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A planar Ising model of self-organized criticality

Nicolas Forien^{*†}

February 19, 2020

Abstract

We consider the planar Ising model in a finite square box and we replace the temperature parameter with a function depending on the magnetization. This creates a feedback from the spin configuration onto the parameter, which drives the system towards the critical point. Using the finite-size scaling results of [CM11], we show that, when the size of the box grows to infinity, the temperature concentrates around the critical temperature of the planar Ising model on the square lattice.

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1 Introduction

1.1 Definition of the model and convergence result

In this article, we build a simple variant of the two-dimensional Ising model which presents a phenomenon of “self-organized criticality”. To define this model, we consider square boxes $\Lambda(n) \subset \mathbb{Z}^2$ of side n , we choose a real parameter $a > 0$, and we set, for any spin configuration $\sigma : \Lambda(n) \rightarrow \{-, +\}$,

$$T_n(\sigma) = \frac{(m(\sigma))^2}{n^{2a}} = \frac{1}{n^{2a}} \left(\sum_{x \in \Lambda(n)} \sigma(x) \right)^2.$$

We then define the probability distribution

$$\mu_n : \sigma \in \{-, +\}^{\Lambda(n)} \mapsto \frac{1}{Z_n} \mu_{n, T_n(\sigma)}^+(\sigma),$$

where Z_n is the appropriate normalization constant and $\mu_{n, T}^+$ is the standard Ising measure at temperature T and $+$ boundary conditions on the box $\Lambda(n)$ with no external magnetic field (see section 2.3 for the precise definition). In our model, the fixed temperature T of the Ising model is replaced with this function T_n of the configuration itself, creating a feedback from the configuration onto the temperature parameter. The goal is to obtain a model whose temperature concentrates around the critical temperature T_c of the Ising model when the size of the box grows to infinity, without having to tune a parameter to a precise critical value. We prove, in dimension 2, the following convergence result:

Theorem 1. *If the parameter a is chosen such that $81/41 < a < 2$, then the law of T_n under μ_n converges to δ_{T_c} when $n \rightarrow \infty$, and we have the following estimate on the convergence speed:*

$$\forall \varepsilon > 0 \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mu_n \left(|T_n - T_c| \geq \varepsilon \right) < 0. \tag{1}$$

We do not think that this constant 81/41 is optimal, since it comes from the hypotheses of [CM11], which are not deemed to be optimal. We explain in paragraph 1.4 how the exponent 81/41 would evolve if the results of [CM11] were to be improved.

1.2 Self-organized criticality

Most lattice models which present a phase transition have a particularly interesting behaviour at their critical point, with (conjectured or proven) properties of conformal invariance, self-similarity and power-law correlations, which turn out to be common within a large class of similar models in statistical mechanics. The physicists Per Bak, Chao Tang and Kurt Wiesenfeld noted in [BTW88] that these universal features can be observed in various physical or biological systems. But, these properties being very specific to the critical point in a phase transition, it seems surprising to meet

them by chance in nature, since it should be very unlikely to have the parameters of a physical system precisely tuned to their critical value.

According to [BTW88], an explanation is that some physical systems tend to be naturally attracted towards a critical state: this phenomenon is called “self-organized criticality”. Examples of self-critical models include the sandpile model [JR08], forest fires [Ber12], avalanche processes [BF09], biological evolution [dBDF⁺94], neural systems [HG14], or sociology [SWdA⁺00], and self-organized criticality was recently experimentally observed in an ultra-cold atomic gas [HAL⁺20]. But these systems turn out to be very difficult to analyze rigorously and there are few self-critical models which are simple enough to be amenable to mathematical study but complex enough to enclose the relevant features of self-organized criticality.

A natural idea to obtain a self-critical model, described in [Sor06], is to start with a model which exhibits a phase transition, and to introduce a feedback by replacing the control parameter (e.g., temperature) with a function of the configuration. If this feedback function is well chosen, it can drive the system towards its critical point. In [CG16], a simple model of self-organized criticality was built from the generalized Curie-Weiss Ising model by using this technique. In [CF19], we have defined a similar model constructed from Bernoulli percolation. Therefore, it was natural to try to extend this result to the Ising model, using the random-cluster representation. But the general technique of [CF19] turned out to be hard to apply in the more general setting of FK-percolation, because of the non-poissonian way the edges become open when the percolation parameter is raised to the critical point from below [DCGP14].

Therefore, we adopt a slightly different approach, and we use the results of [CM11] about the near-critical regime of the planar FK-Ising model. Hence, this proof is very specific to the two-dimensional setting, and we are only able to study a very small window for the parameter a . An improvement of the hypotheses of [CM11] would enlarge our window, but since it only deals with the slightly supercritical regime (when $p - p_c$ tends to zero but is positive), a different method would be needed to study the case of lower parameters a .

1.3 Outline of the paper

To control the deviations of the random variable T_n , we write, for $\varepsilon > 0$,

$$\begin{aligned} \mu_n\left(T_n \geq T_c + \varepsilon\right) &= \frac{1}{Z_n} \sum_{\sigma \in \{-,+\}^{\Lambda(n)}} \mathbb{1}[T_n(\sigma) \geq T_c + \varepsilon] \mu_{n, T_n(\sigma)}^+(\sigma) \\ &= \frac{1}{Z_n} \sum_{b=0}^{n^2} \mathbb{1}\left[b^2/n^{2a} \geq T_c + \varepsilon\right] \sum_{\sigma : |m(\sigma)|=b} \mu_{n, b^2/n^{2a}}^+(\sigma) \\ &= \frac{1}{Z_n} \sum_{b=0}^{n^2} \mathbb{1}\left[b \geq n^a \sqrt{T_c + \varepsilon}\right] \mu_{n, b^2/n^{2a}}^+(|m| = b) \\ &\leq \frac{n^2 + 1}{Z_n} \sup_{T \geq T_c + \varepsilon} \mu_{n, T}^+(|m| \geq n^a \sqrt{T_c + \varepsilon}). \end{aligned} \tag{2}$$

Similarly, we have, for $\varepsilon > 0$,

$$\mu_n\left(T_n \leq T_c - \varepsilon\right) \leq \frac{n^2 + 1}{Z_n} \sup_{T \leq T_c - \varepsilon} \mu_{n, T}^+(|m| \leq n^a \sqrt{T_c - \varepsilon}). \tag{3}$$

Therefore, our strategy consists in proving exponential decay results for

$$\sup_{T \geq T_c + \varepsilon} \mu_{n, T}^+(|m| \geq An^a) \quad \text{and} \quad \sup_{T \leq T_c - \varepsilon} \mu_{n, T}^+(|m| \leq An^a) \tag{4}$$

for fixed $A > 0$ and $\varepsilon > 0$, when the size n of the box tends to infinity. This is done in sections 3 and 4, after some general definitions and notations are introduced in section 2. These exponential

estimates are quite standard for a fixed temperature $T \neq T_c$, but since we do not have a monotony property with respect to T for the magnetization, uniform estimates on $[0, T_c - \varepsilon]$ and on $[T_c + \varepsilon, +\infty)$ cannot be deduced from the pointwise exponential decay. Therefore, we work with the random-cluster representation of the Ising model, whose monotony property helps us to obtain uniform exponential bounds for (4).

But these exponential decay results are not enough to prove theorem 1, since it could be the case that the denominator Z_n in (2) and (3) also decays exponentially. Thus, we need to show at least that Z_n does not decay as fast as the two quantities in (4). This is the key point of our proof, detailed in section 6. In paragraph 1.5 below, we present the strategy to obtain this lower bound on the partition function, which is close to the strategy followed in [CF19]. But here, instead of building a monotone coupling of configurations and looking for a fixed point, we guess the value of this fixed point, because the results of finite-size scaling in [CM11] indicate which speed of convergence to p_c is required to obtain a given magnetization. Therefore, our method is very specific to the two-dimensional Ising model, and an extension to higher dimensions for example would require other ingredients.

In all three regimes (supercritical in section 3, subcritical in section 4, and near-critical in section 6), we control the magnetization of the Ising model with the help of the random-cluster model, which is linked to the Ising model through the Edwards-Sokal coupling (see paragraph 2.5). In an Ising configuration obtained from a FK-percolation configuration with this coupling, the magnetization is the result of two factors: on the one hand, the number of vertices connected to the boundary of the box, and on the other hand, the fluctuations coming from the spins attributed to the clusters which do not touch the boundary of the box. Therefore, we have to monitor both factors in the random-cluster model to obtain a control on the magnetization in the related Ising model.

To control the dependence of our result on the hypotheses of [CM11], we prove our lower bound on Z_n in section 6 under some finite-size scaling assumptions, which are proved in [CM11] but with hypotheses which are not deemed optimal. We show in section 5 how these finite-size scaling assumptions follow from [CM11]. This allows us to discuss how our admissible range for the parameter a would improve if the finite-size scaling postulates were extended to a broader window of percolation parameters.

1.4 Improvement of our exponent under finite-size scaling assumptions

Our proof relies on the results of [CM11], which give information on the set of the sites connected to the boundary in the planar FK-Ising model in a joint regime where $p \rightarrow p_c$ and $n \rightarrow \infty$ simultaneously, with

$$p - p_c \gg \frac{1}{n^{8/41}}. \quad (5)$$

This value of $8/41$ is not deemed optimal, since it is believed that the FK-percolation model should have a supercritical behaviour as long as n is much larger than the correlation length, which scales like $(p - p_c)^{-1}$ [DCGP14]. As we will see in paragraph 1.5, we will need to study FK-percolation in the regime $p - p_c \sim 1/n^{16-8a}$, hence our assumption $a > 81/41$, to fall in the regime (5). To keep track of the influence of this condition on our exponent a , we quote below which of the results of [CM11] are needed for our proof.

Definition 1. *We say that an exponent $s > 0$ satisfies the finite-size scaling assumptions, which we will denote by $\mathcal{FSS}(s)$, if for all $K, \delta > 0$ and for any real sequence $p_n \in [0, 1]$ satisfying*

$$p_n - p_c(2) \underset{n \rightarrow \infty}{\sim} \frac{K}{n^s},$$

we have

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1(\mathcal{F}_n) = 1,$$

with the event \mathcal{F}_n given by

$$\mathcal{F}_n = \left\{ \begin{array}{l} |\mathcal{M}_n| \leq (1 + \delta)\theta(p_n) |\Lambda(n)|, \quad |\mathcal{M}_n \cap \Lambda(n_1)| \geq (1 - \delta)\theta(p_n) |\Lambda(n_1)| \\ \text{and} \quad \max \{ |C(x)| : x \leftrightarrow \partial\Lambda(n) \} \leq n^{s+1/2} \end{array} \right\},$$

where $n_1 = \lfloor 5n/6 \rfloor$, $\phi_{n,p,q}^1$ is the finite-volume random-cluster measure with wired boundary conditions on the box $\Lambda(n)$ and \mathcal{M}_n denotes the set of vertices connected to the boundary $\partial\Lambda(n)$ of the box.

We translate the results of [CM11] into the following proposition:

Proposition 1. *We have $\mathcal{FSS}(s)$ for all $s < 8/41$.*

The next theorem shows how the constant $81/41$ would evolve if the results of [CM11] were to be improved. Together with proposition 1, it easily implies theorem 1. We do not know whether the value $31/16$ is optimal.

Theorem 2. *If $a \in (31/16, 2)$ is such that the finite-size scaling assumptions $\mathcal{FSS}(16 - 8a)$ hold, then the law of T_n under μ_n converges to δ_{T_c} when $n \rightarrow \infty$, and the estimate (1) holds.*

1.5 Heuristics for the lower bound on the partition function

We explain here the strategy to obtain a lower bound on the partition function Z_n of our model. This is the key step of our proof, detailed in section 6. We take $a \in (31/16, 2)$ such that the assumptions $\mathcal{FSS}(16 - 8a)$ hold, and we start by rewriting Z_n as

$$Z_n = \sum_{\sigma \in \{-,+\}^{\Lambda(n)}} \mu_{n,T_n(\sigma)}^+(\sigma) = \sum_{b=-n^2}^{n^2} \mu_{n,b^2/n^{2a}}^+(m=b). \quad (6)$$

Quest for the fixed point b_n : To obtain a lower bound on Z_n , we search for a value of b_n such that, at temperature $T = b_n^2/n^{2a}$, the magnetization is exactly b_n with probability high enough. If we choose b_n such that b_n^2/n^{2a} does not converge towards the critical temperature T_c , then we cannot obtain a slower decay than the decays proved in the subcritical regime (in section 3) and in the supercritical regime (in section 4). Thus, we will choose b_n such that $b_n \sim n^a \sqrt{T_c}$. Thanks to the Edwards-Sokal coupling (see paragraph 2.5), the Ising model at temperature $T = b_n^2/n^{2a}$ can be recovered from the random-cluster model with parameters $q = 2$ and $p = \varphi_n(b_n)$, where φ_n is the function defined by

$$\varphi_n : b \in \{-n^2, \dots, n^2\} \mapsto \begin{cases} 1 & \text{if } b = 0, \\ 1 - \exp\left(-\frac{2n^{2a}}{b^2}\right) & \text{otherwise.} \end{cases} \quad (7)$$

For any percolation configuration $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$, the set of vertices connected to the boundary is denoted by $\mathcal{M}_n(\omega)$ (see section 2 for all these general definitions). The idea is to choose b_n such that, under the law $\phi_{n,\varphi_n(b_n),2}^1$, the number of vertices connected to the boundary is typically of the order of b_n , and then to control the magnetization of the clusters which do not touch the boundary of the box. Therefore, a natural strategy consists in proving a lower bound on the probability that, on the one hand, $|\mathcal{M}_n(\omega)| = b_n$, and on the other hand, the contribution of the other clusters to the magnetization cancels out, so as to attain a magnetization exactly equal to b_n . To obtain a percolation configuration ω such that $|\mathcal{M}_n(\omega)| = b_n$, it is simpler to require as a first step that $|\mathcal{M}_n|$ be between b_n and λn^a , with $\lambda > \sqrt{T_c}$, and in a second step to close some edges to reach exactly b_n . Therefore, we take b_n satisfying

$$\theta(\varphi_n(b_n)) |\Lambda(n)| \stackrel{n \rightarrow \infty}{\sim} \mu n^a, \quad \text{with} \quad \sqrt{T_c} < \mu < \lambda.$$

According to the asymptotics for $\theta(p)$ given by the exact computations of Onsager [Ons44] and Yang [Yan52], we need that

$$\varphi_n(b_n) - p_c(2) \stackrel{n \rightarrow \infty}{\sim} \frac{p_c(2)}{8} \left(\frac{\mu}{n^{2-a}} \right)^8.$$

We will then have, under the assumptions $\mathcal{FSS}(16 - 8a)$,

$$\lim_{n \rightarrow \infty} \phi_{n, \varphi_n(b_n), 2}^1 \left(b_n \leq |\mathcal{M}_n| \leq \lambda n^a \right) = 1. \quad (8)$$

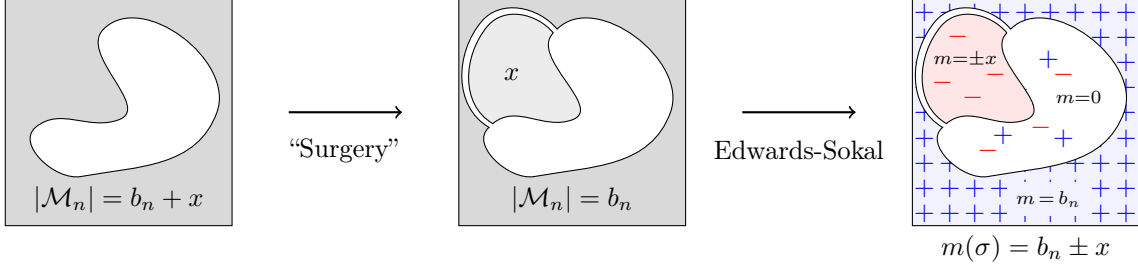


Figure 1: A strategy which seems natural is, when there are $b_n + x$ vertices connected to the boundary, to disconnect a piece of size x to obtain $|\mathcal{M}_n| = b_n$. But, doing so, the disconnected piece makes it harder to bound the magnetization in the corresponding Ising model, and hence we do not proceed this way.

Trick of the halfway cut: Starting from a configuration satisfying (8), a natural idea is to close a certain set of edges to disconnect exactly $|\mathcal{M}_n| - b_n$ vertices from the boundary, in order to obtain $|\mathcal{M}_n| = b_n$. But a problem arises when trying to control the fluctuations of the magnetization of the clusters which do not touch the boundary, because the set we have disconnected from the boundary has a size of order n^a . Hence, even if this set is not necessarily connected, its contribution to the magnetization might be too high to be compensated. Therefore, we bypass this problem by disconnecting twice less vertices than needed to reach $|\mathcal{M}_n| = b_n$, and then forcing the set of the vertices thus disconnected to choose a negative spin. In that way, the overall magnetization resulting from the vertices connected to the boundary and from this set of disconnected vertices altogether will be b_n (or $b_n + 1$), and the only thing left to do will be to force the magnetization of the other clusters to cancel out.

