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Regular variation of fixed points of the smoothing transform

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Abstract

Let (N, A_1, A_2, \dots) be a sequence of random variables with $N \in \mathbb{N} \cup \{\infty\}$ and $A_i \in \mathbb{R}_+$. We are interested in asymptotic properties of non-negative solutions of the distributional equation $Z \stackrel{(d)}{=} \sum_{i=1}^N A_i Z_i$, where Z_i are non-negative random variables independent of each other and independent of (N, A_1, A_2, \dots) , each having the same distribution as Z which is unknown. For a solution Z with finite mean, we prove that for a given $\alpha > 1$, $\mathbb{P}(Z > x)$ is a function regularly varying at ∞ of index $-\alpha$ if and only if the same is true for $\mathbb{P}(Y_1 > x)$, where $Y_1 = \sum_{i=1}^N A_i$. The result completes the sufficient condition obtained by Iksanov & Polotskiy (2006) on the branching random walk. A similar result on sufficient condition is also established for the case where $\alpha = 1$.

AMS Subject Classification: 60K37, 60J80

Keywords: tail behavior, regular variation, smoothing transform, branching random walk, Mandelbrot's martingale

1 Introduction and main results

Let us introduce the smoothing transform that interests us. As usual, we write $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{N}^* = \{1, 2, \dots\}$, $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $\mathbb{R}_+ = [0, \infty)$. Let (N, A_1, A_2, \dots) be a random variable taking values in $\bar{\mathbb{N}} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \dots$. Assume that

$$\mathbb{E} \sum_{i=1}^N A_i = 1. \quad (1.1)$$

Let \mathcal{M}_1 be the set of probability measures on \mathbb{R}_+ with finite mean. The smoothing transform \mathcal{T} on \mathcal{M}_1 is defined by letting $\mathcal{T}\mu$ be the distribution of $\sum_{i=1}^N A_i Z_i$, where Z_i are independent and identically distributed (i.i.d.) random variables with law μ , independent of (N, A_1, A_2, \dots) . It is convenient to write the fixed point equation $\mu = \mathcal{T}\mu$ in the equivalent form

$$Z \stackrel{(d)}{=} \sum_{i=1}^N A_i Z_i, \quad (E)$$

where Z_i are non-negative i.i.d. random variables independent of (N, A_1, A_2, \dots) , each having the same law as Z (which is unknown). By convention, the empty sum is taken to be 0, so that $\sum_{i=1}^N A_i = \sum_{i=1}^N A_i Z_i = 0$ if $N = 0$. The smoothing transform \mathcal{T} will be identified with the corresponding transform on Laplace transforms of elements of \mathcal{M}_1 , so that by writing $\phi(s) = \mathbb{E}e^{-sZ}$, the equation (E) reads

$$\phi(s) = (\mathcal{T}\phi)(s), \quad \text{where } (\mathcal{T}\phi)(s) = \mathbb{E} \prod_{i=1}^N \phi(sA_i), \quad s \geq 0, \quad (E')$$

with the convention that the empty product is taken to be 1.

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The study of fixed points of the smoothing transform is interesting due to a large number of applications in a variety of applied probability settings, including branching processes, self-similar cascades, infinite particles systems, branching random walks, random fractals, the quicksort algorithm and the Pagerank algorithm (which is in the heart of the Google engine): see for example Bingham & Doney (1974, 1975), Kahane & Peyrière (1976), Biggins (1977), Durrett & Liggett (1983), Rösler (1992), Liu (1998), Aldous & Bandyopadhyay (2005), Biggins & Kyprianou (2005), Iksanov & Polotskiy (2006), Buraczewski (2009), Hu & Shi (2009), Volkovich & Litvak (2010), Barral & Jin (2014), Buraczewski & Kolesko (2014), Chen (2015) and Shi (2015).

A typical example for a solution of (E) is the limit of a generalized Mandelbrot's martingale (see Liu (2000)) constructed as follows. Let $U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$ be the set of all finite sequences, where $(\mathbb{N}^*)^0 = \{\emptyset\}$ contains the null sequence \emptyset . Let $(N_u, A_{u1}, A_{u2}, \dots)$, indexed by $u \in U$, be a sequence of independent copies of (N, A_1, A_2, \dots) . Let \mathbb{T} be the Galton-Watson tree with defining elements $\{N_u : u \in U\}$, that is: (i) $\emptyset \in \mathbb{T}$, (ii) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$, (iii) if $u \in \mathbb{T}$ and $i \in \mathbb{N}$, then $ui \in \mathbb{T}$ if and only if $1 \leq i \leq N_u$. Let $\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$ be the set of individuals in generation n , where $|u|$ denotes the length of the sequence u . Set

$$Y_n = \sum_{u_1 \dots u_n \in \mathbb{T}_n} A_{u_1} \cdots A_{u_1 \dots u_n} \quad \text{for } n \geq 1. \quad (1.2)$$

Then $(Y_n)_{n \geq 1}$ is a non-negative martingale with respect to the filtration $\mathcal{E}_n = \sigma\{(N_u, A_{u1}, A_{u2}, \dots) : |u| < n\}$, $n \geq 1$, so that the limit

$$Y_\infty = \lim_{n \rightarrow \infty} Y_n \quad (1.3)$$

exists a.s. by the martingale convergence theorem, with $\mathbb{E}Y_\infty \leq 1$ by Fatou's lemma. By self-similarity of the construction, it can be easily seen that $Z = Y_\infty$ is a solution of (E). We shall see in Lemma 1.1 that any solution of (E) with finite mean can be obtained in this way.

Another typical example concerns the natural martingale occurring in the branching random walk (BRW) on the real line \mathbb{R} , described as follows. The initial particle \emptyset of the 0-th generation is located at $S_\emptyset = 0 \in \mathbb{R}$; it gives birth to $N_\emptyset = N$ new particles $\emptyset i = i$ ($1 \leq i \leq N$) of the first generation with displacements $L_{\emptyset i} = L_i$. In general, each particle $u = u_1 u_2 \dots u_n$ of n -th generation gives birth to N_u new particles ui ($1 \leq i \leq N_u$) of $(n+1)$ -th generation, with displacements (relative to their parent) L_{ui} on the real line, so that their positions are $S_{ui} = S_u + L_{ui}$. The random variables $(N_u, L_{u1}, L_{u2}, \dots)$ are independent and identically distributed. Let Z_n be the counting measure on the real line \mathbb{R} which counts the number of particles of the n -th generation in a given set of \mathbb{R} :

$$Z_n(A) = \sum_{u \in \mathbb{T}_n} \mathbf{1}_A(S_u), \quad A \subset \mathbb{R}, \quad (1.4)$$

where $\mathbf{1}_A$ denotes the indicator function of A , and \mathbb{T}_n is the set of individuals of generation n as defined before. Let $t \in \mathbb{R}$ be fixed such that $m := \mathbb{E} \sum_{i=1}^N e^{-tL_i} < \infty$. Then the Laplace transform of Z_n at t is $\tilde{Z}_n(t) = \int e^{-tx} dZ_n(x) = \sum_{u \in \mathbb{T}_n} e^{-tS_u}$, and the sequence

$$W_n = \frac{\tilde{Z}_n(t)}{\mathbb{E} \tilde{Z}_n(t)} = \sum_{u \in \mathbb{T}_n} \frac{e^{-tS_u}}{m^n} \quad (1.5)$$

(the normalized Laplace transform of Z_n) forms a non-negative martingale with respect to the filtration $\mathcal{E}_n = \sigma\{(N_u, L_{u1}, L_{u2}, \dots) : |u| < n\}$, $n \geq 1$. Notice that $\{W_n\}$ is just $\{Y_n\}$ with $A_u = e^{-tL_u}/m$, so that the a.s. limit variable $Z = \lim_n W_n$ is a solution of (E) with $A_i = e^{-tL_i}/m$.

For $x \geq 0$, write

$$\rho(x) = \mathbb{E} \sum_{i=1}^N A_i^x \quad \text{and} \quad \rho'(x) = \mathbb{E} \sum_{i=1}^N A_i^x \ln A_i \quad (1.6)$$

if the expectation exists with value in the extended real line $\bar{\mathbb{R}} = [-\infty, \infty]$. The following result concerns the existence and uniqueness of nontrivial solutions with finite mean, and the existence of moments of order $\alpha > 1$. As usual, we write

$$\ln^+ x = \max(\ln x, 0) \quad \text{and} \quad \ln^- x = \max(-\ln x, 0) \quad \text{for } x \geq 0.$$

We always assume (1.1) and write $Y_1 = \sum_{i=1}^N A_i$, in accordance with (1.2). Define for $x > 0$,

$$J(x) := \frac{x}{A(x)} \quad \text{with} \quad A(x) = \int_0^x \mathbb{E} \sum_{i=1}^N A_i \mathbf{1}_{\{-\ln A_i > t\}} dt = \mathbb{E} \sum_{i=1}^N A_i \min(\ln^- A_i, x),$$

where the last equality holds by Fubini's theorem. Let $J(0) = \lim_{x \downarrow 0} J(x) = 1/\mathbb{E} \sum_{i=1}^N A_i \mathbf{1}_{\{-\ln A_i > 0\}}$. Notice that for any $x > 0$, $A(x) = 0$ if and only if a.s. $A_i \geq 1$ on $\{A_i > 0\}$, for all $i = 1, \dots, N$. Consequently, if $\rho'(1) < 0$, then $A(1) > 0$, and for any $x \geq 1$, $A(x) \geq A(1) > 0$ and $J(x) \leq x/A(1)$. Notice also that $J(x) = 1/\int_0^1 \mathbb{E} \sum_{i=1}^N A_i \mathbf{1}_{\{-\ln A_i > xu\}} du$ is increasing in x .

Lemma 1.1 (i) *Assume that $\rho'(1)$ is well defined whose value may be infinite. Then the equation (E) has a solution Z with $\mathbb{E}Z = 1$ if and only if either*

$$\rho'(1) \in (-\infty, 0) \quad \text{and} \quad \mathbb{E}[Y_1 \ln^+ Y_1] < \infty, \quad (1.7)$$

or

$$\rho'(1) = -\infty \quad \text{and} \quad \mathbb{E}[Y_1 J(\ln^+ Y_1)] < \infty. \quad (1.8)$$

Moreover, if Z is a solution of (E) with $\mathbb{E}Z = 1$, then $\mathbb{E}Y_1 = 1$ and $Z = Y_\infty$ in law, where Y_∞ is defined by (1.3). (Consequently, there is only one solution Z satisfying $\mathbb{E}Z = 1$.)

(ii) *Let Z be a solution of (E) with $\mathbb{E}Z = 1$. For $\alpha > 1$, $\mathbb{E}Z^\alpha < \infty$ if and only if $\rho(\alpha) < 1$ and $\mathbb{E}Y_1^\alpha < \infty$.*

The first conclusion in Part (i) follows from a result of Alsmeyer and Iksanov (2009). In fact, by Theorem 1.3 of Alsmeyer and Iksanov (2009), when $\rho'(1)$ is well defined, $\mathbb{E}Y_\infty = 1$ (see (1.3) for the definition) if and only if (1.7) or (1.8) holds (actually their result is established in the context of branching random walks, but can also be formulated in the context of Mandelbrot's martingales). So to prove the conclusion we just need to show that the equation (E) has a solution Z with $\mathbb{E}Z = 1$ if and only if $\mathbb{E}Y_\infty = 1$. The "if" part is evident. To see the "only if" part, it suffices to notice that by Theorem 7.1 of Liu (1998), if Z is a solution of (E) with $\mathbb{E}Z = 1$, then

$$Y_n \rightarrow Z \quad \text{in law.}$$

(To see this, it suffices to take $\eta(t) = e^{-t}$ and $\phi(t) = \mathbb{E}e^{-tZ}$ in Theorem 7.1 of Liu (1998); in this case, the condition (H1) therein is not needed). Hence $Z = Y_\infty$ in law. This argument also shows the second conclusion of Part (i).

Part (ii) is just Theorem 2.1 of Liu (2000) (although Liu only considered the case where $N < \infty$ a.s., this condition was not needed in the proof therein).

