligned}$$

Let

$$\mathbb{Y}^{(2)}(\theta_2) = -\frac{1}{2} \int C(z, \theta_1^*)^{-1} [(A(z, \theta_2) - A(z, \theta_2^*))^{\otimes 2}] \nu(dz) \quad (2.2)$$

Let

$$\mathbb{Y}^{(3)}(\theta_3) = - \int 6V(z, \theta_1^*, \theta_3)^{-1} [(H(z, \theta_3) - H(z, \theta_3^*))^{\otimes 2}] \nu(dz).$$

The random field $\mathbb{Y}^{(3)}$ is well defined under [A1] and [A2]. Let

$$\mathbb{Y}^{(J,3)}(\theta_1, \theta_3) = - \int 6V(z, \theta_1, \theta_3)^{-1} [(H(z, \theta_3) - H(z, \theta_3^*))^{\otimes 2}] \nu(dz).$$

We will assume all or some of the following identifiability conditions

[A3] (i) There exists a positive constant χ_1 such that

$$\mathbb{Y}^{(1)}(\theta_1) \leq -\chi_1 |\theta_1 - \theta_1^*|^2 \quad (\theta_1 \in \Theta_1).$$

(i') There exists a positive constant χ'_1 such that

$$\mathbb{Y}^{(J,1)}(\theta_1) \leq -\chi'_1 |\theta_1 - \theta_1^*|^2 \quad (\theta_1 \in \Theta_1).$$

(ii) There exists a positive constant χ_2 such that

$$\mathbb{Y}^{(2)}(\theta_2) \leq -\chi_2 |\theta_2 - \theta_2^*|^2 \quad (\theta_2 \in \Theta_2).$$

(iii) There exists a positive constant χ_3 such that

$$\mathbb{Y}^{(3)}(\theta_3) \leq -\chi_3 |\theta_3 - \theta_3^*|^2 \quad (\theta_3 \in \Theta_3).$$

(iii') There exists a positive constant χ_3 such that

$$\mathbb{Y}^{(J,3)}(\theta_1, \theta_3) \leq -\chi_3 |\theta_3 - \theta_3^*|^2 \quad (\theta_1 \in \Theta_1, \theta_3 \in \Theta_3).$$

3 Basic estimation of the increments

We denote $U^{\otimes k}$ for $U \otimes \dots \otimes U$ (k -times) for a tensor U . For tensors $S^1 = (S^1_{i_{1,1}, \dots, i_{1,d_1}; j_{1,1}, \dots, j_{1,k_1}})$, ..., $S^m = (S^m_{i_{m,1}, \dots, i_{m,d_m}; j_{m,1}, \dots, j_{m,k_m}})$ and a tensor $T = (T^{i_{1,1}, \dots, i_{1,d_1}, \dots, i_{m,1}, \dots, i_{m,d_m}})$, we write

$$\begin{aligned} T[S^1, \dots, S^m] &= T[S^1 \otimes \dots \otimes S^m] \\ &= \left(\sum_{i_{1,1}, \dots, i_{1,d_1}, \dots, i_{m,1}, \dots, i_{m,d_m}} T^{i_{1,1}, \dots, i_{1,d_1}, \dots, i_{m,1}, \dots, i_{m,d_m}} S^1_{i_{1,1}, \dots, i_{1,d_1}; j_{1,1}, \dots, j_{1,k_1}} \right. \\ &\quad \left. \dots S^m_{i_{m,1}, \dots, i_{m,d_m}; j_{m,1}, \dots, j_{m,k_m}} \right)_{j_{1,1}, \dots, j_{1,k_1}, \dots, j_{m,1}, \dots, j_{m,k_m}}. \end{aligned}$$

This notation will be applied for a tensor-valued tensor T as well.

We have

$$\begin{aligned} h^{-1/2} \Delta_j X &= h^{-1/2} \int_{t_{j-1}}^{t_j} B(Z_t, \theta_1^*) dw_t + h^{-1/2} \int_{t_{j-1}}^{t_j} A(Z_t, \theta_2^*) dt \\ &= h^{-1/2} B(Z_{t_{j-1}}, \theta_1^*) \Delta_j w + r_j^{(3.2)} \end{aligned} \quad (3.1)$$

where

$$r_j^{(3.2)} = h^{-1/2} \int_{t_{j-1}}^{t_j} (B(Z_t, \theta_1^*) - B(Z_{t_{j-1}}, \theta_1^*)) dw_t + h^{-1/2} \int_{t_{j-1}}^{t_j} A(Z_t, \theta_2^*) dt \quad (3.2)$$

Lemma 3.1. (a) Under [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (0, 0, 0, 0, 0, 0)$ and [A2] (i),

$$\sup_{s, t \in \mathbb{R}_+, |s-t| \leq \Delta} \|Z_s - Z_t\|_p = O(\Delta^{1/2}) \quad (\Delta \downarrow 0) \quad (3.3)$$

for every $p > 1$.

(b) Under [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (0, 0, 1, 0, 0, 0)$ and [A2] (i), $r_j^{(3.2)} = \overline{O}_{L^\infty-}(h^{1/2})$, i.e.,

$$\sup_n \sup_j \|r_j^{(3.2)}\|_p = O(h^{1/2})$$

for every $p > 1$.

Proof. (a) is trivial. For (b), the first term on the right-hand side of (3.2) can be estimated by the Burkholder-Davis-Gundy inequality, Taylor's formula for $B(Z_t, \theta_1^*) - B(Z_{t_{j-1}}, \theta_1^*)$ and by (3.3). $\square$

We have

$$\begin{aligned} h^{-1/2} \Delta_j X &= h^{-1/2} \int_{t_{j-1}}^{t_j} B(Z_t, \theta_1^*) dw_t + h^{-1/2} \int_{t_{j-1}}^{t_j} A(Z_t, \theta_2^*) dt \\ &= h^{-1/2} \int_{t_{j-1}}^{t_j} B(Z_t, \theta_1^*) dw_t + h^{1/2} A(Z_{t_{j-1}}, \theta_2^*) + r_j^{(3.4)} \end{aligned}$$

where

$$r_j^{(3.4)} = h^{-1/2} \int_{t_{j-1}}^{t_j} (A(Z_t, \theta_2^*) - A(Z_{t_{j-1}}, \theta_2^*)) dt \quad (3.4)$$

Then

Lemma 3.2. $r_j^{(3.4)} = \overline{O}_{L^\infty-}(h)$, i.e.,

$$\sup_n \sup_j \|r_j^{(3.4)}\|_p = O(h) \quad (3.5)$$

for every $p > 1$ if [A1] for $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 0, 0, 0, 0, 0)$ and [A2] (i) hold.

Proof. Thanks to (3.3). □

Let

$$L_H(z, \theta_1, \theta_2, \theta_3) = H_x(z, \theta_3)[A(z, \theta_2)] + \frac{1}{2}H_{xx}(z, \theta_3)[C(z, \theta_1)] + H_y(z, \theta_3)[H(z, \theta_3)].$$

Define the $\mathbb{R}^{dy}$ -valued function $G_n(z, \theta_1, \theta_2, \theta_3)$ by

$$G_n(z, \theta_1, \theta_2, \theta_3) = H(z, \theta_3) + \frac{h}{2}L_H(z, \theta_1, \theta_2, \theta_3).$$

Write

$$\zeta_j = \sqrt{3} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt$$

Then $E[\zeta_j^{\otimes 2}] = h^3 I_r$ for the r -dimensional identity matrix I_r .

We have

$$\begin{aligned} & \Delta_j Y - hG_n(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \\ &= \Delta_j Y - hH(Z_{t_{j-1}}, \theta_3) - \frac{h^2}{2}L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \\ &= hH(Z_{t_{j-1}}, \theta_3^*) - hH(Z_{t_{j-1}}, \theta_3) \\ & \quad + H_x(Z_{t_{j-1}}, \theta_3^*)B(Z_{t_{j-1}}, \theta_1^*) \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt \\ & \quad + \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \{H_x(Z_s, \theta_3^*)B(Z_s, \theta_1^*) - H_x(Z_{t_{j-1}}, \theta_3^*)B(Z_{t_{j-1}}, \theta_1^*)\} dw_s dt \\ & \quad + \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t (L_H(Z_s, \theta_1^*, \theta_2^*, \theta_3^*) - L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3)) ds dt \\ &= \{hH(Z_{t_{j-1}}, \theta_3^*) - hH(Z_{t_{j-1}}, \theta_3)\} + \kappa(Z_{t_{j-1}}, \theta_1^*, \theta_3^*)\zeta_j + \rho_j(\theta_1, \theta_2, \theta_3) \end{aligned} \quad (3.6)$$

where

$$\kappa(Z_{t_{j-1}}, \theta_1^*, \theta_3^*) = 3^{-1/2}H_x(Z_{t_{j-1}}, \theta_3^*)B(Z_{t_{j-1}}, \theta_1^*)$$

and

$$\begin{aligned} \rho_j(\theta_1, \theta_2, \theta_3) &= \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \{H_x(Z_s, \theta_3^*)B(Z_s, \theta_1^*) - H_x(Z_{t_{j-1}}, \theta_3^*)B(Z_{t_{j-1}}, \theta_1^*)\} dw_s dt \\ & \quad + \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t (L_H(Z_s, \theta_1^*, \theta_2^*, \theta_3^*) - L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3)) ds dt. \end{aligned} \quad (3.7)$$

Let

$$\mathcal{D}_j(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} h^{-1/2}(\Delta_j X - hA(Z_{t_{j-1}}, \theta_2)) \\ h^{-3/2}(\Delta_j Y - hG_n(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3)) \end{pmatrix}. \quad (3.8)$$

Lemma 3.3. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 0, 1, 0, 3, 0)$ and [A2] (i) are satisfied. Then

(a) $\sup_n \sup_j \|\rho_j(\theta_1^*, \theta_2^*, \theta_3^*)\|_p = O(h^2)$ for every $p > 1$.

(b) $\sup_n \sup_j \|\mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*)\|_p < \infty$ for every $p > 1$.

Proof. It is possible to show (a) by (3.7) and using the estimate (3.3) with the help of Taylor's formula. Additionally to the representation (3.6), by using (3.1) and (3.2), we obtain (b). $\square$

We denote by $(B_x B)(z, \theta_2)$ the tensor defined by $(B_x B)(z, \theta_2)[u_1 \otimes u_2] = B_x(z, \theta_2)[u_2, B(z, \theta_2)[u_1]]$ for $u_1, u_2 \in \mathbb{R}^r$. Moreover, we write $dw_s dw_t$ for $dw_s \otimes dw_t$, and $(B_x B)(Z_{t_{j-1}}, \theta_2^*) \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dw_t$ for $(B_x B)(Z_{t_{j-1}}, \theta_2^*) [\int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dw_t]$. We will apply this rule in similar situations. Let

$$L_B(z, \theta_1, \theta_2, \theta_3) = B_x(z, \theta_1)[A(z, \theta_2)] + \frac{1}{2} B_{xx}(z, \theta_3)[C(z, \theta_1)] + B_y(z, \theta_3)[H(z, \theta_3)]. \quad (3.9)$$

Lemma 3.4. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 0, 0, 0)$ and [A2] (i) are satisfied. Then

$$h^{-1/2}(\Delta_j X - hA(Z_{t_{j-1}}, \theta_2)) = \xi_j^{(3.11)} + \xi_j^{(3.12)} + r_j^{(3.13)}(\theta_2) \quad (3.10)$$

where

$$\xi_j^{(3.11)} = h^{-1/2} B(Z_{t_{j-1}}, \theta_1^*) \Delta_j w, \quad (3.11)$$

$$\xi_j^{(3.12)} = h^{-1/2} (B_x B)(Z_{t_{j-1}}, \theta_1^*) \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dw_t, \quad (3.12)$$

and

$$\begin{aligned} r_j^{(3.13)}(\theta_2) &= h^{-1/2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t ((B_x B)(Z_s, \theta_1^*) - (B_x B)(Z_{t_{j-1}}, \theta_1^*)) dw_s dw_t \\ &\quad + h^{-1/2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t L_B(Z_s, \theta_1^*, \theta_2^*, \theta_3^*) ds dw_t \\ &\quad + h^{-1/2} \int_{t_{j-1}}^{t_j} (A(Z_t, \theta_2^*) - A(Z_{t_{j-1}}, \theta_2)) dt. \end{aligned} \quad (3.13)$$

Moreover,

$$\sup_n \sup_j \|r_j^{(3.13)}(\theta_2^*)\|_p = O(h) \quad (3.14)$$

for every $p > 1$, and

$$|r_j^{(3.13)}(\theta_2)| \leq r_{n,j}^{(3.16)} \{h^{1/2} |\theta_2 - \theta_2^*| + h\} \quad (3.15)$$

with some random variables $r_{n,j}^{(3.16)}$ satisfying

$$\sup_n \sup_j \|r_{n,j}^{(3.16)}\|_p < \infty \quad (3.16)$$

for every $p > 1$.

Proof. The decomposition (3.10) is obtained by Itô's formula. The estimate (3.14) is verified by (3.3) since $\partial_z(B_x B)$ and $\partial_z A$ are bound by a polynomial in z uniformly in θ . The estimate (3.15) uses $\partial_2 A$ for θ_2 near θ_2^* as well as $\partial_z A$ evaluated at θ_2^* :

$$|r_j^{(3.13)}(\theta_2)|1_{\{|\theta_2 - \theta_2^*| < r\}} \leq r_{n,j}^{(3.16)} \{h^{1/2}|\theta_2 - \theta_2^*| + h\}1_{\{|\theta_2 - \theta_2^*| < r\}}$$

with some positive constant r and some random variables $r_{n,j}^{(3.16)}$ satisfying (3.16). The small number r was taken to ensure convexity of the vicinity of θ_2^* . For θ_2 such that $|\theta_2 - \theta_2^*| \geq r$, the estimate (3.15) is valid by enlarging $r_{n,j}^{(3.16)}$ if necessary. $\square$

Lemma 3.5. (a) Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 0)$ and [A2] (i) are satisfied. Then

$$\Delta_j Y - hG_n(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3^*) = \xi_j^{(3.18)} + \xi_j^{(3.19)} + h^{3/2}r_j^{(3.20)}(\theta_1, \theta_2) + h^{3/2}r_j^{(3.21)}(\theta_1, \theta_2) \quad (3.17)$$

where

$$\xi_j^{(3.18)} = \kappa(Z_{t_{j-1}}, \theta_1^*, \theta_3^*)\zeta_j, \quad (3.18)$$

$$\xi_j^{(3.19)} = ((H_x B)_x B)(Z_{t_{j-1}}, \theta_1^*, \theta_3^*) \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \int_{t_{j-1}}^s dw_r dw_s dt, \quad (3.19)$$

$$\begin{aligned} r_j^{(3.20)}(\theta_1, \theta_2) &= h^{-3/2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \int_{t_{j-1}}^s \{ ((H_x B)_x B)(Z_r, \theta_1^*, \theta_3^*) \\ &\quad - ((H_x B)_x B)(Z_{t_{j-1}}, \theta_1^*, \theta_3^*) \} dw_r dw_s dt \\ &\quad + h^{-3/2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \int_{t_{j-1}}^s L_{H_x B}(Z_r, \theta_1^*, \theta_2^*, \theta_3^*) dr dw_s dt \\ &\quad + h^{-3/2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t (L_H(Z_s, \theta_1, \theta_2, \theta_3^*) - L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3^*)) ds dt \end{aligned} \quad (3.20)$$

with

$$\begin{aligned} L_{H_x B}(z, \theta_1, \theta_2, \theta_3) &= (H_x B)_x(z, \theta_1, \theta_3)[A(z, \theta_2)] + \frac{1}{2}(H_x B)_{xx}(z, \theta_1, \theta_3)[C(z, \theta_1)] \\ &\quad + (H_x B)_y(z, \theta_1, \theta_3)[H(z, \theta_3)], \end{aligned}$$

and

$$r_j^{(3.21)}(\theta_1, \theta_2) = h^{-3/2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t (L_H(Z_s, \theta_1^*, \theta_2^*, \theta_3^*) - L_H(Z_s, \theta_1, \theta_2, \theta_3^*)) ds dt. \quad (3.21)$$

Moreover,

$$\sup_n \sup_j \left\| \sup_{(\theta_1, \theta_2) \in \bar{\Theta}_1 \times \bar{\Theta}_2} |r_j^{(3.20)}(\theta_1, \theta_2)| \right\|_p = O(h) \quad (3.22)$$

for every $p > 1$, and

$$|r_j^{(3.21)}(\theta_1, \theta_2)| \leq h^{1/2} r_{n,j}^{(3.24)} \{|\theta_1 - \theta_1^*| + |\theta_2 - \theta_2^*|\} \quad (3.23)$$

for all $(\theta_1, \theta_2) \in \overline{\Theta}_1 \times \overline{\Theta}_2$ with some random variables $r_{n,j}^{(3.24)}$ satisfying

$$\sup_n \sup_j \|r_{n,j}^{(3.24)}\|_p < \infty \quad (3.24)$$

for every $p > 1$.

(b) Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 2, 0)$ and [A2] (i) are satisfied. Then there exist random variables $r_{n,j}^{(3.25)}$ and a number ρ such that

$$\sup_{\theta_3 \in \overline{\Theta}_3} |\mathcal{D}_j(\theta_1, \theta_2, \theta_3) - \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3)| \leq h^{1/2} r_{n,j}^{(3.25)} \{|\theta_1 - \theta_1^*| + |\theta_2 - \theta_2^*|\}$$

for all $(\theta_1, \theta_2) \in B((\theta_1^*, \theta_2^*), \rho)$ and that

$$\sup_n \sup_j \|r_{n,j}^{(3.25)}\|_p < \infty \quad (3.25)$$

for every $p > 1$.

Proof. By (3.6), we have

$$\Delta_j Y - hG_n(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3^*) = \xi_j^{(3.18)} + \rho_j(\theta_1, \theta_2, \theta_3^*) \quad (3.26)$$

and

$$\begin{aligned} \rho_j(\theta_1, \theta_2, \theta_3^*) &= \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \{H_x(Z_s, \theta_3^*)B(Z_s, \theta_1^*) - H_x(Z_{t_{j-1}}, \theta_3^*)B(Z_{t_{j-1}}, \theta_1^*)\} dw_s dt \\ &\quad + \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t (L_H(Z_s, \theta_1^*, \theta_2^*, \theta_3^*) - L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3^*)) ds dt. \end{aligned}$$

Then the decomposition (3.17) is obvious. The first and third terms on the right-hand side of (3.20) can be estimated with Taylor's formula and (3.3), and the second term is easy to estimate. Thus, we obtain (3.22). Since $\partial_{(\theta_1, \theta_2)} L_H(z, \theta_1, \theta_2, \theta_3^*)$ is bound by a polynomial in z uniformly in (θ_1, θ_2) , there exist random variables $r_{n,j}^{(3.24)}$ that satisfy (3.23) and (3.24). [First show (3.23) on the set $\{|\theta_1, \theta_2 - (\theta_1^*, \theta_2^*)| < r\}$, next see this estimate is valid on $(\overline{\Theta}_1 \times \overline{\Theta}_2) \setminus \{|\theta_1, \theta_2 - (\theta_1^*, \theta_2^*)| < r\}$ by redefining $r_{n,j}^{(3.24)}$ if necessary. We obtained (a). The assertion (b) is easy to verify with (3.6), (3.7) and Lemma 3.4. $\square$

Lemma 3.6. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (0, 0, 0, 0, 2, 1)$ and [A2] (i) are satisfied. Then

$$\sup_{(\theta_1, \theta_2) \in \overline{\Theta}_1 \times \overline{\Theta}_2} |\mathcal{D}_j(\theta_1, \theta_2, \theta_3) - \mathcal{D}_j(\theta_1, \theta_2, \theta_3')| \leq h^{-1/2} r_{n,j}^{(3.27)} |\theta_3 - \theta_3'| \quad (\theta_3, \theta_3' \in \overline{\Theta}_3)$$

for some random variables $r_{n,j}^{(3.27)}$ such that

$$\sup_n \sup_j \|r_{n,j}^{(3.27)}\|_p < \infty \quad (3.27)$$

for every $p > 1$.

Proof.

$$\begin{aligned} & \mathcal{D}_j(\theta_1, \theta_2, \theta_3) - \mathcal{D}_j(\theta_1, \theta_2, \theta'_3) \\ = & \left(h^{-1/2} (H(Z_{t_{j-1}}, \theta'_3) - H(Z_{t_{j-1}}, \theta_3)) + \frac{h^{1/2}}{2} (L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta'_3) - L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3)) \right) \end{aligned}$$

Therefore the lemma is obvious. Apply the Taylor formula for the argument θ_3 if θ_3 and θ'_3 are close, otherwise and if necessary, redefine $r_{n,j}^{(3.27)}$. $\square$

4 An adaptive estimator for θ_3

We will work with some initial estimators $\hat{\theta}_1^0$ for θ_1^0 and $\hat{\theta}_2^0$ for θ_2 . The following standard convergence rates, in part or fully, will be assumed for these estimators:

[A4] (i) $\hat{\theta}_1^0 - \theta_1^* = O_p(n^{-1/2})$ as $n \rightarrow \infty$

(ii) $\hat{\theta}_2^0 - \theta_2^* = O_p(n^{-1/2}h^{-1/2})$ as $n \rightarrow \infty$

Sections 7 and 8 recall certain standard estimators for θ_1 and θ_2 , respectively. The expansions (3.1) and (3.6) with Lemma 3.5 suggest two approaches for estimating θ_3 . The first approach is based on the likelihood of $h^{-3/2}(\Delta_j Y - hG_n(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3))$ only. The second one uses the likelihood corresponding to $\mathcal{D}_j(\theta_1, \theta_2, \theta_3)$. However, it is possible to show that the first approach gives less optimal asymptotic variance; see Remark 4.6. So, we will treat the second approach here.