Construction of the cutting: How can we find a set of edges whose closure disconnects from the boundary exactly

$$\left\lceil \frac{|\mathcal{M}_n(\omega)| - b_n}{2} \right\rceil$$

vertices, and which is not too big, for the closure of these edges not to be too "expensive"? The idea is to consider a sub-box of side $n_1 = \lfloor 5n/6 \rfloor$, and to use again the hypotheses $\mathcal{FSS}(16 - 8a)$, which ensure that

$$\lim_{n \rightarrow \infty} \phi_{n, \varphi_n(b_n), 2}^1 \left(|\mathcal{M}_n \cap \Lambda(n_1)| \geq \nu n^a \right) = 1, \quad \text{with} \quad \nu < \left(\frac{5}{6} \right)^2 \mu.$$

If ν also satisfies

$$\nu > \frac{\lambda - \sqrt{T_c}}{2},$$

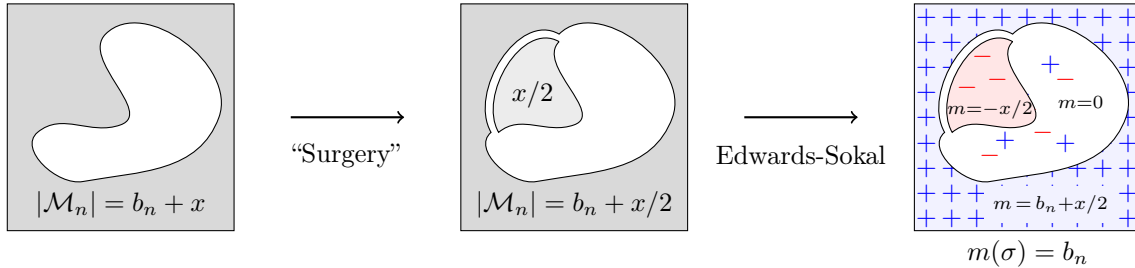


Figure 2: Our strategy to prove the lower bound on Z_n , with the surgery step implemented in section 6.3 and the forced colouring of section 6.4.

then we will have, with probability converging to one,

$$|\mathcal{M}_n \setminus \Lambda(n_1)| = |\mathcal{M}_n| - |\mathcal{M}_n \cap \Lambda(n_1)| \leq \frac{|\mathcal{M}_n|}{2} + \frac{\lambda n^a}{2} - \nu n^a \leq \frac{|\mathcal{M}_n|}{2} + \frac{\sqrt{T_c}}{2} n^a \simeq \frac{|\mathcal{M}_n| + b_n}{2},$$

meaning that it is enough to disconnect the smaller box $\Lambda(n_1)$ in order to go half the way separating $|\mathcal{M}_n|$ from b_n . In what follows, we will take $\lambda = 4$, $\mu = 3$ and $\nu = 2$. Using the pigeonhole principle, we will show that it does not cost more than $O(n^{a-1})$ edges to disconnect this sub-box from the boundary (this will be done in lemma 15). But doing this, we may have cut too much, so we use the geometrical lemma of [CF19] to adjust the cutting to obtain the desired result, for a cost of $O(n^{a/2})$ edges.

Control of the fluctuations of the magnetization: We then construct an Ising configuration from the percolation configuration we have obtained with the cutting procedure. It does not cost more than $2^{-O(n^{a/2})}$ to force the set of the vertices we have disconnected to choose a minus spin, because this set is made up of at most $O(n^{a/2})$ clusters. Regarding the other clusters which were already not connected to the boundary of the box before the cutting operation, we have a bound N on their size given by the finite-size assumptions $\mathcal{FSS}(16 - 8a)$. This allows us to show that, with sufficient probability, the magnetization resulting from these clusters will fall inside the range $[-N, N]$. To compensate this contribution, we will force the spins of the clusters of size 1, after having shown that, with high probability, there are at least N such unit clusters. All this will allow us to build an Ising configuration with a magnetization exactly equal to b_n , with a small parity issue that we will dodge by choosing b_n with the right parity, the magnetization being constrained to have the same parity as $|\Lambda(n)|$.

1.6 A more natural model?

The probability density of our model is

$$\mu_n : \sigma \mapsto \frac{1}{Z_n} \frac{1}{Z_{n, T_n(\sigma)}^+} \exp\left(-\frac{\mathcal{H}_n^+(\sigma)}{T_n(\sigma)}\right), \quad (9)$$

where \mathcal{H}_n^+ is the Hamiltonian of the standard Ising model, defined in (13). An other distribution which may look more natural to consider is

$$\mu'_n : \sigma \mapsto \frac{1}{Z'_n} \exp\left(-\frac{\mathcal{H}_n^+(\sigma)}{T_n(\sigma)}\right), \quad (10)$$

where Z'_n is the required normalization constant. Most surprisingly, simulations seem to indicate that this other model might be too simple to exhibit the self-critical behaviour we are looking for. We give

here a simple heuristics to understand the difference between these two models. When simulating the Ising model with the Glauber dynamics (see section 8.2 of [Gri06]), the configuration is updated one spin after another, by looking at the impact of the spin flip on the energy. In the standard Ising model, the “energy” function writes

$$E_{Ising} = \frac{\mathcal{H}_n^+}{T}.$$

Therefore, when flipping one spin, the variation of energy is

$$\Delta E_{Ising} = \frac{\Delta \mathcal{H}_n^+}{T},$$

and the spin flip is more likely if it leads to a lower value for the Hamiltonian \mathcal{H}_n^+ . However, in the model (10), due to the fact that the temperature is no longer a constant, a spin flip leads to a change in energy given by

$$\Delta E_{\mu_n} = \Delta \left(\frac{\mathcal{H}_n^+}{T_n} \right) \approx \frac{\Delta \mathcal{H}_n^+}{T_n} - \frac{\mathcal{H}_n^+ \Delta T_n}{T_n^2} = \frac{\mathcal{H}_n^+}{T_n} \left(\frac{\Delta \mathcal{H}_n^+}{\mathcal{H}_n^+} - \frac{\Delta T_n}{T_n} \right). \quad (11)$$

There is a competition between the two terms in (11) to influence the spin flip: the first term favours the spin flips which minimize the Hamiltonian, whereas the second term encourages the spin flips which lead to an increase of the temperature (or a decrease, if the Hamiltonian is negative). Therefore, the dynamics of the Hamiltonian is perturbed by the dynamics on the temperature. For our feedback temperature function to create a self-organized behaviour, we need the dynamics on the Hamiltonian to outweigh the drift force in temperature. If this is the case, then the configuration has the time to reach a typical configuration at fixed temperature, and only once this equilibrium is reached, on a longer time scale, the temperature evolves and slowly drives the system towards criticality. Computer simulations confirm this idea that the dynamics on the Hamiltonian and on temperature must not compete but should occur on different time scales, in order to reach a self-critical state. Indeed, if one simulates the model (10) with the Glauber dynamics but without taking into account the change in temperature when deciding the spin flip, one gets a much more promising output, which is improved if the temperature parameter is updated only once in a while, leaving the configuration some time to reach equilibrium at fixed temperature before evaluating a new temperature parameter.

One may wonder why there is not the same unfortunate competition phenomenon with our model μ_n given by equation (9). In this model, if we take into account the influence of the partition function in the denominator, then the energy function writes

$$E_{\mu_n} = \ln Z_{n,T_n}^+ + \frac{\mathcal{H}_n^+}{T_n}.$$

Therefore, a small change in the configuration leads to a change in energy given by

$$\Delta E_{\mu_n} = \Delta \left(\ln Z_{n,T_n}^+ + \frac{\mathcal{H}_n^+}{T_n} \right) \approx \frac{\partial \ln Z_{n,T}^+}{\partial T} \Delta T_n + \frac{\Delta \mathcal{H}_n^+}{T_n} - \frac{\mathcal{H}_n^+ \Delta T_n}{T_n^2}.$$

Now recall that the partition function of the Ising model is such that

$$\frac{\partial \ln Z_{n,T}^+}{\partial T} = \frac{1}{Z_{n,T}^+} \frac{\partial Z_{n,T}^+}{\partial T} = \frac{1}{Z_{n,T}^+} \sum_{\sigma \in \{-,+\}^{\Lambda(n)}} \frac{\mathcal{H}_n^+}{T^2} \exp \left(-\frac{\mathcal{H}_n^+}{T} \right) = \frac{1}{T^2} \mu_{n,T}^+(\mathcal{H}_n^+),$$

which yields

$$\Delta E_{\mu_n} \approx \frac{\Delta \mathcal{H}_n^+}{T_n} + \left(\mu_{n,T_n}^+(\mathcal{H}_n^+) - \mathcal{H}_n^+ \right) \frac{\Delta T_n}{T_n^2}.$$

Hence, the factor in front of ΔT_n is tailored to be smaller than in (11), thanks to the compensation coming from the term $\mu_{n,T_n}^+(\mathcal{H}_n^+)$. This explains why, in our model, the effect of the Hamiltonian overcomes the temperature effect, which ensures that our self-tuning of the temperature parameter is delicate enough to preserve the equilibrium properties of the model.

2 Definitions and notations

2.1 Edges and boxes

The entire article takes place in dimension 2. The square box of side n centered at 0 is denoted

$$\Lambda(n) = \left[-\frac{n}{2}, \frac{n}{2} \right] \cap \mathbb{Z}^2 = \left\{ -\left\lfloor \frac{n}{2} \right\rfloor, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor \right\}^2.$$

We say that two points $x, y \in \mathbb{Z}^2$ are neighbours if $\|x - y\|_1 = 1$, which is denoted $x \sim y$. For any set of vertices $V \subset \mathbb{Z}^2$, we write

$$\mathbb{E}[V] = \left\{ \{x, y\} \subset V : x \sim y \right\}$$

for the set of edges between nearest neighbours of V . We define in this way $\mathbb{E}^2 = \mathbb{E}[\mathbb{Z}^2]$, as well as the edges of the box

$$\mathbb{E}_n = \mathbb{E}[\Lambda(n)].$$

For any set of vertices $V \subset \mathbb{Z}^2$, we define its interior boundary to be

$$\partial V = \left\{ x \in V : \exists y \in \mathbb{Z}^2 \setminus V \quad x \sim y \right\},$$

while its exterior boundary will be written

$$\partial^e V = \left\{ \{x, y\} \in \mathbb{E}^2 : x \in V \quad \text{and} \quad y \in \mathbb{Z}^2 \setminus V \right\}.$$

By a slight abuse of notations, we will say that a subset $V \subset \mathbb{Z}^2$ is connected if all the pairs of vertices of V are connected by an open path whose intermediate vertices all belong to V . For any finite set $V \subset \mathbb{Z}^2$, we define its “external boundary”, denoted by $\partial^{ext}V$, to be the set of the edges of the boundary $\partial^e V$ which connect a vertex of V to a vertex in the infinite connected component of $\mathbb{Z}^2 \setminus V$. The diameter of a finite non-empty set $V \subset \mathbb{Z}^2$ is defined by

$$\text{diam } V = \max_{x, y \in V} \|x - y\|_\infty.$$

2.2 Percolation configurations

An element $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$ is called a percolation configuration. Edges $e \in \mathbb{E}_n$ such that $\omega(e) = 1$ are said open in ω , while the other edges are said closed in ω . The space of configurations is endowed with a natural partial order defined by $\omega_1 \leq \omega_2$ if $\omega_1(e) \leq \omega_2(e)$ for all edges $e \in \mathbb{E}_n$. If μ_1 and μ_2 are two probability distributions on $\{0, 1\}^{\mathbb{E}_n}$, we say that μ_2 stochastically dominates μ_1 , which will be denoted $\mu_1 \preceq \mu_2$, if we have $\mu_1(X) \leq \mu_2(X)$ for every increasing random variable $X : \{0, 1\}^{\mathbb{E}_n} \rightarrow \mathbb{R}$. Given two vertices $x, y \in \Lambda(n)$, we write

$$x \xleftrightarrow{\omega} y$$

when there exists a path from x to y whose edges are all open in the configuration ω . For every vertex $x \in \Lambda(n)$, we write

$$C(x) = C(x, \omega) = \left\{ y \in \Lambda(n) : x \xleftrightarrow{\omega} y \right\}$$

for the connected component of x , or “cluster” of x , in the configuration ω . If $x \in \Lambda(n)$ and $Y \subset \Lambda(n)$, we write

$$x \xleftrightarrow{\omega} Y \quad \Leftrightarrow \quad \exists y \in Y \quad x \xleftrightarrow{\omega} y.$$

For any set of edges $H \subset \mathbb{E}_n$, we define the configuration ω_H obtained from ω by closing all the edges of H :

$$\omega_H : f \in \mathbb{E}_n \longmapsto \begin{cases} 0 & \text{if } f \in H, \\ \omega(f) & \text{otherwise.} \end{cases}$$

We define $\mathcal{M}_n(\omega)$ to be the set of the vertices connected to the boundary in the configuration ω , namely

$$\mathcal{M}_n(\omega) = \left\{ x \in \Lambda(n) : x \overset{\omega}{\longleftrightarrow} \partial\Lambda(n) \right\}.$$

The set of the open clusters in the configuration ω will be denoted

$$\mathcal{C}_n(\omega) = \left\{ C(x) : x \in \Lambda(n) \right\},$$

while $\mathcal{C}_n^-(\omega)$ indicates the set of the open clusters which do not touch the boundary, that is to say

$$\mathcal{C}_n^-(\omega) = \left\{ C(x) : x \in \Lambda(n) \text{ such that } x \not\leftrightarrow \partial\Lambda(n) \right\} = \left\{ C(x) : x \in \Lambda(n) \setminus \mathcal{M}_n(\omega) \right\}.$$

This provides us with a partition of the box $\Lambda(n)$ given by

$$\Lambda(n) = \bigsqcup_{C \in \mathcal{C}_n(\omega)} C = \mathcal{M}_n(\omega) \sqcup \bigsqcup_{C \in \mathcal{C}_n^-(\omega)} C.$$

For every integer $k \in \{1, \dots, n^2\}$, we will write

$$\mathcal{C}_n^-(k) = \mathcal{C}_n^-(k, \omega) = \left\{ C \in \mathcal{C}_n^-(\omega) : |C| = k \right\}$$

for the set of the open clusters which contain exactly k vertices and do not touch the boundary. Finally, let us note

$$k^0(\omega) = |\mathcal{C}_n(\omega)| \quad \text{and} \quad k^1(\omega) = |\mathcal{C}_n^-(\omega)| + 1 \quad (12)$$

for, respectively, the number of open clusters in the configuration ω , and the number of open clusters in ω when all the clusters touching the boundary are counted as one single cluster.

2.3 The Ising model

Fix an integer $n \geq 1$ and a real parameter $T > 0$. The Ising model in the box $\Lambda(n)$ at temperature T and with boundary condition $+$ is defined as the probability measure

$$\mu_{n,T}^+ : \sigma \in \{-, +\}^{\Lambda(n)} \mapsto \frac{1}{Z_{n,T}^+} \exp\left(-\frac{\mathcal{H}_n^+(\sigma)}{T}\right)$$

where $Z_{n,T}^+$ is the appropriate normalization constant, and the Hamiltonian \mathcal{H}_n^+ is given by

$$\mathcal{H}_n^+(\sigma) = \begin{cases} - \sum_{\{x,y\} \in \mathbb{E}_n} \sigma(x)\sigma(y) = -\frac{1}{2} \sum_{\substack{x,y \in \Lambda(n) \\ x \sim y}} \sigma(x)\sigma(y) & \text{if } \sigma(x) = + \text{ for all } x \in \partial\Lambda(n), \\ +\infty & \text{otherwise.} \end{cases} \quad (13)$$

We extend the above definition to the case of zero-temperature by setting

$$\mu_{n,0}^+ : \sigma \in \{-, +\}^{\Lambda(n)} \mapsto \begin{cases} 1 & \text{if } \sigma(x) = + \text{ for all } x \in \Lambda(n), \\ 0 & \text{otherwise.} \end{cases}$$

The magnetization of a configuration $\sigma : \Lambda(n) \rightarrow \{-, +\}$ is defined by

$$m(\sigma) = \sum_{x \in \Lambda(n)} \sigma(x).$$

We write $T_c = 2/\ln(\sqrt{2} + 1)$ for the critical temperature of the Ising model on the two-dimensional square lattice.

2.4 The random-cluster model

The random-cluster model on the box $\Lambda(n)$ with parameters $p \in [0, 1]$ and $q > 0$ and with boundary conditions $\xi \in \{0, 1\}$ is defined by the following probability density:

$$\phi_{n,p,q}^\xi(\omega) : \omega \in \{0, 1\}^{\mathbb{E}_n} \mapsto \frac{1}{Z_{n,p,q}^\xi} q^{k^\xi(\omega)} \prod_{e \in \mathbb{E}_n} p^{\omega(e)} (1-p)^{1-\omega(e)}$$

where the counting functions k^ξ were defined in (12) and $Z_{n,p,q}^\xi$ is the appropriate normalization constant. The boundary conditions $\xi = 1$ are called “wired”, while the boundary conditions $\xi = 0$ are called “free”. If $p \in [0, 1]$, $q \geq 1$ and $\xi \in \{0, 1\}$, then when n tends to infinity, the measure $\phi_{n,p,q}^\xi$ converges weakly to a probability distribution $\phi_{p,q}^\xi$ on the space $\{0, 1\}^{\mathbb{E}^2}$, equipped with the σ -algebra generated by the events depending on finitely many edges (see theorem 4.19 in [Gri06]). We write, for $p \in [0, 1]$, $q \geq 1$ and $\xi \in \{0, 1\}$,

$$\theta^\xi(p, q) = \phi_{p,q}^\xi \left(|C(0)| = \infty \right)$$

for the probability that the origin lies in an infinite open cluster, and $p_c(q)$ will denote the critical point of the random-cluster model, defined by

$$p_c(q) = \inf \left\{ p \in [0, 1] : \theta^1(p, q) > 0 \right\}.$$

We will sometimes omit the parameter q in the notation, which will mean that $q = 2$. In this particular case of $q = 2$, the measures $\phi_{p,2}^0$ and $\phi_{p,2}^1$ turn out to be equal (by corollary 3 in [Rao17]), thus we will just write $\theta(p) = \theta^1(p, 2)$. Finally, let us remark that, for $q = 1$, we recover Bernoulli percolation, where the states of different edges are independent.