The criterion (1.7) has been well-known in the literature. For example, Biggins (1977) proved it under the additional condition $\mathbb{E} \sum_{i=1}^\infty A_i (\ln A_i)^2 < \infty$, Liu (1997) proved it when $\mathbb{E}N < \infty$, Lyons (1997) proved it assuming only $\rho'(1)$ is finite. In fact Biggins (1977) and Lyons (1997) also shown the sufficiency when $\rho'(1) = -\infty$ and $\mathbb{E}[Y_1 \ln^+ Y_1] < \infty$. Notice that, by the discussion before Lemma 1.1, if $\rho'(1) \in [-\infty, 0)$, then $\mathbb{E}[Y_1 \ln^+ Y_1] < \infty$ implies $\mathbb{E}[Y_1 J(\ln^+ Y_1)] < \infty$.

A fundamental problem on the smoothing transform concerns the tail behavior of fixed points. Many authors have studied this problem in various contexts, see for example: Bingham & Doney (1974, 1975) for Galton-Watson processes and Crump-Mode-Jirina processes; Kahane & Peyrière (1976), Guivarc'h (1990), Barral (1999), Liu (2000), Barral & Jin (2014), and Liang & Liu (2015) for Mandelbrot's cascades; Durrett & Liggett (1983) for some infinite particles systems; Biggins & Kyprianou (1997,2005), Iksanov & Polotskiy (2006) and Alsmeyer & Kuhlbusch (2010) for branching random walks; Volkovich & Litvak (2010) for the Pagerank algorithm (which is in the heart of the Google engine); Liu (1997, 1998), Aldous & Bandyopadhyay (2005), Buraczewski(2009), Buraczewski & Kolesko (2014) and Shi (2015) for related topics and many other references. Here we shall study the regular variation of the tail probability $\mathbb{P}(Z > x)$ when $x \rightarrow \infty$, for a solution Z of (E) with finite mean.

Let $\alpha > 1$ be such that $\rho(\alpha) < 1$. Lemma 1.1 (ii) shows that Z and Y_1 would have similar asymptotic properties. We shall show this by establishing comparison theorems between the tail behavior of Z and that of Y_1 . Throughout the paper, let

$$R_0 = \left\{ \ell : [0, \infty) \rightarrow [0, \infty) : \ell \text{ is measurable and } \lim_{x \rightarrow \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \forall \lambda > 0 \right\} \quad (1.9)$$

be the set of functions slowly varying at ∞ (also called the class of regularly varying functions of index 0). Recall that a function $\ell \in R_0$ can be represented in the form (cf. [11], p.12)

$$\ell(x) = c(x) \exp \int_a^x \frac{\epsilon(u)}{u} du \quad (x \geq a)$$

for some $a > 0$, where $c(\cdot)$ and $\epsilon(\cdot)$ are measurable with $c(x) \rightarrow c \in (0, \infty)$ and $\epsilon(x) \rightarrow 0$ as $x \rightarrow \infty$. Without loss of generality, we can take $c(x) \equiv c$. For simplicity, we take $c = 1$. Also, as the value of $\ell(x)$ on finite intervals have no influence on our purpose, we can take $a = 1$, and

$$\ell(x) = \begin{cases} \exp \int_1^x \frac{\epsilon(u)}{u} du & \text{if } x > 1, \\ 1 & \text{if } x \leq 1, \end{cases} \quad (1.10)$$

where $\epsilon(\cdot)$ is measurable and bounded with $\epsilon(x) \rightarrow 0$ as $x \rightarrow \infty$.

As usual, we write $f(s) = o(g(s))$ if $f(s)/g(s) \rightarrow 0$, and $f(s) \sim g(s)$ if $f(s)/g(s) \rightarrow 1$, where $s \rightarrow 0$ or ∞ according to the context. We use

$$f(s) = O(g(s)), \quad s \in I \quad (\text{resp. } s \rightarrow 0),$$

to mean $|f(s)| \leq C|g(s)|$ on I (resp. for $s > 0$ small enough), where $C > 0$ is a constant. We emphasize that $C > 0$ is a deterministic constant when g is a random function. For a set B we use $\text{Int } B$ to denote the interior of B .

In what follows we always assume that $Z \geq 0$ is a solution of (E) with $\mathbb{E}Z = 1$. Our main results are Theorems 1.1 and 1.2 below about the regular variation of $\mathbb{P}(Z > x)$.

Theorem 1.1 *Let $\alpha > 1$ be such that $\rho(\alpha) < 1$ and that*

$$\rho(\alpha + \delta_0) < \infty \quad \text{for some } \delta_0 > 0. \quad (1.11)$$

Then the following two assertions are equivalent for $\ell \in R_0$:

$$\mathbb{P}\{Y_1 > x\} \sim x^{-\alpha} \ell(x) \quad \text{as } x \rightarrow \infty, \quad (1.12)$$

$$\mathbb{P}\{Z > x\} \sim (1 - \rho(\alpha))^{-1} x^{-\alpha} \ell(x) \quad \text{as } x \rightarrow \infty. \quad (1.13)$$

In the context of the Galton-Watson process (where $A_i = 1/\mathbb{E}N < 1$ for all i) and the Crump-Mode-Jirina process (where $A_i \leq 1$ for all i), the same result was established by Bingham & Doney (1974, 1975) when $\alpha > 1$ is not an integer, and by de Meyer (1982) when $\alpha > 1$ is an integer. Their argument was based on the functional equation (E') and a powerful Tauberian theorem on Laplace transforms. The implication "(1.12) \Rightarrow (1.13)" has been extended by Iksanov & Polotskiy (2006) to the general case where A_i are not necessarily bounded, using an elegant martingale argument. The analytic argument of Bingham & Doney (1975) depends heavily on the boundedness of A_i , while the probabilistic argument of Iksanov & Polotskiy (2006) does not give the converse part. Notice that the boundedness condition on A_i restricts much the applications: for example, in the typical case of the canonical Mandelbrot's cascade and in the branching Brownian motion, where A_i are not bounded as $\ln A_i$ are Gaussian variables. Our theorem shows that the results of Bingham & Doney (1974, 1975) and de Meyer (1982) remain true in the general case (where A_i are not necessarily bounded), so that they can be applied for example to general Mandelbrot's martingales and branching random walks; in our approach we extend the analytic arguments of Bingham & Doney (1974, 1975) to the general case. While A_i are not bounded, the arguments become much more delicate; we use delicate truncating technics to overcome the difficulty. Theorem 1.1 also improves the result of Iksanov & Polotskiy (2006) in the sense that we obtain a necessary and sufficient condition, not just a sufficient condition.

When $\alpha \in (1, 2)$, the implication "(1.12) \Rightarrow (1.13)" has also been shown by Rösler, Topchii & Vatutin (2002, 2003) for weighted branching processes with real weights and by Iksanov, Kolesko & Meiners (2019) for a more general functional of the underlying branching random walk. Moreover, in the same work Iksanov et al. pointed out that when $\alpha \in (1, 2)$ and when $\lim_{x \rightarrow \infty} \ell(x)$ exists and is strictly positive and finite, the condition (1.11) can be removed. For a similar situation, see Mikosch and Samorodnitsky (2000). In this paper we need the condition (1.11) to establish Theorem 2.1 about the relations between the reminders of the Taylor expansion of Laplace transforms of Y_1 and Z (see e.g. the proof of Lemma 4.1 which serves

for the proof of Theorem 2.1), and to secure a proper use of Potter's bound while using Theorem 2.1 to prove Theorem 1.1. We mention that we can avoid the use of Potter's bound (so that the condition (1.11) will not be used for this purpose), if ℓ is sub-multiplicative in the sense that $\ell(xy) \leq c \ell(x) \ell(y)$ for some constants $c, x_0 > 0$ and all $x, y \geq x_0$ (which holds in particular if $\lim_{x \rightarrow \infty} \ell(x) \in (0, \infty)$); however we still need (1.11) in the proof of Theorem 2.1, so that we fail to remove it in Theorem 1.1.

Closely related results on weighted moments of the form $\mathbb{E}(Z^\alpha \ell(Z))$ have been established for example in Alsmeyer & Kuhlbusch (2010) and Liang & Liu (2015).

Remark 1.1 Notice that under (1.12) or (1.13), the condition (1.11) is equivalent to the condition

$$\mathbb{E}A^{\alpha+\varepsilon} < \infty \quad \text{for some } \varepsilon > 0, \quad \text{where } A = \sup_{1 \leq i \leq N} A_i, \quad (1.14)$$

which will be used in the following. In fact, if (1.11) holds, then it is clear that (1.14) holds for $\varepsilon = \delta_0$. So we only need to show that (1.14) implies (1.11). Relation (1.12) implies $\mathbb{E}Y_1^{\alpha-\delta_0} < \infty$ and (1.13) implies $\mathbb{E}Z^{\alpha-\delta_0} < \infty$ for $\delta_0 \in (0, \alpha - 1]$. So by Lemma 1.1(b), under (1.12) or (1.13), we have $\mathbb{E}Y_1^{\alpha-\delta_0} < \infty$. By Hölder's inequality,

$$\rho(\alpha + \delta_0) = \mathbb{E} \sum_{i=1}^N A_i^{\alpha+\delta_0} \leq \mathbb{E}[Y_1 A^{\alpha-1+\delta_0}] \leq \left(\mathbb{E}Y_1^{\alpha-\delta_0} \right)^{1/(\alpha-\delta_0)} \cdot \mathbb{E}(A^{\alpha+\delta})^{(\alpha-1-\delta_0)/(\alpha-\delta_0)}, \quad (1.15)$$

where $\delta = \frac{(\alpha+1-\delta_0)\delta_0}{\alpha-1-\delta_0}$. When $\delta_0 > 0$ is small enough, we have $\delta \leq \varepsilon$, so that the right side of (1.15) is finite by (1.14).

We shall prove the equivalence between (1.12) and (1.13) under the condition (1.14), using Tauberian theorems and truncating techniques. Our approach also gives a new proof of the result of Iksanov & Polotskiy (2006).

Let $\alpha_c \geq 1$ be the critical value for the existence of moments of Y_1 :

$$\alpha_c = \sup\{a \geq 1 : \mathbb{E}Y_1^a < \infty\}. \quad (1.16)$$

Notice that Theorem 1.1 is applicable only in the case where $\alpha_c > 1$. When $\alpha_c = 1$, the following result shows that the situation is different. For $\ell \in R_0$ with $\int_1^\infty \ell(t)dt/t < \infty$, set $\tilde{\ell}(x) = 0$ for $x \in [0, 1)$, and

$$\tilde{\ell}(x) = \int_x^\infty \frac{\ell(t)}{t} dt \quad \text{for } x \geq 1. \quad (1.17)$$

Theorem 1.2 *Assume that $\rho(1 + \delta_0) < \infty$ for some $\delta_0 > 0$, and that $\mu := -\rho'(1) \in (0, \infty)$. Let $\ell \in R_0$ be such that $\int_1^\infty \ell(x)dx/x < \infty$. If*

$$\mathbb{E}Y_1 \mathbf{1}_{\{Y_1 > x\}} \sim \ell(x) \quad (x \rightarrow \infty), \quad (1.18)$$

then

$$\mathbb{E}Z \mathbf{1}_{\{Z > x\}} \sim \mu^{-1} \tilde{\ell}(x) \quad (x \rightarrow \infty). \quad (1.19)$$

The same result was established by Bingham & Doney (1974, 1975) for the Galton-Watson process, the Crump-Mode and Jirina processes. Notice that, just as in the result of Bingham & Doney (1974, 1975), here we have not established the equivalence between (1.18) and (1.19). An explanation on this point will be given in Remark 7.1.

Theorems 1.1 and 1.2, without detailed proofs, have been announced in Liang & Liu (2011). In both Theorems 1.1 and 1.2, we are in the case where $\mathbb{E}[Y_1 \ln^+ Y_1] < \infty$. In fact, in Theorem 1.1, each of the conditions (1.12) and (1.13) implies that $\mathbb{E}[Y_1^a] < \infty$ for $1 < a < \alpha$, so that $\mathbb{E}[Y_1 \ln^+ Y_1] < \infty$; in Theorem 1.2, the conditions $\int_1^\infty \ell(x)dx/x < \infty$ and (1.18) also imply that $\mathbb{E}[Y_1 \ln^+ Y_1] < \infty$, because, as mentioned in Bingham & Doney (1974),

$$\int_1^\infty \mathbb{E}[Y_1 \mathbf{1}_{\{Y_1 > x\}}] \frac{dx}{x} = \mathbb{E}[Y_1 \ln^+ Y_1].$$

As information, we mention that when $\mathbb{E}[Y_1 \ln^+ Y_1] = \infty$, the asymptotic behavior of Z has also been studied in the literature, see for example Alsmeyer & Iksanov (2009) about the existence of moments of Z .