4.1 Adaptive quasi-likelihood function for θ_3

Let

$$S(z, \theta_1, \theta_3) = \begin{pmatrix} C(z, \theta_1) & 2^{-1}C(z, \theta_1)H_x(z, \theta_3)^* \\ 2^{-1}H_x(z, \theta_3)C(z, \theta_1) & 3^{-1}H_x(z, \theta_3)C(z, \theta_1)H_x(z, \theta_3)^* \end{pmatrix}$$

Then

$$\begin{aligned} & S(z, \theta_1, \theta_3)^{-1} \\ = & \begin{pmatrix} C(z, \theta_1)^{-1} + 3H_x(z, \theta_3)^*V(z, \theta_1, \theta_3)^{-1}H_x(z, \theta_3) & -6H_x(z, \theta_3)^*V(z, \theta_1, \theta_3)^{-1} \\ -6V(z, \theta_1, \theta_3)^{-1}H_x(z, \theta_3) & 12V(z, \theta_1, \theta_3)^{-1} \end{pmatrix}. \end{aligned} \quad (4.1)$$

Recall that

$$V(z, \theta_1, \theta_3) = H_x(z, \theta_3)C(z, \theta_1)H_x(z, \theta_3)^*.$$

Let

$$\hat{S}(z, \theta_3) = S(z, \hat{\theta}_1^0, \theta_3).$$

We define a log quasi-likelihood function by

$$\mathbb{H}_n^{(3)}(\theta_3) = -\frac{1}{2} \sum_{j=1}^n \left\{ \hat{S}(Z_{t_{j-1}}, \theta_3)^{-1} [\mathcal{D}_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3)^{\otimes 2}] + \log \det \hat{S}(Z_{t_{j-1}}, \theta_3) \right\}. \quad (4.2)$$

Let $\hat{\theta}_3^0$ be a quasi-maximum likelihood estimator (QMLE) for θ_3 for $\mathbb{H}_n^{(3)}$, that is, $\hat{\theta}_3^0$ is a $\bar{\Theta}_3$ -valued measurable mapping satisfying

$$\mathbb{H}_n^{(3)}(\hat{\theta}_3^0) = \max_{\theta_3 \in \bar{\Theta}_3} \mathbb{H}_n^{(3)}(\theta_3).$$

The QMLE $\hat{\theta}_3^0$ for $\mathbb{H}_n^{(3)}$ depends on n as it does on the data $(Z_{t_j})_{j=0,1,\dots,n}$; $\hat{\theta}_1^0$ in the function $\hat{S}$ also depends on $(Z_{t_j})_{j=0,1,\dots,n}$.

We introduce the following random fields depending on n .

$$\begin{aligned} \Psi_1(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1} \partial_1 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right] \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1} H_{xx}(z, \theta_3) [\partial_1 C(Z_{t_{j-1}}, \theta_1)] \end{pmatrix} \right], \end{aligned}$$

$$\begin{aligned} \Psi_2(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_1, \theta_2) \\ 2^{-1} \partial_2 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right] \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_1, \theta_2) \\ 2^{-1} H_x(z, \theta_3) [\partial_2 A(Z_{t_{j-1}}, \theta_2)] \end{pmatrix} \right] \end{aligned}$$

$$\begin{aligned} \tilde{\Psi}_2(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2) \\ 2^{-1} \partial_2 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right] \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2) \\ 2^{-1} H_x(z, \theta_3) [\partial_2 A(Z_{t_{j-1}}, \theta_2)] \end{pmatrix} \right], \end{aligned}$$

where

$$\tilde{\mathcal{D}}_j(\theta_1^*, \theta_2^*, \theta_3^*) = \begin{pmatrix} \xi_j^{(3.11)} + \xi_j^{(3.12)} \\ h^{-3/2} (\xi_j^{(3.18)} + \xi_j^{(3.19)}) \end{pmatrix},$$

and

$$\begin{aligned}\Psi_3(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3) \right. \\ &\quad \left. \otimes \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right]\end{aligned}$$

$$\begin{aligned}\tilde{\Psi}_3(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3) \right. \\ &\quad \left. \otimes \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right]\end{aligned}$$

$$\Psi_{3,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) \end{pmatrix} \right]$$

$$\tilde{\Psi}_{3,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) \end{pmatrix} \right]$$

$$\Psi_{3,2}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right]$$

$$\Psi_{3,3}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \frac{1}{2} \sum_{j=1}^n (S^{-1}(\partial_3 S) S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2} - S(Z_{t_{j-1}}, \theta_1, \theta_3)]$$

$$\Psi_{33,1}(\theta_1, \theta_2, \theta_3) = - \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix}^{\otimes 2} \right]$$

$$\begin{aligned}\Psi_{33,2}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3) \right. \\ &\quad \left. \otimes \begin{pmatrix} 0 \\ \partial_3^2 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3^2 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right]\end{aligned}$$

$$\Psi_{33,3}(\theta_1, \theta_3) = -\frac{1}{2} \sum_{j=1}^n \{ (S^{-1}(\partial_3 S) S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) [\partial_3 S(Z_{t_{j-1}}, \theta_1, \theta_3)] \}$$

$$\begin{aligned}
\Psi_{33,4}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= -2 \sum_{j=1}^n S^{-1}(\partial_3 S) S^{-1}(Z_{t_{j-1}}, \theta_1, \theta_3) [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3) \\
&\quad \otimes \left(\begin{matrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{matrix} \right)] \\
\Psi_{33,5}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \frac{1}{2} \sum_{j=1}^n \partial_3 \{ (S^{-1}(\partial_3 S) S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) \} [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2} - S(Z_{t_{j-1}}, \theta_1, \theta_3)].
\end{aligned}$$

4.2 Consistency of $\hat{\theta}_3^0$

Lemma 4.1. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (0, 0, 0, 1, 1, 1)$ and [A2] (i), (iii) and (iv) are fulfilled. Then*

$$\sup_{t \in \mathbb{R}_+} \left\| \sup_{(\theta_1, \theta_3) \in \overline{\Theta}_1 \times \overline{\Theta}_3} \{ |S(Z_t, \theta_1, \theta_3)| + \det S(Z_t, \theta_1, \theta_3)^{-1} + |S(Z_t, \theta_1, \theta_3)^{-1}| \} \right\|_p < \infty$$

for every $p > 1$

Proof. By [A2] (iii) and (iv), $\det S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1}$ as well as $S(Z_{t_{j-1}}, \theta_1, \theta_3)$ is continuous on $\overline{\Theta}_1 \times \overline{\Theta}_3$ a.s., and continuously differentiable on $\Theta_1 \times \Theta_3$. Moreover we see

$$\sup_{t \in \mathbb{R}_+} \sum_{i=0,1} \sup_{(\theta_1, \theta_3) \in \Theta_1 \times \Theta_3} \left\| \partial_{(\theta_1, \theta_3)}^i (\det S(Z_t, \theta_1, \theta_3)^{-1}) \right\|_p < \infty$$

for every $p > 1$ from (4.1). This implies that

$$\sup_{t \in \mathbb{R}_+} \left\| \sup_{(\theta_1, \theta_3) \in \Theta_1 \times \Theta_3} (\det S(Z_t, \theta_1, \theta_3)^{-1}) \right\|_p < \infty$$

for every $p > 1$ by Sobolev's inequality. The inequality

$$\sup_{t \in \mathbb{R}_+} \left\| \sup_{(\theta_1, \theta_3) \in \Theta_1 \times \Theta_3} |S(Z_t, \theta_1, \theta_3)| \right\|_p < \infty$$

for every $p > 1$ is rather easy to show. □

Let

$$\mathbb{Y}_n^{(3)}(\theta_3) = n^{-1} h \{ \mathbb{H}_n^{(3)}(\theta_3) - \mathbb{H}_n^{(3)}(\theta_3^*) \}.$$

Theorem 4.2. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 1)$ and [A2] are satisfied. Then*

$$\sup_{\theta_3 \in \overline{\Theta}_3} |\mathbb{Y}_n^{(3)}(\theta_3) - \mathbb{Y}^{(3)}(\theta_3)| \xrightarrow{p} 0 \quad (4.3)$$

as $n \rightarrow \infty$, if $\hat{\theta}_1^0 \rightarrow^p \theta_1^*$ and $\hat{\theta}_2^0 \rightarrow^p \theta_2^*$. Moreover, $\hat{\theta}_3^0 \rightarrow^p \theta_3^*$ if [A3] (iii) is additionally satisfied.

Proof of Theorem 4.2. We have

$$\begin{aligned}\mathbb{Y}_n^{(3)}(\theta_3) &= n^{-1}h^{1/2} \sum_{j=1}^n \hat{S}(Z_{t_{j-1}}, \theta_3)^{-1} [h^{1/2} \delta_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3), \mathcal{D}_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3^*)] \\ &\quad - \frac{1}{2n} \sum_{j=1}^n \hat{S}(Z_{t_{j-1}}, \theta_3)^{-1} [(h^{1/2} \delta_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3))^{\otimes 2}] + n^{-1}h R_n^{(4.4)}(\theta_3)\end{aligned}$$

where

$$\delta_j(\theta_1, \theta_2, \theta_3) = -\mathcal{D}_j(\theta_1, \theta_2, \theta_3) + \mathcal{D}_j(\theta_1, \theta_2, \theta_3^*)$$

and

$$\begin{aligned}R_n^{(4.4)}(\theta_3) &= -\frac{1}{2} \sum_{j=1}^n (\hat{S}(Z_{t_{j-1}}, \theta_3)^{-1} - \hat{S}(Z_{t_{j-1}}, \theta_3^*)^{-1}) [\mathcal{D}_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3^*)^{\otimes 2}] \\ &\quad - \frac{1}{2} \sum_{j=1}^n \log \frac{\det \hat{S}(Z_{t_{j-1}}, \theta_3)}{\det \hat{S}(Z_{t_{j-1}}, \theta_3^*)}\end{aligned}\tag{4.4}$$

By Lemma 3.3 (b), Lemma 3.5 (b) and Lemma 4.1, we obtain

$$n^{-1}h \sup_{\theta_3 \in \Theta_3} |R_n^{(4.4)}(\theta_3)| = O_p(h).$$

By definition,

$$h^{1/2} \delta_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3) = \begin{pmatrix} 0 \\ H(Z_{t_{j-1}}, \theta_3) - H(Z_{t_{j-1}}, \theta_3^*) + \frac{h}{2} (L_H(Z_{t_{j-1}}, \hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3) - L_H(Z_{t_{j-1}}, \hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3^*)) \end{pmatrix}$$

Since the functions $A(z, \theta_2)$, $H(z, \theta_3)$ and $L_H(z, \theta_1, \theta_2, \theta_3)$ are dominated by a polynomial in z uniformly in θ , by using the above formula, it is easy to show

$$\sup_{\theta_3 \in \Theta_3} \left| \mathbb{Y}_n^{(3)}(\theta_3) - \mathbb{Y}_n^{(4.6)}(\hat{\theta}_1^0, \theta_3) \right| = O_p(h^{1/2})\tag{4.5}$$

for

$$\mathbb{Y}_n^{(4.6)}(\theta_1, \theta_3) = -\frac{1}{2n} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ H(Z_{t_{j-1}}, \theta_3) - H(Z_{t_{j-1}}, \theta_3^*) \end{pmatrix}^{\otimes 2} \right].\tag{4.6}$$

The derivative $\partial_1 S_x(z, \theta_1, \theta_3)$ is dominated by a polynomial in z uniformly in θ . Therefore

$$\sup_{\theta_3 \in \Theta_3} \left| \mathbb{Y}_n^{(4.6)}(\hat{\theta}_1^0, \theta_3) - \mathbb{Y}_n^{(4.6)}(\theta_1^*, \theta_3) \right| \rightarrow^p 0.\tag{4.7}$$

Finally, the estimate (3.3) gives

$$\sup_{\theta_3 \in \Theta_3} \left| \mathbb{Y}_n^{(4.6)}(\theta_1^*, \theta_3) + \frac{1}{2nh} \int_0^{nh} S(Z_t, \theta_1^*, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ H(Z_t, \theta_3) - H(Z_t, \theta_3^*) \end{pmatrix}^{\otimes 2} \right] dt \right| \rightarrow^p 0.\tag{4.8}$$

Now (4.3) follows from (4.5), (4.7), (4.8) and [A2] (ii) since $\partial_3^i H(z, \theta_1, \theta_3)$ ($i = 0, 1$) are dominated by a polynomial in z uniformly in θ_3 . Then the convergence $\hat{\theta}_3^0 \rightarrow^p \theta_3$ as $n \rightarrow \infty$ is obvious under Condition [A3] (iii). $\square$

4.3 Asymptotic normality of $\hat{\theta}_3^0$

Let

$$M_n^{(3)} = n^{-1/2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1^*, \theta_3^*)^{-1} \left[\begin{pmatrix} h^{-1/2} B(Z_{t_{j-1}}, \theta_2^*) \Delta_j w \\ h^{-3/2} \kappa(Z_{t_{j-1}}, \theta_1^*, \theta_3^*) \zeta_j \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3^*) \end{pmatrix} \right].$$

Let

$$\begin{aligned} \Gamma_{33} &= \int S(z, \theta_1^*, \theta_3^*)^{-1} \left[\begin{pmatrix} 0 \\ \partial_3 H(z, \theta_3^*) \end{pmatrix}^{\otimes 2} \right] \nu(dz) \\ &= \int 12V(z, \theta_1^*, \theta_3^*)^{-1} [(\partial_3 H(z, \theta_3^*))^{\otimes 2}] \nu(dz) \\ &= \int 12\partial_3 H(z, \theta_3^*)^* V(z, \theta_1^*, \theta_3^*)^{-1} \partial_3 H(z, \theta_3^*) \nu(dz). \end{aligned} \quad (4.9)$$

Lemma 4.3. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 1)$, [A2], [A3] (iii) and [A4] are satisfied. Then*

$$n^{-1/2} h^{1/2} \partial_3 \mathbb{H}_n^{(3)}(\theta_3^*) - M_n^{(3)} = o_p(1)$$

as $n \rightarrow \infty$.

Proof. From [A3] (iii), Γ_{33} is non-singular. From (4.2) and (3.8), we have

$$n^{-1/2} h^{1/2} \partial_3 \mathbb{H}_n^{(3)}(\theta_3^*) = R_n^{(4.11)}(\hat{\theta}_1^0, \hat{\theta}_2^0) + R_n^{(4.12)}(\hat{\theta}_1^0, \hat{\theta}_2^0) + R_n^{(4.13)}(\hat{\theta}_1^0, \hat{\theta}_2^0) \quad (4.10)$$

where

$$R_n^{(4.11)}(\hat{\theta}_1^0, \hat{\theta}_2^0) = n^{-1/2} \Psi_{3,1}(\hat{\theta}_1^0, \theta_3^*, \hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3^*), \quad (4.11)$$

$$R_n^{(4.12)}(\hat{\theta}_1^0, \hat{\theta}_2^0) = n^{-1/2} \Psi_{3,2}(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3^*, \hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3^*) \quad (4.12)$$

and

$$R_n^{(4.13)}(\hat{\theta}_1^0, \hat{\theta}_2^0) = n^{-1/2} h^{1/2} \Psi_{3,3}(\hat{\theta}_1^0, \theta_3^*, \hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3^*). \quad (4.13)$$

We have

$$\mathcal{D}_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3^*) - \mathcal{D}_j(\hat{\theta}_1^0, \theta_2^*, \theta_3^*) = -h^{1/2} \left[\begin{array}{c} (A(Z_{t_{j-1}}, \hat{\theta}_2^0) - A(Z_{t_{j-1}}, \theta_2^*)) \\ 2^{-1} H_x(Z_{t_{j-1}}, \theta_3^*) [A(Z_{t_{j-1}}, \hat{\theta}_2^0) - A(Z_{t_{j-1}}, \theta_2^*)] \end{array} \right],$$

and so only by algebraic computation we obtain

$$\hat{S}(Z_{t_{j-1}}, \theta_3^*)^{-1} \left[\mathcal{D}_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3^*) - \mathcal{D}_j(\hat{\theta}_1^0, \theta_2^*, \theta_3^*), \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3^*) \end{pmatrix} \right] = 0. \quad (4.14)$$

Applying Lemma 3.5 (b) under [A4], and next using the results in Lemmas 3.4 and 3.5, we see

$$\begin{aligned} R_n^{(4.11)}(\hat{\theta}_1^0, \hat{\theta}_2^0) &= n^{-1/2} \Psi_{3,1}(\hat{\theta}_1^0, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(h^{1/2}) \\ &= n^{-1/2} \tilde{\Psi}_{3,1}(\hat{\theta}_1^0, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1) \end{aligned} \quad (4.15)$$

since $(nh^2)^{1/2} = o(1)$. Consider the random field

$$\Phi_n^{(4.16)}(u_1) = n^{-1/2} \{ \tilde{\Psi}_{3,1}(\theta_1^* + r_n u_1, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) - \tilde{\Psi}_{3,1}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) \} \quad (4.16)$$

on $\{u_1 \in \mathbb{R}^{p_1}; |u_1| < 1\}$ for any sequence of positive numbers $r_n \rightarrow 0$, Sobolev's inequality gives

$$\sup_{u_1: |u_1| < 1} |\Phi_n^{(4.16)}(u_1)| = o_p(1)$$

with the help of orthogonality. In particular,

$$R_n^{(4.11)}(\hat{\theta}_1^0, \hat{\theta}_2^0) = n^{-1/2} \tilde{\Psi}_{3,1}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1). \quad (4.17)$$

This implies

$$R_n^{(4.11)}(\hat{\theta}_1^0, \hat{\theta}_2^0) = M_n^{(3)} + o_p(1).$$

Simpler is that $R_n^{(4.12)}(\hat{\theta}_1^0, \hat{\theta}_2^0) = O_p(n^{1/2}h)$. Similarly,

$$\begin{aligned} R_n^{(4.13)}(\hat{\theta}_1^0, \hat{\theta}_2^0) &= n^{-1/2} h^{1/2} \Psi_{3,3}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(h^{1/2}) \\ &= O_p(h^{1/2}). \end{aligned}$$

Thus, we obtained the result. $\square$

In what follows, we quite often use the estimates in Lemma 4.1 without mentioning it explicitly.

Lemma 4.4. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 2)$, [A2] and [A4] are satisfied. Then*

$$\sup_{\theta_3 \in B_n} |n^{-1} h \partial_3^2 \mathbb{H}_n^{(3)}(\theta_3) + \Gamma_{33}| \rightarrow^p 0$$

for any sequence of balls B_n in $\mathbb{R}^{p_3}$ shrinking to θ_3^* .

Proof. We have

$$\begin{aligned} n^{-1} h \partial_3^2 \mathbb{H}_n^{(3)}(\theta_3) &= n^{-1} \Psi_{33,1}(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3) \\ &\quad + n^{-1} h^{1/2} \Psi_{33,2}(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3, \hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3) \\ &\quad + n^{-1} h \Psi_{33,3}(\hat{\theta}_1^0, \theta_3) \\ &\quad + n^{-1} h^{1/2} \Psi_{33,4}(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3, \hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3) \\ &\quad + n^{-1} h \Psi_{33,5}(\hat{\theta}_1^0, \theta_3, \hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3). \end{aligned}$$

For $\mathcal{D}_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3)$ in the above expression, we use Lemma 3.5 (b) to replace $\hat{\theta}_i^0$ by θ_i^* for $i = 1, 2$, and Lemma 3.6 to replace $\theta_3 \in B_n$ by θ_3^* with an error uniform in $\theta_3 \in B_n$. Next we use Lemma 3.3 (b). Then

$$n^{-1}h \partial_3^2 \mathbb{H}_n^{(3)}(\theta_3) = -n^{-1} \sum_{j=1}^n \hat{S}(Z_{t_{j-1}}, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) \end{pmatrix}^{\otimes 2} \right] + r_n^{(4.18)}(\theta_3)$$

where

$$\sup_{\theta_3 \in \bar{\Theta}_3} |r_n^{(4.18)}(\theta_3)| = o_p(1). \quad (4.18)$$

Now we obtain the result by using [A2] and estimating the functions $\partial_3 S$ and $\partial_3^2 H$ uniformly in (θ_1, θ_3) . $\square$

Theorem 4.5. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 2)$, [A2], [A3] (iii) and [A4] are satisfied. Then*

$$n^{1/2}h^{-1/2}(\hat{\theta}_3^0 - \theta_3^*) - \Gamma_{33}^{-1}M_n^{(3)} \rightarrow^p 0$$

as $n \rightarrow \infty$. In particular,

$$n^{1/2}h^{-1/2}(\hat{\theta}_3^0 - \theta_3^*) \rightarrow^d N(0, \Gamma_{33}^{-1})$$

as $n \rightarrow \infty$.

Proof. Use Lemmas 4.3 and 4.4. $\square$

Remark 4.6. It is possible to construct a QMLE $\hat{\vartheta}_3$ for θ_3 based on the quasi-log likelihood function

$$\begin{aligned} \mathcal{H}_n^{(3)}(\theta_3) = & -\frac{1}{2} \sum_{j=1}^n \left\{ 3V(Z_{t_{j-1}}, \hat{\theta}_1^0, \theta_3) [\{h^{-3/2}(\Delta_j Y - hG_n(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3))\}^{\otimes 2}] \right. \\ & \left. + \log(3^{-1}V(Z_{t_{j-1}}, \hat{\theta}_1^0, \theta_3)) \right\}. \end{aligned}$$

Then, under a certain set of conditions, we have

$$n^{1/2}h^{-1/2}(\hat{\vartheta}_3 - \theta_3^*) \rightarrow^d N(0, 4\Gamma_{33}^{-1}).$$

Therefore $\hat{\theta}_3^0$ is superior to $\hat{\vartheta}_3$.