2.5 Edwards-Sokal coupling

The random-cluster model with $q = 2$ is related to the Ising model through the Edwards-Sokal coupling. We reproduce here the result presented in section 1.4 of [Gri06]. Let $p \in (0, 1]$ and $T \geq 0$ be such that $p = 1 - e^{-2/T}$ (with the convention $e^{-2/0} = 0$). Let us consider the following probability distribution:

$$\mu : (\sigma, \omega) \in \{-, +\}^{\Lambda(n)} \times \{0, 1\}^{\mathbb{E}_n} \mapsto \frac{1}{Z_\mu} \prod_{e=\{x,y\} \in \mathbb{E}_n} (p \delta_{\sigma(x), \sigma(y)})^{\omega(e)} (1-p)^{1-\omega(e)} \prod_{x \in \partial \Lambda(n)} \delta_{\sigma(x), +}$$

where Z_μ is the appropriate normalizing constant. The marginal law of μ on $\{-, +\}^{\Lambda(n)}$ is the Ising measure $\mu_{n,T}^+$, while the marginal of μ on $\{0, 1\}^{\mathbb{E}_n}$ is the random-cluster measure $\phi_{n,p,2}^1$. For any percolation configuration $\omega \in \{0, 1\}^{\mathbb{E}_n}$, the conditional measure $\mu(\cdot | \omega)$ is obtained by letting $\sigma(x) = +$ for all $x \in \mathcal{M}_n(\omega)$ and by assigning constant spins on the other clusters, independent between different clusters, each spin being equally distributed on $\{-, +\}$. For any $\sigma : \Lambda(n) \rightarrow \{-, +\}$ satisfying $\sigma(x) = +$ for all $x \in \partial \Lambda(n)$, the conditional measure $\mu(\cdot | \sigma)$ is obtained by taking $\omega(e) = 0$ for all edges $e = \{x, y\}$ which are such that $\sigma(x) \neq \sigma(y)$, and by drawing $\omega(e)$ according to a Bernoulli law of parameter p for the other edges e , the state of these edges being conditionally independent. This coupling implies that the critical temperature of the Ising model and the critical probability of the random-cluster model are related by

$$p_c(2) = 1 - e^{-2/T_c} = \frac{\sqrt{2}}{1 + \sqrt{2}}. \quad (14)$$

2.6 Duality

The planar random-cluster model enjoys a useful duality property that we present here. We fix an integer $n \geq 2$, and we define the dual of the box $\Lambda(n)$ to be

$$\Lambda^*(n) = \Lambda(n-1) \pm \left(\frac{1}{2}, \frac{1}{2} \right) = \left\{ \left(x_1 \pm \frac{1}{2}, x_2 \pm \frac{1}{2} \right) : (x_1, x_2) \in \Lambda(n-1) \right\},$$

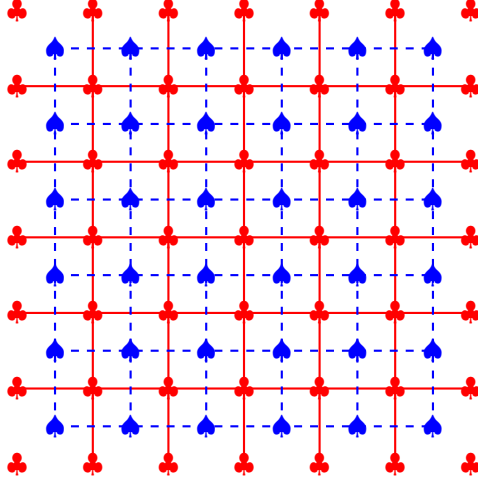


Figure 3: The box $(\Lambda(n), \mathbb{E}_n^{int})$ in solid red lines with clovers and its dual $(\Lambda^*(n), \mathbb{E}_n^*)$ in blue dashed lines with spades.

with a $-$ sign for even n and a $+$ sign otherwise, in such a way that all the vertices of $\Lambda^*(n)$ lie in the middle of the faces of the graph $(\Lambda(n), \mathbb{E}_n)$, as represented on figure 3. With the same convention \pm , let us define the dual edges to be

$$\mathbb{E}_n^* = \mathbb{E}_{n-1} \pm \left(\frac{1}{2}, \frac{1}{2} \right) = \left\{ \{x, y\} \subset \Lambda^*(n) : \|x - y\|_1 = 1 \right\}.$$

The interior edges of the box $\Lambda(n)$ are

$$\mathbb{E}_n^{int} = \left\{ \{x, y\} \in \mathbb{E}_n : \{x, y\} \not\subset \partial\Lambda(n) \right\}. \quad (15)$$

For any interior edge $e \in \mathbb{E}_n^{int}$, we write e^* for the edge of \mathbb{E}_n^* which intersects e perpendicularly in its middle, that is to say

$$\left\{ \begin{array}{l} \left\{ (x_1, x_2), (x_1 + 1, x_2) \right\}^* = \left\{ \left(x_1 + \frac{1}{2}, x_2 - \frac{1}{2} \right), \left(x_1 + \frac{1}{2}, x_2 + \frac{1}{2} \right) \right\} \\ \left\{ (x_1, x_2), (x_1, x_2 + 1) \right\}^* = \left\{ \left(x_1 - \frac{1}{2}, x_2 + \frac{1}{2} \right), \left(x_1 + \frac{1}{2}, x_2 + \frac{1}{2} \right) \right\} \end{array} \right.$$

We then have

$$\mathbb{E}_n^* = \left\{ e^* : e \in \mathbb{E}_n^{int} \right\}.$$

Thus, in the ordinary sense of duality for graphs, the graph $(\Lambda^*(n), \mathbb{E}_n^*)$ is the dual of the graph obtained from $(\Lambda(n), \mathbb{E}_n^{int})$ by identifying all the vertices of the boundary $\partial\Lambda(n)$. If $F \subset \mathbb{E}_n^{int}$ is a set of interior edges, we define $F^* = \{e^* : e \in F\}$. To any configuration $\omega \in \{0, 1\}^{\mathbb{E}_n}$, we associate a dual configuration $\omega^* \in \{0, 1\}^{\mathbb{E}_n^*}$ given by

$$\forall e \in \mathbb{E}_n^{int} \quad \omega^*(e^*) = 1 - \omega(e),$$

as represented on figure 4. If the graph $(\Lambda^*(n), \mathbb{E}_n^*)$ is identified with the box $(\Lambda(n-1), \mathbb{E}_{n-1})$, which amounts to applying a translation of vector $(\mp 1/2, \mp 1/2)$, we obtain a configuration $\omega^* \in \{0, 1\}^{\mathbb{E}_{n-1}}$. Following equation 6.12 in [Gri06], we have the following duality property:

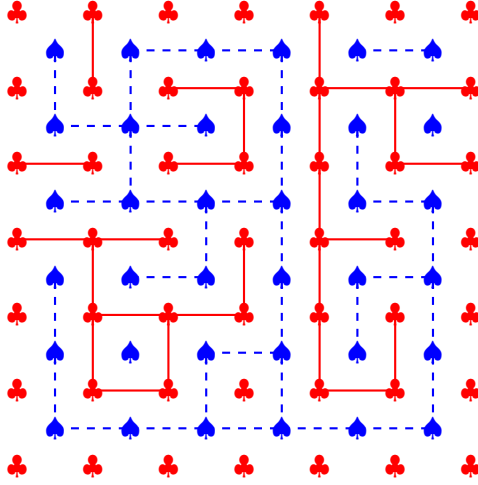


Figure 4: A configuration $\omega \in \{0, 1\}^{\mathbb{E}_n}$ in solid red lines (with only interior edges being represented) and its dual configuration $\omega^* \in \{0, 1\}^{\mathbb{E}_n^*}$ in blue dashed lines.

Proposition 2. *Take $p \in [0, 1]$, $q \geq 1$ and $n \geq 2$. Let $\omega \in \{0, 1\}^{\mathbb{E}_n}$ be a percolation configuration distributed according to $\phi_{n,p,q}^1$. Then, the associated dual configuration ω^* is distributed according to $\phi_{n-1,p^*,q}^0$, where the parameter p^* is given by the duality relation*

$$pp^* = q(1-p)(1-p^*).$$

3 Exponential decay for supercritical temperatures

The aim of this section is to prove the following result:

Lemma 1. *For all $a > 3/2$, we have*

$$\forall T_0 > T_c \quad \forall A > 0 \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{T \geq T_0} \ln \mu_{n,T}^+ \left(|m| \geq An^a \right) < 0.$$

We will use estimates for the subcritical random-cluster model concerning the number of vertices connected to the boundary and the cardinality of the clusters. We will then show in section 3.6 how to deduce from these estimates a control of the magnetization in the Ising model with supercritical temperatures.

3.1 Exponential decay of cluster sizes

We have the following exponential estimate at our disposal, valid with free boundary conditions.

Lemma 2. *For all $q \geq 1$ and for any $p < p_c(q)$, there exists $\psi = \psi(p, q) > 0$ such that*

$$\forall n, k \geq 1 \quad \forall v \in \Lambda(n) \quad \phi_{n,p,q}^0 \left(|C(v)| \geq k \right) \leq e^{-\psi k}.$$

Proof. Let $q \geq 1$. According to theorem 1.2 of [DCRT19], for all $p < p_c(q)$, there exists $c = c(p, q) > 0$ such that, for all $n \geq 1$,

$$\phi_{n,p,q}^1 \left(0 \longleftrightarrow \partial \Lambda(n) \right) \leq e^{-cn}.$$

Using theorem 5.86 in [Gri06], this implies that, for all $p < p_c(q)$, there exists $\psi = \psi(p, q) > 0$ such that

$$\forall n, k \geq 1 \quad \forall v \in \Lambda(n) \quad \phi_{p,q}^0(|C(v)| \geq k) \leq e^{-\psi k},$$

where $C(v)$ is the cluster of v in the finite box $\Lambda(n)$, as defined in paragraph 2.2. The variables $|C(v)|$ being increasing, it follows that

$$\forall n, k \geq 1 \quad \forall v \in \Lambda(n) \quad \phi_{n,p,q}^0(|C(v)| \geq k) \leq \phi_{p,q}^0(|C(v)| \geq k) \leq e^{-\psi k},$$

which is the required inequality. \square

3.2 Control of the number of vertices connected to the boundary

We give here an upper bound for the number of vertices connected to the boundary of the box.

Lemma 3. *For all $a > 1$, we have the following upper bound:*

$$\forall q \geq 1 \quad \forall p < p_c(q) \quad \forall A > 0 \quad \limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \phi_{n,p,q}^1(|\mathcal{M}_n| \geq An^a) < 0.$$

Proof. Let $a > 1$, $q \geq 1$, $p < p_c(q)$, $A > 0$ and $n \geq 1$. By the nesting property of random-cluster measures (lemma 4.13 in [Gri06]), we have

$$\phi_{n,p,q}^1(|\mathcal{M}_n| \geq An^a) = \phi_{n+2,p,q}^0(|\mathcal{M}_n| \geq An^a \mid \mathcal{E}_n) = \frac{\phi_{n+2,p,q}^0(\{|\mathcal{M}_n| \geq An^a\} \cap \mathcal{E}_n)}{\phi_{n+2,p,q}^0(\mathcal{E}_n)}, \quad (16)$$

where \mathcal{E}_n is the event

$$\mathcal{E}_n = \left\{ \forall e \in \mathbb{E}_{n+2} \setminus \mathbb{E}_n \quad \omega(e) = 1 \right\}.$$

Yet, all the vertices of $\partial\Lambda(n)$ are connected by paths using only edges of $\mathbb{E}_{n+2} \setminus \mathbb{E}_n$ (see figure 5). Therefore, if the event \mathcal{E}_n occurs, then all the vertices of \mathcal{M}_n belong to the same open cluster in the enlarged box $\Lambda(n+2)$. Fixing an arbitrary vertex $v_0 \in \partial\Lambda(n)$, we deduce that

$$\phi_{n+2,p,q}^0(\{|\mathcal{M}_n| \geq An^a\} \cap \mathcal{E}_n) \leq \phi_{n+2,p,q}^0(|C(v_0)| \geq An^a) \leq \exp(-\psi An^a), \quad (17)$$

where $\psi = \psi(p, q)$ is the constant given by lemma 2. Note now that the set $\mathbb{E}_{n+2} \setminus \mathbb{E}_n$ contains $8n+4$ edges, which implies by the finite-energy property (see theorem 3.1 in [Gri06]) that

$$\phi_{n+2,p,q}^0(\mathcal{E}_n) \geq \left(\frac{p}{p+q(1-p)} \right)^{8n+4}. \quad (18)$$

Combining (16), (17) and (18), we obtain

$$\phi_{n,p,q}^1(|\mathcal{M}_n| \geq An^a) \leq \left(\frac{p+q(1-p)}{p} \right)^{8n+4} \exp(-\psi An^a).$$

Given that $a > 1$, this entails that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^a} \ln \phi_{n,p,q}^1(|\mathcal{M}_n| \geq An^a) \leq -\psi A < 0,$$

which concludes the proof. \square

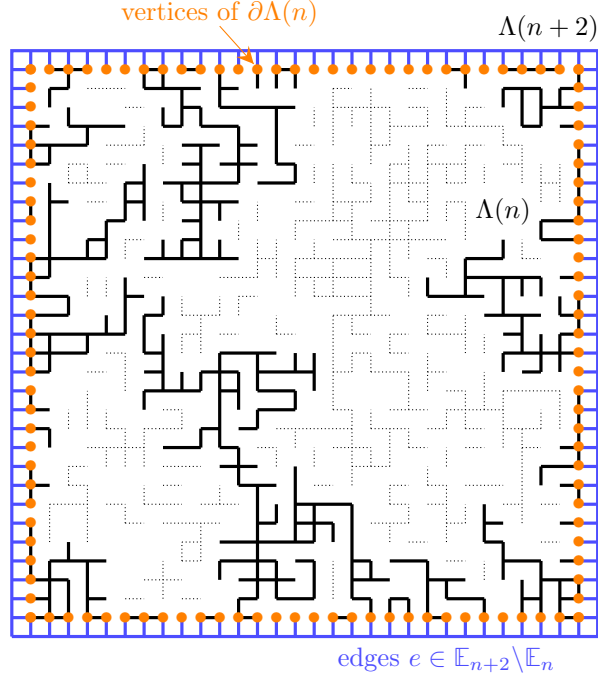


Figure 5: Illustration of the proof of lemma 3: to take wired boundary conditions at the borders of $\Lambda(n)$ amounts to conditioning on the event that the edges of $\mathbb{E}_{n+2} \setminus \mathbb{E}_n$ are all open. But, if these edges are open, then all the vertices of $\partial\Lambda(n)$ are connected inside the box $\Lambda(n+2)$, and hence all the vertices of \mathcal{M}_n end up belonging to a single open cluster inside $\Lambda(n+2)$.

3.3 The cost of a change of boundary conditions

We show here that it does not “cost” more than a factor q^{4n} to change the boundary conditions.