The rest of the paper is organized as follows. In Section 2, we explain the main ideas of the approach, and the relations between the Laplace transforms of Z and Y_1 (cf. Theorems 2.1 and 2.2). Section 3 contains some preliminary results about the equation (E) and its solutions. Sections 4 and 5 are devoted to the proof of auxiliary theorems announced in Section 2. Theorems 1.1 and 1.2 are proved respectively in Sections 6 and 7.

In the following sections, $C > 0$ always stands for a deterministic constant whose value may differ from line to line.

2 Key ideas of the approach

2.1 Tauberian theorems

For a non-negative random variable X , we write $f(s) = \mathbb{E}e^{-sX}$ for its Laplace transform and $\mu_r = \mathbb{E}X^r$ for its r -moment ($r = 0, 1, 2, \dots$). If X has at least finite n -th moment, we set

$$f_n(s) = (-1)^{n+1} \mathbb{E} \left\{ e^{-sX} - \sum_{r=0}^n \frac{(-sX)^r}{r!} \right\}, \quad n \geq 0, \quad (2.1)$$

and

$$\begin{aligned} f_n^{(n)}(s) &= \frac{d^n f_n(s)}{ds^n} = (-1)^n \{f^{(n)}(0) - f^{(n)}(s)\} \\ &= \mathbb{E}X^n(1 - e^{-sX}), \quad n \geq 0. \end{aligned} \quad (2.2)$$

We shall use the following two Tauberian theorems which give necessary and sufficient conditions for $\mathbb{P}(X > x) \sim x^{-\alpha} \ell(x)$, as $x \rightarrow \infty$: see Proposition 2.1 for $\alpha \notin \mathbb{N}$, and Proposition 2.2 for $\alpha \in \mathbb{N}$. As usual, denote by $\Gamma(\cdot)$ the gamma function.

Proposition 2.1 ([9], Theorem A) *Let X be a non-negative random variable with Laplace transform $f(s) = \mathbb{E}e^{-sX}$. Let $n \in \mathbb{N}^*$. If $\mu_r = \mathbb{E}X^r < \infty$ for $r = 0, 1, \dots, n$, then for $\alpha = n + \beta$ with $0 \leq \beta \leq 1$, the following statements are equivalent for $\ell \in R_0$:*

$$\begin{aligned} f_n(s) &\sim s^\alpha \ell(1/s) && (s \rightarrow 0); \\ f_n^{(n)}(s) &\sim \frac{\Gamma(\alpha + 1)}{\Gamma(\beta + 1)} s^\beta \ell(1/s) && (s \rightarrow 0); \\ \mathbb{E}X^n \mathbf{1}_{\{X > x\}} &\sim n! \ell(x) && (x \rightarrow \infty) \text{ when } \beta = 0; \\ \mathbb{P}(X > x) &\sim \frac{(-1)^n}{\Gamma(1 - \alpha)} x^{-\alpha} \ell(x) && (x \rightarrow \infty) \text{ when } 0 < \beta < 1; \\ \mathbb{E}X^{n+1} \mathbf{1}_{\{X \leq x\}} &\sim (n + 1)! \ell(x) && (x \rightarrow \infty) \text{ when } \beta = 1. \end{aligned}$$

Proposition 2.2 *Let X be a non-negative random variable with Laplace transform $f(s) = \mathbb{E}e^{-sX}$. Let $n \geq 0$ be an integer, and assume that $\mu_n = \mathbb{E}X^n < \infty$. Define f_n by (2.1) and*

$$\hat{f}_n(s) = f_n(s)/s^n, \quad s > 0.$$

Then the following two statements are equivalent for $\ell \in R_0$:

$$\mathbb{P}(X > x) \sim x^{-(n+1)} \ell(x) \quad (x \rightarrow \infty); \quad (2.3)$$

$$\frac{\hat{f}_n(t) - \hat{f}_n(\lambda t)}{\ell(1/t)} \sim \frac{1}{n!} \ln \lambda \quad (t \rightarrow 0) \quad \forall \lambda > 1. \quad (2.4)$$

Proposition 2.1 was established by Bingham & Doney (1974), using Karamata's theorem (cf. [11] or [19]). Proposition 2.2 can be obtained from the argument by de Meyer and Teugels (1980, Section 3) via de Haan's theorem (cf. [11], [15] or [16]), and was used in de Meyer (1982) for extending the result of Bingham and Doney (1974) to integer orders. However this result was not explicitly stated neither in de Meyer and Teugels (1980) nor in de Meyer (1982). For reader's convenience, we give the statement here whose proof is postponed in the Appendix.

2.2 Relations between the Laplace transforms of Y_1 and Z

Assume that $Z \geq 0$ is a solution of (E) with $\mathbb{E}Z = 1$. In view of Propositions 2.1 and 2.2, to obtain comparison results for the regular variations of Y_1 and Z , we first show relations between h_n and ϕ_n defined below: for $n \in \mathbb{N}$,

$$h_n(s) = (-1)^{n+1} \mathbb{E} \left\{ e^{-sY_1} - \sum_{r=0}^n \frac{(-sY_1)^r}{r!} \right\}, \quad (2.5)$$

$$\phi_n(s) = (-1)^{n+1} \mathbb{E} \left\{ e^{-sZ} - \sum_{r=0}^n \frac{(-sZ)^r}{r!} \right\}. \quad (2.6)$$

Let $\phi(s) = \mathbb{E}e^{-sZ}$ ($s \geq 0$) be the Laplace transform of Z . Set

$$t(s) = -\ln \phi(s) \quad \text{and} \quad T(s) = \sum_{i=1}^N t(sA_i). \quad (2.7)$$

Since $\mathbb{E}Z = 1$, by Jensen's inequality, we have $\phi(s) = \mathbb{E}e^{-sZ} \geq e^{-s\mathbb{E}Z} = e^{-s}$, so

$$0 \leq t(s) \leq s \quad \text{and} \quad 0 \leq T(s) \leq Y_1 s. \quad (2.8)$$

Moreover, as $\mathbb{E}Z = 1$,

$$t(s) \sim s \quad \text{and} \quad T(s) \sim \sum_{i=1}^N sA_i = sY_1 \quad (s \rightarrow 0). \quad (2.9)$$

For $n \in \mathbb{N}$, if $\mathbb{E}Z^n < \infty$, then ϕ has n -th continuous derivatives $\phi^{(n)}$ on $[0, \infty)$. Therefore $t(\cdot)$ and $T(\cdot)$ have also continuous n -th derivatives, denoted respectively by $t^{(n)}(s)$ and $T^{(n)}(s)$. Clearly, for $s \geq 0$,

$$T^{(n)}(s) = \sum_{i=1}^N A_i^n t^{(n)}(sA_i).$$

Let M be a random variable whose distribution is determined by

$$\mathbb{E}f(M) = \mathbb{E} \sum_{i=1}^N A_i f(A_i) \quad (2.10)$$

for any measurable function $f : [0, \infty) \rightarrow [0, \infty)$. Set

$$g_n(s) = \mathbb{E}Y_1^n (1 - e^{-sY_1}) \quad \text{and} \quad \psi_n(s) = \mathbb{E}Z^n (1 - e^{-sZ}). \quad (2.11)$$

Then $g_n(s) = h_n^{(n)}(s)$ and $\psi_n(s) = \phi_n^{(n)}(s)$, where $h_n^{(n)}$ and $\phi_n^{(n)}$ are the n -th derivatives of h_n and ϕ_n , respectively. We have the following comparison theorem between g_n and ψ_n .

Theorem 2.1 *Let $Z \geq 0$ be a solution of (E) with $\mathbb{E}Z = 1$, and define g_n and ψ_n by (2.11).*

- (i) *Assume that $\rho(\alpha) < 1$ for some $\alpha > 1$, and that $\rho(\alpha + \delta_0) < \infty$ for some $\delta_0 > 0$. Assume also that $\mathbb{E}Y_1^\gamma < \infty$ and $\mathbb{E}Z^\gamma < \infty$ for all $\gamma \in [1, \alpha)$. Then writing $\alpha = n + \beta$ with $n \in \mathbb{N}^*$ and $\beta \in (0, 1]$, we have for $\epsilon > 0$ small enough and all $s \geq 0$,*

$$\psi_n(s) - \mathbb{E}\{M^{n-1}\psi_n(sM)\} = g_n(s) + c_n s + O(s^{\beta+\epsilon}), \quad (2.12)$$

where c_n is a constant only depending on n , with $c_n = 0$ when α is not an integer (i.e. $\beta < 1$).

- (ii) *Assume that $\rho(1 + \delta_0) < \infty$ for some $\delta_0 > 0$. Then for $\epsilon > 0$ small enough and all $s \geq 0$,*

$$\psi_1(s) - \mathbb{E}\psi_1(sM) = \mathbb{E}\{T'(s)(1 - e^{-T(s)})\} + O(s^\epsilon). \quad (2.13)$$

Part (i) deals with the case where $\mathbb{E}(Y_1^\gamma) < \infty$ for some $\gamma > 1$, while Part (ii) applies without any moment condition on Y_1 (other than $\mathbb{E}Y_1 = 1$).

To fix the idea, notice that the interesting part in (2.12) is the validity for $s \in [0, 1]$ (or $s \rightarrow 0$), as the validity for $s \in (1, \infty)$ is evident due to the condition that $\mathbb{E}Y_1^n < \infty$ and $\mathbb{E}Z^n < \infty$ (which implies that each of the first three terms in (2.12) is bounded). The situation is similar for (2.13).

Integrating (2.12) and (2.13) n times, and noting that $\psi_k(0) = g_k(0) = 0$ for $k \leq n$, we obtain the following relations between h_n and ϕ_n .

Theorem 2.2 *Let $Z \geq 0$ be a solution of (E) with $\mathbb{E}Z = 1$, and define h_n and ϕ_n by (2.5) and (2.6).*

(i) *Assume that $\rho(\alpha) < 1$ for some $\alpha > 1$, and that $\rho(\alpha + \delta_0) < \infty$ for some $\delta_0 > 0$. Assume also that $\mathbb{E}Y_1^\gamma < \infty$ and $\mathbb{E}Z^\gamma < \infty$ for all $\gamma \in [1, \alpha]$. Then writing $\alpha = n + \beta$ with $n \in \mathbb{N}^*$ and $\beta \in (0, 1]$, we have for $\epsilon > 0$ small enough and all $s \geq 0$,*

$$\phi_n(s) - \mathbb{E}\{M^{-1}\phi_n(sM)\} = h_n(s) + c_n s^{n+1} + O(s^{\alpha+\epsilon}), \quad (2.14)$$

where c_n is a constant only depending on n , with $c_n = 0$ when α is not an integer (i.e. $\beta < 1$).

(ii) *Assume that $\rho(1 + \delta_0) < \infty$ for some $\delta_0 > 0$. Then for $\epsilon > 0$ small enough and all $s \geq 0$,*

$$\phi_1(s) - \mathbb{E}\{M^{-1}\phi_1(sM)\} = \mathbb{E}\{e^{-T(s)} - 1 + T(s)\} + O(s^{1+\epsilon}). \quad (2.15)$$

Just as in (2.12) and (2.13), what is essential in (2.14) and (2.15) is the validity for $s \in [0, 1]$ (or $s \rightarrow 0$). Part (i) will be applied for the proof of Theorem 1.1, while Part (ii) applies for the proof of Theorem 1.2.