5 Adaptive one-step estimator for $(\theta_1, \theta_2, \theta_3)$

In this section, we will consider a one-step estimator for $\theta = (\theta_1, \theta_2, \theta_3)$ given an initial estimators $(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0)$ for $(\theta_1, \theta_2, \theta_3)$ based on $(Z_{t_j})_{j=0,1,\dots,n}$. We will assume the following rate of convergence for each initial estimator

[A4[#]] (i) $\hat{\theta}_1^0 - \theta_1^* = O_p(n^{-1/2})$ as $n \rightarrow \infty$

(ii) $\hat{\theta}_2^0 - \theta_2^* = O_p(n^{-1/2}h^{-1/2})$ as $n \rightarrow \infty$

(iii) $\hat{\theta}_3^0 - \theta_3^* = O_p(n^{-1/2}h^{1/2})$ as $n \rightarrow \infty$.

The initial estimator $\hat{\theta}_3^0$ is not necessarily the one defined in Section 4, though we already know that one satisfies [A4[#]] (iii). That is, the initial estimator $\hat{\theta}_3^0$ used in this section is requested to attain the convergence rate $n^{-1/2}h^{1/2}$ only, not to necessarily achieve the asymptotic variance equal to Γ_{33}^{-1} or less. Thus, the estimator $\hat{\vartheta}_3$ mentioned by Remark 4.6, as well as $\hat{\theta}_3^0$ in Section 4, can serve as the initial estimator of θ_3 . As Section 7 recalls a construction of the initial estimator $\hat{\theta}_1^0$, in estimation of non-degenerate diffusion processes, there is an estimator of θ_1 satisfying Condition [A4[#]] (i) based on only the first equation of (1.1). It is known that its information cannot be greater than the matrix

$$\frac{1}{2} \int \text{Tr}\{ (C^{-1}(\partial_1 C)C^{-1}\partial_1 C)(z, \theta_1^*) \} \nu(dz).$$

It will be turned out that the amount of information is increased by the one-step estimator. We will recall a standard construction of $\hat{\theta}_2^0$ in Section 8.

Let

$$M_n^{(1)} = \frac{1}{2}n^{-1/2} \sum_{j=1}^n (S^{-1}(\partial_1 S)S^{-1})(Z_{t_{j-1}}, \theta_1^*, \theta_3^*) [\tilde{\mathcal{D}}_j(\theta_1^*, \theta_2^*, \theta_3^*)^{\otimes 2} - S(Z_{t_{j-1}}, \theta_1^*, \theta_3^*)].$$

Let

$$\begin{aligned} \Gamma_{11} &= \frac{1}{2} \int \text{Tr}\{ S^{-1}(\partial_1 S)S^{-1}\partial_1 S(z, \theta_1^*, \theta_3^*) \} \nu(dz) \\ &= \frac{1}{2} \int \left[\text{Tr}\{ (C^{-1}(\partial_1 C)C^{-1}\partial_1 C)(z, \theta_1^*) \} \right. \\ &\quad \left. + \text{Tr}\{ (V^{-1}H_x(\partial_1 C)H_x^*V^{-1}H_x(\partial_1 C)H_x^*)(z, \theta_1^*, \theta_3^*) \} \right] \nu(dz). \end{aligned}$$

If H_x is an invertible (square) matrix, then Γ_{11} coincides with

$$\int \text{Tr}\{ (C^{-1}(\partial_1 C)C^{-1}\partial_1 C)(z, \theta_1^*) \} \nu(dz).$$

Otherwise, it is not always true.

Let

$$\begin{aligned} \Gamma_{22} &= \int S(z, \theta_1^*, \theta_3^*)^{-1} \left[\begin{pmatrix} \partial_2 A(z, \theta_2^*) \\ 2^{-1}\partial_2 L_H(z, \theta_1^*, \theta_2^*, \theta_3^*) \end{pmatrix}^{\otimes 2} \right] \nu(dz) \\ &= \int \partial_2 A(z, \theta_2^*)^* C(z, \theta_1^*)^{-1} \partial_2 A(z, \theta_2^*) \nu(dz). \end{aligned} \tag{5.1}$$

Let $\Gamma^J(\theta^*) = \text{diag}[\Gamma_{11}, \Gamma_{22}, \Gamma_{33}]$, where Γ_{33} is defined by (4.9).

We will use the following random fields:

$$\mathbb{H}_n^{(1)}(\theta_1) = -\frac{1}{2} \sum_{j=1}^n \left\{ S(Z_{t_{j-1}}, \theta_1, \hat{\theta}_3^0)^{-1} [\mathcal{D}_j(\theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0)^{\otimes 2}] + \log \det S(Z_{t_{j-1}}, \theta_1, \hat{\theta}_3^0) \right\}. \quad (5.2)$$

and

$$\mathbb{H}_n^{(2,3)}(\theta_2, \theta_3) = -\frac{1}{2} \sum_{j=1}^n \hat{S}(Z_{t_{j-1}}, \hat{\theta}_3^0)^{-1} [\mathcal{D}_j(\hat{\theta}_1^0, \theta_2, \theta_3)^{\otimes 2}]. \quad (5.3)$$

Recall $\hat{S}(z, \theta_3) = S(z, \hat{\theta}_1^0, \theta_3)$. To construct one-step estimators, we consider the functions

$$\mathbb{E}_n(\theta_1) = \theta_1 - [\partial_1^2 \mathbb{H}_n^{(1)}(\theta_1)]^{-1} \partial_1 \mathbb{H}_n^{(1)}(\theta_1)$$

and

$$\mathbb{F}_n(\theta_2, \theta_3) = \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} - [\partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3)]^{-1} \partial_{(\theta_2, \theta_3)} \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3)$$

when both matrices $\partial_1^2 \mathbb{H}_n^{(1)}(\theta_1)$ and $\partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3)$ are invertible. Let

$$\mathcal{X}_n^{(1)} = \{ \omega \in \Omega; \partial_1^2 \mathbb{H}_n^{(1)}(\hat{\theta}_1^0) \text{ is invertible and } \mathbb{E}_n(\hat{\theta}_1^0) \in \Theta_1 \}$$

and

$$\mathcal{X}_n^{(2,3)} = \{ \omega \in \Omega; \partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\hat{\theta}_2^0, \hat{\theta}_3^0) \text{ is invertible and } \mathbb{F}_n(\hat{\theta}_2^0, \hat{\theta}_3^0) \in \Theta_2 \times \Theta_3 \}$$

Let $\mathcal{X}_n = \mathcal{X}_n^{(1)} \cap \mathcal{X}_n^{(2,3)}$. The event $\mathcal{X}_n$ is a statistic because it is determined by the data $(Z_{t_j})_{j=0, \dots, n}$ only. For $(\theta_1, \theta_2, \theta_3)$, the one-step estimator $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ with the initial estimator $(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0)$ is defined by

$$\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \end{pmatrix} = \begin{cases} \begin{pmatrix} \mathbb{E}_n(\hat{\theta}_1^0) \\ \mathbb{F}_n(\hat{\theta}_2^0, \hat{\theta}_3^0) \end{pmatrix} & \text{on } \mathcal{X}_n \\ v & \text{on } \mathcal{X}_n^c \end{cases}$$

where v is an arbitrary value in Θ .

Let $\hat{\gamma} = (\hat{\theta}_2, \hat{\theta}_3)^*$, $\hat{\gamma}^0 = (\hat{\theta}_2^0, \hat{\theta}_3^0)^*$ and $\gamma^* = (\theta_2^*, \theta_3^*)^*$. Let U be an open ball in $\mathbb{R}^{p_2 + p_3}$ centered at γ^* such that $U \subset \Theta_2 \times \Theta_3$. Let $\mathcal{X}_n^{*(2,3)} = \mathcal{X}_n^{(2,3)} \cap \{\hat{\gamma}^0 \in U\}$.

Lemma 5.1. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 2, 2, 1, 3, 1)$, [A2] (i), (iii), (iv) and [A4[#]] are satisfied. Then*

$$n^{-1/2} h^{-1/2} \partial_2 \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0) = O_p(1)$$

as $n \rightarrow \infty$.

Proof. By using Lemma 3.6 and Lemma 3.5 (b) together with the convergence rate of the initial estimators, we have

$$\begin{aligned} n^{-1/2} h^{-1/2} \partial_2 \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0) &= n^{-1/2} \Psi_2(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0) \\ &= n^{-1/2} \Psi_2(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(1) \\ &= n^{-1/2} \tilde{\Psi}_2(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(1) \end{aligned}$$

by Lemma 3.4 and Lemma 3.5 (a).

The open ball of radius r centered at θ is denoted by $U(\theta, r)$. Define the random field

$$\Phi_n^{(5.4)}(\theta) = n^{-1/2} \tilde{\Psi}_2(\theta_1, \theta_2, \theta_3, \theta_1^*, \theta_2^*, \theta_3^*) \quad (5.4)$$

on $\theta = (\theta_1, \theta_2, \theta_3) \in U(\theta^*, r)$ for a small number r such that $U(\theta^*, r) \subset \Theta$. With the Burkholder-Davis-Gundy inequality and in particular twice differentiability of A in θ_2 , we obtain

$$\sup_n \sum_{i=0,1} \sup_{\theta \in U(\theta^*, r)} \|\partial_\theta^i \Phi_n^{(5.4)}(\theta)\|_p < \infty$$

for every $p > 1$. Therefore, Sobolev's inequality ensures

$$\sup_n \left\| \sup_{\theta \in U(\theta^*, r)} |\Phi_n^{(5.4)}(\theta)| \right\|_p < \infty$$

Consequently,

$$\Phi_n^{(5.4)}(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0) 1_{\{(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0) \in U(\theta^*, r)\}} = O_p(1).$$

This completes the proof. $\square$

Lemma 5.2. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 2)$, [A2] (i), (iii), (iv) and [A4 $^\sharp$] are satisfied. Then

$$n^{-1/2} h^{1/2} \partial_3 \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0) = O_p(1)$$

as $n \rightarrow \infty$.

Proof. The proof is similar to that of Lemma 5.1. First,

$$\begin{aligned} n^{-1/2} h^{1/2} \partial_3 \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0) &= n^{-1/2} \Psi_3(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0) \\ &= n^{-1/2} \tilde{\Psi}_3(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(1). \end{aligned}$$

Then we can show the lemma in the same fashion as Lemma 5.1 with a random field. $\square$

Let

$$B_n = U(\theta_1^*, n^{-1/2} \log(nh)) \times U(\theta_2^*, (nh)^{-1/2} \log(nh)) \times U(\theta_3^*, n^{-1/2} h^{1/2} \log(nh)),$$

$$B'_n = U(\theta_2^*, (nh)^{-1/2} \log(nh)) \times U(\theta_3^*, n^{-1/2} h^{1/2} \log(nh))$$

and

$$B_n'' = U(\theta_1^*, n^{-1/2} \log(nh)) \times U(\theta_3^*, n^{-1/2} h^{1/2} \log(nh)).$$

We will use the following random fields.

$$\Phi_{22,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = - \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta'_2) \\ 2^{-1} \partial_2 L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3) \end{pmatrix}^{\otimes 2} \right]$$

$$\begin{aligned} & \Phi_{22,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_2, \theta''_3) \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2^2 A(Z_{t_{j-1}}, \theta''_2) \\ 2^{-1} H_x(Z_{t_{j-1}}, \theta''_3) [\partial_2^2 A(Z_{t_{j-1}}, \theta''_2)] \end{pmatrix} \right] \end{aligned}$$

$$\begin{aligned} & \tilde{\Phi}_{22,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_2, \theta''_3) \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2^2 A(Z_{t_{j-1}}, \theta''_2) \\ 2^{-1} H_x(Z_{t_{j-1}}, \theta''_3) [\partial_2^2 A(Z_{t_{j-1}}, \theta''_2)] \end{pmatrix} \right] \end{aligned}$$

$$\begin{aligned} & \Phi_{23,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_2, \theta''_3) \\ &= - \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ 2^{-1} \partial_3 L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3) \end{pmatrix} \right. \\ & \quad \left. \otimes \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta''_2) \\ 2^{-1} H_x(Z_{t_{j-1}}, \theta''_3) [\partial_2 A(Z_{t_{j-1}}, \theta''_2)] \end{pmatrix} \right] \end{aligned}$$

$$\begin{aligned} & \Phi_{23,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_2, \theta''_3) \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1} \partial_3 H_x(Z_{t_{j-1}}, \theta''_3) [\partial_2 A(Z_{t_{j-1}}, \theta''_2)] \end{pmatrix} \right] \end{aligned}$$

$$\Phi_{33,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = - \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta'_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3) \end{pmatrix}^{\otimes 2} \right]$$

$$\begin{aligned} \Phi_{33,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_1, \theta''_2, \theta''_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3) \right. \\ & \quad \left. \otimes \begin{pmatrix} 0 \\ \partial_3^2 H(Z_{t_{j-1}}, \theta''_3) + 2^{-1} h \partial_3^2 L_H(Z_{t_{j-1}}, \theta''_1, \theta''_2, \theta''_3) \end{pmatrix} \right] \end{aligned}$$

Lemma 5.3. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 1, 3, 1)$, [A2] and [A4[#]] are satisfied. Then

$$\sup_{(\theta_2, \theta_3) \in B'_n} |n^{-1}h^{-1} \partial_2^2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3) + \Gamma_{22}| \rightarrow^p 0$$

as $n \rightarrow \infty$.

Proof. We have

$$\begin{aligned} n^{-1}h^{-1} \partial_2^2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3) &= n^{-1}\Phi_{22,1}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \theta_2, \theta_3) \\ &\quad + n^{-1}h^{-1/2}\Phi_{22,2}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \theta_2, \theta_3, \theta_2, \theta_3) \end{aligned} \quad (5.5)$$

Apply Lemma 3.6 and Lemma 3.5 (b) to obtain

$$\begin{aligned} &\sup_{(\theta_1, \theta_3) \in B''_n} \sup_{(\theta'_1, \theta'_2, \theta'_3) \in B_n} \sup_{(\theta''_2, \theta''_3) \in B'_n} |n^{-1}h^{-1/2}\Phi_{22,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_2, \theta''_3) \\ &\quad - n^{-1}h^{-1/2}\Phi_{22,2}(\theta_1, \theta_3, \theta_1^*, \theta_2^*, \theta_3^*, \theta''_2, \theta''_3)| \\ &= o_p(1). \end{aligned} \quad (5.6)$$

Here we used the assumption that the functions are bound by a polynomial in z uniformly in the parameters, and the count

$$n^{-1}h^{-1/2} \times n \times h^{-1/2} \times n^{-1/2}h^{1/2} \log(nh) = \frac{\log(nh)}{\sqrt{nh}}$$

to estimate the error when replacing θ'_3 by θ_3^* , as well a similar count when replacing (θ'_1, θ'_2) by (θ_1^*, θ_2^*) .

We apply Lemmas 3.4 and 3.5 (a) to obtain

$$\begin{aligned} &\sup_{(\theta_1, \theta_3) \in B''_n} \sup_{(\theta''_2, \theta''_3) \in B'_n} |n^{-1}h^{-1/2}\Phi_{22,2}(\theta_1, \theta_3, \theta_1^*, \theta_2^*, \theta_3^*, \theta''_2, \theta''_3) \\ &\quad - n^{-1}h^{-1/2}\tilde{\Phi}_{22,2}(\theta_1, \theta_3, \theta_1^*, \theta_2^*, \theta_3^*, \theta''_2, \theta''_3)| \\ &= O_p((nh)^{-1/2} \log(nh)) = o_p(1). \end{aligned} \quad (5.7)$$

Since $\tilde{\mathcal{D}}_j(\theta_1^*, \theta_2^*, \theta_3^*)$ in $\tilde{\Phi}_{22,2}$ are martingale differences with respect to a suitable filtration, we can conclude by the random field argument with the Sobolev space of index $(1, p)$, $p > 1$, that

$$\sup_{(\theta_1, \theta_3) \in B''_n} \sup_{(\theta''_2, \theta''_3) \in B'_n} |n^{-1}h^{-1/2}\tilde{\Phi}_{22,2}(\theta_1, \theta_3, \theta_1^*, \theta_2^*, \theta_3^*, \theta''_2, \theta''_3)| = O_p((nh)^{-1/2}) = o_p(1) \quad (5.8)$$

On the other hand,

$$\sup_{(\theta_1, \theta_3) \in B''_n} \sup_{(\theta'_1, \theta'_2, \theta'_3) \in B_n} |n^{-1}\Phi_{22,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) - n^{-1}\Phi_{22,1}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*)| = o_p(1) \quad (5.9)$$

From (5.5)-(5.9) and [A4[#]] (i), (iii), we obtain

$$\sup_{(\theta_2, \theta_3) \in B'_n} |n^{-1} h^{-1} \partial_2^2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3) - n^{-1} \Phi_{22,1}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*)| = o_p(1). \quad (5.10)$$

Now the assertion of the lemma is easy to obtain if one uses [A1], [A2] and Lemma 3.1. $\square$

Let

$$\begin{aligned} i(z, \theta) &= \begin{pmatrix} \partial_2 A(z, \theta_2)^* & 2^{-1} \partial_2 L_H(z, \theta_1, \theta_2, \theta_3)^* \\ O & \partial_3 H(z, \theta_3)^* \end{pmatrix} S(z, \theta_1, \theta_3)^{-1} \\ &\quad \times \begin{pmatrix} \partial_2 A(z, \theta_2) & O \\ 2^{-1} \partial_2 L_H(z, \theta_1, \theta_2, \theta_3) & \partial_3 H(z, \theta_3) \end{pmatrix}. \end{aligned} \quad (5.11)$$

Then simple calculus with (4.1) and

$$\partial_2 L_H(z, \theta_1, \theta_2, \theta_3) = H_x(z, \theta_3) [\partial_2 A(z, \theta_2)]$$

yield

$$i(z, \theta) = \begin{pmatrix} \partial_2 A(z, \theta_2)^* C(z, \theta_1)^{-1} \partial_2 A(z, \theta_2) & O \\ O & 12 \partial_3 H(z, \theta_3)^* V(z, \theta_1, \theta_3)^{-1} \partial_3 H(z, \theta_3) \end{pmatrix}. \quad (5.12)$$

Lemma 5.4. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 1)$ and [A2] are satisfied. Then*

$$\sup_{(\theta_2, \theta_3) \in B'_n} |n^{-1} \partial_3 \partial_2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3)| \xrightarrow{p} 0$$

as $n \rightarrow \infty$.

Proof. From (5.11) and (5.12), we see

$$S(z, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ \partial_3 H(z, \theta_3) \end{pmatrix}, \begin{pmatrix} \partial_2 A(z, \theta_2) \\ 2^{-1} \partial_2 L_H(z, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right] = 0.$$

Then, by definition,

$$\begin{aligned} n^{-1} \partial_3 \partial_2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3) &= n^{-1} h \Phi_{23,1}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \theta_2, \theta_3, \theta_2, \theta_3) \\ &\quad + n^{-1} h^{1/2} \Phi_{23,2}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \theta_2, \theta_3, \theta_2, \theta_3). \end{aligned}$$

Now it is not difficult to show the desired result. $\square$

Lemma 5.5. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 2)$ and [A2] are satisfied. Then*

$$\sup_{(\theta_2, \theta_3) \in B'_n} |n^{-1} h \partial_3^2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3) + \Gamma_{33}| \xrightarrow{p} 0$$

as $n \rightarrow \infty$.

Proof. By definition,

$$\begin{aligned} n^{-1}h \partial_3^2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3) &= n^{-1} \Phi_{33,1}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \theta_2, \theta_3) \\ &\quad + n^{-1}h^{1/2} \Phi_{33,2}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \theta_2, \theta_3, \hat{\theta}_1^0, \theta_2, \theta_3). \end{aligned}$$

$\Phi_{33,1}$ involves the first derivative ∂_3 , and $\Phi_{33,2}$ does the second derivative ∂_3^2 . First applying Lemma 3.6 and Lemma 3.5 (b), and next Lemma 3.3 (b), we have

$$\begin{aligned} &\sup_{(\theta_2, \theta_3) \in B'_n} |n^{-1}h^{1/2} \Phi_{33,2}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \theta_2, \theta_3, \hat{\theta}_1^0, \theta_2, \theta_3)| \\ &\leq \sup_{(\theta_2, \theta_3) \in B'_n} |n^{-1}h^{1/2} \Phi_{33,2}(\hat{\theta}_1^0, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*, \hat{\theta}_1^0, \theta_2, \theta_3)| + O_p(n^{-1/2}h^{1/2} \log(nh)) \\ &= O_p(h^{1/2}). \end{aligned}$$

Moreover, it is easy to show

$$\sup_{(\theta_2, \theta_3) \in B'_n} |n^{-1} \Phi_{33,1}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \theta_2, \theta_3) + \Gamma_{33}| \rightarrow^p 0$$

from [A1], [A2] with the aid of Lemma 3.1. $\square$

Let

$$a_n = \begin{pmatrix} n^{-1/2}h^{-1/2} & 0 \\ 0 & n^{-1/2}h^{1/2} \end{pmatrix}.$$

Lemma 5.6. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 1, 3, 2)$ and [A2] are satisfied. Then*

$$\sup_{(\theta_2, \theta_3) \in B'_n} |a_n \partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\theta_2, \theta_3) a_n + \Gamma^{(2,3)}(\theta^*)| \rightarrow^p 0 \quad (5.13)$$

where

$$\Gamma^{(2,3)}(\theta^*) = \begin{pmatrix} \Gamma_{22} & O \\ O & \Gamma_{33} \end{pmatrix}.$$

Proof. The convergence (5.13) follows from Lemmas 5.3, 5.4 and 5.5. $\square$

Lemma 5.7. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 1, 3, 2)$, [A2] and [A4 $^\sharp$] are satisfied. Then $P[\mathcal{X}_n^{*(2,3)}] \rightarrow 1$ as $n \rightarrow \infty$.*

Proof. By Lemmas 5.1 and 5.2,

$$a_n \partial_{(\theta_2, \theta_3)} \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0) = O_p(1)$$

and by Lemma 5.6,

$$(a_n \partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0) a_n)^{-1} = O_p(1).$$

Therefore,

$$(\partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0))^{-1} \partial_{(\theta_2, \theta_3)} \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0) = O_p((nh)^{-1/2})$$

as $n \rightarrow \infty$. This means $P[\mathcal{X}_n^{*(2,3)}] \rightarrow 1$. $\square$

Let

$$\begin{aligned} M_n^{(2)} &= n^{-1/2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1^*, \theta_3^*)^{-1} \left[\begin{pmatrix} h^{-1/2} B(Z_{t_{j-1}}, \theta_1^*) \Delta_j w \\ h^{-3/2} \kappa(Z_{t_{j-1}}, \theta_1^*, \theta_3^*) \zeta_j \end{pmatrix}, \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2^*) \\ 2^{-1} \partial_2 L_H(Z_{t_{j-1}}, \theta_1^*, \theta_2^*, \theta_3^*) \end{pmatrix} \right] \\ &= n^{-1/2} \sum_{j=1}^n C(Z_{t_{j-1}}, \theta_1^*)^{-1} [h^{-1/2} B(Z_{t_{j-1}}, \theta_1^*) \Delta_j w, \partial_2 A(Z_{t_{j-1}}, \theta_2^*)]. \end{aligned} \quad (5.14)$$

Lemma 5.8. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 1)$, [A2] and [A4] are satisfied. Then*

$$n^{-1/2} h^{-1/2} \partial_2 \mathbb{H}_n^{(2,3)}(\theta_2^*, \theta_3^*) - M_n^{(2)} \xrightarrow{p} 0$$

as $n \rightarrow \infty$.