Lemma 4. *Let $q \geq 1$, $p \in [0, 1]$, $\xi \in \{0, 1\}$, and $n \geq 1$. For any event $\mathcal{A} \subset \{0, 1\}^{\mathbb{E}_n}$, we have*

$$\phi_{n,p,q}^{\xi}(\mathcal{A}) \leq q^{4n} \phi_{n,p,q}^{1-\xi}(\mathcal{A}).$$

Proof. Let $q \geq 1$, $p \in [0, 1]$, $n \geq 1$ and $\mathcal{A} \subset \{0, 1\}^{\mathbb{E}_n}$. Recall that k^0 denotes the number of open clusters in the box $\Lambda(n)$, while k^1 denotes the number of open clusters when all clusters touching the boundary are counted as one single cluster, as defined in (12). There cannot be more clusters touching the boundary than the number of vertices on this boundary, namely $|\partial\Lambda(n)| \leq 4n$, implying that

$$\forall \omega \in \{0, 1\}^{\mathbb{E}_n} \quad k^1(\omega) \leq k^0(\omega) \leq k^1(\omega) + 4n.$$

This entails that, on the one side,

$$\phi_{n,p,q}^0(\mathcal{A}) = \frac{\phi_{n,p,1}^0(q^{k^0} \mathbb{1}_{\mathcal{A}})}{\phi_{n,p,1}^0(q^{k^0})} \leq \frac{\phi_{n,p,1}^0(q^{k^1+4n} \mathbb{1}_{\mathcal{A}})}{\phi_{n,p,1}^0(q^{k^1})} = q^{4n} \frac{\phi_{n,p,1}^1(q^{k^1} \mathbb{1}_{\mathcal{A}})}{\phi_{n,p,1}^1(q^{k^1})} = q^{4n} \phi_{n,p,q}^1(\mathcal{A}),$$

and on the other side,

$$\phi_{n,p,q}^1(\mathcal{A}) = \frac{\phi_{n,p,1}^1(q^{k^1} \mathbb{1}_{\mathcal{A}})}{\phi_{n,p,1}^1(q^{k^1})} \leq \frac{\phi_{n,p,1}^0(q^{k^0} \mathbb{1}_{\mathcal{A}})}{\phi_{n,p,1}^0(q^{k^0-4n})} = q^{4n} \frac{\phi_{n,p,1}^0(q^{k^0} \mathbb{1}_{\mathcal{A}})}{\phi_{n,p,1}^0(q^{k^0})} = q^{4n} \phi_{n,p,q}^0(\mathcal{A}),$$

which concludes the proof. \square

3.4 Negative correlation between cluster sizes

We prove here a correlation inequality between the cardinalities of pairwise disjoint clusters, which is in a way a surrogate of the BK inequality, which is missing in the random-cluster model. Let $n \geq 1$. For $N \in \{1, \dots, n^2\}$, for any pairwise disjoint vertices $v_1, \dots, v_N \in \Lambda(n)$ and for any choice of integers $k_1, \dots, k_N \in \mathbb{N}$, we define the event

$$\mathcal{Q}_N(v_1, \dots, v_N, k_1, \dots, k_N) = \left\{ \forall i \leq N \quad |C(v_i)| \geq k_i \quad \text{and} \quad \forall i \neq j \quad v_i \not\leftrightarrow v_j \right\}.$$

Lemma 5. *Let $q \geq 1$, $p \in [0, 1]$, $\xi \in \{0, 1\}$ and $n \geq 1$. For any integer $N \in \{1, \dots, n^2\}$, for any pairwise distinct vertices $v_1, \dots, v_N \in \Lambda(n)$ and for any $k_1, \dots, k_N \in \mathbb{N}$, we have*

$$\phi_{n,p,q}^\xi \left(\mathcal{Q}_N(v_1, \dots, v_N, k_1, \dots, k_N) \right) \leq \prod_{i=1}^N \phi_{n,p,q}^\xi \left(|C(v_i)| \geq k_i \right). \quad (19)$$

Proof. Having fixed $q \geq 1$, $p \in [0, 1]$, $\xi \in \{0, 1\}$ and $n \geq 1$, we proceed by induction on N . The result is straightforward for $N = 1$. Let $N \in \{1, \dots, n^2 - 1\}$ be such that the inequality holds for N . Let $v_1, \dots, v_{N+1} \in \Lambda(n)$ be pairwise disjoint vertices and let $k_1, \dots, k_{N+1} \in \mathbb{N}$. Let us consider pairwise disjoint connected subsets $C_1, \dots, C_{N+1} \subset \Lambda(n)$ such that $v_i \in C_i$ for all $i \in \{1, \dots, N+1\}$. We consider the set of the edges delimiting the border of the union of the N first clusters, namely

$$F = \partial^e \left(\bigcup_{i=1}^N C_i \right) = \left\{ \{x, y\} \in \mathbb{E}_n : x \in \bigcup_{i=1}^N C_i \quad \text{and} \quad y \in \Lambda(n) \setminus \bigcup_{i=1}^N C_i \right\}.$$

If $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$ is a configuration such that $C(v_i) = C_i$ for all $i \in \{1, \dots, N\}$, then all the edges of F must be closed in ω . This enables us to write

$$\begin{aligned} \phi_{n,p,q}^\xi \left(\forall i \leq N+1 \quad C(v_i) = C_i \right) &= \phi_{n,p,q}^\xi \left(\forall i \leq N \quad C(v_i) = C_i \right) \\ &\quad \times \phi_{n,p,q}^\xi \left(C(v_{N+1}) = C_{N+1} \mid \forall i \leq N \quad C(v_i) = C_i \right) \\ &= \phi_{n,p,q}^\xi \left(\forall i \leq N \quad C(v_i) = C_i \right) \\ &\quad \times \phi_{n,p,q}^\xi \left(C(v_{N+1}) = C_{N+1} \mid \omega|_F = 0 \quad \text{and} \quad \forall i \leq N \quad C(v_i) = C_i \right). \end{aligned} \quad (20)$$

Yet, conditionally on the fact that all the edges of F are closed, the events

$$\left\{ C(v_{N+1}) = C_{N+1} \right\} \quad \text{and} \quad \left\{ \forall i \leq N \quad C(v_i) = C_i \right\}$$

are independent under $\phi_{n,p,q}^\xi$. This implies that

$$\begin{aligned} \phi_{n,p,q}^\xi \left(C(v_{N+1}) = C_{N+1} \mid \omega|_F = 0 \quad \text{and} \quad \forall i \leq N \quad C(v_i) = C_i \right) \\ = \phi_{n,p,q}^\xi \left(C(v_{N+1}) = C_{N+1} \mid \omega|_F = 0 \right), \end{aligned}$$

and thus equation (20) becomes

$$\begin{aligned} \phi_{n,p,q}^\xi \left(\forall i \leq N+1 \quad C(v_i) = C_i \right) &= \phi_{n,p,q}^\xi \left(\forall i \leq N \quad C(v_i) = C_i \right) \\ &\quad \times \phi_{n,p,q}^\xi \left(C(v_{N+1}) = C_{N+1} \mid \omega|_F = 0 \right). \end{aligned}$$

Summing over all the connected sets C_{N+1} such that

$$v_{N+1} \in C_{N+1} \subset \Lambda(n) \setminus \bigcup_{i=1}^N C_i \quad \text{and} \quad |C_{N+1}| \geq k_{N+1},$$

one gets

$$\begin{aligned} \phi_{n,p,q}^\xi \left(|C(v_{N+1})| \geq k_{N+1} \text{ and } \forall i \leq N \ C(v_i) = C_i \right) \\ \leq \phi_{n,p,q}^\xi \left(\forall i \leq N \ C(v_i) = C_i \right) \phi_{n,p,q}^\xi \left(|C(v_{N+1})| \geq k_{N+1} \mid \omega|_F = 0 \right). \end{aligned} \quad (21)$$

Note that the event $\{|C(v_{N+1})| \geq k_{N+1}\}$ is increasing, whereas the event $\{\omega|_F = 0\}$ is decreasing. Thus, it follows from the FKG inequality (see theorem 3.8 in [Gri06]) that

$$\phi_{n,p,q}^\xi \left(|C(v_{N+1})| \geq k_{N+1} \mid \omega|_F = 0 \right) \leq \phi_{n,p,q}^\xi \left(|C(v_{N+1})| \geq k_{N+1} \right).$$

Using this in (21) and summing over all the pairwise disjoint connected sets $C_1, \dots, C_N \subset \Lambda(n)$ such that $v_i \in C_i$ and $|C_i| \geq k_i$ for all $i \in \{1, \dots, N\}$ yields

$$\begin{aligned} \phi_{n,p,q}^\xi \left(Q_{N+1}(v_1, \dots, v_{N+1}, k_1, \dots, k_{N+1}) \right) \\ \leq \phi_{n,p,q}^\xi \left(\mathcal{Q}_N(v_1, \dots, v_N, k_1, \dots, k_N) \right) \phi_{n,p,q}^\xi \left(|C(v_{N+1})| \geq k_{N+1} \right). \end{aligned}$$

The induction hypothesis then allows to conclude, giving inequality (19). \square

3.5 Control of the size of the clusters

Recall that, for any percolation configuration $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$, we denote by $\mathcal{C}_n(\omega)$ the set of the open clusters in the box $\Lambda(n)$ in the configuration ω . We obtain here an exponential inequality on the tail of the distribution of the cardinalities of these clusters, uniformly on any segment included in $[0, p_c)$. This inequality will be useful to control the sum of the squares of cardinalities of clusters in lemma 7, as well as to obtain a similar control in the supercritical regime, thanks to a duality argument.

Lemma 6. *For $A, b, c \geq 0$ and $n \geq 1$, we consider the event*

$$\mathcal{D}_n(A, b, c) = \left\{ \sum_{C \in \mathcal{C}_n : |C| \geq n^b} |C| \geq An^c \right\} = \left\{ \left| \left\{ x \in \Lambda(n) : |C(x)| \geq n^b \right\} \right| \geq An^c \right\}.$$

Then we have the upper bound

$$\begin{aligned} \forall q \geq 1 \quad \forall p_0 < p_c(q) \quad \forall A > 0 \quad \forall b > 0 \quad \forall c \geq 0 \\ \limsup_{n \rightarrow \infty} \frac{1}{n^c} \sup_{p \leq p_0} \ln \phi_{n,p,q}^0 \left(\mathcal{D}_n(A, b, c) \right) < 0. \end{aligned}$$

Proof. Let $q \geq 1$, $p \leq p_0 < p_c(q)$, $A > 0$, $b > 0$, $c \geq 0$ and $n \geq 1$. Defining $M = \lceil An^{c-b} \rceil$, we claim that

$$\mathcal{D}_n(A, b, c) \subset \bigcup_{N=1}^M \bigcup_{\substack{v_1, \dots, v_N \in \Lambda(n) \\ \text{pairwise distinct}}} \bigcup_{\substack{n^b \leq k_1, \dots, k_N \leq n^2 \\ k_1 + \dots + k_N \geq An^c}} \mathcal{Q}_N(v_1, \dots, v_N, k_1, \dots, k_N), \quad (22)$$

where \mathcal{Q}_N is the event defined in section 3.4. Indeed, let $\omega \in \mathcal{D}_n(A, b, c)$, and choose a collection of clusters

$$B \subset \left\{ C \in \mathcal{C}_n : |C| \geq n^b \right\} \quad \text{such that} \quad \sum_{C \in B} |C| \geq An^c, \quad (23)$$

the collection B being chosen minimal, in the sense of inclusion, among the ones satisfying (23). Let $N = |B|$, and pick an arbitrary cluster $C_0 \in B$. Having chosen B minimal, we have

$$(N-1)n^b \leq \sum_{C \in B \setminus \{C_0\}} |C| < An^c,$$

which entails that $N < 1 + An^{c-b}$ and thus $N \leq M$. Now if we write $B = \{C_1, \dots, C_N\}$ and choose a vertex $v_i \in C_i$ and let $k_i = |C_i|$ for every $i \in \{1, \dots, N\}$, then we have

$$\omega \in \mathcal{Q}_N(v_1, \dots, v_N, k_1, \dots, k_N).$$

Hence, we have shown that the inclusion (22) holds. Using the submultiplicativity relation given by lemma 5, it follows that

$$\phi_{n,p,q}^0(\mathcal{D}_n(A, b, c)) \leq \sum_{N=1}^M \sum_{\substack{v_1, \dots, v_N \in \Lambda(n) \\ \text{pairwise distinct}}} \sum_{\substack{n^b \leq k_1, \dots, k_N \leq n^2 \\ k_1 + \dots + k_N \geq An^c}} \prod_{i=1}^N \phi_{n,p,q}^0(|C(v_i)| \geq k_i).$$

Lemma 2 provides us with $\psi = \psi(p_0, q) > 0$ such that

$$\forall n, k \geq 1 \quad \forall v \in \Lambda(n) \quad \phi_{n,p_0,q}^0(|C(v)| \geq k) \leq e^{-\psi k}.$$

Noting that the cardinalities $|C(v)|$ are increasing variables and that $p \leq p_0$, we obtain that

$$\begin{aligned} \phi_{n,p,q}^0(\mathcal{D}_n(A, b, c)) &\leq \sum_{N=1}^M \sum_{\substack{v_1, \dots, v_N \in \Lambda(n) \\ \text{pairwise distinct}}} \sum_{\substack{n^b \leq k_1, \dots, k_N \leq n^2 \\ k_1 + \dots + k_N \geq An^c}} \exp\left(-\psi \sum_{i=1}^N k_i\right) \\ &\leq M (n^2)^{2M} \exp(-\psi An^c). \end{aligned}$$

This being true for all $p \leq p_0$, we get

$$\frac{1}{n^c} \sup_{p \leq p_0} \ln \phi_{n,p,q}^0(\mathcal{D}_n(A, b, c)) \leq \frac{\ln M + 4M \ln n}{n^c} - \psi A = -\psi A + O\left(\frac{\ln n}{n^b}\right).$$

Recalling that we have taken $b > 0$, we deduce that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^c} \sup_{p \leq p_0} \ln \phi_{n,p,q}^0(\mathcal{D}_n(A, b, c)) \leq -\psi A < 0,$$

which concludes our proof. \square

This inequality allows us to obtain a uniform control on the sum of the squares of the cardinalities of the clusters in the subcritical regime, when p does not get too close to p_c . In practice, we could do without uniformity since this variable is an increasing variable (opening an edge connecting two clusters C_1 and C_2 increases this sum by $(|C_1| + |C_2|)^2 - |C_1|^2 - |C_2|^2 \geq 0$), but the uniformity in lemma 6 will be needed for the regime $T < T_c$.

Lemma 7. *For all $a > 3/2$, we have the following control on the cardinalities of the clusters in the subcritical regime:*

$$\forall q \geq 1 \quad \forall p_0 < p_c(q) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{p \leq p_0} \ln \phi_{n,p,q}^1\left(\sum_{C \in \mathcal{C}_n} |C|^2 > 2n^{a+1/2}\right) = -\infty.$$

Proof. Let $a > 3/2$, $q \geq 1$, $p \leq p_0 < p_c(q)$ and $n \geq 1$. Using the notation \mathcal{D}_n introduced in lemma 6, if the event $\mathcal{D}_n(1, a - 3/2, a/2 + 1/4)$ does not occur, then we have

$$\sum_{C \in \mathcal{C}_n : |C| \geq n^{a-3/2}} |C| < n^{a/2+1/4},$$

which implies that

$$\begin{aligned}
\sum_{C \in \mathcal{C}_n} |C|^2 &= \sum_{C \in \mathcal{C}_n : |C| < n^{a-3/2}} |C|^2 + \sum_{C \in \mathcal{C}_n : |C| \geq n^{a-3/2}} |C|^2 \\
&\leq n^{a-3/2} \times \sum_{C \in \mathcal{C}_n : |C| < n^{a-3/2}} |C| + \left(\sum_{C \in \mathcal{C}_n : |C| \geq n^{a-3/2}} |C| \right)^2 \\
&\leq n^{a-3/2} \times n^2 + \left(n^{a/2+1/4} \right)^2 \\
&= 2n^{a+1/2}.
\end{aligned}$$

We can deduce that

$$\phi_{n,p,q}^0 \left(\sum_{C \in \mathcal{C}_n} |C|^2 > 2n^{a+1/2} \right) \leq \phi_{n,p,q}^0 \left(\mathcal{D}_n \left(1, a - \frac{3}{2}, \frac{a}{2} + \frac{1}{4} \right) \right).$$

Lemma 4 allows us to change the boundary conditions, leading to

$$\phi_{n,p,q}^1 \left(\sum_{C \in \mathcal{C}_n} |C|^2 > 2n^{a+1/2} \right) \leq q^{4n} \phi_{n,p,q}^0 \left(\mathcal{D}_n \left(1, a - \frac{3}{2}, \frac{a}{2} + \frac{1}{4} \right) \right). \quad (24)$$

Besides, it follows from lemma 6 that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{a/2+1/4}} \sup_{p \leq p_0} \ln \phi_{n,p,q}^0 \left(\mathcal{D}_n \left(1, a - \frac{3}{2}, \frac{a}{2} + \frac{1}{4} \right) \right) < 0.$$

Yet, the condition $a > 3/2$ ensures that $a/2 + 1/4 > 1$, so that we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sup_{p \leq p_0} \ln \phi_{n,p,q}^0 \left(\mathcal{D}_n \left(1, a - \frac{3}{2}, \frac{a}{2} + \frac{1}{4} \right) \right) = -\infty. \quad (25)$$

Combining (24) and (25), we get

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{p \leq p_0} \ln \phi_{n,p,q}^1 \left(\sum_{C \in \mathcal{C}_n} |C|^2 > 2n^{a+1/2} \right) \\
\leq 4 \ln q + \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{p \leq p_0} \ln \phi_{n,p,q}^0 \left(\mathcal{D}_n \left(1, a - \frac{3}{2}, \frac{a}{2} + \frac{1}{4} \right) \right) = -\infty,
\end{aligned}$$

concluding the proof. \square

3.6 Moving to the Ising model

We are now in a position to prove the exponential decay result above the critical temperature.