Let us explain one of the main difficulties in our approach compared with the work of Bingham and Doney (1975) who considered the case where $A_i \leq 1$. We just consider Part (ii) as an example for illustration. When $\sup_{1 \leq i \leq N} A_i$ is bounded, say $\sup_{1 \leq i \leq N} A_i \leq C$ a.s. for some constant $C > 0$, then it can be easily seen that the function $\mathbb{E}(e^{-T(s)} - 1 + T(s))$ appearing in Theorem 2.2 (ii) is close to $h_1(s)$ as $s \rightarrow 0$, which is essential for the proof of Theorem 1.2 (see Section 7). In fact, as $t(s) \sim s$ ($s \rightarrow 0$), for each $\epsilon \in (0, 1)$, there is $s_0 = s_0(\epsilon) > 0$ such that $(1 - \epsilon)s \leq t(s) \leq (1 + \epsilon)s$, $\forall s \in (0, s_0)$. As $A_i s \leq Cs$ and $H(x) := e^{-x} - 1 + x$ is increasing, it follows that for $s \leq s_0/C$, $T(s) = \sum_{i=1}^N t(A_i s)$ lies between $(1 \pm \epsilon)sY_1$, so that

$$\mathbb{E}(e^{-T(s)} - 1 + T(s)) \text{ lies between } \mathbb{E}H((1 \pm \epsilon)sY_1) = h_1((1 \pm \epsilon)s). \quad (2.16)$$

When $\sup_{1 \leq i \leq N} A_i$ is not bounded, the argument above is no longer valid, so new ideas are needed to arrive at a similar conclusion: cf. Lemma 7.1; this is the typical difficulty that we have compared with the work of Bingham and Doney (1975).

We now summarize the key ideas for proving Theorem 1.1. In the case where $\alpha > 1$ is not an integer, by Proposition 2.1, it suffices to show that the following two assertions are equivalent:

$$h_n(s) \sim s^\alpha \ell(1/s) \quad (s \rightarrow 0), \quad (2.17)$$

$$\phi_n(s) \sim \frac{s^\alpha \ell(1/s)}{1 - \rho(\alpha)} \quad (s \rightarrow 0). \quad (2.18)$$

This equivalence will be shown by means of Theorem 2.2 (i). In the case where $\alpha > 1$ is an integer, the argument is similar, using Proposition 2.2 instead of Proposition 2.1.

To prove Theorem 1.2, again by Proposition 2.1 (with $n = 1$ and $\beta = 0$), we just need to show that

$$h_1(s) \sim s\ell(1/s) \quad (s \rightarrow 0) \quad (2.19)$$

implies

$$\phi_1(s) \sim \mu^{-1} s\tilde{\ell}(1/s) \quad (s \rightarrow 0), \quad (2.20)$$

where $\tilde{\ell}$ is defined in (1.17). This will be done with help of Theorem 2.2 (ii). Unfortunately, we fail to prove the equivalence between (2.19) and (2.20), which would lead to the equivalence between (1.18) and (1.19). See Remark 7.1 on this point.

The details of the proofs of Theorems 1.1 and 1.2 are given respectively in Sections 6 and 7. Both proofs are based on Theorem 2.2 about the comparison between h_n and ϕ_n , which is a direct consequence of Theorems 2.1 whose proof is given in Sections 4 and 5.

3 Asymptotic properties of Laplace transforms

Let $Z \geq 0$ be a random variable with $\mathbb{E}Z = 1$. Set $\phi(s) = \mathbb{E}e^{-sZ}$ and $t(s) = -\ln \phi(s)$ for $s \geq 0$. In this section we present some asymptotic properties of $\phi(s), t(s)$ and their derivatives, as $s \rightarrow 0$.

As in Bingham and Doney (1975), differentiating n times the function $\phi(s)$ in the form

$$\phi(s) = e^{-t(s)}, \quad (3.1)$$

we obtain

$$\phi^{(n)}(s) = e^{-t(s)} \{P_{n,t}(s) + (-t'(s))^n - t^{(n)}(s)\}, \quad (3.2)$$

where $P_{n,t}(s)$ is a polynomial in $t'(s), \dots, t^{(n-1)}(s)$:

$$P_{n,t}(s) = \sum_{a_1, \dots, a_{n-1}} c(a_1, \dots, a_{n-1}) (t'(s))^{a_1} \dots (t^{(n-1)}(s))^{a_{n-1}}, \quad (3.3)$$

$c(a_1, \dots, a_{n-1})$ are integers and $\{a_k\}_{k=1}^{n-1}$ are non-negative integers satisfying

$$\sum_{k=1}^{n-1} ka_k = n \quad \text{and} \quad a_1 \neq n; \quad (3.4)$$

by convention, $P_{1,t}(s) = P_{2,t}(s) = 0$. By iteration, $t^{(n)}(s)$ can be written in the form

$$t^{(n)}(s) = Q_{n,\phi}(s) + \left(\frac{\phi'(s)}{\phi(s)}\right)^n + \left(-\frac{\phi^{(n)}(s)}{\phi(s)}\right), \quad (3.5)$$

where $Q_{n,\phi}(s)$ is a polynomial in $\frac{\phi'(s)}{\phi(s)}, \dots, \frac{\phi^{(n-1)}(s)}{\phi(s)}$:

$$Q_{n,\phi}(s) = \sum_{a_1, \dots, a_{n-1}} d(a_1, \dots, a_{n-1}) \left(\frac{\phi'(s)}{\phi(s)}\right)^{a_1} \dots \left(\frac{\phi^{(n-1)}(s)}{\phi(s)}\right)^{a_{n-1}}, \quad (3.6)$$

$d(a_1, \dots, a_{n-1})$ are integers and $\{a_k\}_{k=1}^{n-1}$ are non-negative integers satisfying (3.4), with the convention that $Q_{1,\phi}(s) = Q_{2,\phi}(s) = 0$.

The following elementary properties of $\frac{\phi^{(k)}(s)}{\phi(s)}$ and $t^{(k)}(s)$ will be frequently used.

Lemma 3.1 *Let $Z \geq 0$ be a non-negative random variable. Write $\phi(s) = \mathbb{E}e^{-sZ}$ and $t(s) = -\ln \phi(s)$.*

(i) *If $\mathbb{E}Z^n < \infty$ for some $n \in \mathbb{N}^*$, then for $1 \leq k \leq n$,*

$$\sup_{s \geq 0} \left| \frac{\phi^{(k)}(s)}{\phi(s)} \right| < \infty, \quad \sup_{s \geq 0} |t^{(k)}(s)| < \infty, \quad (3.7)$$

and for $0 \leq k \leq n-1$,

$$\phi^{(k)}(s) = \phi^{(k)}(0) + O(s), \quad t^{(k)}(s) = t^{(k)}(0) + O(s), \quad s \geq 0. \quad (3.8)$$

(ii) *If $\mathbb{E}Z^{n+\beta} < \infty$ for some $n \in \mathbb{N}^*$ and $\beta \in (0, 1]$, then*

$$\phi^{(n)}(s) = \phi^{(n)}(0) + O(s^\beta) \quad \text{and} \quad t^{(n)}(s) = t^{(n)}(0) + O(s^\beta), \quad s \geq 0. \quad (3.9)$$

Consequently, the function ϕ_n defined by (2.6) satisfies

$$\phi_n(s) = O(s^{n+\beta}), \quad s \geq 0. \quad (3.10)$$

We mention that what is important in (3.8)-(3.10) is the validity for $s \in [0, 1]$ (or $s \rightarrow 0$).

Proof of Lemma 3.1. (i) Without loss of generality, we assume that $\mathbb{E}Z = 1$. So the law μ of Z satisfies $\mu([0, 1]) > 0$. In order to prove (3.7), from the representation form (3.5), it is sufficient to show that $\frac{\phi^{(k)}(s)}{\phi(s)}$

are uniformly bounded on $[0, \infty)$ for $1 \leq k \leq n$. Recalling that ϕ is the Laplace transform of Z , we have for $1 \leq k \leq n$,

$$\begin{aligned} \left| \frac{\phi^{(k)}(s)}{\phi(s)} \right| &= \frac{\int_{[0, \infty)} t^k e^{-ts} \mu(dt)}{\int_{[0, \infty)} e^{-ts} \mu(dt)} \\ &\leq \frac{\int_{[0, 1]} t^k e^{-ts} \mu(dt)}{\int_{[0, 1]} e^{-ts} \mu(dt)} + \frac{\int_{(1, \infty)} t^k e^{-ts} \mu(dt)}{\int_{[0, 1]} e^{-ts} \mu(dt)} \\ &\leq 1 + \frac{e^{-s} \int_{(1, \infty)} t^k \mu(dt)}{e^{-s} \int_{[0, 1]} \mu(dt)} \\ &\leq 1 + \frac{\mathbb{E}Z^k}{\mu([0, 1])}, \end{aligned} \quad (3.11)$$

which is finite by the assumption. This ends the proof of (3.7). (3.8) is a consequence of (3.7) by the mean value theorem.

(ii) Notice that for $n \in \mathbb{N}$,

$$\phi^{(n)}(s) - \phi^{(n)}(0) = (-1)^n \mathbb{E}Z^n (e^{-sZ} - 1).$$

For any $\beta \in (0, 1]$, we have $1 - e^{-x} \leq Cx^\beta$ ($x > 0$), so

$$1 - e^{-sZ} \leq Cs^\beta Z^\beta.$$

It follows that for $n \in \mathbb{N}$ and $s \geq 0$,

$$|\phi^{(n)}(s) - \phi^{(n)}(0)| \leq Cs^\beta \mathbb{E}Z^{n+\beta}.$$

This gives the first conclusion in (3.9). Using this inequality, together with (3.11), and the fact that $|\phi(s) - \phi(0)| \leq Cs \leq Cs^\beta$ for $s \in [0, 1]$, we have

$$\left| \frac{\phi^{(k)}(s)}{\phi(s)} - \frac{\phi^{(k)}(0)}{\phi(0)} \right| \leq \left| \frac{\phi^{(k)}(s)}{\phi(s)} \cdot \frac{\phi(0) - \phi(s)}{\phi(0)} \right| + \left| \frac{\phi^{(k)}(s)}{\phi(0)} - \frac{\phi^{(k)}(0)}{\phi(0)} \right| \leq Cs^\beta, \quad s \in [0, 1], \quad k = 1, \dots, n.$$

It follows from the expression (3.5) of $t^{(n)}(s)$ that $|t^{(n)}(s) - t^{(n)}(0)| \leq Cs^\beta$ for $s \in [0, 1]$; the later holds evidently for $s > 1$ since $\sup_{s \geq 0} |t^{(n)}(s)| < \infty$. This gives the second conclusion in (3.9). The proof of (3.9) is then finished.

Integrating (3.9) n times, we see that (3.10) holds. \square

4 Proof of Theorem 2.1: case where $\alpha \in [1, \infty) \setminus \{2, 3, \dots\}$

Let $Z \geq 0$ be a solution of (E) with $\mathbb{E}Z = 1$. We rewrite (E') in the form

$$\phi(s) = \mathbb{E}e^{-T(s)}. \quad (E'')$$

Differentiating n times the equation (E'') just as we did for (3.1), we obtain

$$\phi^{(n)}(s) = \mathbb{E}e^{-T(s)} \{P_{n,T}(s) + (-T'(s))^n - T^{(n)}(s)\}, \quad (4.1)$$

where $P_{n,T}(s)$ is defined as $P_{n,t}(s)$ (cf. (3.3)) with t replaced by T .

Before giving the proof of Theorem 2.1, we first show two lemmas.