Proof. By using Lemma 3.5 (b) together with the convergence rate of the estimators $\hat{\theta}_1^0$ and $\hat{\theta}_3^0$, and next by Lemma 3.5 (a) and Lemma 3.4, we have

$$\begin{aligned} & n^{-1/2} h^{-1/2} \partial_2 \mathbb{H}_n^{(2,3)}(\theta_2^*, \theta_3^*) \\ &= n^{-1/2} \sum_{j=1}^n \hat{S}(Z_{t_{j-1}}, \hat{\theta}_3^0)^{-1} \left[\mathcal{D}_j(\hat{\theta}_1^0, \theta_2^*, \theta_3^*), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2^*) \\ 2^{-1} \partial_2 L_H(Z_{t_{j-1}}, \hat{\theta}_1^0, \theta_2^*, \theta_3^*) \end{pmatrix} \right] \\ &= n^{-1/2} \sum_{j=1}^n \hat{S}(Z_{t_{j-1}}, \hat{\theta}_3^0)^{-1} \left[\mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2^*) \\ 2^{-1} \partial_2 L_H(Z_{t_{j-1}}, \hat{\theta}_1^0, \theta_2^*, \theta_3^*) \end{pmatrix} \right] + O_p(h^{1/2}). \\ &= n^{-1/2} \sum_{j=1}^n \hat{S}(Z_{t_{j-1}}, \theta_3^*)^{-1} \left[\tilde{\mathcal{D}}_j(\theta_1^*, \theta_2^*, \theta_3^*), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2^*) \\ 2^{-1} \partial_2 L_H(Z_{t_{j-1}}, \hat{\theta}_1^0, \theta_2^*, \theta_3^*) \end{pmatrix} \right] + O_p(\sqrt{nh}) + O_p(h^{1/2}). \end{aligned} \quad (5.15)$$

Here we used the derivative $\partial_1 H$.

We consider the random field

$$\Phi_n^{(5.16)}(u_1) = n^{-1/2} \tilde{\Psi}_2(\theta_1(u_1), \theta_2^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) \quad (5.16)$$

on $\{u_1 \in \mathbb{R}^{p_1}; |u_1| < 1\}$, where $\theta_1(u_1) = \theta_1^* + n^{-1/2}(\log n)u_1$. Then L^p -estimate of

$$\partial_1^i \{\Phi_n^{(5.16)}(u_1) - \Phi_n^{(5.16)}(0)\} \quad (i = 0, 1)$$

yields

$$\sup_{u_1 \in U(0,1)} |\Phi_n^{(5.16)}(u_1) - \Phi_n^{(5.16)}(0)| \xrightarrow{p} 0,$$

in particular,

$$\Phi_n^{(5.16)}(u_1^\dagger) - \Phi_n^{(5.16)}(0) \rightarrow^p 0$$

where $u_1^\dagger = n^{1/2}(\log n)^{-1}(\hat{\theta}_1 - \theta_1^*)$. Obviously, $M_n^{(2)} - \Phi_n^{(5.16)}(0) \rightarrow^p 0$. Since the first term on the right-hand side of (5.15) is nothing but $\Phi_n^{(5.16)}(u_1^\dagger)$ on an event the probability of which goes to 1, we have already obtained the result. $\square$

Lemma 5.9. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 1)$, [A2] and [A4] are satisfied. Then*

$$n^{-1/2}h^{1/2} \partial_3 \mathbb{H}_n^{(2,3)}(\theta_2^*, \theta_3^*) - M_n^{(3)} \rightarrow^p 0$$

as $n \rightarrow \infty$.

Proof. We have

$$\begin{aligned} n^{-1/2}h^{1/2} \partial_3 \mathbb{H}_n^{(2,3)}(\theta_2^*, \theta_3^*) &= n^{-1/2} \sum_{j=1}^n S(Z_{t_{j-1}}, \hat{\theta}_1^0, \hat{\theta}_3^0)^{-1} \left[\mathcal{D}_j(\hat{\theta}_1^0, \theta_2^*, \theta_3^*) \right. \\ &\quad \left. \otimes \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3^*) + 2^{-1}h \partial_3 L_H(Z_{t_{j-1}}, \hat{\theta}_1^0, \theta_2^*, \theta_3^*) \end{pmatrix} \right]. \end{aligned}$$

Then this lemma can be proved in the same way as Lemma 5.8. $\square$

Let

$$M_n^{(2,3)} = \begin{pmatrix} M_n^{(2)} \\ M_n^{(3)} \end{pmatrix}.$$

Combining Lemmas 5.8 and 5.9, we obtain the following lemma.

Lemma 5.10. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 1)$, [A2] and [A4[#]] are satisfied. Then*

$$a_n \partial_{(\theta_2, \theta_3)} \mathbb{H}_n^{(2,3)}(\theta_2^*, \theta_3^*) - M_n^{(2,3)} \rightarrow^p 0$$

and $M_n^{(2,3)} \rightarrow^d N(0, \Gamma^{(2,3)}(\theta^*))$ as $n \rightarrow \infty$. In particular,

$$a_n \partial_{(\theta_2, \theta_3)} \mathbb{H}_n^{(2,3)}(\theta_2^*, \theta_3^*) \rightarrow^d N(0, \Gamma^{(2,3)}(\theta^*))$$

as $n \rightarrow \infty$.

Theorem 5.11. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 1, 3, 2)$, [A2] and [A4[#]] are satisfied. Then*

$$a_n^{-1}(\hat{\gamma} - \gamma^*) - (\Gamma^{(2,3)}(\theta^*))^{-1} M_n^{(2,3)} \rightarrow^p 0 \quad (5.17)$$

as $n \rightarrow \infty$. In particular,

$$a_n^{-1}(\hat{\gamma} - \gamma^*) \rightarrow^d N(0, (\Gamma^{(2,3)}(\theta^*))^{-1}) \quad (5.18)$$

as $n \rightarrow \infty$.

Proof. Let

$$\mathcal{X}_n^{**(2,3)} = \mathcal{X}_n^{*(2,3)} \cap \{(\hat{\theta}_1^0, \hat{\gamma}^0) \in B_n\} \cap \left\{ \sup_{\gamma \in B'_n} |a_n \partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\gamma) a_n + \Gamma^{(2,3)}(\theta^*)| < c \right\}.$$

Here c is a positive constant and we will make it sufficiently small. Then $P[\mathcal{X}_n^{**}] \rightarrow 1$ thanks to Lemmas 5.7 and 5.6. On the event $\mathcal{X}_n^{**(2,3)}$, we apply Taylor's formula to obtain

$$\begin{aligned} & a_n^{-1}(\hat{\gamma} - \gamma^*) \\ &= [a_n \partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0) a_n]^{-1} \left\{ -a_n \partial_{(\theta_2, \theta_3)} \mathbb{H}_n^{(2,3)}(\gamma^*) \right. \\ & \quad \left. + a_n \int_0^1 [\partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\hat{\gamma}^0) - \partial_{(\theta_2, \theta_3)}^2 \mathbb{H}_n^{(2,3)}(\hat{\gamma}(u))] du a_n a_n^{-1}(\hat{\gamma}^0 - \gamma^*) \right\} \end{aligned}$$

where $\hat{\gamma}(u) = \gamma^* + u(\hat{\gamma}^0 - \gamma^*)$. Then Lemmas 5.6 and 5.10 give (5.17). Then the martingale central limit theorem gives (5.18). $\square$

Let

$$b_n = \begin{pmatrix} n^{-1/2} & 0 & 0 \\ 0 & n^{-1/2} h^{-1/2} & 0 \\ 0 & 0 & n^{-1/2} h^{1/2} \end{pmatrix}.$$

The following notation for random fields will be used.

$$\begin{aligned} \Psi_{1,1}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \Psi_1(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1} \partial_1 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right] \end{aligned}$$

$$\Psi_{1,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \frac{1}{2} \sum_{j=1}^n (S^{-1}(\partial_1 S)) S^{-1}(Z_{t_{j-1}}, \theta_1, \theta_3) [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2} - S(Z_{t_{j-1}}, \theta_1, \theta_3)]$$

$$\Psi_{11,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ 2^{-1} \partial_1 L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3) \end{pmatrix}^{\otimes 2} \right]$$

$$\begin{aligned} \Psi_{11,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_1, \theta''_2, \theta''_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \right. \\ & \quad \left. \otimes \begin{pmatrix} 0 \\ 2^{-1} \partial_1^2 L_H(Z_{t_{j-1}}, \theta''_1, \theta''_2, \theta''_3) \end{pmatrix} \right] \end{aligned}$$

$$\Psi_{11,3}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \sum_{j=1}^n \partial_1 \{ S^{-1}(\partial_1 S) S^{-1}(Z_{t_{j-1}}, \theta_1, \theta_3) \} [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2} - S(Z_{t_{j-1}}, \theta'_1, \theta'_3)]$$

$$\Psi_{11,4}(\theta_1, \theta_3) = \sum_{j=1}^n (S^{-1}(\partial_1 S)S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) [\partial_1 S(Z_{t_{j-1}}, \theta_1, \theta_3)]$$

$$\begin{aligned} \Psi_{11,5}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_1, \theta''_2, \theta''_3) &= \sum_{j=1}^n (S^{-1}(\partial_1 S)S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \right. \\ &\quad \left. \otimes \begin{pmatrix} 0 \\ 2^{-1} \partial_1 L_H(Z_{t_{j-1}}, \theta''_1, \theta''_2, \theta''_3) \end{pmatrix} \right]. \end{aligned}$$

Lemma 5.12. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 3, 3, 1)$, [A2] and [A4[#]] are satisfied. Then, for any sequence of positive numbers r_n tending to 0,*

$$\sup_{\theta_1 \in U(\theta_1^*, r_n)} |n^{-1} \partial_1^2 \mathbb{H}_n^{(1)}(\theta_1) + \Gamma_{11}| \rightarrow^p 0 \quad (5.19)$$

as $n \rightarrow \infty$.

Proof. By definition,

$$\begin{aligned} n^{-1} \partial_1^2 \mathbb{H}_n^{(1)}(\theta_1) &= -n^{-1} h \Psi_{11,1}(\theta_1, \hat{\theta}_3^0, \theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0) \\ &\quad + n^{-1} h^{1/2} \Psi_{11,2}(\theta_1, \hat{\theta}_3^0, \theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0) \\ &\quad - \frac{1}{2} n^{-1} \Psi_{11,3}(\theta_1, \hat{\theta}_3^0, \theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0) \\ &\quad - \frac{1}{2} n^{-1} \Psi_{11,4}(\theta_1, \hat{\theta}_3^0) \quad (\text{this term will remain}) \\ &\quad - n^{-1} h^{1/2} \Psi_{11,5}(\theta_1, \hat{\theta}_3^0, \theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0) \end{aligned}$$

We will use Condition [A4] for $\hat{\theta}_2^0$ and $\hat{\theta}_3^0$, and the estimate $|\theta_1 - \theta_1^*| < r_n$ for $\theta_1 \in U(\theta_1^*, r_n)$.

Then

$$\begin{aligned}
& \sup_{\theta_1 \in U(\theta_1^*, r_n)} \left| n^{-1} \partial_1^2 \mathbb{H}_n^{(1)}(\theta_1) + \Gamma_{11} \right| \\
\leq & O_p(h) \\
& + n^{-1} h^{1/2} \sup_{\theta_1 \in U(\theta_1^*, r_n)} \left| \Psi_{11,2}(\theta_1, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*, \theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0) \right| + h^{1/2} O_p(n^{-1/2} + h^{1/2}) \\
& \quad (\text{Lemmas 3.6 and 3.5(b)}) \\
& + n^{-1} \sup_{\theta_1 \in U(\theta_1^*, r_n)} \left| \Psi_{11,3}(\theta_1, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*) \right| + O_p(h^{1/2} + n^{-1/2} h^{1/2}) \\
& \quad (\text{Lemmas 3.6, 3.5(b) and 3.3(b)}) \\
& + \left(-\frac{1}{2} n^{-1} \Psi_{11,4}(\theta_1^*, \theta_3^*) + \Gamma_{11} \right) + O_p(r_n) \\
& + n^{-1} h^{1/2} \sup_{\theta_1 \in U(\theta_1^*, r_n)} \left| \Psi_{11,5}(\theta_1, \hat{\theta}_3^0, \theta_1, \theta_2^*, \theta_3^*, \theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0) \right| + O_p(h^{1/2} + n^{-1/2}) \\
& \quad (\text{Lemmas 3.6 and 3.5(b)}) \\
= & O_p(h) \\
& + O_p(h^{1/2}) \quad (\text{Lemma 3.3(b)}) \\
& + O_p(n^{-1/2}) + O_p(r_n) \quad (\text{random field argument with orthogonality}) \\
& + o_p(1) \quad (\text{Lemma 3.1(a)}) \\
& + O_p(h^{1/2}) \quad (\text{Lemma 3.5(b)}) \\
= & o_p(1)
\end{aligned}$$

We remark that the used lemmas and appearing functions here require the regularity indices $(i_A, j_A, i_B, j_B, i_H, j_H)$ for [A1] as follows: $(1, 0, 1, 0, 3, 0)$ for Lemma 3.3(b); $(1, 1, 2, 1, 2, 0)$ for Lemma 3.5(b); $(0, 0, 0, 0, 2, 1)$ for Lemma 3.6; $j_B = 3, j_H = 1$ for random field argument for $\Psi_{11,3}$. $\square$

Lemma 5.13. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 2, 1)$, [A2] and [A4^{\#}] are satisfied. Then*

$$n^{-1/2} \partial_1 \mathbb{H}_n^{(1)}(\hat{\theta}_1^0) = O_p(1) \quad (5.20)$$

as $n \rightarrow \infty$.

Proof. We have the expression

$$n^{-1/2} \partial_1 \mathbb{H}_n^{(1)}(\hat{\theta}_1^0) = n^{-1/2} h^{1/2} \Psi_{1,1}(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0) + n^{-1/2} \Psi_{1,2}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0).$$

We use [A4^{\#}] together with Lemmas 3.6 and 3.5 (b) to show

$$\begin{aligned}
n^{-1/2} h^{1/2} \Psi_{1,1}(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0) &= n^{-1/2} h^{1/2} \Psi_{1,1}(\hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1) \\
&= o_p(1) = O_p(1)
\end{aligned}$$

and

$$\begin{aligned} n^{-1/2} \Psi_{1,2}(\hat{\theta}_1^0, \hat{\theta}_3^0, \hat{\theta}_1^0, \hat{\theta}_2^0, \hat{\theta}_3^0) &= n^{-1/2} \Psi_{1,2}(\hat{\theta}_1^0, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(1) \\ &= O_p(1) \end{aligned}$$

as $n \rightarrow \infty$. Here random field argument was used. $\square$

Lemma 5.14. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 1)$, [A2] and [A4[#]] are satisfied. Then*

$$n^{-1/2} \partial_1 \mathbb{H}_n^{(1)}(\theta_1^*) - M_n^{(1)} \rightarrow^p 0 \quad (5.21)$$

as $n \rightarrow \infty$. In particular,

$$n^{-1/2} \partial_1 \mathbb{H}_n^{(1)}(\theta_1^*) \rightarrow^d N(0, \Gamma_{11}) \quad (5.22)$$

as $n \rightarrow \infty$.

Proof. We have

$$\begin{aligned} \mathbb{E}_j(\theta_2, \theta_3) &:= \mathcal{D}_j(\theta_1^*, \theta_2, \theta_3) - \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*) \\ &= \left(\begin{array}{c} h^{1/2} (A(Z_{t_{j-1}}, \theta_2^*) - A(Z_{t_{j-1}}, \theta_2)) \\ h^{-1/2} (H(Z_{t_{j-1}}, \theta_3^*) - H(Z_{t_{j-1}}, \theta_3)) \\ + 2^{-1} h^{1/2} (L_H(Z_{t_{j-1}}, \theta_1^*, \theta_2^*, \theta_3^*) - L_H(Z_{t_{j-1}}, \theta_1^*, \theta_2, \theta_3)) \end{array} \right) \end{aligned}$$

Define the random field $\Xi_n(u_2, u_3)$ on $(u_2, u_3) \in U(0, 1)^2$ by

$$\begin{aligned} &\Xi_n(u_2, u_3) \\ &= n^{-1/2} \sum_{j=1}^n (S^{-1}(\partial_1 S) S^{-1})(Z_{t_{j-1}}, \theta_1^*, \theta_3^* + r_n^{(3)} u_3) \left[\mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*) \otimes \mathbb{E}_j(\theta_2^* + r_n^{(2)} u_2, \theta_3^* + r_n^{(3)} u_3) \right] \end{aligned}$$

where $r_n^{(2)} = (nh)^{-1/2} \log(nh)$ and $r_n^{(3)} = n^{-1/2} h^{1/2} \log(nh)$. Then the Burkholder-Davis-Gundy inequality gives

$$\lim_{n \rightarrow \infty} \sup_{(u_2, u_3) \in U(0, 1)^2} \sum_{i=0,1} \left\| \partial_{(u_2, u_3)}^i \Xi_n(u_2, u_3) \right\|_p = 0,$$

which implies

$$\sup_{(u_2, u_3) \in U(0, 1)^2} |\Xi_n(u_2, u_3)| \rightarrow^p 0$$

under [A4[#]], and hence

$$n^{-1/2} \sum_{j=1}^n (S^{-1}(\partial_1 S) S^{-1})(Z_{t_{j-1}}, \theta_1^*, \hat{\theta}_3^0) \left[\mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*) \otimes \mathbb{E}_j(\hat{\theta}_2^0, \hat{\theta}_3^0) \right] \rightarrow^p 0 \quad (5.23)$$

as $n \rightarrow \infty$. It is easier to see

$$n^{-1/2} \sum_{j=1}^n (S^{-1}(\partial_1 S)S^{-1})(Z_{t_{j-1}}, \theta_1^*, \hat{\theta}_3^0) [\mathbf{E}_j(\hat{\theta}_2^0, \hat{\theta}_3^0)^{\otimes 2}] \rightarrow^p 0 \quad (5.24)$$

as $n \rightarrow \infty$. From (5.23) and (5.24),

$$\begin{aligned} n^{-1/2} \Psi_{1,2}(\theta_1^*, \hat{\theta}_3^0, \theta_1^*, \hat{\theta}_2^0, \hat{\theta}_3^0) &= n^{-1/2} \Psi_{1,2}(\theta_1^*, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1) \\ &= n^{-1/2} \Psi_{1,2}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1) \end{aligned} \quad (5.25)$$

as $n \rightarrow \infty$, where the last equality is by [A4[#]].

On the other hand, by [A4[#]] and Lemmas 3.6 and 3.5 (b), we obtain

$$n^{-1/2} h^{1/2} \Psi_{1,1}(\theta_1^*, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1^*, \hat{\theta}_2^0, \hat{\theta}_3^0) = n^{-1/2} h^{1/2} \Psi_{1,1}(\theta_1^*, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1) \quad (5.26)$$

By random field argument applied to the first term on the right-hand side of (5.26),

$$n^{-1/2} h^{1/2} \Psi_{1,1}(\theta_1^*, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1^*, \hat{\theta}_2^0, \hat{\theta}_3^0) = o_p(1). \quad (5.27)$$

Consequently, from (5.25) and (5.27), we obtain the convergence (5.21) since

$$\begin{aligned} n^{-1/2} \partial_1 \mathbb{H}_n^{(1)}(\theta_1^*) &= n^{-1/2} h^{1/2} \Psi_{1,1}(\theta_1^*, \hat{\theta}_2^0, \hat{\theta}_3^0, \theta_1^*, \hat{\theta}_2^0, \hat{\theta}_3^0) + n^{-1/2} \Psi_{1,2}(\theta_1^*, \hat{\theta}_3^0, \theta_1^*, \hat{\theta}_2^0, \hat{\theta}_3^0) \\ &= n^{-1/2} \Psi_{1,2}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1) \\ &= M_n^{(1)} + o_p(1) \end{aligned}$$

by using Lemmas 3.10 and 3.5 (a). Convergence (5.22) follows from this fact and Lemma 3.1 with [A2], $\square$

Finally, we obtain a limit theorem for the joint adaptive one-step estimator.