Proof of lemma 1. Let $a > 3/2$, $A > 0$ and $T \geq T_0 > T_c$. To handle the magnetization in the Ising model at temperature T , we use the Edwards-Sokal coupling (see paragraph 2.5) to build an Ising spin configuration from a random-cluster configuration. Thus, we set $p = 1 - e^{-2/T}$ and $p_0 = 1 - e^{-2/T_0}$, so that we have $p \leq p_0 < p_c(2)$. Let ω be a percolation configuration distributed according to $\phi_{n,p,2}^1$. For each subset $C \subset \Lambda(n)$, we draw a random variable ε_C equally distributed on $\{-, +\}$, the variables $(\varepsilon_C)_{C \subset \Lambda(n)}$ being mutually independent and independent of ω . This represents many more

variables than necessary, but it makes notations more concise. We write \mathbb{P} for the joint law of ω and $(\varepsilon_C)_{C \subset \Lambda(n)}$. Recalling that $\mathcal{C}_n^-(\omega)$ is the set of open clusters in ω which do not touch the boundary of the box, we define an Ising configuration by setting

$$\sigma : \begin{cases} \Lambda(n) \longrightarrow \{-, +\} \\ x \longmapsto \begin{cases} + & \text{if } x \in \mathcal{M}_n(\omega), \\ \varepsilon_C & \text{if } x \in C \in \mathcal{C}_n^-(\omega). \end{cases} \end{cases} \quad (26)$$

The configuration σ is then distributed according to $\mu_{n,T}^+$, and the magnetization of this configuration is

$$m(\sigma) = |\mathcal{M}_n(\omega)| + \sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C.$$

Therefore, we have the inclusion

$$\left\{ |m(\sigma)| \geq An^a \right\} \subset \left\{ |\mathcal{M}_n(\omega)| \geq \frac{A}{2}n^a \right\} \cup \left\{ \left| \sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C \right| \geq \frac{A}{2}n^a \right\},$$

which implies that

$$\mu_{n,T}^+ \left(|m| \geq An^a \right) \leq \phi_{n,p,2}^1 \left(|\mathcal{M}_n| \geq \frac{A}{2}n^a \right) + \mathbb{P} \left(\left| \sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C \right| \geq \frac{A}{2}n^a \right).$$

On the one hand, the number of vertices connected to the boundary being an increasing variable, we have

$$\forall p \leq p_0 \quad \phi_{n,p,2}^1 \left(|\mathcal{M}_n| \geq \frac{A}{2}n^a \right) \leq \phi_{n,p_0,2}^1 \left(|\mathcal{M}_n| \geq \frac{A}{2}n^a \right).$$

On the other hand, by conditioning on $\sum_{C \in \mathcal{C}_n^-} |C|^2$, we get

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C \right| \geq \frac{A}{2}n^a \right) &\leq \phi_{n,p,2}^1 \left(\sum_{C \in \mathcal{C}_n^-} |C|^2 > 2n^{a+1/2} \right) \\ &\quad + \mathbb{P} \left(\left| \sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C \right| \geq \frac{A}{2}n^a \mid \sum_{C \in \mathcal{C}_n^-} |C|^2 \leq 2n^{a+1/2} \right). \end{aligned}$$

It follows from Hoeffding's inequality (see [Hoe63]) that

$$\mathbb{P} \left(\left| \sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C \right| \geq \frac{A}{2}n^a \mid \sum_{C \in \mathcal{C}_n^-} |C|^2 \leq 2n^{a+1/2} \right) \leq 2 \exp \left(-\frac{A^2 n^{2a}}{16n^{a+1/2}} \right) \leq 2 \exp \left(-\frac{A^2}{16}n \right),$$

where we have used the fact that $a > 3/2$. Therefore, we obtain

$$\begin{aligned} \mu_{n,T}^+ \left(|m| \geq An^a \right) &\leq \phi_{n,p_0,2}^1 \left(|\mathcal{M}_n| \geq \frac{A}{2}n^a \right) \\ &\quad + \phi_{n,p,2}^1 \left(\sum_{C \in \mathcal{C}_n^-} |C|^2 > 2n^{a+1/2} \right) + 2 \exp \left(-\frac{A^2}{16}n \right). \end{aligned}$$

Taking the supremum over $T \geq T_0$ leads to

$$\begin{aligned} \sup_{T \geq T_0} \mu_{n,T}^+ \left(|m| \geq An^a \right) &\leq \phi_{n,p_0,2}^1 \left(|\mathcal{M}_n| \geq \frac{A}{2} n^a \right) \\ &\quad + \sup_{p \leq p_0} \phi_{n,p,2}^1 \left(\sum_{C \in \mathcal{C}_n^-} |C|^2 > 2n^{a+1/2} \right) + 2 \exp \left(-\frac{A^2}{16} n \right). \end{aligned}$$

Combining this with the exponential estimates of lemmas 3 and 7 yields the desired result. \square

4 Exponential decay for subcritical temperatures

The goal of this section is to prove the following estimate:

Lemma 8. *For all $a < 2$, we have*

$$\forall T_0 < T_c \quad \forall A > 0 \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{T \leq T_0} \ln \mu_{n,T}^+ \left(m \leq An^a \right) < 0.$$

Note that, compared to (4), we have replaced $|m|$ with m . For a fixed temperature $T < T_c$, this exponential decay follows from theorem 5.2 in [Cer06]. But we need a control which is uniform on any segment included in $[0, T_c)$. We obtain this uniformity thanks to the random-cluster representation. In this model, the number of vertices connected to the boundary is an increasing variable, so an estimate for a particular parameter $p > p_c(q)$ suffices to obtain a uniform bound. To have a uniform control of the cardinality of clusters which do not touch the boundary, we use a duality argument to deduce it from the results of the previous section about the regime $p < p_c(q)$.

4.1 Control of the number of vertices connected to the boundary

We state here the counterpart of lemma 3 in the supercritical regime.

Lemma 9. *We have the following upper bound:*

$$\forall q \geq 1 \quad \forall p > p_c(q) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \phi_{n,p,q}^1 \left(|\mathcal{M}_n| \leq \frac{\theta(p)n^2}{2} \right) < 0.$$

Proof. This result is a consequence of theorem 5.5 of [Cer06], which proves it for any p larger than the threshold of exponential decay for dual connections, which is equal to $p_c(q)$ by lemma 2. \square

4.2 Control of the size of the clusters which do not touch the boundary

We prove here the analog of lemma 6 in the supercritical regime $p > p_c(q)$, concerning this time the size of the clusters which do not touch the boundary of the box. Recall that the collection of these clusters is denoted by \mathcal{C}_n^- . We want a control which is uniform on any segment included in $(p_c, 1]$ of the variable

$$\sum_{C \in \mathcal{C}_n^- : |C| \geq n} |C| = \left| \left\{ x \in \Lambda(n) \setminus \mathcal{M}_n : |C(x)| \geq n \right\} \right|.$$

To this end, we use duality (see paragraph 2.6) to convert large clusters which do not touch the boundary into large connected contours in the dual configuration. This will allow us to use the estimate given by lemma 6 on cluster sizes in the regime $p < p_c(q)$. This is why lemma 6 was stated with \mathcal{C}_n rather than \mathcal{C}_n^- , even though a control of the clusters which do not touch the boundary would have been enough to obtain lemma 1.

Lemma 10. *We have the following estimate:*

$$\forall q \geq 1 \quad \forall p_0 > p_c(q) \quad \forall A > 0 \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{p \geq p_0} \ln \phi_{n,p,q}^1 \left(\sum_{C \in \mathcal{C}_n^- : |C| \geq n} |C| \geq An^2 \right) < 0.$$

Proof. Let $q \geq 1$, $p \geq p_0 > p_c(q)$, $A > 0$ and $n \geq 2$. Let us consider the set

$$B = \left\{ C \in \mathcal{C}_n^- : |C| \geq n \right\}.$$

For any cluster $C \in B$, we consider its “external boundary” $\partial^{ext}C$, as defined in paragraph 2. Because the clusters $C \in B$ do not touch the boundary of the box, all the edges in $\partial^{ext}C$ are internal edges of the box $\Lambda(n)$, meaning that $\partial^{ext}C \subset \mathbb{E}_n^{int}$, where \mathbb{E}_n^{int} was defined by (15). We consider the dual $(\partial^{ext}C)^*$ of this external boundary (see paragraph 2.6 for the definition of dual edges). This set $(\partial^{ext}C)^*$ forms a connected closed contour surrounding C (see figure 6), which implies that

$$\left| (\partial^{ext}C)^* \right| = |\partial^{ext}C| \geq 1 + \text{diam } C \geq \sqrt{|C|} \geq \sqrt{n}. \quad (27)$$

We now define the set of edges

$$F = \bigcup_{C \in B} \partial^{ext}C.$$

Because one edge can connect at most two different clusters, we have

$$|F| \geq \frac{1}{2} \sum_{C \in B} |\partial^{ext}C|.$$

Combining this with (27) leads to

$$|F| \geq \frac{1}{2} \sum_{C \in B} \sqrt{|C|} \geq \frac{1}{2} \sqrt{\sum_{C \in B} |C|}. \quad (28)$$

Let us now consider the dual configuration ω^* associated with ω , as defined in paragraph 2.6. The edges of F are all closed in ω , thus the edges of F^* are all open in ω^* . Also, it follows from equation (28) that

$$\sum_{C \in B} |C| \geq An^2 \quad \Rightarrow \quad |F^*| = |F| \geq \frac{\sqrt{A}}{2} n.$$

What’s more, the connected components of F^* are unions of $(\partial^{ext}C)^*$ for a certain number of clusters $C \in B$, which implies, given (27), that they all contain at least \sqrt{n} vertices. All this leads to

$$\sum_{C \in B} |C| \geq An^2 \quad \Rightarrow \quad \omega^* \in \mathcal{D}_{n-1} \left(\frac{\sqrt{A}}{2}, \frac{1}{2}, 1 \right),$$

where \mathcal{D}_{n-1} is the event defined in lemma 6. We can now deduce, by the duality property (see proposition 2), that

$$\phi_{n,p,q}^1 \left(\sum_{C \in B} |C| \geq An^2 \right) \leq \phi_{n-1,p^*,q}^0 \left(\mathcal{D}_{n-1} \left(\frac{\sqrt{A}}{2}, \frac{1}{2}, 1 \right) \right),$$

and thus the result stems from the estimate provided by lemma 6. □

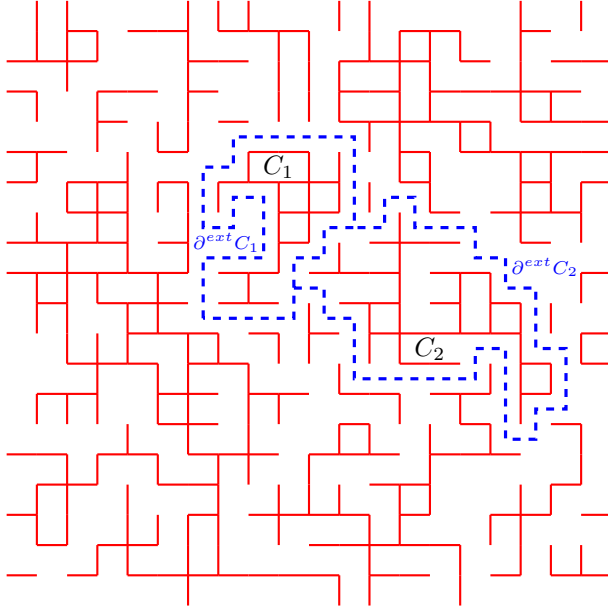


Figure 6: Construction of the dual circuits surrounding the large clusters which do not touch the boundary of the box.

4.3 Moving to the Ising model

We prove here the uniform exponential decay on any segment included in $[0, T_c)$ for the Ising model.

Proof of lemma 8. Let $a < 2$, $T_0 < T_c$ and $A > 0$. As in the previous section, we take $T \leq T_0$ and we set $p = 1 - e^{-2/T}$ and $p_0 = 1 - e^{-2/T_0}$, so that $p \geq p_0 > p_c(2)$. Let ω be a percolation configuration distributed according to $\phi_{n,p,2}^1$, and let $(\varepsilon_C)_{C \subset \Lambda(n)}$ be i.i.d. random variables equally distributed on $\{-, +\}$ and independent of ω . We build a spin configuration σ like in (26). Then we have

$$\mu_{n,T}^+(m \leq An^a) \leq \phi_{n,p,2}^1 \left(|\mathcal{M}_n| \leq \frac{\theta(p_0)n^2}{2} \right) + \mathbb{P} \left(\sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C \leq -\frac{\theta(p_0)n^2}{2} + An^a \right).$$

Given that $a < 2$, we have, for n large enough,

$$\frac{\theta(p_0)n^2}{2} - An^a \geq \frac{\theta(p_0)n^2}{4}.$$

The variable $|\mathcal{M}_n|$ being increasing, we can write, for n large enough,

$$\mu_{n,T}^+(m \leq An^a) \leq \phi_{n,p_0,2}^1 \left(|\mathcal{M}_n| \leq \frac{\theta(p_0)n^2}{2} \right) + \mathbb{P} \left(\sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C \leq -\frac{\theta(p_0)n^2}{4} \right).$$

In the second term, we split the summation in two parts as follows:

$$\sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C = \sum_{\substack{C \in \mathcal{C}_n^-(\omega) \\ |C| \geq n}} |C| \varepsilon_C + \sum_{\substack{C \in \mathcal{C}_n^-(\omega) \\ |C| < n}} |C| \varepsilon_C.$$

Concerning the first sum, we just write

$$\mathbb{P}\left(\sum_{C \in \mathcal{C}_n^-(\omega) : |C| \geq n} |C| \varepsilon_C \leq -\frac{\theta(p_0)n^2}{8}\right) \leq \phi_{n,p,2}^1\left(\sum_{C \in \mathcal{C}_n^-(\omega) : |C| \geq n} |C| \geq \frac{\theta(p_0)n^2}{8}\right).$$

To deal with the second sum, note that

$$\sum_{\substack{C \in \mathcal{C}_n^-(\omega) \\ |C| < n}} |C|^2 \leq n \times \sum_{\substack{C \in \mathcal{C}_n^-(\omega) \\ |C| < n}} |C| \leq n |\Lambda(n)| = n^3,$$

which implies by Hoeffding's inequality that

$$\mathbb{P}\left(\sum_{C \in \mathcal{C}_n^-(\omega) : |C| < n} |C| \varepsilon_C \leq -\frac{\theta(p_0)n^2}{8}\right) \leq \exp\left[-\frac{2}{4n^3} \left(\frac{\theta(p_0)n^2}{8}\right)^2\right] = \exp\left(-\frac{\theta(p_0)^2 n}{128}\right).$$

Therefore, we have, for n large enough,

$$\begin{aligned} \mu_{n,T}^+(m \leq An^a) &\leq \phi_{n,p_0,2}^1\left(|\mathcal{M}_n| \leq \frac{\theta(p_0)n^2}{2}\right) \\ &\quad + \phi_{n,p,2}^1\left(\sum_{C \in \mathcal{C}_n^-(\omega) : |C| \geq n} |C| \geq \frac{\theta(p_0)n^2}{8}\right) + \exp\left(-\frac{\theta(p_0)^2 n}{128}\right). \end{aligned}$$

Taking the supremum over $T \leq T_0$ then yields to (still for n large enough)

$$\begin{aligned} \sup_{T \leq T_0} \mu_{n,T}^+(m \leq An^a) &\leq \phi_{n,p_0,2}^1\left(|\mathcal{M}_n| \leq \frac{\theta(p_0)n^2}{2}\right) \\ &\quad + \sup_{p \geq p_0} \phi_{n,p,2}^1\left(\sum_{C \in \mathcal{C}_n^-(\omega) : |C| \geq n} |C| \geq \frac{\theta(p_0)n^2}{8}\right) + \exp\left(-\frac{\theta(p_0)^2 n}{128}\right). \end{aligned}$$

Combining this with the exponential estimates of lemmas 9 and 10, we obtain the proclaimed result. \square

5 Finite-size scaling results for near-critical FK-percolation

This section is devoted to the proof of proposition 1, namely that the property $\mathcal{FSS}(s)$ holds for any exponent $s < 8/41$. We have to show that, for all $s \in (0, 8/41)$, for all $K, \delta > 0$ and for any sequence $p_n \in [0, 1]$ satisfying

$$p_n - p_c(2) \stackrel{n \rightarrow \infty}{\sim} \frac{K}{n^s}, \quad (29)$$

we have

$$\lim_{n \rightarrow \infty} \phi_{n,p_n,2}^1\left(|\mathcal{M}_n| \leq (1 + \delta)\theta(p_n) |\Lambda(n)|\right) = 1, \quad (30a)$$

$$\lim_{n \rightarrow \infty} \phi_{n,p_n,2}^1\left(\max_{C \in \mathcal{C}_n^-} |C| \leq n^{s+1/2}\right) = 1, \quad (30b)$$

$$\lim_{n \rightarrow \infty} \phi_{n,p_n,2}^1\left(|\mathcal{M}_n \cap \Lambda(n_1)| \geq (1 - \delta)\theta(p_n) |\Lambda(n_1)|\right) = 1, \quad (30c)$$

where $n_1 = \lfloor 5n/6 \rfloor$. Let us first write down the following formula, which stems from the exact computations of Onsager [Ons44] and Yang [Yan52]:

$$\theta(p) \stackrel{p \downarrow p_c(2)}{\sim} \left[8 \left(\frac{p}{p_c(2)} - 1\right)\right]^{1/8}. \quad (31)$$

Proof of proposition 1. Let s be such that $0 < s < 8/41$, let $K, \delta > 0$ and $p_n \in [0, 1]$ be a sequence satisfying (29). Thanks to

$$5 + \frac{1}{8} = \frac{41}{8} < \frac{1}{s},$$

we can choose a' such that

$$5 < a' < \frac{1}{s} - \frac{1}{8}. \quad (32)$$

Proof of (30a): We apply theorem 2 of [CM11] with $a', p = p_n$ and δ . Let $c = c(a', \delta)$ be the constant provided by this theorem. The condition

$$n > c(p_n - p_c(2))^{-a'}$$

is satisfied for n large enough, because

$$(p_n - p_c(2))^{-a'} = O(n^{a's}) = o(n).$$

Therefore, it follows from theorem 2 of [CM11] that

$$\limsup_{n \rightarrow \infty} \frac{1}{(p_n - p_c(2))^{2a'+1/4} n^2} \ln \phi_{n, p_n, 2}^1 \left(|\mathcal{M}_n| > (1 + \delta)\theta(p_n) |\Lambda(n)| \right) < 0.$$

Yet, we know by (32) that

$$2a' + \frac{1}{4} < \frac{2}{s},$$

which implies, using the asymptotics of $p_n - p_c(2)$ given by (29), that

$$(p_n - p_c(2))^{2a'+1/4} n^2 \stackrel{n \rightarrow \infty}{\sim} K^{2a'+1/4} \times n^{2-(2a'+1/4)s} \xrightarrow{n \rightarrow \infty} +\infty.$$

Therefore, we have

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1 \left(|\mathcal{M}_n| > (1 + \delta)\theta(p_n) |\Lambda(n)| \right) = 0,$$

which proves (30a).