Lemma 4.1 *If $\mathbb{E}Z^{n+\beta-\epsilon} < \infty$ and $\mathbb{E} \sum_{i=1}^N A_i^{n+\beta+\delta_0} < \infty$ for some $n \in \mathbb{N}^*$, $\beta \in (0, 1]$, $\delta_0 > 0$ and $\epsilon \in (0, \beta)$, then we have for $\epsilon > 0$ small enough,*

$$0 \leq \mathbb{E}Y_1^n \left(e^{-T(s)} - e^{-sY_1} \right) = O(s^{\beta+\epsilon}), \quad s \geq 0. \quad (4.2)$$

Proof. Notice that $t'(0) = -\frac{\phi'(0)}{\phi(0)} = 1$ and that $\mathbb{E}Z^{1+\beta-\epsilon} < \infty$ implies $|t'(s) - t'(0)| \leq Cs^{\beta-\epsilon}$ (by (3.9) of Lemma 3.1), so $|t(s) - s| \leq C\frac{s^{1+\beta-\epsilon}}{1+\beta-\epsilon}$. Hence

$$0 \leq sY_1 - T(s) \leq Cs^{1+\beta-\epsilon}A^{\beta-\epsilon}Y_1, \quad (4.3)$$

so

$$\begin{aligned} T(s) &\geq sY_1 - Cs^{1+\beta-\epsilon}A^{\beta-\epsilon}Y_1 \\ &= sY_1(1 - Cs^{\beta-\epsilon}A^{\beta-\epsilon}) \\ &\geq sY_1/2 \quad \text{if } Cs^{\beta-\epsilon}A^{\beta-\epsilon} \leq 1/2. \end{aligned} \quad (4.4)$$

Now by the mean value theorem, $e^{-T(s)} - e^{-sY_1} = e^{-\xi}(sY_1 - T(s))$ for some $\xi \in [T(s), sY_1]$, so that

$$e^{-T(s)} - e^{-sY_1} \leq e^{-T(s)}(sY_1 - T(s)).$$

Hence from (4.3) and (4.4), we see that if $Cs^{\beta-\epsilon}A^{\beta-\epsilon} \leq 1/2$, then

$$e^{-T(s)} - e^{-sY_1} \leq e^{-sY_1/2}(sY_1 - T(s)) \leq Ce^{-sY_1/2}s^{1+\beta-\epsilon}A^{\beta-\epsilon}Y_1.$$

Using this and the fact that $e^{-x/2}x^{1-2\epsilon}$ is bounded for $x \geq 0$, we have

$$\begin{aligned} \left(e^{-T(s)} - e^{-sY_1}\right) Y_1^n \mathbf{1}_{\{Cs^{\beta-\epsilon}A^{\beta-\epsilon} \leq 1/2\}} &\leq Ce^{-sY_1/2}s^{1+\beta-\epsilon}A^{\beta-\epsilon}Y_1 \cdot Y_1^n \cdot 1 \\ &= Ce^{-sY_1/2}(sY_1)^{1-2\epsilon} \cdot s^{\beta+\epsilon}A^{\beta-\epsilon}Y_1^{n+2\epsilon} \\ &\leq Cs^{\beta+\epsilon}A^{\beta-\epsilon}Y_1^{n+2\epsilon}. \end{aligned} \quad (4.5)$$

On the other hand, as $\mathbf{1}_{\{Cs^{\beta-\epsilon}A^{\beta-\epsilon} > 1/2\}} \leq 2^b C^b s^{\beta+\epsilon}A^{\beta+\epsilon}$ with $b = (\beta + \epsilon)/(\beta - \epsilon)$, we have

$$\begin{aligned} \left(e^{-T(s)} - e^{-sY_1}\right) Y_1^n \mathbf{1}_{\{Cs^{\beta-\epsilon}A^{\beta-\epsilon} > 1/2\}} &\leq Y_1^n \mathbf{1}_{\{Cs^{\beta-\epsilon}A^{\beta-\epsilon} > 1/2\}} \\ &\leq 2^b C^b s^{\beta+\epsilon}A^{\beta+\epsilon}Y_1^n. \end{aligned} \quad (4.6)$$

It follows from (4.5) and (4.6) that

$$\left(e^{-T(s)} - e^{-sY_1}\right) Y_1^n \leq Cs^{\beta+\epsilon} \left(A^{\beta-\epsilon}Y_1^{n+2\epsilon} + A^{\beta+\epsilon}Y_1^n\right). \quad (4.7)$$

Noting that $\mathbb{E}Z^{n+\beta-\epsilon} < \infty$ implies $\mathbb{E}Y_1^{n+\beta-\epsilon} < \infty$, by Hölder's inequality,

$$\mathbb{E}A^{\beta-\epsilon}Y_1^{n+2\epsilon} \leq \left(\mathbb{E}A^{(\beta-\epsilon)p}\right)^{1/p} \cdot \left(\mathbb{E}Y_1^{(n+2\epsilon)p^*}\right)^{1/p^*} < \infty, \quad (4.8)$$

$$\mathbb{E}A^{\beta+\epsilon}Y_1^n \leq \left(\mathbb{E}A^{(\beta+\epsilon)q}\right)^{1/q} \cdot \left(\mathbb{E}Y_1^{nq^*}\right)^{1/q^*} < \infty, \quad (4.9)$$

where $p = \frac{n+\beta-\epsilon}{\beta-3\epsilon}$, $p^* = \frac{n+\beta-\epsilon}{n+2\epsilon}$, $q = \frac{n+\beta-\epsilon}{\beta-\epsilon}$, $q^* = \frac{n+\beta-\epsilon}{n}$ and $\epsilon > 0$ is small enough such that $(\beta - \epsilon)p \leq n + \beta + \delta_0$ and $(\beta + \epsilon)q \leq n + \beta + \delta_0$. Therefore (4.7) implies (4.2). \square

Further, when $n = 1$, we also have the following lemma.

Lemma 4.2 *If $\mathbb{E}Z^{1+\beta-\epsilon} < \infty$ and $\mathbb{E}\sum_{i=1}^N A_i^{1+\beta+\delta_0} < \infty$ for some $\beta \in (0, 1]$, $\delta_0 > 0$ and $\epsilon \in (0, \beta)$, then we have for $\epsilon > 0$ small enough,*

$$\mathbb{E}T'(s) \left(1 - e^{-T(s)}\right) = g_1(s) + O(s^{\beta+\epsilon}), \quad s \geq 0. \quad (4.10)$$

Proof. Recall that $g_1(s) = \mathbb{E}Y_1(1 - e^{-sY_1})$, so (4.10) is equivalent to

$$\mathbb{E}(K_1 + K_2) = O(s^{\beta+\epsilon}), \quad s \geq 0, \quad (4.11)$$

where

$$\begin{aligned} K_1 &= (T'(s) - Y_1) \left(1 - e^{-T(s)}\right), \\ K_2 &= Y_1 \left(e^{-sY_1} - e^{-T(s)}\right). \end{aligned}$$

In Lemma 4.1, we have shown that

$$0 \geq \mathbb{E}K_2 \geq Cs^{\beta+\epsilon}, \quad s \geq 0,$$

so we just need to show that

$$0 \geq \mathbb{E}K_1 \geq Cs^{\beta+\epsilon}, \quad s \geq 0. \quad (4.12)$$

Notice that $K_1 \leq 0$. In fact, Recalling the definition of $T'(s)$, we have $T'(s) - Y_1 = \sum_{i=1}^N A_i [t'(sA_i) - t'(0)]$. Since $t'(s)$ is non-increasing, it follows that $T'(s) - Y_1 \leq 0$, so that $K_1 \leq 0$.

As $\mathbb{E}Z^{1+\beta-\epsilon} < \infty$, we have $|t'(s) - t'(0)| \leq Cs^{\beta-\epsilon}$ for $s \geq 0$ (by (3.9) of Lemma 3.1), so that

$$|T'(s) - Y_1| \leq Cs^{\beta-\epsilon} \sum_{i=1}^N A_i^{1+\beta-\epsilon} \leq Cs^{\beta-\epsilon} A^{\beta-\epsilon} Y_1. \quad (4.13)$$

Noting that $1 - e^{-x} \leq x^{2\epsilon}$ for $2\epsilon \leq 1$ and $x > 0$ and that $T(s) \leq sY_1$, we obtain

$$\begin{aligned} |K_1| &\leq Cs^{\beta-\epsilon} A^{\beta-\epsilon} Y_1 (T(s))^{2\epsilon} \\ &\leq Cs^{\beta-\epsilon} A^{\beta-\epsilon} Y_1 (sY_1)^{2\epsilon} \\ &= Cs^{\beta+\epsilon} A^{\beta-\epsilon} Y_1^{1+2\epsilon}. \end{aligned}$$

As $\mathbb{E}A^{\beta-\epsilon} Y_1^{1+2\epsilon} < \infty$ (by (4.8)) for $\epsilon > 0$ small enough, this implies (4.12). So the proof of (4.10) is finished. \square

Proof of Theorem 2.1: case where $\alpha \in [1, \infty) \setminus \{2, 3, \dots\}$. For $\alpha > 1$ not an integer, write $\alpha = n + \beta$, where $n \in \mathbb{N}^*$ and $\beta \in (0, 1)$. For $\alpha = 1$, define $\beta = 0$. We distinguish two cases according to $\alpha \in [1, 2)$ or $\alpha > 2$. In the following, ϵ always stands for a number in $(0, 1 - \beta)$ small enough.

(a) **Case $\alpha \in [1, 2)$.** When $\alpha = 1$ (so $\beta = 0$), we suppose that $\rho(1 + \delta_0) < \infty$ for some $\delta_0 > 0$, and we need to prove (2.13), that is,

$$\psi_1(s) - \mathbb{E}\psi_1(sM) = \mathbb{E}\{T'(s)(1 - e^{-T(s)})\} + O(s^\epsilon), \quad \text{where } \psi_1(s) = \phi'(s) + 1; \quad (4.14)$$

when $\alpha \in (1, 2)$ (so $\beta \in (0, 1)$), we assume that $\rho(\alpha) < 1$, $\rho(\alpha + \delta_0) < \infty$ for some $\delta_0 > 0$, and that $\mathbb{E}Y_1^\gamma < \infty$ and $\mathbb{E}Z^\gamma < \infty$ for all $\gamma \in [1, \alpha)$, and we need to prove (2.12) with $n = 1$, that is

$$\psi_1(s) - \mathbb{E}\{\psi_1(sM)\} = g_1(s) + O(s^{\beta+\epsilon}). \quad (4.15)$$

To prove (4.14) and (4.15), we only need to prove that

$$\phi'(s) - \mathbb{E}\phi'(sM) = \mathbb{E}T'(s)(1 - e^{-T(s)}) + O(s^{\beta+\epsilon}), \quad s \geq 0. \quad (4.16)$$

In fact, since $\psi_1(s) = \phi'(s) + 1$, in (4.16) we can replace ϕ' by ψ_1 to obtain

$$\psi_1(s) - \mathbb{E}\psi_1(sM) = \mathbb{E}T'(s)(1 - e^{-T(s)}) + O(s^{\beta+\epsilon}). \quad (4.17)$$

When $\beta = 0$, this gives (4.14). When $\beta \in (0, 1)$, (4.17) together with Lemma 4.2 imply (4.15).

It remains to prove (4.16). Since $\phi'(s) = \mathbb{E}\{-T'(s)e^{-T(s)}\}$ (cf. (E')), (4.16) is equivalent to

$$\mathbb{E}T'(s) + \mathbb{E}\phi'(sM) = O(s^{\beta+\epsilon}), \quad s \geq 0. \quad (4.18)$$

As $T'(s) = \sum_{i=1}^N A_i t'(sA_i)$ and $t'(s) = -\frac{\phi'(s)}{\phi(s)}$, by the definition of M (cf. (2.10)), we obtain

$$\begin{aligned} \mathbb{E}T'(s) + \mathbb{E}\phi'(sM) &= \mathbb{E}\left\{\sum_{i=1}^N A_i t'(sA_i) + \sum_{i=1}^N A_i \phi'(sA_i)\right\} \\ &= \mathbb{E}\left\{\sum_{i=1}^N A_i (e^{t(sA_i)} - 1)(-\phi'(sA_i))\right\} \\ &= \mathbb{E}\left\{\sum_{i=1}^N A_i (1 - \phi(sA_i)) \cdot \left[\frac{-\phi'(sA_i)}{\phi(sA_i)}\right]\right\}. \end{aligned} \quad (4.19)$$

Since $\frac{1-\phi(x)}{x} \rightarrow 1$ ($x \rightarrow 0$) and $0 < \beta + \epsilon \leq 1$, we have

$$1 - \phi(x) \leq Cx^{\beta+\epsilon}, \quad x > 0$$

(the inequality is evident for $x > 1$; if $x \leq 1$, then $1 - \phi(x) \leq Cx \leq Cx^{\beta+\epsilon}$). So with the fact that $\sup_{s \geq 0} \left| \frac{-\phi'(s)}{\phi(s)} \right| < \infty$ (by Lemma 3.1(i)) and $\mathbb{E}Z < \infty$, we have

$$\begin{aligned} |\mathbb{E}T'(s) + \mathbb{E}\phi'(sM)| &\leq Cs^{\beta+\epsilon} \mathbb{E} \sum_{i=1}^N A_i^{1+\beta+\epsilon} \\ &= Cs^{\beta+\epsilon} \rho(\alpha + \epsilon). \end{aligned}$$

This gives (4.18) and therefore (4.16).