Theorem 5.15. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 3, 3, 2)$, [A2], [A3] and [A4] are satisfied. Then*

$$b_n^{-1}(\hat{\theta} - \theta^*) \rightarrow^d N(0, (\Gamma^J(\theta^*))^{-1})$$

as $n \rightarrow \infty$.

Proof. Let

$$\mathcal{X}_n^{***} = \mathcal{X}_n^{(1)} \cap \mathcal{X}_n^{**(2,3)} \cap \{(\hat{\theta}_1^0, \hat{\gamma}^0) \in B_n\} \cap \left\{ \sup_{\theta_1 \in B_n'''} |n^{-1} \partial_1^2 \mathbb{H}_n^{(1)}(\theta_1) + \Gamma_{11}| < c_1 \right\}$$

where $B_n''' = U(\theta_1^*, n^{-1/2} \log n)$, and c_1 is a sufficiently small number such that $|\mathbf{A} + \Gamma_{11}| < c_1$ implies $\det \mathbf{A} \neq 0$ any $\mathbf{p}_1 \times \mathbf{p}_1$ matrix $\mathbf{A}$. We obtain $P[\mathcal{X}_n^{***}] \rightarrow 1$ from Lemmas 5.13 and 5.12.

On the event $\mathcal{X}_n^{***}$, we apply Taylor's formula to obtain

$$\begin{aligned} &n^{1/2}(\hat{\theta}_1 - \theta_1^*) \\ &= [n^{-1} \partial_{\theta_1}^2 \mathbb{H}_n^{(1)}(\hat{\theta}_1^0)]^{-1} \left\{ -n^{-1/2} \partial_{\theta_1} \mathbb{H}_n^{(1)}(\theta_1^*) \right. \\ &\quad \left. + n^{-1} \int_0^1 [\partial_1^2 \mathbb{H}_n^{(1)}(\hat{\theta}_1^0) - \partial_1^2 \mathbb{H}_n^{(1)}(\hat{\theta}_1(u))] du n^{1/2}(\hat{\theta}_1^0 - \theta_1^*) \right\} \end{aligned}$$

where $\hat{\theta}_1(u) = \theta_1^* + u(\hat{\theta}_1^0 - \theta^*)$. Then we obtain

$$n^{1/2}(\hat{\theta}_1 - \theta_1^*) - \Gamma_{11}^{-1} M_n^{(1)} \xrightarrow{p} 0 \quad (5.28)$$

as $n \rightarrow \infty$ from Lemmas 5.12 and 5.14. Therefore the convergence of $b_n^{-1}(\hat{\theta} - \theta^*)$ follows from the martingale central limit theorem and the relations (5.17) and (5.28) $\square$

6 Non-adaptive estimator

In this section, we consider a non-adaptive joint quasi-maximum likelihood estimator. This method does not require initial estimators. From computational point of view, adaptive methods often have merits but the non-adaptive method is still theoretically interesting. We will work with the quasi-log likelihood function $\mathbb{H}_n(\theta)$ given by

$$\mathbb{H}_n(\theta) = -\frac{1}{2} \sum_{j=1}^n \left\{ S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} [\mathcal{D}_j(\theta_1, \theta_2, \theta_3)^{\otimes 2}] + \log \det S(Z_{t_{j-1}}, \theta_1, \theta_3) \right\} \quad (6.1)$$

for $\theta = (\theta_1, \theta_2, \theta_3)$. Suppose that a function $\hat{\theta}^J = (\hat{\theta}_1^J, \hat{\theta}_2^J, \hat{\theta}_3^J)$ of the data maximizes $\mathbb{H}_n(\theta)$ in $\overline{\Theta}$. Let

$$\mathbb{D}_n(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \mathbb{H}_n(\theta_1, \theta_2, \theta_3) - \mathbb{H}_n(\theta'_1, \theta'_2, \theta'_3).$$

Let

$$\begin{aligned} & \mathbb{D}_n^{[1]}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) \\ &= -\frac{1}{2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} [(\mathcal{D}_j(\theta_1, \theta_2, \theta_3) - \mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3))^{\otimes 2}] \\ &= -\frac{1}{2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\left(\begin{array}{c} h^{1/2}(A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta'_2)) \\ h^{-1/2}(H(Z_{t_{j-1}}, \theta_3) - H(Z_{t_{j-1}}, \theta'_3)) \\ + 2^{-1} h^{1/2}(L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) - L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3)) \end{array} \right) \right]^{\otimes 2} \right], \\ \mathbb{D}_n^{[2]}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= h^{-1/2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ H(Z_{t_{j-1}}, \theta_3) - H(Z_{t_{j-1}}, \theta'_3) \end{pmatrix} \right], \end{aligned}$$

$$\begin{aligned} & \mathbb{D}_n^{[3]}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) \\ &= h^{1/2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta'_2) \\ 2^{-1}(L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) - L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3)) \end{pmatrix} \right], \end{aligned}$$

and

$$\begin{aligned} \mathbb{D}_n^{[4]}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= -\frac{1}{2} \sum_{j=1}^n \left\{ (S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} - S(Z_{t_{j-1}}, \theta'_1, \theta'_3)^{-1}) [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2}] \right. \\ & \quad \left. + \log \frac{\det S(Z_{t_{j-1}}, \theta_1, \theta_3)}{\det S(Z_{t_{j-1}}, \theta'_1, \theta'_3)} \right\}. \end{aligned}$$

Then

$$\begin{aligned}\mathbb{D}_n(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \mathbb{D}_n^{[1]}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) + \mathbb{D}_n^{[2]}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) \\ &\quad + \mathbb{D}_n^{[3]}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) + \mathbb{D}_n^{[4]}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3).\end{aligned}\quad (6.2)$$

Lemma 6.1. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (0, 0, 1, 1, 2, 1)$ and [A2] are satisfied. Then*

(a) *As $n \rightarrow \infty$,*

$$\sup_{\theta \in \bar{\Theta}} |n^{-1}h\{\mathbb{H}_n(\theta_1, \theta_2, \theta_3) - \mathbb{H}_n(\theta_1, \theta_2, \theta_3^*)\} - \mathbb{Y}^{(3)}(\theta_3)| \rightarrow^p 0 \quad (6.3)$$

(b) *If [A3] (iii) is satisfied, then $\hat{\theta}_3^J \rightarrow^p \theta_3^*$ as $n \rightarrow \infty$.*

Proof. We have

$$\begin{aligned}n^{-1}h\{\mathbb{H}_n(\theta_1, \theta_2, \theta_3) - \mathbb{H}_n(\theta_1, \theta_2, \theta_3^*)\} &= n^{-1}h\mathbb{D}_n(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3^*) \\ &= n^{-1}h\mathbb{D}_n^{[1]}(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3^*) + n^{-1}h\mathbb{D}_n^{[2]}(\theta_1, \theta_3, \theta_1, \theta_2, \theta_3^*) \\ &\quad + n^{-1}h\mathbb{D}_n^{[3]}(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3^*) + n^{-1}h\mathbb{D}_n^{[4]}(\theta_1, \theta_3, \theta_1, \theta_2, \theta_3^*).\end{aligned}$$

By definition,

$$\begin{aligned}&n^{-1}h\mathbb{D}_n^{[1]}(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3^*) \\ &= -\frac{1}{2}n^{-1}\sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\left(\begin{array}{c} 0 \\ H(Z_{t_{j-1}}, \theta_3) - H(Z_{t_{j-1}}, \theta_3^*) \\ +2^{-1}h(L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) - L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3^*)) \end{array} \right) \right]^{\otimes 2}.\end{aligned}$$

We apply Sobolev's inequality to uniformly estimate the “ S ”-part and the “ H ”-part; these estimates involve $\partial_1^i \partial_z^j B$ and $\partial_3^i \partial_z^j H$ for $i \in \{0, 1\}$ and $j \in \{0, 1\}$. For the “ L_H ”-part, we use the assumption that the function is bound by a polynomial in $Z_{t_{j-1}}$ uniformly in θ . More precisely, we obtain

$$\sup_{\theta \in \bar{\Theta}} |n^{-1}h\mathbb{D}_n^{[1]}(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3^*) - \Phi_n^{(6.4)}(\theta_1, \theta_3)| = o_p(1)$$

where

$$\Phi_n^{(6.4)}(\theta_1, \theta_3) = -\frac{1}{2}n^{-1}\sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\left(\begin{array}{c} 0 \\ H(Z_{t_{j-1}}, \theta_3) - H(Z_{t_{j-1}}, \theta_3^*) \end{array} \right) \right]^{\otimes 2}. \quad (6.4)$$

With the help of Lemma 3.1 (a), Taylor's formula and [A2] give

$$\sup_{(\theta_1, \theta_3) \in \bar{\Theta}_1 \times \bar{\Theta}_3} \left| \Phi_n^{(6.4)}(\theta_1, \theta_3) + \int 6V(z, \theta_1, \theta_3)^{-1} [(H(z, \theta_3) - H(z, \theta_3^*))^{\otimes 2}] \nu(dz) \right| = o_p(1).$$

The uniform-in- (θ_1, θ_3) convergence follows from the point-wise convergence with the aid of the derivatives with respect to (θ_1, θ_3) . Remark that $\partial_x V$ therefore $\partial_x B$ is used, and L_H has H_{xx} in its expression.

It is easy to see

$$\sup_{\theta \in \Theta} |n^{-1} h \mathbb{D}_n^{[2]}(\theta_1, \theta_3, \theta_1, \theta_2, \theta_3^*)| = O_p(h^{1/2}),$$

$$\sup_{\theta \in \Theta} |n^{-1} h \mathbb{D}_n^{[3]}(\theta_1, \theta_2, \theta_3, \theta_1, \theta_2, \theta_3^*)| = O_p(h^{3/2})$$

and

$$\sup_{\theta \in \Theta} |n^{-1} h \mathbb{D}_n^{[4]}(\theta_1, \theta_3, \theta_1, \theta_2, \theta_3^*)| = O_p(h).$$

This completes the proof of (a). The assertion (b) is a consequence of (a). In fact, for $\epsilon > 0$,

$$\begin{aligned} \{|\hat{\theta}_3^J - \theta_3^*| > \epsilon\} &\subset \{\mathbb{Y}^{(3)}(\hat{\theta}_3^J) < -\chi_3 \epsilon^2\} \\ &\subset \left\{ \sup_{\theta \in \Theta} |n^{-1} h \{\mathbb{H}_n(\theta_1, \theta_2, \theta_3) - \mathbb{H}_n(\theta_1, \theta_2, \theta_3^*)\} - \mathbb{Y}^{(3)}(\theta_3)| > \chi_3 \epsilon^2 / 2 \right\} \end{aligned}$$

since $\mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \hat{\theta}_3^J) - \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) \geq 0$. $\square$

We will derive a rate of convergence of $\hat{\theta}_3^J$ by the random field $\mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3)$.

Lemma 6.2. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 1)$ and [A2] are satisfied. Then*

$$n^{-1/2} h^{1/2} \partial_3 \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) = O_p(n^{1/2} h^{1/2})$$

as $n \rightarrow \infty$.

Proof. We first use Lemmas 3.5 (b) and 3.3 (b), next take out the principal part of $\mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*)$, and apply argument with a random field and the Burkholder-Davis-Gundy inequality. By this procedure,

$$\begin{aligned} &n^{-1/2} h^{1/2} \partial_3 \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) \\ &= n^{-1/2} \Psi_3(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) + n^{-1/2} h^{1/2} \Psi_{3,3}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) \\ &= n^{-1/2} \Psi_3(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(n^{1/2} h^{1/2}) \\ &= n^{-1/2} \tilde{\Psi}_3(\theta_1^*, \theta_2^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(n^{1/2} h^{1/2}) \\ &= O_p(n^{1/2} h^{1/2}). \end{aligned}$$

$\square$

Lemma 6.3. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 2)$ and [A2] are satisfied. Then, for any sequence of positive numbers r_n tending to 0,*

$$\sup_{\theta_3 \in U(\theta_3^*, r_n)} |n^{-1} h \partial_3^2 \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3) + \Gamma_{33}(\hat{\theta}_1^J, \theta_3^*)| \rightarrow^p 0 \quad (6.5)$$

as $n \rightarrow \infty$, where

$$\Gamma_{33}(\theta_1, \theta_3^*) = \int 12V(z, \theta_1, \theta_3^*)^{-1} [\partial_3 H(z, \theta_3^*)^{\otimes 2}] \nu(dz).$$

If [A3](iii') is satisfied, then $\Gamma_{33}(\theta_1, \theta_3^*)$ is non-degenerate uniformly in θ_1 and $\hat{\theta}_3^J - \theta_3^* = O_p(h)$.

Proof. By definition,

$$\begin{aligned} n^{-1}h \partial_3^2 \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3) &= n^{-1}\Psi_{33,1}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3) + n^{-1}h^{1/2}\Psi_{33,2}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3) \\ &\quad + n^{-1}h\Psi_{33,3}(\hat{\theta}_1^J, \theta_3) + n^{-1}h^{1/2}\Psi_{33,4}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3) \\ &\quad + n^{-1}h\Psi_{33,5}(\hat{\theta}_1^J, \theta_3, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3). \end{aligned}$$

By Lemmass 3.5 (b) and 3.6, we have

$$\begin{aligned} n^{-1}h^{1/2}\Psi_{33,4}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3) &= n^{-1}h^{1/2}\Psi_{33,4}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(r_n) + O_p(h) \\ &= O(h^{1/2} + r_n) \end{aligned}$$

and this error is uniform in $\theta_3 \in B(\theta_3^*, r_n)$. Here Lemma 4.1 was applied to estimate the factor $(S^{-1}(\partial_3 S)S^{-1})(Z_{t_{j-1}}, \hat{\theta}_1^J, \theta_3)$. Estimation of the term involving $\Psi_{33,2}$ is similar. Estimation of other terms is simpler. The term $n^{-1}\Psi_{33,1}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3)$ is approximated by $\Gamma_{33}(\hat{\theta}_1^J, \theta_3^*)$ uniformly in $B(\theta_3^*, r_n)$. Remark that ∂_3^2 appears in $\Psi_{33,2}$ and $\Psi_{33,5}$. We do not need further differentiation with respect to θ_3 to estimate them, because they are accompanied with the factor h and the uniform-in- θ_3 estimate for each term is carried out by simple L^p estimate without random field argument.

Condition [A3] (iii') implies [A3] (iii). We obtain the rate of convergence of $\hat{\theta}_3^J$ from the consistency given in Lemma 6.1 (b), Lemma 6.2 and (6.5), if applying the Taylor formula and $\partial_3 \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \hat{\theta}_3^J) = 0$ on an event with probability tending to 1. $\square$

Lemma 6.4. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 2)$, [A2] and [A3](iii') are satisfied. Then*

(a) *As $n \rightarrow \infty$,*

$$\sup_{(\theta_1, \theta_2) \in \overline{\Theta}_1 \times \overline{\Theta}_2} \left| n^{-1} \{ \mathbb{H}_n(\theta_1, \theta_2, \hat{\theta}_3^J) - \mathbb{H}_n(\theta_1^*, \theta_2, \hat{\theta}_3^J) \} - \mathbb{Y}^{(J,1)}(\theta_1) \right| \rightarrow^p 0 \quad (6.6)$$

(b) *If [A3] (i') is satisfied, then $\hat{\theta}_1^J \rightarrow^p \theta_1^*$ as $n \rightarrow \infty$.*

Proof. We have

$$\begin{aligned} &n^{-1} \{ \mathbb{H}_n(\theta_1, \theta_2, \hat{\theta}_3^J) - \mathbb{H}_n(\theta_1^*, \theta_2, \hat{\theta}_3^J) \} \\ &= n^{-1} \mathbb{D}_n(\theta_1, \theta_2, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J) \\ &= n^{-1} \mathbb{D}_n^{[1]}(\theta_1, \theta_2, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J) + n^{-1} \mathbb{D}_n^{[2]}(\theta_1, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J) \\ &\quad + n^{-1} \mathbb{D}_n^{[3]}(\theta_1, \theta_2, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J) + n^{-1} \mathbb{D}_n^{[4]}(\theta_1, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J). \end{aligned}$$

We have

$$\begin{aligned} &n^{-1} \mathbb{D}_n^{[1]}(\theta_1, \theta_2, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J) \\ &= -\frac{1}{2} n^{-1} h \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \hat{\theta}_3^J)^{-1} \left[\begin{pmatrix} 0 \\ 2^{-1}(L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \hat{\theta}_3^J) - L_H(Z_{t_{j-1}}, \theta_1^*, \theta_2, \hat{\theta}_3^J)) \end{pmatrix}^{\otimes 2} \right]. \end{aligned}$$

Therefore,

$$\sup_{(\theta_1, \theta_2) \in \overline{\Theta}_1 \times \overline{\Theta}_2} |n^{-1} \mathbb{D}_n^{[1]}(\theta_1, \theta_2, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J)| = O_p(h).$$

By definition, $\mathbb{D}_n^{[2]}(\theta_1, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J) = 0$. Moreover, by using the preliminary estimate $\hat{\theta}_3^J - \theta_3^* = O_p(h)$ provided by Lemma 6.3, and the expression

$$\begin{aligned} & n^{-1} \mathbb{D}_n^{[3]}(\theta_1, \theta_2, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J) \\ &= n^{-1} h^{1/2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3^J)^{-1} \left[\mathcal{D}_j(\theta_1^*, \theta_2, \hat{\theta}_3^J), \begin{pmatrix} 0 \\ 2^{-1}(L(Z_{t_{j-1}}, \theta_1, \theta_2, \hat{\theta}_3^J) - L(Z_{t_{j-1}}, \theta_1^*, \theta_2, \hat{\theta}_3^J)) \end{pmatrix} \right], \end{aligned}$$

we obtain

$$\sup_{(\theta_1, \theta_2) \in \overline{\Theta}_1 \times \overline{\Theta}_2} |n^{-1} \mathbb{D}_n^{[3]}(\theta_1, \theta_2, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J)| = O_p(h)$$

by using Lemmas 3.6, 3.5 (b) and 3.3 (b).

Now

$$\begin{aligned} \mathbb{D}_n^{[4]}(\theta_1, \hat{\theta}_3^J, \theta_1^*, \theta_2, \hat{\theta}_3^J) &= -\frac{1}{2} \sum_{j=1}^n \left\{ (S(Z_{t_{j-1}}, \theta_1, \hat{\theta}_3^J)^{-1} - S(Z_{t_{j-1}}, \theta_1^*, \hat{\theta}_3^J)^{-1}) [\mathcal{D}_j(\theta_1^*, \theta_2, \hat{\theta}_3^J)^{\otimes 2}] \right. \\ &\quad \left. + \log \frac{\det S(Z_{t_{j-1}}, \theta_1, \hat{\theta}_3^J)}{\det S(Z_{t_{j-1}}, \theta_1^*, \hat{\theta}_3^J)} \right\}. \end{aligned}$$

Once again by using $\hat{\theta}_3^J - \theta_3^* = O_p(h)$ provided by Lemma 6.3, we obtain the result with the help of Taylor's formula and Lemma 3.1. $\square$

We shall deduce a tentative rough estimate $o_p(n^{-1/2}h^{-1/2})$ for the error of $\hat{\theta}_1^J$.

Lemma 6.5. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 2)$, [A2] and [A3](iii') are satisfied. Then*

$$n^{-1/2} h^{1/2} \partial_1 \mathbb{H}_n(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) = o_p(1)$$

as $n \rightarrow \infty$.

Proof. We use the tentative estimate of $\hat{\theta}_3^J - \theta_3^*$ given by Lemma 6.3. Then

$$\begin{aligned} & n^{-1/2} h^{1/2} \partial_1 \mathbb{H}_n(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) \\ &= n^{-1/2} h \Psi_{1,1}(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J, \theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) + n^{-1/2} h^{1/2} \Psi_{1,2}(\theta_1^*, \hat{\theta}_3^J, \theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) \\ &= O_p(n^{1/2}h) + O_p(h^{1/2}) \\ &= o_p(1) \end{aligned} \tag{6.7}$$

since $nh^2 \rightarrow 0$. In the equality (6.7), we used the following estimates for the second term:

$$\begin{aligned} & n^{-1/2} h^{1/2} \Psi_{1,2}(\theta_1^*, \hat{\theta}_3^J, \theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) \\ &= n^{-1/2} h^{1/2} \Psi_{1,2}(\theta_1^*, \hat{\theta}_3^J, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(n^{1/2}h) \quad (\because \text{Lemmas 6.3, 3.6, 3.5(b) and 3.3(b)}) \\ &= n^{-1/2} h^{1/2} \Psi_{1,2}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(n^{1/2}h) \quad (\because \text{Lemmas 6.3 and random field argument}) \\ &= O_p(n^{1/2}h) + O_p(h^{1/2}) \quad (\because \text{orthogonality}). \end{aligned}$$

A similar estimate applies to the first term on (6.7). $\square$

Recall

$$\begin{aligned}\Gamma_{11} &= \frac{1}{2} \int \text{Tr}\{S^{-1}(\partial_1 S)S^{-1}\partial_1 S(z, \theta_1^*, \theta_3^*)\} \nu(dz) \\ &= \frac{1}{2} \int \left[\text{Tr}\{(C^{-1}(\partial_1 C)C^{-1}\partial_1 C)(z, \theta_1^*)\} \right. \\ &\quad \left. + \text{Tr}\{(V^{-1}H_x(\partial_1 C)H_x^*V^{-1}H_x(\partial_1 C)H_x^*)(z, \theta_1^*, \theta_3^*)\} \right] \nu(dz).\end{aligned}$$

Lemma 6.6. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 3, 3, 2)$, [A2], [A3](i') and [A3](iii') are satisfied. Then, for any sequence of positive numbers r_n tending to 0,*

$$\sup_{\theta_1 \in U(\theta_1^*, r_n)} \left| n^{-1} \partial_1^2 \mathbb{H}_n(\theta_1, \hat{\theta}_2^J, \hat{\theta}_3^J) + \Gamma_{11} \right| \rightarrow^p 0 \quad (6.8)$$

as $n \rightarrow \infty$. In particular, $\hat{\theta}_1^J - \theta_1^* = o_p(n^{-1/2}h^{-1/2})$.