Proof of (30b): We wish to apply theorem 1 of [CM11] with the same parameter a' as above, and with $M = \lfloor n^{s/2+1/4} \rfloor$. We say that a cluster $C \subset \Lambda(n)$ “crosses” a sub-box $B \subset \Lambda(n)$ if there exists in $C \cap B$ an open path from the bottom side to the top side of B and an open path from the left side to the right side of B (this condition is in fact stronger than the one of [CM11], which only requests C to intersect all the faces of B , but the same proof works with our definition). We have already checked that the condition $n > c(p_n - p_c(2))^{-a'}$ of the theorem is satisfied for n large enough. Regarding the condition on M , we have on the one hand $M \leq n$ because $s/2 + 1/4 < 1$, and on the other hand,

$$\frac{\ln n}{p_n - p_c(2)} \stackrel{n \rightarrow \infty}{\sim} (\ln n)n^s = o(M), \quad (33)$$

because $s < s/2 + 1/4$. Therefore, we can apply theorem 1 of [CM11], which gives us a constant $K_1 > 0$ such that, for n large enough,

$$\phi_{m_n, p_n, 2}^1(\mathcal{E}_n^C) \leq \exp\left(-K_1(p_n - p_c(2))M\right),$$

where the probability considered is the random-cluster measure on a larger box of side $m_n = \lfloor 6n/5 \rfloor$, and where \mathcal{E}_n is the event “in the box $\Lambda(n)$, there exists a cluster C_0 crossing every sub-box of $\Lambda(n)$ with diameter M ”. Yet, by (33), we know that

$$\lim_{n \rightarrow \infty} (p_n - p_c(2))M = +\infty,$$

whence

$$\lim_{n \rightarrow \infty} \phi_{m_n, p_n, 2}^1(\mathcal{E}_n) = 1. \quad (34)$$

We now want to recover the usual boundary conditions on the box $\Lambda(n)$, instead of the larger box $\Lambda(m_n)$. By the nesting property of random-cluster measures, we have

$$\phi_{n, p_n, 2}^1(\mathcal{E}_n) = \phi_{m_n, p_n, 2}^1\left(\mathcal{E}_n \mid \omega = 1 \text{ on } \mathbb{E}_{m_n} \setminus \mathbb{E}_n\right). \quad (35)$$

The event \mathcal{E}_n being increasing, the FKG inequality ensures that

$$\phi_{m_n, p_n, 2}^1\left(\mathcal{E}_n \mid \omega = 1 \text{ on } \mathbb{E}_{m_n} \setminus \mathbb{E}_n\right) \geq \phi_{m_n, p_n, 2}^1(\mathcal{E}_n).$$

Combining this with equations (34) and (35) leads to

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1(\mathcal{E}_n) = 1. \quad (36)$$

If this event \mathcal{E}_n occurs, then every open path of diameter at least M must be included into C_0 , because otherwise there would be a sub-box of side M that C_0 does not cross. Yet the cluster C_0 touches the boundary $\partial\Lambda(n)$, so all open paths of diameter larger than M are connected to the boundary because included in M . Therefore, we have the implication

$$\mathcal{E}_n \Rightarrow \max_{C \in \mathcal{C}_n^-} \text{diam } C \leq M - 1.$$

The diameter of a cluster is related to its cardinality by

$$|C| \leq (1 + \text{diam } C)^2,$$

hence

$$\mathcal{E}_n \Rightarrow \max_{C \in \mathcal{C}_n^-} |C| \leq M^2 \leq n^{s+1/2}.$$

Therefore, equation (36) implies (30b).

Proof of (30c): We now wish to apply theorem 3 of [CM11] to the smaller box $\Lambda(n_1)$, with wired boundary conditions on the box $\Lambda(n)$. With again the same parameter a' , we have

$$a's \left(1 + \frac{1}{8a'}\right) = a's + \frac{s}{8} < 1 - \frac{s}{8} + \frac{s}{8} = 1,$$

which enables us to choose α such that

$$a's < \alpha < \frac{8a'}{8a' + 1}.$$

Let $c = c(a', \alpha)$ be the constant given by the theorem applied with a' , α and δ . The required condition that $n^\alpha(p_n - p_c(2))^{a'} > c$ is satisfied for n large enough, because

$$n^\alpha(p_n - p_c(2))^{a'} \stackrel{n \rightarrow \infty}{\sim} K^{a'} \times n^{\alpha - a's} \xrightarrow{n \rightarrow \infty} +\infty.$$

Recall now that \mathcal{C}_{n_1} denotes the collection of the open clusters in the sub-box $\Lambda(n_1)$. By theorem 3 of [CM11] we have

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1\left(\max_{C \in \mathcal{C}_{n_1}} |C| \geq (1 - \delta)\theta(p_n) |\Lambda(n_1)|\right) = 1.$$

Combining this with (30b), we get

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1 \left(\max_{C \in \mathcal{C}_{n_1}} |C| \geq (1 - \delta)\theta(p_n) |\Lambda(n_1)| \quad \text{and} \quad \max_{C \in \mathcal{C}_n^-} |C| \leq n^{s+1/2} \right) = 1. \quad (37)$$

Yet, according to the asymptotics for $\theta(p)$ given by (31), we know that

$$(1 - \delta)\theta(p_n) |\Lambda(n_1)| \stackrel{n \rightarrow \infty}{\sim} (1 - \delta) \left(\frac{5}{6} \right)^2 \left(\frac{8}{p_c(2)} \right)^{1/8} n^{2-s/8}.$$

As $s < 4/3$, we have $2 - s/8 > s + 1/2$ and thus, for n large enough, we have

$$(1 - \delta)\theta(p_n) |\Lambda(n_1)| > n^{s+1/2}.$$

Hence, if the event in (37) occurs, then there is in the smaller box $\Lambda(n_1)$ a cluster containing strictly more than $n^{s+1/2}$ vertices, which must therefore be connected to the boundary of the larger box $\Lambda(n)$. Thus, we have

$$\max_{C \in \mathcal{C}_{n_1}} |C| \geq (1 - \delta)\theta(p_n) |\Lambda(n_1)| \quad \text{and} \quad \max_{C \in \mathcal{C}_n^-} |C| \leq n^{s+1/2} \Rightarrow |\mathcal{M}_n \cap \Lambda(n_1)| \geq (1 - \delta)\theta(p_n) |\Lambda(n_1)|,$$

so the result follows from (37). \square

6 Lower bound on the partition function

The goal of this section is to show the following lower bound on the partition function of our model:

Lemma 11. *For every $a \in (31/16, 2)$ such that the assumptions $\mathcal{FSS}(16 - 8a)$ hold, we have*

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{(\ln n)n^\rho} > -\infty,$$

where ρ is given by

$$\rho = \max \left(\frac{a}{2}, \frac{33}{2} - 8a \right).$$

We follow the strategy presented in paragraph 1.5.

6.1 The price for closing edges

We start by stating a useful lemma to estimate the probability for closing a random set of edges.

Lemma 12. *Let $n \geq 1$, $p \in]0, 1[$, $q \geq 1$ and $\mathcal{A} \subset \{0, 1\}^{\mathbb{E}_n}$. Let H be an arbitrary application which associates to any configuration $\omega \in \mathcal{A}$ a certain set of edges $H(\omega) \subset \mathbb{E}_n$. We consider the following application*

$$\psi: \begin{cases} \mathcal{A} & \longrightarrow \{0, 1\}^{\mathbb{E}_n} \\ \omega & \longmapsto \omega_{H(\omega)} \end{cases}$$

which, to every configuration ω , associates the configuration obtained from ω by closing the edges of $H(\omega)$. Letting

$$N = \max_{\omega \in \mathcal{A}} |H(\omega)|,$$

we have

$$\phi_{n, p, q}^1(\psi(\mathcal{A})) \geq \phi_{n, p, q}^1(\mathcal{A}) \left[\frac{1}{3|\Lambda(n)|} \left(1 \wedge \frac{1-p}{p} \right) \right]^N.$$

Proof. This is a straightforward consequence of lemma 6.3 in [CP00]. To apply this lemma, note that, for every configuration $\omega \in \psi(\mathcal{A})$, we have

$$|\psi^{-1}(\{\omega\})| \leq \left| \left\{ H \subset \mathbb{E}_n : |H| \leq N \right\} \right| \leq 1 + |\mathbb{E}_n|^N \leq 1 + (2|\Lambda(n)|)^N \leq (3|\Lambda(n)|)^N,$$

and the result then stems from the aforementioned lemma. \square

6.2 Construction of the fixed point and preliminary estimates

We now turn to the construction of the parameter b_n . We define (see paragraph 1.5 for the heuristics leading to this definition)

$$b'_n = n^a \sqrt{2} \left[-\ln \left(1 - p_c(2) - \frac{3^8 p_c(2)}{8n^{16-8a}} \right) \right]^{-1/2},$$

along with

$$b_n = \begin{cases} \lfloor b'_n \rfloor & \text{if } \lfloor b'_n \rfloor \equiv n^2 \pmod{2}, \\ \lfloor b'_n \rfloor - 1 & \text{otherwise,} \end{cases}$$

so that we always have

$$b_n \equiv n^2 \pmod{2}. \quad (38)$$

Our final aim is to get a lower bound on the probability that, at temperature $T = b_n^2/n^{2a}$, the magnetization is exactly equal to b_n . For this, we need b_n to have the same parity as $|\Lambda(n)| = n^2$, hence the above definition. We also set $p_n = \varphi_n(b_n)$, where the function φ_n is the one defined by (7). Recall that, if ω is a percolation configuration in the box $\Lambda(n)$, then $\mathcal{C}_n^-(\omega)$ denotes the set of the open clusters in ω which do not touch the boundary $\partial\Lambda(n)$, and $\mathcal{C}_n^-(1, \omega)$ denotes the subset of these clusters which contain only one vertex. The present section is devoted to the proof of the following statement, which is a direct outcome of the assumptions $\mathcal{FSS}(16 - 8a)$.

Lemma 13. *Let $a \in (31/16, 2)$. Setting $n_1 = \lfloor 5n/6 \rfloor$, we define the event*

$$\mathcal{G}_n = \left\{ |\mathcal{M}_n| \leq 4n^a, \quad |\mathcal{M}_n \cap \Lambda(n_1)| \geq 2n^a \quad \text{and} \quad \max_{C \in \mathcal{C}_n^-} |C| \leq n^{33/2-8a} \leq |\mathcal{C}_n^-(1)| - 1 \right\}. \quad (39)$$

If the hypotheses $\mathcal{FSS}(16 - 8a)$ hold, then we have

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1(\mathcal{G}_n) = 1.$$

Before proving this lemma, we check that our definition of b_n leads to the right convergence speed towards the critical point.

Lemma 14. *For any $a > 16/9$, we have the following estimates:*

$$b_n \stackrel{n \rightarrow \infty}{\sim} n^a \sqrt{T_c}, \quad (40)$$

$$p_n - p_c(2) \stackrel{n \rightarrow \infty}{\sim} \frac{3^8 p_c(2)}{8n^{16-8a}}, \quad (41)$$

$$\theta(p_n) |\Lambda(n)| \stackrel{n \rightarrow \infty}{\sim} 3n^a. \quad (42)$$

Proof. Equation (40) is a straightforward consequence of the definition of b_n and of the relation (14) between the critical temperature T_c and the critical point $p_c(2)$. To show (41), note on the one hand that, the function φ_n being decreasing on $\{0, \dots, n^2\}$, we have

$$p_n = \varphi_n(b_n) \geq \varphi_n(b'_n) = p_c(2) + \frac{3^8 p_c(2)}{8n^{16-8a}}.$$

On the other hand, we can write

$$\begin{aligned}
p_n - \left(p_c(2) + \frac{3^8 p_c(2)}{8n^{16-8a}} \right) &\leq \varphi_n(b'_n - 2) - \varphi_n(b'_n) \\
&= \exp\left(-\frac{2n^{2a}}{b'_n{}^2}\right) - \exp\left(-\frac{2n^{2a}}{(b'_n - 2)^2}\right) \\
&\leq \frac{2n^{2a}}{(b'_n - 2)^2} - \frac{2n^{2a}}{b'_n{}^2} \\
&= \frac{2n^{2a}(4b'_n - 4)}{(b'_n - 2)^2 b'_n{}^2}.
\end{aligned}$$

Combining this with the asymptotics for b_n given by equation (40), we get, using the fact that $a > 16/9$,

$$p_n - p_c(2) - \frac{3^8 p_c(2)}{8n^{16-8a}} = O\left(\frac{1}{n^a}\right) = o\left(\frac{1}{n^{16-8a}}\right).$$

Therefore, equation (41) is satisfied, and (42) can be deduced from it, using the expansion of $\theta(p)$ given by (31). \square

We now prove that the conditions $\mathcal{FSS}(16-8a)$ imply the result about the event \mathcal{G}_n .

Proof of lemma 13. Let $a \in (31/16, 2)$ be such that the hypotheses $\mathcal{FSS}(16-8a)$ hold. Then the three conditions (30a), (30b) and (30c) are satisfied with $s = 16-8a$. Applying condition (30a) with $\delta = 1/6$ leads to

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1 \left(|\mathcal{M}_n| \leq \frac{7}{6} \theta(p_n) |\Lambda(n)| \right) = 1. \quad (43)$$

Yet, from the expansion of $\theta(p_n) |\Lambda(n)|$ given by (42), we know that, for n large enough,

$$\theta(p_n) |\Lambda(n)| < \frac{8}{7} \times 3n^a,$$

which implies that

$$\frac{7}{6} \theta(p_n) |\Lambda(n)| < \frac{7}{6} \times \frac{8}{7} \times 3n^a = 4n^a.$$

Combining this with (43) yields

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1 \left(|\mathcal{M}_n| \leq 4n^a \right) = 1. \quad (44)$$

It follows from condition (30b) that

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1 \left(\max_{C \in \mathcal{C}_n^-} |C| \leq n^{33/2-8a} \right) = 1. \quad (45)$$

We now show that there are in the sub-box $\Lambda(n_1)$ at least $2n^a$ vertices connected to the boundary of the large box $\Lambda(n)$. Using condition (30c) with $\delta = 1/50$, we get

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1 \left(|\mathcal{M}_n \cap \Lambda(n_1)| \geq \frac{49}{50} \theta(p_n) |\Lambda(n_1)| \right) = 1. \quad (46)$$

Yet, following the expansion of $\theta(p_n) |\Lambda(n)|$ given by (42), we know that

$$\frac{49}{50} \theta(p_n) |\Lambda(n_1)| \stackrel{n \rightarrow \infty}{\sim} \frac{49}{50} \theta(p_n) \frac{25}{36} |\Lambda(n)| = \frac{49}{72} \theta(p_n) |\Lambda(n)| \stackrel{n \rightarrow \infty}{\sim} \frac{49}{24} n^a.$$

Noting that $49/24 > 2$ and plugging this into (46), we obtain

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1 \left(|\mathcal{M}_n \cap \Lambda(n_1)| \geq 2n^a \right) = 1. \quad (47)$$

Eventually, we show a lower bound on the number of clusters of size 1 which do not touch the boundary of the box. These clusters will be useful when passing to the Ising model (in section 6.4), in order to tune precisely the value of the magnetization by choosing the spins of these unit clusters. Following theorem 3.21 in [Gri06], the random-cluster measure is stochastically dominated by the corresponding Bernoulli percolation measure, that is to say $\phi_{n, p_n, 2}^1 \preceq \phi_{n, p_n, 1}^1$. Recall now that $|\mathcal{C}_n^-(1)|$ denotes the number of clusters of size 1 which do not touch the boundary. This variable is decreasing, so we have

$$\phi_{n, p_n, 2}^1 \left(|\mathcal{C}_n^-(1)| \geq 1 + n^{33/2-8a} \right) \geq \phi_{n, p_n, 1}^1 \left(|\mathcal{C}_n^-(1)| \geq 1 + n^{33/2-8a} \right). \quad (48)$$

Considering one half of the vertices inside the box, we define

$$U_n = U_n(\omega) = \sum_{\substack{x \in \Lambda(n) \setminus \partial \Lambda(n) \\ \|x\|_1 \equiv 0 \pmod{2}}} \mathbb{1}_{C(x)=\{x\}}, \quad (49)$$

which is such that $|\mathcal{C}_n^-(1)| \geq U_n$. Taking the expectation, we can deduce that

$$\phi_{n, p_n, 1}^1(U_n) \geq \left\lfloor \frac{(n-2)^2}{2} \right\rfloor (1-p)^4 \stackrel{n \rightarrow \infty}{\sim} \frac{n^2(1-p)^4}{2}.$$

Therefore, using the fact that $a > 31/16 > 29/16$, we have

$$1 + n^{33/2-8a} = o(n^2) = o\left(\phi_{n, p_n, 1}^1(U_n)\right).$$

Yet, under the law $\phi_{n, p_n, 1}^1$, the variables appearing in the sum (49) are mutually independent, whence by Hoeffding's inequality,

$$\phi_{n, p_n, 1}^1 \left(U_n < 1 + n^{33/2-8a} \right) \leq \exp \left(- \frac{2(\phi_{n, p_n, 1}^1(U_n) - 1 - n^{33/2-8a})^2}{[(n-2)^2/2]} \right) \stackrel{n \rightarrow \infty}{\rightarrow} 0.$$

Given that $|\mathcal{C}_n^-(1)| \geq U_n$, it follows that

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 1}^1 \left(|\mathcal{C}_n^-(1)| < 1 + n^{33/2-8a} \right) = 0.$$

Using (48), this leads to

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1 \left(|\mathcal{C}_n^-(1)| \geq 1 + n^{33/2-8a} \right) = 1. \quad (50)$$

Combining (44), (45), (47) and (50), we obtain lemma 13. \square

6.3 Surgery on the set of vertices connected to the boundary

We explain here the surgery step on \mathcal{M}_n , starting from a configuration realizing the event \mathcal{G}_n .