(b) **Case** $\alpha > 2$ not an integer. So $\alpha = n + \beta$ with $n \in \{2, 3, \dots\}$ and $\beta \in (0, 1)$. We assume the conditions of Theorem 2.1 with the given α , and we need to prove (2.12). Notice that

$$\psi_n(s) = (-1)^{n+1} [\phi^{(n)}(s) - \phi^{(n)}(0)] \quad \text{and} \quad g_n(s) = -\mathbb{E}[Y_1^n (e^{-sY_1} - 1)].$$

So (2.12) is equivalent to

$$[\phi^{(n)}(s) - \phi^{(n)}(0)] - \mathbb{E}M^{n-1}[\phi^{(n)}(sM) - \phi^{(n)}(0)] = \mathbb{E}[(-Y_1)^n (e^{-sY_1} - 1)] + O(s^{\beta+\epsilon}). \quad (4.20)$$

To prove (4.20), we only need to show that for all $s \geq 0$,

$$\phi^{(n)}(s) - \mathbb{E}M^{n-1}\phi^{(n)}(sM) = \mathbb{E}e^{-sY_1}(-Y_1)^n + \mathbb{E}e^{-T(0)}P_{n,T}(0) - D\mathbb{E}M^{n-1} + O(s^{\beta+\epsilon}), \quad (4.21)$$

where D is a constant. In fact, putting $s = 0$ in the above equality and subtracting, we obtain (4.20).

It remains to prove (4.21). By (4.1), we can decompose the left hand side of (4.21) into three parts:

$$\phi^{(n)}(s) - \mathbb{E}M^{n-1}\phi^{(n)}(sM) = \mathbb{E}e^{-T(s)}P_{n,T}(s) + \mathbb{E}e^{-T(s)}(-T'(s))^n - [\mathbb{E}e^{-T(s)}T^{(n)}(s) + \mathbb{E}M^{n-1}\phi^{(n)}(sM)]. \quad (4.22)$$

In the following, we will show that the three terms on the right hand side of (4.22) are equal to

$$\mathbb{E}e^{-T(0)}P_{n,T}(0) + O(s^{\beta+\epsilon}), \quad \mathbb{E}e^{-sY_1}(-Y_1)^n + O(s^{\beta+\epsilon}) \quad \text{and} \quad D\mathbb{E}M^{n-1} + O(s^{\beta+\epsilon}),$$

respectively. This together with (4.22) implies (4.21), which ends the proof of (4.20). The proof for the above three terms will be done in three steps respectively.

Step 1. We first show that

$$\mathbb{E} \left\{ e^{-T(s)}P_{n,T}(s) - e^{-T(0)}P_{n,T}(0) \right\} = O(s^{\beta+\epsilon}), \quad s \geq 0. \quad (4.23)$$

Recall that $P_{n,T}(s)$ is of the form

$$P_{n,T}(s) = \sum_{a_1, \dots, a_{n-1}} c(a_1, \dots, a_{n-1}) (T'(s))^{a_1} \dots (T^{(n-1)}(s))^{a_{n-1}},$$

where $a_k \in \mathbb{N}$ satisfy (3.4). So we first estimate

$$e^{-T(s)} \prod_{k=1}^{n-1} (T^{(k)}(s))^{a_k} - e^{-T(0)} \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k} = I_1 + I_2, \quad (4.24)$$

where

$$\begin{aligned} I_1 &= e^{-T(s)} \left[\prod_{k=1}^{n-1} (T^{(k)}(s))^{a_k} - \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k} \right], \\ I_2 &= \left[e^{-T(s)} - e^{-T(0)} \right] \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k}. \end{aligned}$$

Let us show that $|\mathbb{E}I_1| \leq Cs^{\beta+\epsilon}$. As

$$\prod_{k=1}^{n-1} a_k - \prod_{k=1}^{n-1} b_k = \sum_{l=1}^{n-1} (a_l - b_l) \left(\prod_{k=1}^{l-1} b_k \right) \left(\prod_{k=l+1}^{n-1} a_k \right)$$

(with the convention that the empty product is taken to be 1), we have

$$\prod_{k=1}^{n-1} (T^{(k)}(s))^{a_k} - \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k} = \sum_{l=1}^{n-1} \left[(T^{(l)}(s))^{a_l} - (T^{(l)}(0))^{a_l} \right] \prod_{k=1}^{l-1} (T^{(k)}(0))^{a_k} \cdot \prod_{k=l+1}^{n-1} (T^{(k)}(s))^{a_k}. \quad (4.25)$$

Notice that for any integer $l \geq 1$, $a^l - b^l = (a - b) \left(\sum_{x+y=l-1} a^x b^y \right)$ (with an evident interpretation when $l = 1$), where x and y are strictly positive integers. So we get

$$(T^{(l)}(s))^{a_l} - (T^{(l)}(0))^{a_l} = \left[T^{(l)}(s) - T^{(l)}(0) \right] \sum_{x+y=a_l-1} (T^{(l)}(s))^x (T^{(l)}(0))^y. \quad (4.26)$$

Since $T^{(k)}(s) = \sum_{i=1}^N A_i^k t^{(k)}(sA_i)$ and $\sup_{s \geq 0} |t^{(k)}(s)| < \infty$ for $k = 1, \dots, n-1$ (by Lemma 3.1(i)), we have

$$|T^{(k)}(s)| \leq \sum_{i=1}^N A_i^k |t^{(k)}(sA_i)| \leq C \sum_{i=1}^N A_i^k \leq CY_1^k, \quad 1 \leq k \leq n-1. \quad (4.27)$$

As $|t^{(l)}(s) - t^{(l)}(0)| \leq Cs^{\beta+\epsilon}$ for $1 \leq l \leq n-1$ and $s \geq 0$ (it follows from (3.8) for $s \in [0, 1]$, and from (3.7) for $s \geq 1$), we see that

$$\begin{aligned} |T^{(l)}(s) - T^{(l)}(0)| &\leq \sum_{i=1}^N A_i^l \left| t^{(l)}(sA_i) - t^{(l)}(0) \right| \\ &\leq Cs^{\beta+\epsilon} \sum_{i=1}^N A_i^{l+\beta+\epsilon}, \quad s \geq 0, \quad 1 \leq l \leq n-1. \end{aligned} \quad (4.28)$$

From (4.25)-(4.28), we obtain

$$\begin{aligned} \prod_{k=1}^{n-1} (T^{(k)}(s))^{a_k} - \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k} &= \sum_{l=1}^{n-1} O(s^{\beta+\epsilon}) \sum_{i=1}^N A_i^{l+\beta+\epsilon} \cdot (Y_1)^{(a_l-1)l} \cdot Y_1^{n-a_l} \\ &= O(s^{\beta+\epsilon}) A^{\beta+\epsilon} \cdot Y_1^n, \end{aligned}$$

and

$$|I_1| \leq Ce^{-T(s)} s^{\beta+\epsilon} A^{\beta+\epsilon} Y_1^n \leq Cs^{\beta+\epsilon} A^{\beta+\epsilon} Y_1^n, \quad s \geq 0. \quad (4.29)$$

Notice that by Hölder's inequality,

$$\mathbb{E}A^{\beta+\epsilon} Y_1^n \leq \left(\mathbb{E}A^{(\beta+\epsilon)p} \right)^{\frac{1}{p}} \cdot \left(\mathbb{E}Y_1^{np^*} \right)^{\frac{1}{p^*}} < \infty, \quad (4.30)$$

where $p = \frac{n+\beta-\epsilon}{\beta-\epsilon}$, $p^* = \frac{n+\beta-\epsilon}{n}$ and $\epsilon > 0$ is small such that $(\beta+\epsilon)p = (\beta+\epsilon)(n+\beta-\epsilon)/(\beta-\epsilon) < n+\beta+\delta_0$ (recall that $\mathbb{E}Y_1^{n+\beta-\epsilon} < \infty$ and $\mathbb{E}A^{n+\beta+\delta_0} < \infty$). So it follows from (4.29) and (4.30) that

$$|\mathbb{E}I_1| \leq Cs^{\beta+\epsilon}, \quad s \geq 0. \quad (4.31)$$

We now consider I_2 . As $a_1 \neq n$, there exists $l \geq 2$ with $a_l \geq 1$, so that

$$|T^{(l)}(0)| \leq \sum_{i=1}^N A_i^l |t^{(l)}(0)| \leq C \sum_{i=1}^N A_i^{l-1} A \leq CAY_1^{l-1}.$$

Noting that $|T^{(k)}(s)| = CY_1^k$ for $1 \leq k \leq n-1$ (cf. (4.27)), we have

$$\begin{aligned} \prod_{k=1}^{n-1} |T^{(k)}(0)|^{a_k} &= |T^{(l)}(0)|^{a_l} \cdot \prod_{\substack{k=1 \\ k \neq l}}^{n-1} |T^{(k)}(0)|^{a_k} \\ &\leq CA^{a_l} Y_1^{(l-1)a_l} Y_1^{n-la_l} \\ &= CA^{a_l} Y_1^{n-a_l}. \end{aligned}$$

On the other hand, for $\beta + \epsilon \in (0, 1]$, as $1 - e^{-x} \leq x^{\beta+\epsilon}$ ($\forall x > 0$) and $T(s) \leq sY_1$, we have

$$1 - e^{-T(s)} \leq (sY_1)^{\beta+\epsilon}, \quad s \geq 0.$$

So

$$|I_2| \leq C(sY_1)^{\beta+\epsilon} A^{a_l} Y_1^{n-a_l} \leq Cs^{\beta+\epsilon} A^{\beta+\epsilon} Y_1^n. \quad (4.32)$$

Since $\mathbb{E}A^{\beta+\epsilon}Y_1^n < \infty$ (by (4.30)), we see that

$$|\mathbb{E}I_2| \leq Cs^{\beta+\epsilon}, \quad s \geq 0. \quad (4.33)$$

From (4.24), (4.31) and (4.33), we get

$$\mathbb{E} \left\{ e^{-T(s)} \prod_{k=1}^{n-1} (T^{(k)}(s))^{a_k} - e^{-T(0)} \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k} \right\} = O(s^{\beta+\epsilon}), \quad s \geq 0. \quad (4.34)$$

This gives (4.23).

Step 2. We next show that

$$\mathbb{E}e^{-T(s)}(-T'(s))^n = \mathbb{E}e^{-sY_1}(-Y_1)^n + O(s^{\beta+\epsilon}), \quad s \geq 0. \quad (4.35)$$

To this end, we use the decomposition

$$\mathbb{E}e^{-T(s)}(-T'(s))^n - \mathbb{E}e^{-sY_1}(-Y_1)^n = \mathbb{E} \left\{ e^{-T(s)} [(-T'(s))^n - (-Y_1)^n] \right\} + \mathbb{E} \left\{ (e^{-T(s)} - e^{-sY_1})(-Y_1)^n \right\} \quad (4.36)$$

We shall prove that each of the right terms is of order $O(s^{\beta+\epsilon})$.

As $\mathbb{E}Z^2 < \infty$ and $\beta + \epsilon \leq 1$, we have $|t'(s) - t'(0)| \leq Cs^{\beta+\epsilon}$ (by (3.9) of Lemma 3.1), so that

$$\begin{aligned} |T'(s) - Y_1| &\leq \sum_{i=1}^N A_i |t'(sA_i) - t'(0)| \\ &\leq Cs^{\beta+\epsilon} \sum_{i=1}^N A_i^{1+\beta+\epsilon} \leq Cs^{\beta+\epsilon} A^{\beta+\epsilon} Y_1. \end{aligned} \quad (4.37)$$

Since $|T'(s)| = \left| \sum_{i=1}^N A_i t'(sA_i) \right| \leq Y_1 \sup_{s \geq 0} |t'(s)|$, this implies that

$$\begin{aligned} e^{-T(s)} |(-T'(s))^n - (-Y_1)^n| &\leq |T'(s) - Y_1| \sum_{k=1}^{n-1} |T'(s)|^k Y_1^{n-1-k} \\ &\leq C \left(\sup_{s \geq 0} |t'(s)| + 1 \right)^n \cdot s^{\beta+\epsilon} A^{\beta+\epsilon} Y_1^n. \end{aligned} \quad (4.38)$$

Recalling that $\mathbb{E}A^{\beta+\epsilon}Y_1^n < \infty$, we thus obtain that

$$\mathbb{E}e^{-T(s)} |(-T'(s))^n - (-Y_1)^n| \leq Cs^{\beta+\epsilon}, \quad s \geq 0. \quad (4.39)$$

By Lemma 4.1,

$$\mathbb{E} \left(e^{-T(s)} - e^{-sY_1} \right) Y_1^n \leq Cs^{\beta+\epsilon}, \quad s \geq 0. \quad (4.40)$$

Combining (4.39) and (4.40), we get (4.35).