Proof. By definition,

$$\begin{aligned}n^{-1} \partial_1^2 \mathbb{H}_n(\theta_1, \hat{\theta}_2^J, \hat{\theta}_3^J) &= -n^{-1}h \Psi_{11,1}(\theta_1, \hat{\theta}_3^J, \theta_1, \hat{\theta}_2^J, \hat{\theta}_3^J) \\ &\quad + n^{-1}h^{1/2} \Psi_{11,2}(\theta_1, \hat{\theta}_3^J, \theta_1, \hat{\theta}_2^J, \hat{\theta}_3^J, \theta_1, \hat{\theta}_2^J, \hat{\theta}_3^J) \\ &\quad - \frac{1}{2}n^{-1} \Psi_{11,3}(\theta_1, \hat{\theta}_3^J, \theta_1, \hat{\theta}_2^J, \hat{\theta}_3^J) \\ &\quad - \frac{1}{2}n^{-1} \Psi_{11,4}(\theta_1, \hat{\theta}_3^J) \quad (\text{this term will remain}) \\ &\quad - n^{-1}h^{1/2} \Psi_{11,5}(\theta_1, \hat{\theta}_3^J, \theta_1, \hat{\theta}_2^J, \hat{\theta}_3^J, \theta_1, \hat{\theta}_2^J, \hat{\theta}_3^J).\end{aligned}$$

If we apply the same machinery as in the proof of Lemma 6.5, it is easy to obtain the result. [It is remarked that ∂_1^2 appears in $\Psi_{11,2}$ and $\Psi_{11,3}$. Uniform-in- θ_1 estimate for $\Psi_{11,2}$ is simple since it has the factor $h^{1/2}$ in front of it. On the other hand, we use random field argument for $\Psi_{11,3}$ after making the martingale differences. We need ∂_1^3 at this stage.] For the second assertion, the argument becomes local by Lemma 6.4, then Lemma 6.5 and the convergence (6.8) gives it by Taylor's formula. $\square$

Lemma 6.7. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 3, 3, 2)$, [A2], [A3](i') and [A3](iii') are satisfied. Then*

$$n^{-1/2}h^{1/2} \partial_3 \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) = O_p(1)$$

as $n \rightarrow \infty$. In particular, $\hat{\theta}_3^J - \theta_3^* = O_p(n^{-1/2}h^{1/2})$.

Proof. First using an algebraic identity similar to (4.14), next using Lemma 6.6 and once again using Lemma 6.6 with Lemma 3.5(b), we have

$$\begin{aligned}n^{-1/2} \Psi_{3,1}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) &= n^{-1/2} \Psi_{3,1}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \theta_2^*, \theta_3^*) \\ &= n^{-1/2} \Psi_{3,1}(\hat{\theta}_1^J, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(1) \\ &= n^{-1/2} \Psi_{3,1}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(1).\end{aligned}$$

Then, from the representation of $\mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*)$ given by Lemmas 3.4 and 3.5 (a) with the aid of the orthogonality of the martingale parts, we obtain

$$n^{-1/2} \Psi_{3,1}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) = O_p(1).$$

Lemmas 3.5(b) and 3.3 easily ensures

$$n^{-1/2} \Psi_{3,2}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) = O_p(1)$$

Lemmas 3.5(b), 3.3 and 6.6 give

$$n^{-1/2} h^{1/2} \Psi_{3,3}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) = n^{-1/2} h^{1/2} \Psi_{3,3}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(n^{1/2} h) + O_p(1),$$

and the representation of $\mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*)$ in Lemmas 3.4 and 3.5 (a) and the orthogonality between the martingale differences, we see

$$n^{-1/2} h^{1/2} \Psi_{3,3}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) = O_p(1).$$

Consequently,

$$\begin{aligned} n^{-1/2} h^{1/2} \partial_3 \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) &= n^{-1/2} \Psi_{3,1}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) + n^{-1/2} \Psi_{3,2}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) \\ &\quad + n^{-1/2} h^{1/2} \Psi_{3,3}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) \\ &= O_p(1). \end{aligned}$$

For the last assertion, we may apply Lemma 6.3. □

Recall

$$M_n^{(1)} = \frac{1}{2} n^{-1/2} \sum_{j=1}^n (S^{-1}(\partial_1 S) S^{-1})(Z_{t_{j-1}}, \theta_1^*, \theta_3^*) [\tilde{\mathcal{D}}_j(\theta_1^*, \theta_2^*, \theta_3^*)^{\otimes 2} - S(Z_{t_{j-1}}, \theta_1^*, \theta_3^*)].$$

Lemma 6.8. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 3, 3, 2)$, [A2], [A3](i') and [A3](iii') are satisfied. Then*

$$n^{-1/2} \partial_1 \mathbb{H}_n(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) = O_p(1)$$

as $n \rightarrow \infty$. Moreover,

$$n^{1/2}(\hat{\theta}_1^J - \theta_1^*) - \Gamma_{11}^{-1} M_n^{(1)} \xrightarrow{p} 0$$

as $n \rightarrow \infty$. In particular, $\hat{\theta}_1^J - \theta_1^* = O_p(n^{-1/2})$ as $n \rightarrow \infty$.

Proof. We are in the same situation as Lemma 6.5 but we can use the convergence rate $\hat{\theta}_3^J - \theta_3^* = O_p(n^{-1/2} h^{1/2})$ elaborated by Lemma 6.7. Then

$$\begin{aligned} &n^{-1/2} \partial_1 \mathbb{H}_n(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) \\ &= n^{-1/2} h^{1/2} \Psi_{1,1}(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J, \theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) + n^{-1/2} \Psi_{1,2}(\theta_1^*, \hat{\theta}_3^J, \theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) \\ &= n^{-1/2} h^{1/2} \Psi_{1,1}(\theta_1^*, \theta_2^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(n^{1/2} h) + O_p(h^{1/2}) \quad (\text{Lemmas 3.5(b), 3.6 and 6.7}) \\ &\quad + n^{-1/2} \Psi_{1,2}(\theta_1^*, \theta_3^*, \theta_1^*, \hat{\theta}_2^J, \theta_3^*) + O_p(1) \quad (\text{Lemmas 3.6, 3.3(b) and 6.7}) \end{aligned}$$

For the last term, we can use the decomposition

$$\begin{aligned} \mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J)^{\otimes 2} - \mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \theta_3^*)^{\otimes 2} &= \{ \mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) - \mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \theta_3^*) \}^{\otimes 2} \\ &\quad + 2 \left\{ \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*) + (\mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \theta_3^*) - \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*)) \right\} \\ &\quad \otimes_{sym} \{ \mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) - \mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \theta_3^*) \}, \end{aligned}$$

where $\otimes_{sym}$ means the symmetrized tensor product.

We have

$$\begin{aligned} &\mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \theta_3^*)^{\otimes 2} - \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*)^{\otimes 2} \\ &= 2(\mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \theta_3^*) - \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*)) \otimes_{sym} \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*) + (\mathcal{D}_j(\theta_1^*, \hat{\theta}_2^J, \theta_3^*) - \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*))^{\otimes 2} \\ &= 2h^{1/2} \left(\begin{array}{c} A(Z_{t_{j-1}}, \theta_2^*) - A(Z_{t_{j-1}}, \hat{\theta}_2^J) \\ 2^{-1}(L_H(Z_{t_{j-1}}, \theta_1^*, \theta_2^*, \theta_3^*) - L_H(Z_{t_{j-1}}, \theta_1^*, \hat{\theta}_2^J, \theta_3^*)) \end{array} \right) \otimes_{sym} \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*) + O_{L^\infty}(h) \end{aligned}$$

To estimate $n^{-1/2}\Psi_{1,2}(\theta_1^*, \theta_3^*, \theta_1^*, \hat{\theta}_2^J, \theta_3^*)$, we introduce the random field

$$\begin{aligned} \Xi_n^{(6.9)}(\theta_2) &= n^{-1/2}h^{1/2} \sum_{j=1}^n (S^{-1}(\partial_1 S))S^{-1}(Z_{t_{j-1}}, \theta_1^*, \theta_3^*) \\ &\quad \cdot \left[\left(\begin{array}{c} A(Z_{t_{j-1}}, \theta_2^*) - A(Z_{t_{j-1}}, \theta_2) \\ 2^{-1}(L_H(Z_{t_{j-1}}, \theta_1^*, \theta_2^*, \theta_3^*) - L_H(Z_{t_{j-1}}, \theta_1^*, \theta_2, \theta_3^*)) \end{array} \right) \otimes_{sym} \mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*) \right]. \end{aligned} \tag{6.9}$$

With the aid of the representation of $\mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*)$ and the orthogonality between martingale differences, a random field argument concludes

$$\sup_{\theta_2 \in \Theta_2} |\Xi_n^{(6.9)}(\theta_2)| = O_p(h^{1/2}),$$

in particular,

$$n^{-1/2}\Psi_{1,2}(\theta_1^*, \theta_3^*, \theta_1^*, \hat{\theta}_2^J, \theta_3^*) = n^{-1/2}\Psi_{1,2}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1).$$

The orthogonality further applied gives

$$n^{-1/2}h^{1/2}\Psi_{1,1}(\theta_1^*, \theta_2^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) = O_p(h^{1/2}).$$

Consequently,

$$\begin{aligned} n^{-1/2} \partial_1 \mathbb{H}_n(\theta_1^*, \hat{\theta}_2^J, \hat{\theta}_3^J) &= n^{-1/2}\Psi_{1,2}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1) \\ &= M_n^{(1)} + o_p(1) \\ &= O_p(1) \end{aligned} \tag{6.10}$$

as $n \rightarrow \infty$.

Since $\hat{\theta}_1^J \rightarrow^p \theta_1^*$ by e.g. Lemma 6.4, we can show the first order efficiency of $\hat{\theta}_1^J$ by using Taylor's formula combined with (6.10) and Lemma 6.6. $\square$

Lemma 6.9. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 3, 3, 2)$, [A2], [A3](i') and [A3](iii') are satisfied. Then

$$n^{-1/2}h^{1/2} \partial_3 \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) - M_n^{(3)} \xrightarrow{p} 0 \quad (6.11)$$

as $n \rightarrow \infty$. In particular,

$$n^{1/2}h^{-1/2}(\hat{\theta}_3^J - \theta_3^*) \xrightarrow{d} N(0, \Gamma_{33}^{-1}) \quad (6.12)$$

as $n \rightarrow \infty$.

Proof. We elaborate the estimate in the proof of Lemma 6.7. Taking advantage of the convergence rate of $\hat{\theta}_1^J$ given by Lemma 6.8, we see

$$\begin{aligned} & n^{-1/2}h^{1/2} \partial_3 \mathbb{H}_n(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) \\ = & n^{-1/2}\Psi_{3,1}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \theta_2^*, \theta_3^*) + n^{-1/2}\Psi_{3,2}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) \\ & + n^{-1/2}h^{1/2}\Psi_{3,3}(\hat{\theta}_1^J, \theta_3^*, \hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*) \\ = & n^{-1/2}\Psi_{3,1}(\hat{\theta}_1^J, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(h^{1/2}) \quad (\text{Lemmas 6.8 and 3.5(b)}) \\ & + n^{-1/2}\Psi_{3,2}(\hat{\theta}_1^J, \hat{\theta}_2^J, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(h^{1/2}) \quad (\text{Lemma 3.5(b)}) \\ & + n^{-1/2}h^{1/2}\Psi_{3,3}(\hat{\theta}_1^J, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1) \quad (\text{Lemmas 3.5(b) and 3.3(b)}). \end{aligned}$$

By Lemma 6.8, the representation of $\mathcal{D}_j(\theta_1^*, \theta_2^*, \theta_3^*)$ and the orthogonality, we obtain

$$\begin{aligned} n^{-1/2}h^{1/2}\Psi_{3,3}(\hat{\theta}_1^J, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) &= n^{-1/2}h^{1/2}\Psi_{3,3}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + O_p(h^{1/2}) \\ &= O_p(h^{1/2}). \end{aligned}$$

We have

$$\sup_{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2} |n^{-1/2}\Psi_{3,2}(\theta_1, \theta_2, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*)| = O_p(h).$$

Next, we consider

$$\Xi_n^{(6.13)}(u_1) = n^{-1/2} \left\{ \Psi_{3,1}(\theta_1^* + r_n u_1, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) - \Psi_{3,1}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) \right\} \quad (6.13)$$

for any sequence r_n of positive numbers such that $r_n \rightarrow 0$. Then a random field argument with Sobolev's inequality ensures the convergence

$$\sup_{u_1 \in B(0,1)} |\Xi_n^{(6.13)}(u_1)| = o_p(1).$$

Therefore,

$$\begin{aligned} n^{-1/2}\Psi_{3,1}(\hat{\theta}_1^J, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) &= n^{-1/2}\Psi_{3,1}(\theta_1^*, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1) \\ &= M_n^{(3)} + o_p(1). \end{aligned}$$

From the above estimates, we already have (6.11). Moreover, Lemmas 6.3 and the martingale central limit theorem gives (6.12). $\square$

Lemma 6.10. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 3, 3, 2)$, [A2], [A3](i') and [A3](iii') are satisfied. Then

(a) As $n \rightarrow \infty$,

$$\sup_{\theta_2 \in \bar{\Theta}_2} |n^{-1}h^{-1} \{ \mathbb{H}_n(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J) - \mathbb{H}_n(\hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) \} - \mathbb{Y}^{(2)}(\theta_2)| \rightarrow^p 0 \quad (6.14)$$

(b) If [A3] (ii) is satisfied, then $\hat{\theta}_2^J \rightarrow^p \theta_2^*$ as $n \rightarrow \infty$.

Proof. We have

$$\begin{aligned} & n^{-1}h^{-1} \{ \mathbb{H}_n(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J) - \mathbb{H}_n(\hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) \} \\ &= n^{-1}h^{-1} \mathbb{D}_n(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) \\ &= n^{-1}h^{-1} \mathbb{D}_n^{[1]}(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) + n^{-1}h^{-1} \mathbb{D}_n^{[2]}(\hat{\theta}_1^J, \hat{\theta}_3^J, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) \\ &\quad + n^{-1}h^{-1} \mathbb{D}_n^{[3]}(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) + n^{-1}h^{-1} \mathbb{D}_n^{[4]}(\hat{\theta}_1^J, \hat{\theta}_3^J, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J). \end{aligned}$$

We have

$$\begin{aligned} & n^{-1}h^{-1} \mathbb{D}_n^{[1]}(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) \\ &= -n^{-1}h^{-1} \frac{1}{2} \sum_{j=1}^n S(Z_{t_{j-1}}, \hat{\theta}_1^J, \hat{\theta}_3^J)^{-1} [(\mathcal{D}_j(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J) - \mathcal{D}_j(\hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J))^{\otimes 2}] \\ &= -\frac{1}{2}n^{-1} \sum_{j=1}^n S(Z_{t_{j-1}}, \hat{\theta}_1^J, \hat{\theta}_3^J)^{-1} \left[\begin{pmatrix} A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta_2^*) \\ 2^{-1}(L_H(Z_{t_{j-1}}, \hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J) - L_H(Z_{t_{j-1}}, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J)) \end{pmatrix}^{\otimes 2} \right] \\ &= -\frac{1}{2}n^{-1} \sum_{j=1}^n S(Z_{t_{j-1}}, \hat{\theta}_1^J, \hat{\theta}_3^J)^{-1} \left[\begin{pmatrix} A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta_2^*) \\ 2^{-1}H_x(Z_{t_{j-1}}, \hat{\theta}_3^J)[A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta_2^*)] \end{pmatrix}^{\otimes 2} \right] \\ &= -\frac{1}{2}n^{-1} \sum_{j=1}^n C(Z_{t_{j-1}}, \hat{\theta}_1^J)^{-1} [(A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta_2^*))^{\otimes 2}], \end{aligned}$$

$$\mathbb{D}_n^{[2]}(\hat{\theta}_1^J, \hat{\theta}_3^J, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) = 0, \mathbb{D}_n^{[4]}(\hat{\theta}_1^J, \hat{\theta}_3^J, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) = 0 \text{ and}$$

$$n^{-1}h^{-1} \mathbb{D}_n^{[3]}(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J, \hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) = n^{-1}h^{-1} \mathbb{D}_n^{[3]}(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1)$$

by Lemmas 6.7, 6.8, 3.6 and 3.5 (b), where the order $o_p(1)$ is uniform in $\theta_2 \in \Theta_2$. The last expression is

$$n^{-1}h^{-1} \mathbb{D}_n^{[3]}(\theta_1^*, \theta_2, \theta_3^*, \theta_1^*, \theta_2^*, \theta_3^*) + o_p(1)$$

by using the exact convergence rate of $\hat{\theta}_1^J$ and $\hat{\theta}_3^J$, where $o_p(1)$ is uniform in $\theta_2 \in \Theta_2$. Random field argument shows that the last one converges in probability to zero uniformly in θ_2 . This shows (a). The property (b) is now easy to deduce from (a). $\square$

We will derive a convergence rate of $\hat{\theta}_2^J$.

Lemma 6.11. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 3, 2)$, [A2], [A3](i') and [A3](iii') are satisfied. Then

$$n^{-1/2}h^{-1/2}\partial_2\mathbb{H}_n(\hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) - M_n^{(2)} \rightarrow^p 0$$

as $n \rightarrow \infty$.

Proof. By simple algebra and Lemma 3.2,

$$\begin{aligned} & n^{-1/2}h^{-1/2}\partial_2\mathbb{H}_n(\hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J) \\ = & n^{-1/2} \sum_{j=1}^n S(Z_{t_{j-1}}, \hat{\theta}_1^J, \hat{\theta}_3^J)^{-1} \left[\mathcal{D}_j(\hat{\theta}_1^J, \theta_2^*, \hat{\theta}_3^J), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2^*) \\ 2^{-1}H_x(Z_{t_{j-1}}, \hat{\theta}_3^J)[\partial_2 A(Z_{t_{j-1}}, \theta_2^*)] \end{pmatrix} \right] \\ = & n^{-1/2} \sum_{j=1}^n C(Z_{t_{j-1}}, \hat{\theta}_1^J)^{-1} [\partial_2 A(Z_{t_{j-1}}, \theta_2^*), h^{-1/2}(\Delta_j X - hA(Z_{t_{j-1}}, \theta_2^*))] \\ = & n^{-1/2} \sum_{j=1}^n C(Z_{t_{j-1}}, \hat{\theta}_1^J)^{-1} [\partial_2 A(Z_{t_{j-1}}, \theta_2^*), h^{-1/2}B(Z_{t_{j-1}}, \theta_1^*)\Delta_j w] + o_p(1) \\ = & M_n^{(2)} + o_p(1). \end{aligned}$$

Here the last equation can be verified by a θ_1 -random field argument using the consistency of $\hat{\theta}_1^J$ obtained in Lemma 6.4. Remark that $M_n^{(2)}$ is defined by (5.14) on p.26. $\square$

Lemma 6.12. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 3, 3, 2)$, [A2], [A3](i'), [A3](ii) and [A3](iii') are satisfied. Then

$$\sup_{\theta_2 \in U(\theta_2^*, r_n)} |n^{-1}h^{-1} \partial_2^2 \mathbb{H}_n(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J) + \Gamma_{22}| \rightarrow^p 0 \quad (6.15)$$

as $n \rightarrow \infty$, where r_n is any sequence of positive numbers such that $r_n \rightarrow 0$ and

$$\begin{aligned} \Gamma_{22} &= \int S(z, \theta_1^*, \theta_3^*)^{-1} \left[\begin{pmatrix} \partial_2 A(z, \theta_2^*) \\ 2^{-1}\partial_2 L_H(z, \theta_1^*, \theta_2^*, \theta_3^*) \end{pmatrix}^{\otimes 2} \right] \nu(dz) \\ &= \int C(z, \theta_1^*)^{-1} [(\partial_2 A(z, \theta_2^*)^{\otimes 2}] \nu(dz). \end{aligned}$$

Moreover,

$$n^{1/2}h^{1/2}(\hat{\theta}_2^J - \theta_2^*) - \Gamma_{22}^{-1}M_n^{(2)} \rightarrow^p 0 \quad (6.16)$$

as $n \rightarrow \infty$. In particular,

$$n^{1/2}h^{1/2}(\hat{\theta}_2^J - \theta_2^*) \rightarrow^d N(0, \Gamma_{22}^{-1})$$

as $n \rightarrow \infty$.