Lemma 15. *Let $K > 0$, and let us consider the event*

$$\mathcal{R}_n = \left\{ \begin{array}{l} \exists H \subset \mathbb{E}_n, \quad |H| \leq Kn^{a/2}, \quad |\mathcal{M}_n(\omega_H)| = \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil \\ \text{and} \quad \max_{C \in \mathcal{C}_n} |C| \leq n^{33/2-8a} \leq |\mathcal{C}_n^-(1)| - 1 \end{array} \right\}. \quad (51)$$

For all $a \in (31/16, 2)$ such that the postulates $\mathcal{FSS}(16-8a)$ hold, with p_n defined as in section 6.2, there exists $K > 0$ such that

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1(\mathcal{R}_n) = 1.$$

Proof. Let $a \in (31/16, 2)$ be such that $\mathcal{FSS}(16 - 8a)$ holds. Let $n \geq 12$ and $n_1 = \lfloor 5n/6 \rfloor$, and let p_n be defined as in section 6.2. Let $\omega : \mathbb{E}_n \rightarrow \{0, 1\}$ be a configuration realizing the event \mathcal{G}_n defined by (39). We are going to construct a set of edges $H \subset \mathbb{E}_n$ such that

$$|\mathcal{M}_n(\omega_H)| = \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil.$$

Recall that $\mathbb{E}[\mathcal{M}_n(\omega)]$ denotes the set of the edges of \mathbb{E}_n connecting two vertices of $\mathcal{M}_n(\omega)$. For every $j \geq 1$, we consider the set $E_j = \partial^e \Lambda(j)$ of the edges which delimit the boundary of the box $\Lambda(j)$. We have $E_j \subset \mathbb{E}_{j+2}$ and $E_j \cap \mathbb{E}_j = \emptyset$. Therefore, if $j, k \geq 1$ are such that $|j - k| \geq 2$, then the sets E_j and E_k are disjoint. From this we deduce by the pigeonhole principle that there exists an integer $j(\omega)$ satisfying

$$n_1 \leq 2j(\omega) \leq n - 2 \quad \text{and} \quad \left| E_{2j(\omega)} \cap \mathbb{E}[\mathcal{M}_n(\omega)] \right| \leq \frac{|\mathbb{E}[\mathcal{M}_n(\omega)]|}{L_n},$$

where

$$L_n = \left\lfloor \frac{n-2}{2} \right\rfloor - \left\lceil \frac{n_1}{2} \right\rceil + 1 \stackrel{n \rightarrow \infty}{\sim} \frac{n}{12}.$$

We choose such an integer $j(\omega)$ and we let $H_0(\omega) = E_{2j(\omega)} \cap \mathbb{E}[\mathcal{M}_n(\omega)]$. Recall that, by the definition (39) of the event \mathcal{G}_n , we have $|\mathcal{M}_n(\omega)| \leq 4n^a$, which entails that

$$|H_0(\omega)| \leq \frac{2|\mathcal{M}_n(\omega)|}{L_n} \leq \frac{8n^a}{L_n}.$$

Again by the definition of the event \mathcal{G}_n , we know that

$$|\mathcal{M}_n(\omega)| \geq |\mathcal{M}_n(\omega) \cap \Lambda(n_1)| \geq 2n^a > b_n + 1,$$

for n large enough, because $b_n + 1 \sim n^a \sqrt{T_c}$ and $\sqrt{T_c} < 2$. It follows that, for n large enough,

$$|\mathcal{M}_n(\omega)| > \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil.$$

Thus, we may consider a subset $H_1(\omega) \subset H_0(\omega)$, maximal in the sense of inclusion among the subsets satisfying

$$|\mathcal{M}_n(\omega_{H_1(\omega)})| \geq \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil. \quad (52)$$

Due to the fact that the set of edges $H_0(\omega)$ separates the sub-box $\Lambda(n_1)$ from the boundary of the large box $\partial\Lambda(n)$, we have

$$|\mathcal{M}_n(\omega_{H_0(\omega)})| \leq |\mathcal{M}_n(\omega)| - |\mathcal{M}_n(\omega) \cap \Lambda(n_1)| \leq |\mathcal{M}_n(\omega)| - 2n^a.$$

Yet, because ω realizes the event \mathcal{G}_n , we have $|\mathcal{M}_n(\omega)| \leq 4n^a$, whence

$$|\mathcal{M}_n(\omega_{H_0(\omega)})| \leq \frac{|\mathcal{M}_n(\omega)|}{2} + \frac{4n^a}{2} - 2n^a = \frac{|\mathcal{M}_n(\omega)|}{2} < \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil.$$

It follows that the inclusion $H_1(\omega) \subset H_0(\omega)$ is a strict inclusion, and hence, $H_1(\omega)$ being maximal, there exists an edge $e \in H_0(\omega) \setminus H_1(\omega)$ such that

$$|\mathcal{M}_n(\omega_{H_1(\omega) \cup \{e\}})| < \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil. \quad (53)$$

Thus, closing the edge e in the configuration $\omega_{H_1(\omega)}$ strictly decreases the number $|\mathcal{M}_n|$ of vertices connected to the boundary, hence one of the endpoints of e , which we will call v , must end up disconnected from the boundary $\partial\Lambda(n)$ when closing the edge e in the configuration $\omega_{H_1(\omega)}$. Let us write C_v

for the cluster of v in the configuration $\omega_{H_1(\omega) \cup \{e\}}$, and let E_v be the set of the edges of $\mathbb{E}[C_v]$ which are open in the configuration $\omega_{H_1(\omega) \cup \{e\}}$. Then, because C_v is the piece which is disconnected when closing the edge e in $\omega_{H_1(\omega)}$, we have

$$|C_v| = |\mathcal{M}_n(\omega_{H_1(\omega)})| - |\mathcal{M}_n(\omega_{H_1(\omega) \cup \{e\}})|. \quad (54)$$

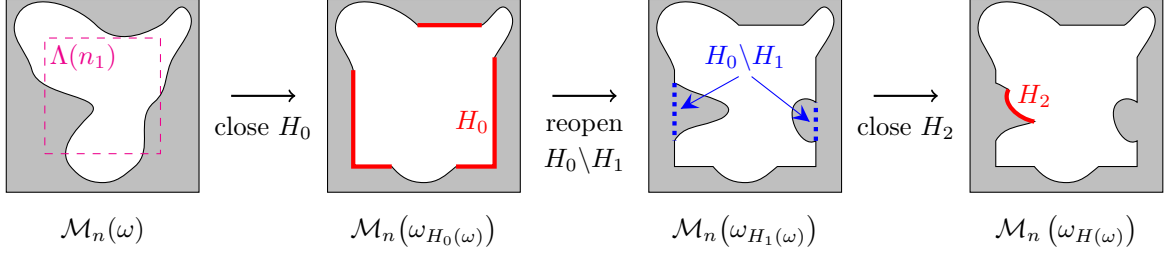


Figure 7: Steps of the surgical procedure to bring the number of vertices connected to the boundary from $|\mathcal{M}_n| = b_n + x$ to $|\mathcal{M}_n| = b_n + x/2$.

Defining

$$m = \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil - |\mathcal{M}_n(\omega_{H_1(\omega) \cup \{e\}})|,$$

it follows from equations (52), (53) and (54) that

$$1 \leq m \leq |C_v|.$$

Therefore, according to lemma 1 of [CF19], there exists a subset $H_2(\omega) \subset E_v$ with cardinality

$$|H_2(\omega)| \leq K_0 \sqrt{|C_v|},$$

where K_0 is a fixed constant, such that the connected component of v in the graph $(C_v, E_v \setminus H_2(\omega))$ contains exactly m vertices. We then let $H(\omega) = H_1(\omega) \cup H_2(\omega)$, whose cardinality is

$$|H(\omega)| \leq |H_0(\omega)| + |H_2(\omega)| \leq \frac{8n^a}{L_n} + K_0 \sqrt{4n^a} = O(n^{a-1}) + O(n^{a/2}) = O(n^{a/2}).$$

Now take $K > 0$ (independent of n and ω) such that, for every $n \geq 1$,

$$\frac{8n^a}{L_n} + K_0 \sqrt{4n^a} \leq Kn^{a/2}.$$

Then, we have

$$|H(\omega)| \leq Kn^{a/2}.$$

Besides, by construction of H , we have

$$|\mathcal{M}_n(\omega_H(\omega))| = |\mathcal{M}_n(\omega_{H_1(\omega) \cup \{e\}})| + m = \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil.$$

At the end of the day, we have proved the inclusion $\mathcal{G}_n \subset \mathcal{R}_n$, and consequently, the result follows from lemma 13. \square

What do we get by closing the edges of $H(\omega)$ for $\omega \in \mathcal{R}_n$? We will show that the resulting configurations $\omega_{H(\omega)}$ fall into the event

$$\mathcal{S}_n = \left\{ \begin{array}{l} \exists \mathcal{C}_0 \subset \mathcal{C}_n^- : |\mathcal{C}_0| \leq 2Kn^{a/2}, \quad |\mathcal{M}_n| - \sum_{C \in \mathcal{C}_0} |C| = b_n \\ \text{and} \quad \max_{C \in \mathcal{C}_n^- \setminus \mathcal{C}_0} |C| \leq n^{33/2-8a} \leq |\mathcal{C}_n^-(1) \setminus \mathcal{C}_0| \end{array} \right\}. \quad (55)$$

The set \mathcal{C}_0 corresponds to the piece that was disconnected from the boundary to make half the way from $|\mathcal{M}_n|$ to b_n , as explained in section 1.5. This event will allow us in section 6.4 to build an Ising configuration with magnetization of exactly b_n , through the Edwards-Sokal coupling. To this end, we will force the clusters $C \in \mathcal{C}_0$ to be assigned a negative spin and we will hope for the contribution of the other clusters to cancel out. The conditions appearing in the event \mathcal{S}_n about the size of the clusters which do not belong to \mathcal{C}_0 and about the number of unit clusters will enable us to control this contribution. But first of all, we show the following estimate:

Lemma 16. *For every $a \in (31/16, 2)$ satisfying $\mathcal{FSS}(16-8a)$, with p_n defined as in section 6.2, there exists a constant $K > 0$ such that*

$$\liminf_{n \rightarrow \infty} \frac{1}{(\ln n)n^{a/2}} \ln \phi_{n, p_n, 2}^1(\mathcal{S}_n) > -\infty.$$

To construct a configuration realizing this event \mathcal{S}_n , we start with a configuration realizing the event \mathcal{R}_n and we close the set of edges H given by the definition (51) of \mathcal{R}_n . This allows us to derive a lower bound on the probability of the event \mathcal{S}_n , thanks to lemma 12 which controls the price for closing edges.

Proof of lemma 16. Let $a \in (31/16, 2)$ such that $\mathcal{FSS}(16-8a)$ holds, let p_n be defined as in section 6.2, and take $K > 0$ given by lemma 15, such that

$$\lim_{n \rightarrow \infty} \phi_{n, p_n, 2}^1(\mathcal{R}_n) = 1. \quad (56)$$

Let ω be a configuration realizing the event \mathcal{R}_n . By the definition (51) of \mathcal{R}_n , we can take $H(\omega) \subset \mathbb{E}_n$ such that

$$|H(\omega)| \leq Kn^{a/2} \quad \text{and} \quad |\mathcal{M}_n(\omega_{H(\omega)})| = \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil.$$

What's more, by shrinking the set $H(\omega)$ if necessary, we can assume that $H(\omega) \subset \mathbb{E}[\mathcal{M}_n(\omega)]$, which ensures that the clusters which do not touch the boundary in ω are left undamaged in $\omega_{H(\omega)}$, so that $\mathcal{C}_n^-(\omega) \subset \mathcal{C}_n^-(\omega_{H(\omega)})$. We then want to show that $\omega_{H(\omega)}$ realizes the event \mathcal{S}_n . To this end, we have to build a set $\mathcal{C}_0 \subset \mathcal{C}_n^-(\omega_{H(\omega)})$ which satisfies the conditions of the definition (55) of \mathcal{S}_n . A natural candidate is the set of the clusters in $\omega_{H(\omega)}$ which were connected to the boundary before the closure of the edges of $H(\omega)$, that is to say in the configuration ω , but are not anymore in the modified configuration $\omega_{H(\omega)}$. Thus, we define

$$\mathcal{C}_0 = \mathcal{C}_0(\omega) = \mathcal{C}_n^-(\omega_{H(\omega)}) \setminus \mathcal{C}_n^-(\omega).$$

Because we have taken $H(\omega) \subset \mathbb{E}[\mathcal{M}_n(\omega)]$, the clusters of \mathcal{C}_0 must be included in \mathcal{M}_n , whence

$$\mathcal{C}_0 = \left\{ C \in \mathcal{C}_n^-(\omega_{H(\omega)}) : C \subset \mathcal{M}_n(\omega) \right\}.$$

Therefore, we have

$$\bigcup_{C \in \mathcal{C}_0} C = \mathcal{M}_n(\omega) \setminus \mathcal{M}_n(\omega_{H(\omega)}),$$

which implies that

$$\begin{aligned} |\mathcal{M}_n(\omega_{H(\omega)})| - \sum_{C \in \mathcal{C}_0} |C| &= |\mathcal{M}_n(\omega_{H(\omega)})| - \left(|\mathcal{M}_n(\omega)| - |\mathcal{M}_n(\omega_{H(\omega)})| \right) \\ &= 2 |\mathcal{M}_n(\omega_{H(\omega)})| - |\mathcal{M}_n(\omega)|. \end{aligned}$$

Using the fact that ω realizes the event \mathcal{R}_n , we deduce that

$$|\mathcal{M}_n(\omega_{H(\omega)})| - \sum_{C \in \mathcal{C}_0} |C| = 2 \left\lceil \frac{|\mathcal{M}_n(\omega)| + b_n}{2} \right\rceil - |\mathcal{M}_n(\omega)|.$$

If $|\mathcal{M}_n(\omega)| + b_n$ is even, then we have

$$|\mathcal{M}_n(\omega_{H(\omega)})| - \sum_{C \in \mathcal{C}_0} |C| = b_n,$$

and

$$|\mathcal{C}_n^-(1, \omega_{H(\omega)}) \setminus \mathcal{C}_0| = |\mathcal{C}_n^-(1, \omega)| \geq n^{33/2-8a} + 1.$$

Assume now that $|\mathcal{M}_n(\omega)| + b_n$ is odd. In this case, we have

$$|\mathcal{M}_n(\omega_{H(\omega)})| - \sum_{C \in \mathcal{C}_0} |C| = b_n + 1.$$

Therefore, we need to add a unit cluster in \mathcal{C}_0 . By the definition of the event \mathcal{R}_n , the number of unit clusters in ω which do not touch the boundary is

$$|\mathcal{C}_n^-(1, \omega)| \geq n^{33/2-8a} + 1 \geq 1,$$

which allows us to choose a cluster $C_1 \in \mathcal{C}_n^-(1, \omega)$. We then let $\mathcal{C}'_0 = \mathcal{C}_0 \cup \{C_1\}$, and we have

$$|\mathcal{M}_n(\omega_{H(\omega)})| - \sum_{C \in \mathcal{C}'_0} |C| = b_n.$$

Besides, we have

$$|\mathcal{C}_n^-(1, \omega_{H(\omega)}) \setminus \mathcal{C}'_0| = |\mathcal{C}_n^-(1, \omega) \setminus \{C_1\}| = |\mathcal{C}_n^-(1, \omega)| - 1 \geq n^{33/2-8a}.$$

This is why we gave ourselves a margin of 1 between the condition on the unit clusters in \mathcal{R}_n and the one in \mathcal{S}_n . Thus, if we set $\mathcal{C}'_0 = \mathcal{C}_0$ in the case where $|\mathcal{M}_n(\omega)| + b_n$ even, then whatever the parity of $|\mathcal{M}_n(\omega)| + b_n$, we have

$$|\mathcal{M}_n(\omega_{H(\omega)})| - \sum_{C \in \mathcal{C}'_0} |C| = b_n \quad \text{and} \quad |\mathcal{C}_n^-(1, \omega_{H(\omega)}) \setminus \mathcal{C}'_0| \geq n^{33/2-8a}.$$

In addition to that, due to the fact that $\mathcal{C}_n^-(\omega_{H(\omega)}) \setminus \mathcal{C}'_0 \subset \mathcal{C}_n^-(\omega)$, we know that

$$\max_{C \in \mathcal{C}_n^-(\omega_{H(\omega)}) \setminus \mathcal{C}'_0} |C| \leq \max_{C \in \mathcal{C}_n^-(\omega)} |C| \leq n^{33/2-8a}.$$

Eventually, note that the closure of one edge cannot increase the number of open clusters by more than 1, whence $|\mathcal{C}_0| \leq |H(\omega)|$ and thus

$$|\mathcal{C}'_0| \leq |\mathcal{C}_0| + 1 \leq |H(\omega)| + 1 \leq Kn^{a/2} + 1 \leq 2Kn^{a/2},$$

upon increasing the constant K if necessary, to ensure that $K \geq 1$. Thus, we have proved that the application

$$\psi: \begin{cases} \mathcal{R}_n & \longrightarrow \{0, 1\}^{\mathbb{E}_n} \\ \omega & \longmapsto \omega_{H(\omega)} \end{cases}$$

takes its values in in \mathcal{S}_n . According to lemma 12 which controls the cost for closing edges, we deduce that

$$\begin{aligned} \phi_{n, p_n, 2}^1(\mathcal{S}_n) &\geq \phi_{n, p_n, 2}^1(\psi(\mathcal{R}_n)) \geq \phi_{n, p_n, 2}^1(\mathcal{R}_n) \left[\frac{1}{3n^2} \left(1 \wedge \frac{1-p_n}{p_n} \right) \right]^{2Kn^{a/2}} \\ &\geq \phi_{n, p_n, 2}^1(\mathcal{R}_n) \left(\frac{1-p_n}{3n^2} \right)^{2Kn^{a/2}}. \end{aligned}$$

We deduce that

$$\liminf_{n \rightarrow \infty} \frac{\ln \phi_{n, p_n, 2}^1(\mathcal{S}_n)}{(\ln n)n^{a/2}} \geq \liminf_{n \rightarrow \infty} \frac{\ln \phi_{n, p_n, 2}^1(\mathcal{R}_n)}{(\ln n)n^{a/2}} + \liminf_{n \rightarrow \infty} \frac{2K}{\ln n} \left(\ln(1-p_n) - \ln 3 - 2 \ln n \right).$$

It follows from (56) that

$$\liminf_{n \rightarrow \infty} \frac{\ln \phi_{n, p_n, 2}^1(\mathcal{R}_n)}{(\ln n)n^{a/2}} = 0.$$

Yet, we have $p_n \rightarrow p_c(2) \in (0, 1)$, which implies that $1-p_n$ is bounded away from 0. Consequently, we obtain

$$\liminf_{n \rightarrow \infty} \frac{\ln \phi_{n, p_n, 2}^1(\mathcal{S}_n)}{(\ln n)n^{a/2}} \geq -4K > -\infty,$$

which is the desired result. \square

6.4 Moving to the Ising model

Armed with the estimate given by lemma 16, we are now in a position to prove our lower bound on the partition function.