Step 3. We then show that for some constant D ,

$$\mathbb{E}e^{-T(s)}T^{(n)}(s) = -\mathbb{E}M^{n-1}\phi^{(n)}(sM) + D\mathbb{E}M^{n-1} + O(s^{\beta+\epsilon}), \quad s \geq 0. \quad (4.41)$$

In fact, as $0 \leq 1 - e^{-T(s)} \leq (T(s))^{\beta+\epsilon} \leq s^{\beta+\epsilon}Y_1^{\beta+\epsilon}$ ($\forall s > 0$ and $\beta + \epsilon \leq 1$) and $\sup_{s \geq 0} |t^{(n)}(s)| < \infty$, we see that

$$\begin{aligned} e^{-T(s)}T^{(n)}(s) &= \left\{ (1 + O(s^{\beta+\epsilon})Y_1^{\beta+\epsilon}) \sum_{i=1}^N A_i^n t^{(n)}(sA_i) \right\} \\ &= \left\{ \sum_{i=1}^N A_i^n t^{(n)}(sA_i) \right\} + O(s^{\beta+\epsilon})A^{\beta+\epsilon}Y_1^n. \end{aligned} \quad (4.42)$$

Since $\mathbb{E}Z^n < \infty$ and $t^{(k)}(s) = t^{(k)}(0) + O(s)$ for $1 \leq k \leq n-1$ (by (3.8) of Lemma 3.1), from the representation form (3.2) we get

$$\phi^{(n)}(s) = -t^{(n)}(s) + D + O(s), \quad s > 0,$$

for some constant D (independent of s). Since $\phi^{(n)}(s)$ and $t^{(n)}(s)$ are bounded, it follows that

$$\phi^{(n)}(s) = -t^{(n)}(s) + D + O(s^{\beta+\epsilon})$$

for $\beta + \epsilon \in (0, 1]$ and $s \geq 0$. Therefore,

$$e^{-T(s)}T^{(n)}(s) = \sum_{i=1}^N A_i^n \left(-\phi^{(n)}(sA_i) + D \right) + O(s^{\beta+\epsilon})A^{\beta+\epsilon}Y_1^n, \quad s \geq 0. \quad (4.43)$$

As $\mathbb{E}A^{\beta+\epsilon}Y_1^n < \infty$ (cf. (4.30)), by the dominated convergence theorem and (4.43), together with the definition of M (cf. (2.10)), we obtain that

$$\begin{aligned} \mathbb{E}e^{-T(s)}T^{(n)}(s) &= \mathbb{E} \sum_{i=1}^N A_i^n \left(-\phi^{(n)}(sA_i) + D \right) + O(s^{\beta+\epsilon})\mathbb{E}A^{\beta+\epsilon}Y_1^n \\ &= -\mathbb{E}M^{n-1}\phi^{(n)}(sM) + D\mathbb{E}M^{n-1} + O(s^{\beta+\epsilon})\mathbb{E}A^{\beta+\epsilon}Y_1^n \\ &= -\mathbb{E}M^{n-1}\phi^{(n)}(sM) + D\mathbb{E}M^{n-1} + O(s^{\beta+\epsilon}), \quad s \geq 0. \end{aligned} \quad (4.44)$$

This ends the proof of (4.41).

From (4.23), (4.35) and (4.41), together with (4.22), we obtain (4.21). This ends the proof of (4.20), which is equivalent to (2.12) according to the the analysis at the beginning of Part (b).

The proof of Theorem 2.1 for $\alpha \in [1, \infty) \setminus \{2, 3, \dots\}$ is thus finished. \square

5 Proof of Theorem 2.1: case where $\alpha \in \{2, 3, \dots\}$

The proof of Theorem 2.1 in the case where $\alpha \in \{2, 3, \dots\}$ is similar to that in the case where α is not an integer, but is more delicate.

Proof of Theorem 2.1: case where $\alpha = n+1 \in \{2, 3, \dots\}$. We also distinguish two cases according to $n = 1$ or $n > 1$.

(a) **Case $n = 1$.** As in the case where $\alpha \in (1, 2)$, it is sufficient to prove that

$$\mathbb{E}T'(s) + \mathbb{E}\phi'(sM) - \rho(2)s = O(s^{1+\epsilon}), \quad s \geq 0. \quad (5.1)$$

Using (2.10) with $f(x) = -\phi'(sx)$, we have

$$\begin{aligned} \mathbb{E}T'(s) + \mathbb{E}\phi'(sM) - \rho(2)s &= \mathbb{E} \left\{ \sum_{i=1}^N A_i (e^{t(sA_i)} - 1)(-\phi'(sA_i)) - \sum_{i=1}^N sA_i^2 \right\} \\ &= s\mathbb{E} \sum_{i=1}^N A_i^2 \left\{ \frac{1 - \phi(sA_i)}{sA_i} \left[-\frac{\phi'(sA_i)}{\phi(sA_i)} \right] - 1 \right\} \\ &= K_3 + K_4, \end{aligned}$$

where

$$\begin{aligned} K_3 &= s\mathbb{E} \sum_{i=1}^N A_i^2 \left\{ \frac{1 - \phi(sA_i) - sA_i}{sA_i} \left[-\frac{\phi'(sA_i)}{\phi(sA_i)} \right] \right\}, \\ K_4 &= s\mathbb{E} \sum_{i=1}^N A_i^2 \left\{ -\frac{\phi'(sA_i)}{\phi(sA_i)} - 1 \right\} = s\mathbb{E} \sum_{i=1}^N A_i^2 (t'(sA_i) - 1). \end{aligned}$$

As $|1 - e^{-x} - x| \leq Cx^{1+\epsilon}$ ($x \geq 0$) and $\mathbb{E}Z^{1+\epsilon} < \infty$, we see that $|1 - \phi(s) - s| \leq C\mathbb{E}Z^{1+\epsilon}s^{1+\epsilon}$, and

$$|1 - \phi(sA_i) - sA_i| \leq C\mathbb{E}Z^{1+\epsilon}s^{1+\epsilon}A_i^{1+\epsilon}.$$

Combining this with the fact that $\sup_{s \geq 0} \left| -\frac{\phi'(s)}{\phi(s)} \right| < \infty$ (by Lemma 3.1(i)), we have

$$|K_3| \leq Cs^{1+\epsilon} \mathbb{E} \sum_{i=1}^N A_i^{2+\epsilon} = Cs^{1+\epsilon} \rho(2+\epsilon), \quad s \geq 0. \quad (5.2)$$

We now consider K_4 . Recalling that $|t'(s) - t'(0)| \leq Cs^\epsilon$, we have

$$|t'(sA_i) - 1| \leq Cs^\epsilon A_i^\epsilon,$$

and

$$|K_4| \leq Cs^{1+\epsilon} \mathbb{E} \sum_{i=1}^N A_i^{2+\epsilon} = Cs^{1+\epsilon} \rho(2+\epsilon), \quad s \geq 0. \quad (5.3)$$

It follows from (5.2) and (5.3) that (5.1) holds.

(b) **Case $n > 1$.** As in the case where $\alpha > 2$ is not an integer, it is sufficient to prove that

$$\phi^{(n)}(s) - \mathbb{E}M^{n-1}\phi^{(n)}(sM) = (-1)^n \mathbb{E}e^{-sY_1} Y_1^n + c_0 + c_n s + O(s^{1+\epsilon}), \quad s \geq 0, \quad (5.4)$$

where c_0 and c_n are non-negative constants. In fact, putting $s = 0$ in the above equality and subtracting, and noticing that $\psi_n(s) = (-1)^{n+1}[\phi^{(n)}(s) - \phi^{(n)}(0)]$ and $g_n(s) = -\mathbb{E}[Y_1^n(e^{-sY_1} - 1)]$, we obtain (2.12).

In the following, we will show that the three terms on the right hand side of (4.22) are respectively

$$\mathbb{E}e^{-T(0)}P_{n,T}(0) + D_1s + O(s^{1+\epsilon}), \quad \mathbb{E}e^{-sY_1}(-Y_1)^n + (-1)^n D_2s + O(s^{1+\epsilon}) \quad \text{and} \quad D_3 + D_4s + O(s^{1+\epsilon}),$$

where D_i ($i = 1, 2, 3, 4$) are non-negative constants. This together with (4.22) imply (5.4). The proof for the above three terms will be done in three steps respectively.

Step 1. We first show that for some constant D_1 ,

$$\mathbb{E} \left\{ e^{-T(s)}P_{n,T}(s) - e^{-T(0)}P_{n,T}(0) \right\} = D_1s + O(s^{1+\epsilon}), \quad s \geq 0. \quad (5.5)$$

We begin with the estimation of

$$e^{-T(s)} \prod_{k=1}^{n-1} (T^{(k)}(s))^{a_k} - e^{-T(0)} \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k} = J_1 + J_2, \quad (5.6)$$

where

$$\begin{aligned} J_1 &= e^{-T(s)} \left[\prod_{k=1}^{n-1} (T^{(k)}(s))^{a_k} - \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k} \right], \\ J_2 &= \left[e^{-T(s)} - e^{-T(0)} \right] \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k}. \end{aligned}$$

We first consider J_1 . As $t^{(l)}(s) = t^{(l)}(0) + t^{(l+1)}(0)s + O(s^{1+\epsilon})$ for $1 \leq l \leq n-1$ (by (3.8) of Lemma 3.1), we see that

$$\begin{aligned} T^{(l)}(s) - T^{(l)}(0) &= \sum_{i=1}^N A_i^l \left[t^{(l)}(sA_i) - t^{(l)}(0) \right] \\ &= \sum_{i=1}^N A_i^l \left[t^{(l+1)}(0)sA_i + O(s^{1+\epsilon})A_i^{1+\epsilon} \right] \\ &= t^{(l+1)}(0)s \sum_{i=1}^N A_i^{l+1} + O(s^{1+\epsilon}) \sum_{i=1}^N A_i^{l+1+\epsilon}. \end{aligned}$$

Since $t^{(k)}(s) = t^{(k)}(0) + O(s^\epsilon)$ for $k = 1, 2, \dots, n-1$, we have

$$T^{(k)}(s) = \sum_{i=1}^N A_i^k t^{(k)}(sA_i) = t^{(k)}(0) \sum_{i=1}^N A_i^k + O(s^\epsilon) \sum_{i=1}^N A_i^{k+\epsilon}.$$

Therefore, by (4.25), we obtain

$$\prod_{k=1}^{n-1} (T^{(k)}(s))^{a_k} - \prod_{k=1}^{n-1} (T^{(k)}(0))^{a_k} = J_3 s + O(s^{1+\epsilon}) A^{1+\epsilon} Y_1^n,$$

where

$$J_3 = \sum_{l=1}^{n-1} \left[t^{(l+1)}(0) \sum_{i=1}^N A_i^{l+1} \right] \left[t^{(l)}(0) \sum_{i=1}^N A_i^l \right]^{a_l-1} \prod_{\substack{k=1 \\ k \neq l}}^{n-1} \left[t^{(k)}(0) \sum_{i=1}^N A_i^k \right]^{a_k}.$$

By Hölder's inequality,

$$\mathbb{E} A^{1+\epsilon} Y_1^{n+\epsilon} \leq \left(\mathbb{E} A^{(1+\epsilon)p} \right)^{1/p} \left(\mathbb{E} Y_1^{np^*} \right)^{1/p^*} < \infty, \quad (5.7)$$

where $p = \frac{\alpha-\epsilon}{1-2\epsilon}$, $p^* = \frac{\alpha-\epsilon}{n+\epsilon}$ and $\epsilon > 0$ small enough such that $(1+\epsilon)p = (1+\epsilon)(\alpha-\epsilon)/(1-2\epsilon) \leq \alpha + \delta_0$. As J_3 is dominated by $O(1)AY_1^n$ with $\mathbb{E}[AY_1^n] < \infty$ (by (5.7)) and $e^{-T(s)} = 1 + O(s^\epsilon)Y_1^\epsilon$, $J_1 = J_3 s + O(s^{1+\epsilon})(AY_1^{n+\epsilon} + A^{1+\epsilon}Y_1^{n+\epsilon})$, we see that

$$\mathbb{E} J_1 = s \mathbb{E} J_3 + O(s^{1+\epsilon}), \quad s \geq 0. \quad (5.8)$$

Since $1 - e^{-x} - x = O(x^{1+\epsilon})$ ($0 < \epsilon \leq 1$, $x \geq 0$) and $T(s) = sY_1 + O(s^{1+\epsilon})Y_1^{1+\epsilon}$, we have

$$e^{-T(s)} - 1 = -sY_1 + O(s^{1+\epsilon})Y_1^{1+\epsilon}.$$

Using this together with $T^{(k)}(0) = t^{(k)}(0) \sum_{i=1}^N A_i^k$, we get

$$\begin{aligned} J_2 &= [-sY_1 + O(s^{1+\epsilon})Y_1^{1+\epsilon}] \prod_{k=1}^{n-1} \left(t^{(k)}(0) \sum_{i=1}^N A_i^k \right)^{a_k} \\ &= -sJ_4 + O(s^{1+\epsilon})Y_1^n A^{1+\epsilon}, \end{aligned}$$

where

$$J_4 = Y_1 \prod_{k=0}^{n-1} \left(t^{(k)}(0) \sum_{i=1}^N A_i^k \right)^{a_k}$$

is dominated by $O(1)AY_1^n$ with $\mathbb{E}AY_1^n < \infty$ (by (5.7)). So by (5.7),

$$\mathbb{E} J_2 = -s \mathbb{E} J_4 + O(s^{1+\epsilon}), \quad s \geq 0. \quad (5.9)$$

It follows from (5.8) and (5.9) that (5.5) holds with $D_1 = \mathbb{E}J_3 - \mathbb{E}J_4$.