Proof. We see

$$\begin{aligned}
& n^{-1}h^{-1}\partial_2^2\mathbb{H}_n(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J) \\
&= -n^{-1}\sum_{j=1}^n S(Z_{t_{j-1}}, \hat{\theta}_1^J, \hat{\theta}_3^J)^{-1} \left[\begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2) \\ 2^{-1}H_x(Z_{t_{j-1}}, \hat{\theta}_3^J)[\partial_2 A(Z_{t_{j-1}}, \theta_2)] \end{pmatrix}^{\otimes 2} \right] \\
&\quad + n^{-1}h^{-1/2}\sum_{j=1}^n S(Z_{t_{j-1}}, \hat{\theta}_1^J, \hat{\theta}_3^J)^{-1} \left[\mathcal{D}_j(\hat{\theta}_1^J, \theta_2, \hat{\theta}_3^J), \begin{pmatrix} \partial_2^2 A(Z_{t_{j-1}}, \theta_2) \\ 2^{-1}H_x(Z_{t_{j-1}}, \hat{\theta}_3^J)[\partial_2^2 A(Z_{t_{j-1}}, \theta_2)] \end{pmatrix} \right] \\
&= -n^{-1}\sum_{j=1}^n S(Z_{t_{j-1}}, \hat{\theta}_1^J, \hat{\theta}_3^J)^{-1} \left[\begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2) \\ 2^{-1}H_x(Z_{t_{j-1}}, \hat{\theta}_3^J)[\partial_2 A(Z_{t_{j-1}}, \theta_2)] \end{pmatrix}^{\otimes 2} \right] \\
&\quad + n^{-1}h^{-1/2}\sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1^*, \theta_3^*)^{-1} \left[\mathcal{D}_j(\theta_1^*, \theta_2, \theta_3^*), \begin{pmatrix} \partial_2^2 A(Z_{t_{j-1}}, \theta_2) \\ 2^{-1}H_x(Z_{t_{j-1}}, \theta_3^*)[\partial_2^2 A(Z_{t_{j-1}}, \theta_2)] \end{pmatrix} \right] \\
&\quad + o_p(n^{-1/2}h^{-1/2}) \quad (\text{Lemmas 6.7 and 6.8}) \\
&= -n^{-1}\sum_{j=1}^n C(Z_{t_{j-1}}, \hat{\theta}_1)^{-1} [(\partial_2 A(Z_{t_{j-1}}, \theta_2))^{\otimes 2}] \\
&\quad + n^{-1}h^{-1/2}\sum_{j=1}^n C(Z_{t_{j-1}}, \theta_1^*)^{-1} [\partial_2^2 A(Z_{t_{j-1}}, \theta_2), h^{-1/2}(\Delta_j X - hA(Z_{t_{j-1}}, \theta_2^*))] + o_p(1) \\
&= -n^{-1}\sum_{j=1}^n C(Z_{t_{j-1}}, \theta_1^*)^{-1} [(\partial_2 A(Z_{t_{j-1}}, \theta_2))^{\otimes 2}] \quad (\text{Lemma 6.8}) \\
&\quad + n^{-1}h^{-1/2}\sum_{j=1}^n C(Z_{t_{j-1}}, \theta_1^*)^{-1} [\partial_2^2 A(Z_{t_{j-1}}, \theta_2), h^{-1/2}(\Delta_j X - hA(Z_{t_{j-1}}, \theta_2^*))] + o_p(1) \\
&= -n^{-1}\sum_{j=1}^n C(Z_{t_{j-1}}, \theta_1^*)^{-1} [(\partial_2 A(Z_{t_{j-1}}, \theta_2))^{\otimes 2}] + o_p(1)
\end{aligned}$$

The order $o_p(1)$ is uniform in $\theta_2 \in \Theta_2$. The last equation is verified by random field argument with the shrinking $B(\theta_2^*, r_n)$, where we need $\partial_2^3 A$. Since $\hat{\theta}_2^J \rightarrow^p \theta_2^*$ by Lemma 6.10(b), applying Taylor's formula with $\partial_2^2 A$, we obtain (6.15) with the help of Lemma 3.1 (a). Moreover, we obtain (6.16) by combining (6.15) with Lemma 6.11. $\square$

Let

$$\hat{\theta}^J = \begin{pmatrix} \hat{\theta}_1^J \\ \hat{\theta}_2^J \\ \hat{\theta}_3^J \end{pmatrix}.$$

Recall

$$\begin{aligned}\Gamma^J(\theta^*) &= \text{diag} \left[\frac{1}{2} \int \left[\text{Tr} \{ (C^{-1}(\partial_1 C) C^{-1} \partial_1 C)(z, \theta_1^*) \} \right. \right. \\ &\quad \left. \left. + \text{Tr} \{ (V^{-1} H_x(\partial_1 C) H_x^* V^{-1} H_x(\partial_1 C) H_x^*)(z, \theta_1^*, \theta_3^*) \} \right] \nu(dz), \right. \\ &\quad \left. \int \partial_2 A(z, \theta_2^*)^* C(z, \theta_1^*)^{-1} \partial_2 A(z, \theta_2^*) \nu(dz), \right. \\ &\quad \left. \int 12 \partial_3 H(z, \theta_3^*)^* V(z, \theta_1^*, \theta_3^*)^{-1} \partial_3 H(z, \theta_3^*) \nu(dz) \right]\end{aligned}$$

Theorem 6.13. *Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 3, 3, 2)$, [A2], [A3](i'), [A3](ii) and [A3](iii') are satisfied. Then*

$$b_n^{-1}(\hat{\theta}^J - \theta^*) \rightarrow^d N(0, (\Gamma^J(\theta^*))^{-1})$$

as $n \rightarrow \infty$.

Proof. By Lemmas 6.8, 6.12 and 6.9, we obtain the result. By simple linear calculus, we can see that $M_n^{(2)}$ and $M_n^{(3)}$ are asymptotically orthogonal. Since $M_n^{(1)}$ is written by the second Wiener chaos, it is asymptotically orthogonal to $M_n^{(2)}$ and $M_n^{(3)}$. $\square$

7 Estimation of θ_1

The purpose of this section is to recall a standard construction of estimator for θ_1 and to clarify what conditions we mentioned validate its asymptotic properties. Let

$$\mathbb{H}_n^{(1)}(\theta_1) = -\frac{1}{2} \sum_{j=1}^n \left\{ C(Z_{t_{j-1}}, \theta_1)^{-1} [h^{-1}(\Delta_j X)^{\otimes 2}] + \log \det C(Z_{t_{j-1}}, \theta_1) \right\}$$

where $\Delta_j X = X_{t_j} - X_{t_{j-1}}$. It should be remarked that the present $\mathbb{H}_n^{(1)}(\theta_1)$ is different from the one given in (5.2) on p.20. Under [A1] and [A2] (iii), $\mathbb{H}_n^{(1)}$ is a continuous function on $\overline{\Theta}_1$ a.s.

Given the data $(Z_{t_j})_{j=0,1,\dots,n}$, let us consider the quasi-maximum likelihood estimator (QMLE) $\hat{\theta}_1^0 = \hat{\theta}_{1,n}^0$ for θ_1 , that is, $\hat{\theta}_1^0$ is any measurable function of $(Z_{t_j})_{j=0,1,\dots,n}$ satisfying

$$\mathbb{H}_n^{(1)}(\hat{\theta}_1^0) = \max_{\theta_1 \in \overline{\Theta}_1} \mathbb{H}_n^{(1)}(\theta_1) \quad a.s.$$

Routinely, $n^{1/2}$ -consistency and asymptotic normality of $\hat{\theta}_1^0$ can be established. We will give a brief for self-containedness and for the later use. Let

$$\Gamma^{(1)}[u_1^{\otimes 2}] = \frac{1}{2} \int_{\mathbb{R}^{d_Z}} \text{Tr} \{ C^{-1}(\partial_1 C)[u_1] C^{-1}(\partial_1 C)[u_1](z, \theta_1^*) \} \nu(dz) \quad (7.1)$$

for $u_1 \in \mathbb{R}^{p_1}$. We will see the existence and positivity of $\Gamma^{(1)}$ in the following theorem.

Theorem 7.1. (a) Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (0, 0, 1, 1, 0, 0)$, [A2] (i), (ii), (iii), and [A3] (i) are satisfied. Then $\hat{\theta}_1^0 \rightarrow^p \theta_1^*$ as $n \rightarrow \infty$.

(b) Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 0, 2, 3, 0, 0)$, [A2] (i), (ii), (iii), and [A3] (i) are satisfied. Then $\Gamma^{(1)}$ exists and is positive-definite, and

$$\sqrt{n}(\hat{\theta}_1^0 - \theta_1^*) - (\Gamma^{(1)})^{-1} \hat{M}_n^{(1)} \rightarrow^p 0$$

as $n \rightarrow \infty$, where

$$\hat{M}_n^{(1)} = \frac{1}{2} n^{-1/2} \sum_{j=1}^n (C^{-1}(\partial_1 C) C^{-1})(Z_{t_{j-1}}, \theta_1^*) [(h^{-1/2} B(Z_{t_{j-1}}, \theta_1^*) \Delta_j w)^{\otimes 2} - C(Z_{t_{j-1}}, \theta_1^*)].$$

Moreover, $M_n^{(1)} \rightarrow^d N_{\mathbf{p}_1}(0, \Gamma^{(1)})$ as $n \rightarrow \infty$. In particular,

$$\sqrt{n}(\hat{\theta}_1^0 - \theta_1^*) \rightarrow^d N_{\mathbf{p}_1}(0, (\Gamma^{(1)})^{-1})$$

as $n \rightarrow \infty$.

Proof. (a): Let $\mathbb{Y}_n^{(1)}(\theta_1) = n^{-1}(\mathbb{H}_n^{(1)}(\theta_1) - \mathbb{H}_n^{(1)}(\theta_1^*))$. Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (0, 0, 1, 1, 0, 0)$ and [A2] (i), (ii), (iii). Use (3.1) and Lemma 3.1, then

$$\sum_{i=0}^1 \sup_{\theta_1 \in \Theta_1} \left\| |\partial_1^i(\mathbb{Y}_n^{(1)}(\theta_1) - \mathbb{Y}^{(1)}(\theta_1))| \right\|_p \rightarrow 0 \quad (n \rightarrow \infty)$$

for every $p > 1$. By Sobolev's inequality, we obtain

$$\left\| \sup_{\theta_1 \in \Theta_1} |\mathbb{Y}_n^{(1)}(\theta_1) - \mathbb{Y}^{(1)}(\theta_1)| \right\|_p \rightarrow 0$$

for every $p > 1$. Therefore, the identifiability condition [A3] (i) ensures $\hat{\theta}_1^0 \rightarrow^p \theta_1^*$ as $n \rightarrow \infty$.

(b): Under [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (0, 0, 0, 2, 0, 0)$, we have

$$\partial_1 \mathbb{H}_n^{(1)}(\theta_1) = \frac{1}{2} \sum_{j=1}^n (C^{-1}(\partial_1 C) C^{-1})(Z_{t_{j-1}}, \theta_1) [h^{-1}(\Delta_j X)^{\otimes 2} - C(Z_{t_{j-1}}, \theta_1)]$$

and

$$\begin{aligned} \partial_1^2 \mathbb{H}_n^{(1)}(\theta_1) &= \frac{1}{2} \sum_{j=1}^n \partial_1 (C^{-1}(\partial_1 C) C^{-1})(Z_{t_{j-1}}, \theta_1) [h^{-1}(\Delta_j X)^{\otimes 2} - C(Z_{t_{j-1}}, \theta_1)] \\ &\quad - \frac{1}{2} \sum_{j=1}^n (C^{-1}(\partial_1 C) C^{-1})(Z_{t_{j-1}}, \theta_1) [\partial_1 C(Z_{t_{j-1}}, \theta_1)]. \end{aligned}$$

Now, by using orthogonality and the estimate (3.5), if [A1] is satisfied for $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 0, 2, 1, 0, 0)$, then

$$\begin{aligned} & n^{-1/2} \partial_1 \mathbb{H}_n^{(1)}(\theta_1^*) \\ &= \frac{1}{2} n^{-1/2} \sum_{j=1}^n (C^{-1}(\partial_1 C)C^{-1})(Z_{t_{j-1}}, \theta_1^*) [h^{-1}(\Delta_j X)^{\otimes 2} - C(Z_{t_{j-1}}, \theta_1^*)] \\ &= M_n^{(7.2)} + O_{L^\infty-}(n^{-1/2}) + O_{L^\infty-}(n^{1/2}h) \end{aligned}$$

where

$$M_n^{(7.2)} = \frac{1}{2} n^{-1/2} \sum_{j=1}^n (C^{-1}(\partial_1 C)C^{-1})(Z_{t_{j-1}}, \theta_1^*) \left[\left(h^{-1/2} \int_{t_{j-1}}^{t_j} B(Z_t, \theta_1^*) dw_t \right)^{\otimes 2} - C(Z_{t_{j-1}}, \theta_1^*) \right]. \quad (7.2)$$

At the same time Itô's formula gives

$$\begin{aligned} h^{-1/2} \int_{t_{j-1}}^{t_j} B(Z_t, \theta_1^*) dw_t &= h^{-1/2} B(Z_{t_{j-1}}, \theta_1^*) \Delta_j w + h^{-1/2} \int_{t_{j-1}}^{t_j} (B(Z_t, \theta_1^*) - B(Z_{t_{j-1}}, \theta_1^*)) dw_t \\ &= h^{-1/2} B(Z_{t_{j-1}}, \theta_1^*) \Delta_j w + h^{-1/2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t B_x(Z_s, \theta_1^*) [B(Z_s, \theta_1^*) dw_s] dw_t \\ &\quad + h^{-1/2} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t L_B(Z_s, \theta_1^*, \theta_2^*, \theta_3^*) ds dw_t \end{aligned} \quad (7.3)$$

for $L_B(z, \theta_1, \theta_2, \theta_3)$ given by (3.9). The products of the first two terms on the right-hand side of (7.3) form martingale differences, and hence

$$M_n^{(7.2)} = M_n^{(1)} + O_{L^\infty-}(h^{1/2}) + O_{L^\infty-}(n^{1/2}h).$$

Under [A2] (i), (ii), (iii), the martingale central limit theorem gives

$$M_n^{(1)} \rightarrow^d N_{\mathbf{p}_1}(0, \Gamma^{(1)})$$

as $n \rightarrow \infty$. Consequently,

$$n^{-1/2} \partial_1 \mathbb{H}_n^{(1)}(\theta_1^*) \rightarrow^d N(0, \Gamma^{(1)}) \quad (7.4)$$

if [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 0, 2, 1, 0, 0)$ and [A2] (i), (ii), (iii) are fulfilled.

Next, suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (0, 0, 2, 3, 0, 0)$ and [A2] (i), (ii), (iii) are fulfilled. It is rather simple to prove

$$\sum_{i=0}^1 \sup_{u_1: \theta_1^* + \rho_n u_1 \in \Theta_1, |u_1| < 1} \left\| \partial_{u_1}^i \left(n^{-1} \partial_1^2 \mathbb{H}_n^{(1)}(\theta_1^* + \rho_n u_1) + \Gamma^{(1)} \right) \right\|_p \rightarrow 0 \quad (7.5)$$

for every $p > 1$ and any sequence $(\rho_n)_{n \in \mathbb{N}}$ of positive numbers tending to 0 as $n \rightarrow \infty$. We apply Sobolev's embedding inequality to each component of the matrix valued random field $\partial_1^2 \mathbb{H}_n^{(1)}(\theta_1^* + \rho_n u_1)$ on $\{u_1 \in \mathbb{R}^{p_1}; |u_1| < 1\}$ for large n . Then (7.5) gives

$$\left\| \sup_{u_1: \theta_1^* + \rho_n u_1 \in \Theta_1, |u_1| < 1} \left| n^{-1} \partial_1^2 \mathbb{H}_n^{(1)}(\theta_1^* + \rho_n u_1) + \Gamma^{(1)} \right| \right\|_p \rightarrow 0 \quad (7.6)$$

for every $p > 1$.

Suppose that [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 0, 2, 3, 0, 0)$, [A2] (i), (ii), (iii), and [A3] (i) are satisfied. Then differentiating $\mathbb{Y}^{(1)}$ twice, we see, from [A3] (i), that $\Gamma^{(1)}$ is positive-definite. By (a), $\hat{\theta}_1^0 \rightarrow^p \theta_1^*$. With this fact, we obtain (b) from (7.4) and (7.6). $\square$

Remark 7.2. It is possible to show that the quasi-Bayesian estimator (QBE) also enjoys the same asymptotic properties as the QMLE in Theorem 7.1, if we follows the argument in Yoshida [40]. This means we can use both estimators together with the estimator for θ_2 e.g. given in Section 8, to construct a one-step estimator for θ_3 based on the scheme presented in Section 4, and consequently we can construct a one-step estimator for $\theta = (\theta_1, \theta_2, \theta_3)$ by the method in Section 5.

8 Estimation of θ_2

This section will recall a standard construction of estimator for θ_2 . As usual, the scheme is adaptive. Suppose that an estimator $\hat{\theta}_1^0$ based on the data $(Z_{t_j})_{j=0,1,\dots,n}$ satisfies Condition [A4[#]] (i), i.e.,

$$\hat{\theta}_1^0 - \theta_1^* = O_p(n^{-1/2})$$

as $n \rightarrow \infty$. Obviously we can apply the estimator constructed in Section 7, but any estimator satisfying this condition can be used.

Define the random field $\mathbb{H}_n^{(2)}$ on $\bar{\Theta}_2$ by

$$\mathbb{H}_n^{(2)}(\theta_2) = -\frac{1}{2} \sum_{j=1}^n C(Z_{t_{j-1}}, \hat{\theta}_1^0)^{-1} [h^{-1}(\Delta_j X - hA(Z_{t_{j-1}}, \theta_2))^{\otimes 2}]. \quad (8.1)$$

We will denote by $\hat{\theta}_2^0 = \hat{\theta}_{2,n}^0$ any sequence of quasi-maximum likelihood estimator for $\mathbb{H}_n^{(2)}$, that is,

$$\mathbb{H}_n^{(2)}(\hat{\theta}_2^0) = \sup_{\theta_2 \in \bar{\Theta}_2} \mathbb{H}_n^{(2)}(\theta_2).$$

Let $\mathbb{Y}_n^{(2)}(\theta_2) = T^{-1}(\mathbb{H}_n^{(2)}(\theta_2) - \mathbb{H}_n^{(2)}(\theta_2^*))$, where $T = nh$.

Lemma 8.1. *Suppose that Conditions [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 0, 0)$, [A2] and [A4[#]] (i). Then*

$$\sup_{\theta_2 \in \bar{\Theta}_2} |\mathbb{H}_n^{(2)}(\theta_2) - \mathbb{Y}_n^{(2)}(\theta_2)| \rightarrow^p 0$$

as $n \rightarrow \infty$. If additionally [A3] (ii) is satisfied, then $\hat{\theta}_2^0 \rightarrow^p \theta_2^*$ as $n \rightarrow \infty$.

Proof.

$$\mathbb{Y}_n^{(2)}(\theta_2) = \Phi_n^{(8.2)}(\theta_2) + \Phi_n^{(8.3)}(\hat{\theta}_1^0, \theta_2)$$

where

$$\Phi_n^{(8.2)}(\theta_2) = -\frac{1}{2T} \sum_{j=1}^n hC(Z_{t_{j-1}}, \hat{\theta}_1^0)^{-1} [(A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta_2^*))^{\otimes 2}] \quad (8.2)$$

and

$$\Phi_n^{(8.3)}(\theta_1, \theta_2) = T^{-1} \sum_{j=1}^n C(Z_{t_{j-1}}, \theta_1)^{-1} [\Delta_j X - hA(Z_{t_{j-1}}, \theta_2^*), A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta_2^*)]. \quad (8.3)$$

For $\mathbb{Y}^{(2)}$ given by (2.2) on p.4,

$$\sum_{i=0}^1 \sup_{\theta_2 \in \Theta_2} \|\partial_2^i (\Phi_n^{(8.2)}(\theta_2) - \mathbb{Y}^{(2)}(\theta_2))\|_p \rightarrow 0 \quad (8.4)$$

for every $p > 1$. Here Conditions [A1] (i) with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 1, 1, 0, 0)$, [A2] (i)-(iii) and [A4[#]] (i) were used. Then (8.4) implies

$$\sup_{\theta_2 \in \bar{\Theta}_2} |\Phi_n^{(8.2)}(\theta_2) - \mathbb{Y}^{(2)}(\theta_2)| \rightarrow^p 0 \quad (8.5)$$

as $n \rightarrow \infty$.

We have

$$\sum_{i=0}^1 \sup_{(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2} \|\partial_{(\theta_1, \theta_2)}^i \Phi_n^{(8.3)}(\theta_1, \theta_2)\|_p \rightarrow^p 0 \quad (8.6)$$

for every $p > 1$ from Lemma 3.10 applied to $\Delta_j X - hA(Z_{t_{j-1}}, \theta_2^*)$ with the aid of orthogonality. The conditions we used include [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 0, 0)$. The embedding inequality makes

$$\sup_{(\theta_1, \theta_2) \in \bar{\Theta}_1 \times \bar{\Theta}_2} |\Phi_n^{(8.3)}(\theta_1, \theta_2)| \rightarrow^p 0 \quad (8.7)$$

from (8.6).

The proof completes by the estimates (8.5) and (8.7). $\square$

The matrix Γ_{22} is defined by (5.1) on p.19.

Lemma 8.2. *Under Conditions [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 1, 0, 0)$, [A2] and [A4[#]], Then*

$$\sup_{\theta_2 \in U(\theta_2^*, r_n)} |T^{-1} \partial_2^2 \mathbb{H}_n^{(2)}(\theta_2) - \Gamma_{22}| \rightarrow^p 0 \quad (8.8)$$

for any sequence of positive numbers r_n satisfying $r_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. From (8.1),

$$\partial_2 \mathbb{H}_n^{(2)}(\theta_2) = \sum_{j=1}^n C(Z_{t_{j-1}}, \hat{\theta}_1^0)^{-1} [\Delta_j X - hA(Z_{t_{j-1}}, \theta_2), \partial_2 A(Z_{t_{j-1}}, \theta_2)] \quad (8.9)$$

and

$$\begin{aligned} \partial_2^2 \mathbb{H}_n^{(2)}(\theta_2) &= - \sum_{j=1}^n C(Z_{t_{j-1}}, \hat{\theta}_1^0)^{-1} [(\partial_2 A(Z_{t_{j-1}}, \theta_2))^{\otimes 2}] h \\ &\quad + \sum_{j=1}^n C(Z_{t_{j-1}}, \hat{\theta}_1^0)^{-1} [\Delta_j X - hA(Z_{t_{j-1}}, \theta_2), \partial_2^2 A(Z_{t_{j-1}}, \theta_2)] \end{aligned} \quad (8.10)$$

Let

$$\hat{M}_n^{(2)} = T^{-1/2} \sum_{j=1}^n C(Z_{t_{j-1}}, \theta_1^*)^{-1} [B(Z_{t_{j-1}}, \theta_1^*) \Delta_j w, \partial_2 A(Z_{t_{j-1}}, \theta_2^*)] \quad (8.11)$$

By random field argument for (8.9) with $\partial_1 C(Z_{t_{j-1}}, \theta_1)$ and $[A4^\sharp]$, we obtain

$$T^{-1/2} \partial_2 \mathbb{H}_n^{(2)}(\theta_2^*) - \hat{M}_n^{(2)} \rightarrow^p 0 \quad (8.12)$$

as $n \rightarrow \infty$, if (3.10) in Lemma 3.4 applied to $\Delta_j X - hA(Z_{t_{j-1}}, \theta_2^*)$.