Proof of lemma 11. Let $a \in (31/16, 2)$ be such that the postulates $\mathcal{FSS}(16-8a)$ hold. Let p_n and b_n be defined as in section 6.2. Let us consider

$$T_n^* = \frac{2}{-\ln(1-p_n)} = \frac{b_n^2}{n^{2a}}.$$

Recall that, by rewriting Z_n in the form (6), we have seen that it suffices to prove a lower bound on the probability that, under the law $\phi_{n, p_n, 2}^1$, the magnetization is exactly b_n . As in the proof of lemmas 1 and 8, we use Edwards-Sokal coupling (see paragraph 2.5) to deduce the lower bound on Z_n from our result on the random-cluster model. Let ω be a percolation configuration distributed according to $\phi_{n, p_n, 2}^1$. For each subset $C \subset \Lambda(n)$, we draw a uniform random variable $\varepsilon_C \in \{-, +\}$, the variables $(\varepsilon_C)_{C \subset \Lambda(n)}$ being mutually independent and independent of ω . Let σ be the spin configuration associated with ω and $(\varepsilon_C)_{C \subset \Lambda(n)}$ as in (26). The configuration σ then follows the law $\mu_{n, T_n^*}^+$, and its magnetization writes

$$m(\sigma) = |\mathcal{M}_n(\omega)| + \sum_{C \in \mathcal{C}_n^-(\omega)} |C| \varepsilon_C.$$

For every configuration ω realizing the event \mathcal{S}_n , we choose a set $\mathcal{C}_0(\omega) \subset \mathcal{C}_n^-(\omega)$ satisfying the properties in the definition (55) of \mathcal{S}_n , namely

$$|\mathcal{C}_0| \leq 2Kn^{a/2}, \quad |\mathcal{M}_n| - \sum_{C \in \mathcal{C}_0} |C| = b_n \quad \text{and} \quad \max_{C \in \mathcal{C}_n^-(\omega) \setminus \mathcal{C}_0} |C| \leq n^{33/2-8a} \leq |\mathcal{C}_n^-(1, \omega) \setminus \mathcal{C}_0|.$$

This enables us to define the event

$$\mathcal{T}_n = \mathcal{S}_n \cap \left\{ \forall C \in \mathcal{C}_0 \quad \varepsilon_C = - \quad \text{and} \quad \sum_{C \in \mathcal{C}_n^- \setminus \mathcal{C}_0} |C| \varepsilon_C = 0 \right\}.$$

Note that, if \mathcal{T}_n occurs, it implies that

$$m(\sigma) = \left(|\mathcal{M}_n(\omega)| - \sum_{C \in \mathcal{C}_0} |C| \right) + \sum_{C \in \mathcal{C}_n^- \setminus \mathcal{C}_0} |C| \varepsilon_C = b_n + 0 = b_n,$$

so that our lower bound (6) on the partition function becomes

$$Z_n \geq \mathbb{P}(m(\sigma) = b_n) \geq \mathbb{P}(\mathcal{T}_n). \quad (57)$$

We fix a configuration $\omega_0 \in \mathcal{S}_n$, and we will reason conditionally on the event $\{\omega = \omega_0\}$. In this context, the variables $\mathcal{C}_n^- = \mathcal{C}_n^-(\omega_0)$ and $\mathcal{C}_0 = \mathcal{C}_0(\omega_0)$ are henceforth fixed. We can write

$$\mathbb{P}(\mathcal{T}_n \mid \omega = \omega_0) = \frac{1}{2^{|\mathcal{C}_0|}} \mathbb{P}\left(\sum_{C \in \mathcal{C}_n^- \setminus \mathcal{C}_0} |C| \varepsilon_C = 0 \right). \quad (58)$$

Let $N = \lfloor n^{33/2-8a} \rfloor$. Recall that, by the definition (55) of the event \mathcal{S}_n , we have

$$\max_{C \in \mathcal{C}_n^- \setminus \mathcal{C}_0} |C| \leq N \quad (59)$$

and

$$|\mathcal{C}_n^-(1) \setminus \mathcal{C}_0| \geq N,$$

which means that there are at least N unit clusters in $\mathcal{C}_n^- \setminus \mathcal{C}_0$. The idea is to leave N of these clusters aside, and to control the magnetization of the other clusters of $\mathcal{C}_n^- \setminus \mathcal{C}_0$ to show that it falls within the range $[-N, N]$ with sufficient probability. We will then be able to use these N unit clusters kept aside to adjust the value of the magnetization and to force it to reach exactly 0. Thus, we consider a set $\mathcal{C}_1 \subset \mathcal{C}_n^-(1) \setminus \mathcal{C}_0$, with cardinality N . To ensure that the overall magnetization of the clusters of $\mathcal{C}_n^- \setminus (\mathcal{C}_0 \cup \mathcal{C}_1)$ falls within the range $[-N, N]$, we sort these clusters depending on their cardinality. Then, for every $j \in \{1, \dots, N\}$, we control separately the contribution of the clusters of $\mathcal{C}_n^- \setminus (\mathcal{C}_0 \cup \mathcal{C}_1)$ which contain exactly j vertices. Therefore, for every $j \in \{0, \dots, N\}$, we write

$$S_j = \sum_{\substack{C \in \mathcal{C}_n^- \setminus (\mathcal{C}_0 \cup \mathcal{C}_1) \\ |C| \leq j}} |C| \varepsilon_C = \sum_{i=1}^j \sum_{C \in \mathcal{C}_n^-(i) \setminus (\mathcal{C}_0 \cup \mathcal{C}_1)} |C| \varepsilon_C.$$

It follows from (59) that

$$S_N = \sum_{C \in \mathcal{C}_n^- \setminus (\mathcal{C}_0 \cup \mathcal{C}_1)} |C| \varepsilon_C.$$

Equation (58) therefore leads to

$$\mathbb{P}(\mathcal{T}_n \mid \omega = \omega_0) = \frac{1}{2^{|\mathcal{C}_0|}} \mathbb{P}\left(S_N + \sum_{C \in \mathcal{C}_1} \varepsilon_C = 0 \right). \quad (60)$$

It follows from Stirling's formula that

$$\binom{2k}{k} \frac{1}{4^k} \underset{k \rightarrow \infty}{\sim} \frac{1}{\sqrt{\pi k}},$$

which allows us to fix $K_2 > 0$ such that, for every $k \geq 1$,

$$\binom{2k}{k} \frac{1}{4^k} \geq \frac{K_2}{\sqrt{2k}}.$$

By diminishing K_2 if necessary, we can assume that $K_2 \leq 1$. We then prove by induction on j that, for all $j \in \{0, \dots, N\}$,

$$\mathbb{P}\left(|S_j| \leq N\right) \geq \left(\frac{K_2}{2n}\right)^j. \quad (61)$$

The result is obvious for $j = 0$, since $S_0 = 0$. Let $j \in \{1, \dots, N\}$ be such that (61) holds for $j - 1$. We consider

$$B_j = \mathcal{C}_n^-(j) \setminus (\mathcal{C}_0 \cup \mathcal{C}_1),$$

which is such that

$$S_j = S_{j-1} + j \sum_{C \in B_j} \varepsilon_C.$$

If the set B_j is empty, then the inequality (61) follows from the induction hypothesis, because we have taken $K_2 \leq 1$. Now suppose that B_j is not empty. If $|B_j|$ is even, then to obtain a null contribution from the clusters $C \in B_j$ to the total magnetization, we need half of these clusters to choose a + spin and the other half to be assigned a - spin. Thus, for $|B_j|$ even, we have

$$\mathbb{P}\left(|S_j| \leq N \mid |S_{j-1}| \leq N\right) \geq \mathbb{P}\left(\sum_{C \in B_j} \varepsilon_C = 0\right) = \binom{|B_j|}{|B_j|/2} \frac{1}{2^{|B_j|}} \geq \frac{K_2}{\sqrt{|B_j|}} \geq \frac{K_2}{n}.$$

Assume now that $|B_j|$ is odd, and choose an arbitrary cluster $C_0 \in B_j$. To control S_j , we will request a null overall contribution from the clusters of $B_j \setminus \{C_0\}$, and we will ask for the extra term coming from C_0 to have a sign opposed to S_{j-1} , which ensures that $|S_j|$ remains in the interval $[-N, N]$. Let us consider the function $\eta : \mathbb{Z} \rightarrow \{-, +\}$ defined by

$$\eta(x) = -\text{sgn}(x) = \begin{cases} + & \text{if } x \leq 0, \\ - & \text{otherwise.} \end{cases}$$

We can write, for $|B_j|$ odd,

$$\begin{aligned} \mathbb{P}\left(|S_j| \leq N \mid |S_{j-1}| \leq N\right) &\geq \mathbb{P}\left(\varepsilon_{C_0} = \eta(S_{j-1}) \quad \text{and} \quad \sum_{C \in B_j \setminus \{C_0\}} \varepsilon_C = 0\right) \\ &= \frac{1}{2} \binom{|B_j| - 1}{(|B_j| - 1)/2} \frac{1}{2^{|B_j| - 1}} \\ &\geq \frac{K_2}{2n}. \end{aligned}$$

This proves the induction step, and thus the lower bound (61) holds for every $j \in \{0, \dots, N\}$, which implies in particular that

$$\mathbb{P}\left(|S_N| \leq N\right) \geq \left(\frac{K_2}{2n}\right)^N. \quad (62)$$

We will now use the variables ε_C for $C \in \mathcal{C}_1$, which we had shelved aside, to compensate S_N and thus attain a magnetization exactly equal to b_n . This is only possible if S_N has the same parity as the

number N of unit clusters in \mathcal{C}_1 . Let us check that this is indeed the case, by writing

$$\begin{aligned} S_N &= \sum_{C \in \mathcal{C}_n^- \setminus (\mathcal{C}_0 \cup \mathcal{C}_1)} |C| \varepsilon_C \\ &\equiv \sum_{C \in \mathcal{C}_n^-} |C| + \sum_{C \in \mathcal{C}_0} |C| + |\mathcal{C}_1| \pmod{2} \\ &= |\Lambda(n)| - |\mathcal{M}_n| + \sum_{C \in \mathcal{C}_0} |C| + N. \end{aligned}$$

Yet, due to the fact that ω_0 realizes the event \mathcal{S}_n , we have

$$|\mathcal{M}_n| - \sum_{C \in \mathcal{C}_0} |C| = b_n,$$

whence

$$S_N \equiv n^2 - b_n + N \equiv N \pmod{2},$$

where we have used equation (38) whereby $b_n \equiv n^2 \pmod{2}$. Hence, there is no parity problem, thanks to the precaution we took when defining b_n . Thus, $N - S_N$ is always even. For every $s \in \{0, \dots, N\}$, we write

$$\mathbb{P}\left(\sum_{C \in \mathcal{C}_1} \varepsilon_C = -S_N \mid S_N = N - 2s\right) = \mathbb{P}\left(\sum_{C \in \mathcal{C}_1} \varepsilon_C = s - (N - s)\right) = \binom{N}{s} \frac{1}{2^N} \geq \frac{1}{2^N}.$$

This being true for every $s \in \{0, \dots, N\}$, and $N - S_N$ being always even, it follows that

$$\mathbb{P}\left(\sum_{C \in \mathcal{C}_1} \varepsilon_C = -S_N \mid |S_N| \leq N\right) \geq \frac{1}{2^N}.$$

From this we deduce, using (62), that

$$\mathbb{P}\left(S_N + \sum_{C \in \mathcal{C}_1} \varepsilon_C = 0\right) \geq \mathbb{P}\left(\sum_{C \in \mathcal{C}_1} \varepsilon_C = -S_N \mid |S_N| \leq N\right) \mathbb{P}\left(|S_N| \leq N\right) \geq \left(\frac{K_2}{4n}\right)^N.$$

Combining this with equation (60), we obtain that

$$\mathbb{P}\left(\mathcal{T}_n \mid \omega = \omega_0\right) \geq \frac{1}{2^{|\mathcal{C}_0|}} \left(\frac{K_2}{4n}\right)^N \geq \left(\frac{1}{2}\right)^{2Kn^{a/2}} \left(\frac{K_2}{4n}\right)^{n^{33/2-8a}}.$$

This being true for all the configurations ω_0 realizing the event \mathcal{S}_n , it follows that

$$\mathbb{P}(\mathcal{T}_n \mid \mathcal{S}_n) \geq \left(\frac{K_2}{8n}\right)^{2Kn^\rho},$$

where we have taken

$$\rho = \max\left(\frac{a}{2}, \frac{33}{2} - 8a\right).$$

We deduce that

$$\frac{\ln \mathbb{P}(\mathcal{T}_n \mid \mathcal{S}_n)}{(\ln n)n^\rho} \geq \frac{2K(\ln K_2 - \ln 8)}{\ln n} - 2K \xrightarrow{n \rightarrow \infty} -2K. \quad (63)$$

Coming back to the lower bound (57) on Z_n , we get

$$Z_n \geq \mathbb{P}(\mathcal{T}_n) = \mathbb{P}(\mathcal{T}_n \mid \mathcal{S}_n) \phi_{n, p_n, 2}^1(\mathcal{S}_n).$$

Combining this with (63) and with the lower bound on $\phi_{n, p_n, 2}^1(\mathcal{S}_n)$ derived in lemma 16 then yields

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{(\ln n)n^\rho} \geq \liminf_{n \rightarrow \infty} \frac{\ln \mathbb{P}(\mathcal{T}_n \mid \mathcal{S}_n)}{(\ln n)n^\rho} + \liminf_{n \rightarrow \infty} \frac{\ln \phi_{n, p_n, 2}^1(\mathcal{S}_n)}{(\ln n)n^\rho} > -\infty,$$

which concludes the proof. \square

7 Conclusion

We summarize here how the different estimates combine together to give our convergence result.

Proof of theorem 2. Let $a \in (31/16, 2)$ be such that the finite-size scaling assumptions $\mathcal{FSS}(16 - 8a)$ hold, and let $\varepsilon > 0$. Following the computation (2), we have

$$\mu_n(T_n \geq T_c + \varepsilon) \leq \frac{n^2 + 1}{Z_n} \sup_{T \geq T_c + \varepsilon} \mu_{n,T}^+ \left(|m| \geq n^a \sqrt{T_c + \varepsilon} \right),$$

which implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mu_n(T_n \geq T_c + \varepsilon) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{T \geq T_c + \varepsilon} \ln \mu_{n,T}^+ \left(|m| \geq n^a \sqrt{T_c + \varepsilon} \right) - \liminf_{n \rightarrow \infty} \frac{\ln Z_n}{n}.$$

Yet, the lower bound on Z_n obtained in lemma 11 ensures that

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{(\ln n)n^\rho} > -\infty,$$

where

$$\rho = \max \left(\frac{a}{2}, \frac{33}{2} - 8a \right).$$

The condition $a \in (31/16, 2)$ implies that $\rho < 1$, leading to

$$\liminf_{n \rightarrow \infty} \frac{\ln Z_n}{n} \geq 0.$$

Therefore, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mu_n(T_n \geq T_c + \varepsilon) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{T \geq T_c + \varepsilon} \ln \mu_{n,T}^+ \left(|m| \geq n^a \sqrt{T_c + \varepsilon} \right) < 0,$$

where we have used lemma 1. Likewise, we show that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mu_n(T_n \leq T_c - \varepsilon) < 0$$

using lemma 8, which completes the demonstration. \square

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