Under Conditions [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 1, 0, 0)$, [A2] and $[A4^\sharp]$, we obtain the convergence (8.8) for any sequence of positive numbers r_n satisfying $r_n \rightarrow 0$ as $n \rightarrow \infty$. Here we applied random field argument to the second term on the right-hand side of (8.10). $\square$

The matrix Γ_{22} is given by (5.1) on p.19. Form Lemmas 8.1, 8.1 and 8.2, the following theorem follows.

Theorem 8.3. (a) Suppose that Conditions [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 1, 2, 1, 0, 0)$, [A2], [A3] (ii) and $[A4^\sharp]$ (i). Then $\hat{\theta}_2^0 \rightarrow^p \theta_2^*$ as $n \rightarrow \infty$.

(b) Suppose that Conditions [A1] with $(i_A, j_A, i_B, j_B, i_H, j_H) = (1, 3, 2, 1, 0, 0)$, [A2], [A3] (ii) and $[A4^\sharp]$ (i). Then

$$(nh)^{1/2} (\hat{\theta}_2^0 - \theta_2^*) - \Gamma_{22}^{-1} \hat{M}_n^{(2)} \rightarrow^p 0$$

as $n \rightarrow \infty$. In particular,

$$(nh)^{1/2} (\hat{\theta}_2^0 - \theta_2^*) \rightarrow^d N(0, \Gamma_{22}^{-1})$$

as $n \rightarrow \infty$.

Remark 8.4. The estimator $\hat{\theta}_1^0$ in Section 7 is asymptotically orthogonal to $\hat{\theta}_2^0$ constructed in this section. Therefore, for that $\hat{\theta}_1^0$, we obtain the joint convergence

$$(n^{1/2}(\hat{\theta}_1^0 - \theta_1^*), (nh)^{1/2}(\hat{\theta}_2^0 - \theta_2^*)) \rightarrow^d N(0, \text{diag}[(\Gamma^{(1)})^{-1}, \Gamma_{22}^{-1}])$$

as $n \rightarrow \infty$, as is well known.

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9 Symbols and Conditions

9.1 Section 1

$$\begin{cases} dX_t &= A(Z_t, \theta_2)dt + B(Z_t, \theta_1)dw_t \\ dY_t &= H(Z_t, \theta_3)dt \end{cases}$$

9.2 Section 2

[A1] (i) $A \in C_p^{i_A, j_A}(\mathbb{R}^{dz} \times \Theta_2; \mathbb{R}^{dx})$ and $B \in C_p^{i_B, j_B}(\mathbb{R}^{dz} \times \Theta_1; \mathbb{R}^{dx} \otimes \mathbb{R})$.

(ii) $H \in C_p^{i_H, j_H}(\mathbb{R}^{dz} \times \Theta_3; \mathbb{R}^{dy})$.

[A2] (i) $\sup_{t \in \mathbb{R}_+} \|Z_t\|_p < \infty$ for every $p > 1$.

(ii) There exists a probability measure ν on $\mathbb{R}^{dz}$ such that

$$\frac{1}{T} \int_0^T f(Z_t) dt \xrightarrow{p} \int f(z) \nu(dz) \quad (T \rightarrow \infty)$$

for any bounded measurable function $f : \mathbb{R}^{dz} \rightarrow \mathbb{R}$.

(iii) The function $\theta_1 \mapsto C(Z_t, \theta_1)^{-1}$ is continuous on $\overline{\Theta}_1$ a.s., and

$$\sup_{\theta_1 \in \Theta_1} \sup_{t \in \mathbb{R}_+} \|\det C(Z_t, \theta_1)^{-1}\|_p < \infty$$

for every $p > 1$.

(iv) For the $\mathbb{R}^{dy} \otimes \mathbb{R}^{dy}$ valued function $V(z, \theta_1, \theta_3) = H_x(z, \theta_3)C(z, \theta_1)H_x(z, \theta_3)^*$, the function $(\theta_1, \theta_3) \mapsto V(Z_t, \theta_1, \theta_3)^{-1}$ is continuous on $\overline{\Theta}_1 \times \overline{\Theta}_3$ a.s., and

$$\sup_{(\theta_1, \theta_3) \in \Theta_1 \times \Theta_3} \sup_{t \in \mathbb{R}_+} \|\det V(Z_t, \theta_1, \theta_3)^{-1}\|_p < \infty$$

for every $p > 1$.

$$\mathbb{Y}^{(1)}(\theta_1) = -\frac{1}{2} \int \left\{ \text{Tr}(C(z, \theta_1)^{-1}C(z, \theta_1^*)) - \mathbf{d}_X + \log \frac{\det C(z, \theta_1)}{\det C(z, \theta_1^*)} \right\} \nu(dz).$$

Since $|\log x| \leq x + x^{-1}$ for $x > 0$, $\mathbb{Y}^{(1)}(\theta_1)$ is a continuous function on $\overline{\Theta}_1$ well defined under [A1] and [A2]. Let

$$\begin{aligned} \mathbb{Y}^{(J,1)}(\theta_1) &= -\frac{1}{2} \int \left\{ \text{Tr}(C(z, \theta_1)^{-1}C(z, \theta_1^*)) + \text{Tr}(V(z, \theta_1, \theta_3^*)^{-1}V(z, \theta_1^*, \theta_3^*)) - \mathbf{d}_Z \right. \\ &\quad \left. + \log \frac{\det C(z, \theta_1) \det V(\theta_1, \theta_3^*)}{\det C(z, \theta_1^*) \det V(\theta_1^*, \theta_3^*)} \right\} \nu(dz) \end{aligned}$$

Let

$$\mathbb{Y}^{(2)}(\theta_2) = -\frac{1}{2} \int C(z, \theta_1^*) [(A(z, \theta_2) - A(z, \theta_2^*))^{\otimes 2}] \nu(dz)$$

Let

$$\mathbb{Y}^{(3)}(\theta_3) = - \int 6V(z, \theta_1^*, \theta_3)^{-1} [(H(z, \theta_3) - H(z, \theta_3^*))^{\otimes 2}] \nu(dz).$$

The random field $\mathbb{Y}^{(3)}$ is well defined under [A1] and [A2]. Let

$$\mathbb{Y}^{(J,3)}(\theta_1, \theta_3) = - \int 6V(z, \theta_1, \theta_3)^{-1} [(H(z, \theta_3) - H(z, \theta_3^*))^{\otimes 2}] \nu(dz).$$

We will assume all or some of the following identifiability conditions

[A3] (i) There exists a positive constant χ_1 such that

$$\mathbb{Y}^{(1)}(\theta_1) \leq -\chi_1 |\theta_1 - \theta_1^*|^2 \quad (\theta_1 \in \Theta_1).$$

(i') There exists a positive constant χ'_1 such that

$$\mathbb{Y}^{(J,1)}(\theta_1) \leq -\chi'_1 |\theta_1 - \theta_1^*|^2 \quad (\theta_1 \in \Theta_1).$$

(ii) There exists a positive constant χ_2 such that

$$\mathbb{Y}^{(2)}(\theta_1) \leq -\chi_2 |\theta_2 - \theta_2^*|^2 \quad (\theta_2 \in \Theta_2).$$

(iii) There exists a positive constant χ_3 such that

$$\mathbb{Y}^{(3)}(\theta_3) \leq -\chi_3 |\theta_3 - \theta_3^*|^2 \quad (\theta_3 \in \Theta_3).$$

(iii') There exists a positive constant χ_3 such that

$$\mathbb{Y}^{(J,3)}(\theta_1, \theta_3) \leq -\chi_3 |\theta_3 - \theta_3^*|^2 \quad (\theta_1 \in \Theta_1, \theta_3 \in \Theta_3).$$

9.3 Section 3

$$L_H(z, \theta_1, \theta_2, \theta_3) = H_x(z, \theta_3)[A(z, \theta_2)] + \frac{1}{2}H_{xx}(z, \theta_3)[C(z, \theta_1)] + H_y(z, \theta_3)[H(z, \theta_3)].$$

$$G_n(z, \theta_1, \theta_2, \theta_3) = H(z, \theta_3) + \frac{h}{2}L_H(z, \theta_1, \theta_2, \theta_3).$$

$$\zeta_j = \sqrt{3} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt$$

$$\mathcal{D}_j(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} h^{-1/2}(\Delta_j X - hA(Z_{t_{j-1}}, \theta_2)) \\ h^{-3/2}(\Delta_j Y - hG_n(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3)) \end{pmatrix}.$$

9.4 Section 4

$$S(z, \theta_1, \theta_3) = \begin{pmatrix} C(z, \theta_1) & 2^{-1}C(z, \theta_1)H_x(z, \theta_3)^* \\ 2^{-1}H_x(z, \theta_3)C(z, \theta_1) & 3^{-1}H_x(z, \theta_3)C(z, \theta_1)H_x(z, \theta_3)^* \end{pmatrix}$$

$$S(z, \theta_1, \theta_3)^{-1} = \begin{pmatrix} C(z, \theta_1)^{-1} + 3H_x(z, \theta_3)^*V(z, \theta_1, \theta_3)^{-1}H_x(z, \theta_3) & -6H_x(z, \theta_3)^*V(z, \theta_1, \theta_3)^{-1} \\ -6V(z, \theta_1, \theta_3)^{-1}H_x(z, \theta_3) & 12V(z, \theta_1, \theta_3)^{-1} \end{pmatrix}$$

$$\hat{S}(z, \theta_3) = S(z, \hat{\theta}_1^0, \theta_3)$$

$$\mathbb{H}_n^{(3)}(\theta_3) = -\frac{1}{2} \sum_{j=1}^n \left\{ \hat{S}(Z_{t_{j-1}}, \theta_3)^{-1} [\mathcal{D}_j(\hat{\theta}_1^0, \hat{\theta}_2^0, \theta_3)^{\otimes 2}] + \log \det \hat{S}(Z_{t_{j-1}}, \theta_3) \right\}$$

$$\begin{aligned} \Psi_1(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1}\partial_1 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right] \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1}H_{xx}(z, \theta_3)[\partial_1 C(Z_{t_{j-1}}, \theta_1)] \end{pmatrix} \right] \end{aligned}$$

$$\begin{aligned} \Psi_2(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_1, \theta_2) \\ 2^{-1}\partial_2 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right] \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_1, \theta_2) \\ 2^{-1}H_x(z, \theta_3)[\partial_2 A(Z_{t_{j-1}}, \theta_2)] \end{pmatrix} \right] \end{aligned}$$

$$\begin{aligned} \tilde{\Psi}_2(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2) \\ 2^{-1}\partial_2 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right] \\ &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} \partial_2 A(Z_{t_{j-1}}, \theta_2) \\ 2^{-1}H_x(z, \theta_3)[\partial_2 A(Z_{t_{j-1}}, \theta_2)] \end{pmatrix} \right], \end{aligned}$$

where

$$\tilde{\mathcal{D}}_j(\theta_1^*, \theta_2^*, \theta_3^*) = \begin{pmatrix} \xi_j^{(3.11)} + \xi_j^{(3.12)} \\ h^{-3/2}(\xi_j^{(3.18)} + \xi_j^{(3.19)}) \end{pmatrix},$$

$$\begin{aligned}\Psi_3(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3) \right. \\ &\quad \left. \otimes \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right]\end{aligned}$$

$$\begin{aligned}\tilde{\Psi}_3(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3) \right. \\ &\quad \left. \otimes \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right]\end{aligned}$$

$$\Psi_{3,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) \end{pmatrix} \right]$$

$$\tilde{\Psi}_{3,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) \end{pmatrix} \right]$$

$$\Psi_{3,2}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right]$$

$$\Psi_{3,3}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \frac{1}{2} \sum_{j=1}^n (S^{-1}(\partial_3 S) S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2} - S(Z_{t_{j-1}}, \theta_1, \theta_3)]$$

$$\Psi_{33,1}(\theta_1, \theta_2, \theta_3) = - \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix}^{\otimes 2} \right]$$

$$\begin{aligned}\Psi_{33,2}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3) \right. \\ &\quad \left. \otimes \begin{pmatrix} 0 \\ \partial_3^2 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3^2 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right]\end{aligned}$$

$$\Psi_{33,3}(\theta_1, \theta_3) = -\frac{1}{2} \sum_{j=1}^n \{ (S^{-1}(\partial_3 S) S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) [\partial_3 S(Z_{t_{j-1}}, \theta_1, \theta_3)] \}$$

$$\begin{aligned}
\Psi_{33,4}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= -2 \sum_{j=1}^n S^{-1}(\partial_3 S) S^{-1}(Z_{t_{j-1}}, \theta_3) [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3) \\
&\quad \otimes \left(\begin{array}{c} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{array} \right)] \\
\Psi_{33,5}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \frac{1}{2} \sum_{j=1}^n \partial_3 \{ (S^{-1}(\partial_3 S) S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) \} [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2} - S(Z_{t_{j-1}}, \theta_1, \theta_3)].
\end{aligned}$$

9.5 Section 5

$$\mathbb{H}_n^{(1)}(\theta_1) = -\frac{1}{2} \sum_{j=1}^n \left\{ S(Z_{t_{j-1}}, \theta_1, \hat{\theta}_3^0)^{-1} [\mathcal{D}_j(\theta_1, \hat{\theta}_2^0, \hat{\theta}_3^0)^{\otimes 2}] + \log \det S(Z_{t_{j-1}}, \theta_1, \hat{\theta}_3^0) \right\}.$$

$$\mathbb{H}_n^{(2,3)}(\theta_2, \theta_3) = -\frac{1}{2} \sum_{j=1}^n \hat{S}(Z_{t_{j-1}}, \hat{\theta}_3^0)^{-1} [\mathcal{D}_j(\hat{\theta}_1^0, \theta_2, \theta_3)^{\otimes 2}].$$

Recall $\hat{S}(z, \theta_3) = S(z, \hat{\theta}_1^0, \theta_3)$.

$$\Phi_{22,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = -\sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\left(\begin{array}{c} \partial_2 A(Z_{t_{j-1}}, \theta'_2) \\ 2^{-1} \partial_2 L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3) \end{array} \right)^{\otimes 2} \right]$$

$$\begin{aligned}
&\Phi_{22,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_2, \theta''_3) \\
&= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \left(\begin{array}{c} \partial_2^2 A(Z_{t_{j-1}}, \theta''_2) \\ 2^{-1} H_x(Z_{t_{j-1}}, \theta''_3) [\partial_2^2 A(Z_{t_{j-1}}, \theta''_2)] \end{array} \right) \right]
\end{aligned}$$

$$\begin{aligned}
&\tilde{\Phi}_{22,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_2, \theta''_3) \\
&= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\tilde{\mathcal{D}}_j(\theta'_1, \theta'_2, \theta'_3), \left(\begin{array}{c} \partial_2^2 A(Z_{t_{j-1}}, \theta''_2) \\ 2^{-1} H_x(Z_{t_{j-1}}, \theta''_3) [\partial_2^2 A(Z_{t_{j-1}}, \theta''_2)] \end{array} \right) \right]
\end{aligned}$$

$$\begin{aligned}
&\Phi_{23,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_2, \theta''_3) \\
&= -\sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\left(\begin{array}{c} 0 \\ 2^{-1} \partial_3 L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3) \end{array} \right) \right. \\
&\quad \left. \otimes \left(\begin{array}{c} \partial_2 A(Z_{t_{j-1}}, \theta''_2) \\ 2^{-1} H_x(Z_{t_{j-1}}, \theta''_3) [\partial_2 A(Z_{t_{j-1}}, \theta''_2)] \end{array} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& \Phi_{23,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_2, \theta''_3) \\
&= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1} \partial_3 H_x(Z_{t_{j-1}}, \theta''_3) [\partial_2 A(Z_{t_{j-1}}, \theta''_2)] \end{pmatrix} \right] \\
\Phi_{33,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= - \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ \partial_3 H(Z_{t_{j-1}}, \theta'_3) + 2^{-1} h \partial_3 L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3) \end{pmatrix}^{\otimes 2} \right] \\
\Phi_{33,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_1, \theta''_2, \theta''_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3) \right. \\
&\quad \left. \otimes \begin{pmatrix} 0 \\ \partial_3^2 H(Z_{t_{j-1}}, \theta''_3) + 2^{-1} h \partial_3^2 L_H(Z_{t_{j-1}}, \theta''_1, \theta''_2, \theta''_3) \end{pmatrix} \right] \\
\Psi_{1,1}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \Psi_1(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) \\
&= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ 2^{-1} \partial_1 L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) \end{pmatrix} \right] \\
\Psi_{1,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \frac{1}{2} \sum_{j=1}^n (S^{-1}(\partial_1 S)) S^{-1}(Z_{t_{j-1}}, \theta_1, \theta_3) [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2} - S(Z_{t_{j-1}}, \theta_1, \theta_3)] \\
\Psi_{11,1}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\begin{pmatrix} 0 \\ 2^{-1} \partial_1 L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3) \end{pmatrix}^{\otimes 2} \right] \\
\Psi_{11,2}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_1, \theta''_2, \theta''_3) &= \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \right. \\
&\quad \left. \otimes \begin{pmatrix} 0 \\ 2^{-1} \partial_1^2 L_H(Z_{t_{j-1}}, \theta''_1, \theta''_2, \theta''_3) \end{pmatrix} \right] \\
\Psi_{11,3}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= \sum_{j=1}^n \partial_1 \{ S^{-1}(\partial_1 S) S^{-1}(Z_{t_{j-1}}, \theta_1, \theta_3) \} [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2} - S(Z_{t_{j-1}}, \theta'_1, \theta'_3)] \\
\Psi_{11,4}(\theta_1, \theta_3) &= \sum_{j=1}^n (S^{-1}(\partial_1 S) S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) [\partial_1 S(Z_{t_{j-1}}, \theta_1, \theta_3)] \\
\Psi_{11,5}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3, \theta''_1, \theta''_2, \theta''_3) &= \sum_{j=1}^n (S^{-1}(\partial_1 S) S^{-1})(Z_{t_{j-1}}, \theta_1, \theta_3) \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \right. \\
&\quad \left. \otimes \begin{pmatrix} 0 \\ 2^{-1} \partial_1 L_H(Z_{t_{j-1}}, \theta''_1, \theta''_2, \theta''_3) \end{pmatrix} \right].
\end{aligned}$$

9.6 Section 6

$$\mathbb{H}_n(\theta) = -\frac{1}{2} \sum_{j=1}^n \left\{ S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} [\mathcal{D}_j(\theta_1, \theta_2, \theta_3)^{\otimes 2}] + \log \det S(Z_{t_{j-1}}, \theta_1, \theta_3) \right\}$$

for $\theta = (\theta_1, \theta_2, \theta_3)$.

$$\mathbb{D}_n(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) = \mathbb{H}_n(\theta_1, \theta_2, \theta_3) - \mathbb{H}_n(\theta'_1, \theta'_2, \theta'_3).$$

$$\begin{aligned} & \mathbb{D}_n^{[1]}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) \\ &= -\frac{1}{2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\left(\begin{array}{c} h^{1/2}(A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta'_2)) \\ h^{-1/2}(H(Z_{t_{j-1}}, \theta_3) - H(Z_{t_{j-1}}, \theta'_3)) \\ + 2^{-1} h^{1/2}(L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) - L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3)) \end{array} \right) \right]^{\otimes 2} \end{aligned}$$

$$\mathbb{D}_n^{[2]}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) = h^{-1/2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} 0 \\ H(Z_{t_{j-1}}, \theta_3) - H(Z_{t_{j-1}}, \theta'_3) \end{pmatrix} \right]$$

$$\begin{aligned} & \mathbb{D}_n^{[3]}(\theta_1, \theta_2, \theta_3, \theta'_1, \theta'_2, \theta'_3) \\ &= h^{1/2} \sum_{j=1}^n S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} \left[\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3), \begin{pmatrix} A(Z_{t_{j-1}}, \theta_2) - A(Z_{t_{j-1}}, \theta'_2) \\ 2^{-1}(L_H(Z_{t_{j-1}}, \theta_1, \theta_2, \theta_3) - L_H(Z_{t_{j-1}}, \theta'_1, \theta'_2, \theta'_3)) \end{pmatrix} \right] \end{aligned}$$

$$\begin{aligned} \mathbb{D}_n^{[4]}(\theta_1, \theta_3, \theta'_1, \theta'_2, \theta'_3) &= -\frac{1}{2} \sum_{j=1}^n \left\{ (S(Z_{t_{j-1}}, \theta_1, \theta_3)^{-1} - S(Z_{t_{j-1}}, \theta'_1, \theta'_3)^{-1}) [\mathcal{D}_j(\theta'_1, \theta'_2, \theta'_3)^{\otimes 2}] \right. \\ &\quad \left. + \log \frac{\det S(Z_{t_{j-1}}, \theta_1, \theta_3)}{\det S(Z_{t_{j-1}}, \theta'_1, \theta'_3)} \right\} \end{aligned}$$