Step 2. We next show that, for some constant D_2 and $s \geq 0$,

$$\mathbb{E} e^{-T(s)} (T'(s))^n - \mathbb{E} e^{-sY_1} Y_1^n - D_2 s = O(s^{1+\epsilon}). \quad (5.10)$$

Notice that (5.10) holds evidently for $s \in (1, \infty)$ (as $|T'(s)| \leq CY_1$ and $\mathbb{E}Y_1^n < \infty$), so we only need to prove it for $s \in [0, 1]$. To prove this, it is sufficient to show the following two estimations:

$$\mathbb{E}e^{-T(s)} [(T'(s))^n - Y_1^n] = D_2s + O(s^{1+\epsilon}), \quad s \in [0, 1], \quad (5.11)$$

$$\mathbb{E} \left[e^{-T(s)} - e^{-sY_1} \right] Y_1^n = O(s^{1+\epsilon}), \quad s \in [0, 1]. \quad (5.12)$$

We first show (5.11). Notice that

$$(T'(s))^n - Y_1^n = (T'(s) - Y_1) \left[\sum_{k=0}^{n-1} (T'(s))^k Y_1^{n-1-k} \right]. \quad (5.13)$$

Since $T'(s) = Y_1 + O(s^\epsilon)A^\epsilon Y_1$ (cf. (4.37)), writing $\bar{A} = \max(A, 1)$, we have for $s \in [0, 1]$,

$$\begin{aligned} \sum_{k=0}^{n-1} (T'(s))^k Y_1^{n-1-k} &= \sum_{k=0}^{n-1} [1 + O(s^\epsilon)A^\epsilon]^k Y_1^{n-1} \\ &= \sum_{k=0}^{n-1} [1 + O(s^\epsilon)\bar{A}^{k\epsilon}] Y_1^{n-1} \\ &= \left[n + O(s^\epsilon)\bar{A}^{(n-1)\epsilon} \right] Y_1^{n-1}, \end{aligned} \quad (5.14)$$

where the second equality holds since for $0 \leq k \leq n-1$,

$$[1 + O(s^\epsilon)\bar{A}^\epsilon]^k - 1 = \sum_{j=1}^k \binom{k}{j} O(s^{j\epsilon})\bar{A}^{j\epsilon} = O(s^\epsilon)\bar{A}^{k\epsilon},$$

using the fact that $s \in [0, 1]$ and $\bar{A} \geq 1$ imply $s^{j\epsilon} \leq s^\epsilon$ and $\bar{A}^{j\epsilon} \leq \bar{A}^{k\epsilon}$ for each $1 \leq j \leq k$. As $t'(s) = 1 + t''(0)s + O(s^{1+\epsilon})$ ($s \geq 0$), we see that

$$\begin{aligned} T'(s) - Y_1 &= \sum_{i=1}^N A_i (t'(sA_i) - 1) \\ &= t''(0)s \sum_{i=1}^N A_i^2 + O(s^{1+\epsilon}) \sum_{i=1}^N A_i^{2+\epsilon}. \end{aligned} \quad (5.15)$$

From (5.13), (5.14) and (5.15), we obtain that for $s \in [0, 1]$,

$$\begin{aligned} (T'(s))^n - Y_1^n &= \left[t''(0)s \sum_{i=1}^N A_i^2 + O(s^{1+\epsilon}) \sum_{i=1}^N A_i^{2+\epsilon} \right] \left[n + O(s^\epsilon)\bar{A}^{(n-1)\epsilon} \right] Y_1^{n-1} \\ &= nt''(0)s \sum_{i=1}^N A_i^2 Y_1^{n-1} + O(s^{1+\epsilon}) \sum_{i=1}^N A_i^{2+\epsilon} Y_1^{n-1} \\ &\quad + O(s^{1+\epsilon}) \sum_{i=1}^N A_i^2 \bar{A}^{(n-1)\epsilon} Y_1^{n-1} + O(s^{1+2\epsilon}) \sum_{i=1}^N A_i^{2+\epsilon} \bar{A}^{(n-1)\epsilon} Y_1^{n-1}. \end{aligned} \quad (5.16)$$

Recalling that $Y_1 = \sum_{i=1}^N A_i$ and $\bar{A} \geq 1$, we have

$$\begin{aligned} \sum_{i=1}^N A_i^{2+\epsilon} Y_1^{n-1} &\leq A^{1+\epsilon} Y_1^n \leq \bar{A}^{1+n\epsilon} Y_1^n, \\ \sum_{i=1}^N A_i^2 \bar{A}^{(n-1)\epsilon} Y_1^{n-1} &\leq \bar{A}^{1+n\epsilon} Y_1^n, \\ \sum_{i=1}^N A_i^{2+\epsilon} \bar{A}^{(n-1)\epsilon} Y_1^{n-1} &\leq \bar{A}^{1+n\epsilon} Y_1^n. \end{aligned}$$

Hence from (5.16), we get

$$(T'(s))^n - Y_1^n = nt''(0)s \sum_{i=1}^N A_i^2 Y_1^{n-1} + O(s^{1+\epsilon}) \bar{A}^{1+n\epsilon} Y_1^n. \quad (5.17)$$

Since $e^{-x} = 1 + O(x^\epsilon)$ ($0 < \epsilon \leq 1$, $x \geq 0$) and $0 \leq T(s) \leq sY_1$ (cf. (2.8)), together with (5.17), we see that for $s \in [0, 1]$,

$$\begin{aligned} & e^{-T(s)} [(T'(s))^n - Y_1^n] \\ &= [1 + O(s^\epsilon) Y_1^\epsilon] \left[nt''(0)s \sum_{i=1}^N A_i^2 Y_1^{n-1} + O(s^{1+\epsilon}) \bar{A}^{1+n\epsilon} Y_1^n \right] \\ &= nt''(0)s \sum_{i=1}^N A_i^2 Y_1^{n-1} + O(s^{1+\epsilon}) [\bar{A} Y_1^{n+\epsilon} + \bar{A}^{1+n\epsilon} Y_1^n + \bar{A}^{1+n\epsilon} Y_1^{n+\epsilon}] \\ &= nt''(0)s \sum_{i=1}^N A_i^2 Y_1^{n-1} + O(s^{1+\epsilon}) \bar{A}^{1+n\epsilon} [Y_1^n + Y_1^{n+\epsilon}]. \end{aligned} \quad (5.18)$$

Notice that

$$\mathbb{E} \sum_{i=1}^n A_i^2 Y_1^{n-1} \leq \mathbb{E} A Y_1^n \leq (\mathbb{E} A^p)^{1/p} \left(\mathbb{E} Y_1^{np^*} \right)^{1/p^*} < \infty,$$

where $p = \alpha + \delta_0$, $p^* = \frac{\alpha + \delta_0}{\alpha + \delta_0 - 1}$, so that $np^* < \alpha$ and that

$$\begin{aligned} \mathbb{E} \bar{A}^{1+n\epsilon} Y_1^n &\leq C [\mathbb{E} A^{1+n\epsilon} Y_1^n + \mathbb{E} Y_1^n] \\ &\leq C \left\{ [\mathbb{E} A^{[1+n\epsilon]q}]^{1/q} [\mathbb{E} Y_1^{nq^*}]^{1/q^*} + \mathbb{E} Y_1^n \right\} < \infty, \\ \mathbb{E} \bar{A}^{1+n\epsilon} Y_1^{n+\epsilon} &\leq C [\mathbb{E} A^{1+n\epsilon} Y_1^{n+\epsilon} + \mathbb{E} Y_1^{n+\epsilon}] \\ &\leq C \left\{ [\mathbb{E} A^{[1+n\epsilon]q}]^{1/q} [\mathbb{E} Y_1^{(n+\epsilon)q^*}]^{1/q^*} + \mathbb{E} Y_1^{n+\epsilon} \right\} < \infty, \end{aligned}$$

where $q = \frac{\alpha + \delta_0}{1+n\epsilon}$, $q^* = \frac{\alpha + \delta_0}{(\alpha + \delta_0) - [1+n\epsilon]}$, δ_0 and ϵ are small enough. By the dominated convergence theorem, it follows from (5.18) that (5.11) holds with $D_2 = nt''(0) \mathbb{E} \sum_{i=1}^N A_i^2 Y_1^{n-1}$.

We now prove (5.12). By the same argument as in the proof of (4.7) (choosing $\beta = 1$), we see that if $Cs^{1-\epsilon} A^{1-\epsilon} \leq 1/2$, then $T(s) \geq sY_1/2$, so that

$$e^{-T(s)} - e^{-sY_1} \leq C e^{-sY_1/2} s^{2-\epsilon} A^{1-\epsilon} Y_1.$$

Therefore,

$$\begin{aligned} \left(e^{-T(s)} - e^{-sY_1} \right) Y_1^n \mathbf{1}_{\{Cs^{1-\epsilon} A^{1-\epsilon} \leq 1/2\}} &\leq C e^{-sY_1/2} (sY_1)^{1-2\epsilon} s^{1+\epsilon} A^{1-\epsilon} Y_1^{n+2\epsilon} \\ &\leq C s^{1+\epsilon} A^{1-\epsilon} Y_1^{n+2\epsilon}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left(e^{-T(s)} - e^{-sY_1} \right) Y_1^n \mathbf{1}_{\{Cs^{1-\epsilon} A^{1-\epsilon} > 1/2\}} &\leq C Y_1^n (s^{1-\epsilon} A^{1-\epsilon})^{(1+\epsilon)/(1-\epsilon)} \\ &\leq C s^{1+\epsilon} A^{1-\epsilon} Y_1^{n+2\epsilon}. \end{aligned}$$

Hence, we have shown that

$$\left(e^{-T(s)} - e^{-sY_1} \right) Y_1^n \leq C s^{1+\epsilon} A^{1-\epsilon} Y_1^{n+2\epsilon}.$$

As $\mathbb{E} A^{1-\epsilon} Y_1^{n+\epsilon} < \infty$ (cf. (5.7)), this gives (5.12) which ends the proof of (5.10).

Step 3. We then show that, for some constants D_3 and D_4 ,

$$\mathbb{E} e^{-T(s)} T^{(n)}(s) + \mathbb{E} M^{n-1} \phi^{(n)}(sM) = D_3 + D_4 s + O(s^{1+\epsilon}), \quad s \geq 0. \quad (5.19)$$

Notice that $0 < x - 1 + e^{-x} \leq Cx^{1+\epsilon}$ ($\forall x > 0$) and $T(s) \leq sY_1$, we have

$$e^{-T(s)} = 1 - T(s) + O(s^{1+\epsilon})Y_1^{1+\epsilon}.$$

As $\mathbb{E}Z^n < \infty$, we have $\sup_{s \geq 0} |t^{(k)}(s)| < \infty$ for $0 \leq k \leq n$, and $t^{(k)}(s) = t^{(k)}(0) + t^{(k+1)}(0)s + O(s^{1+\epsilon})$ for $0 \leq k \leq n-1$ (by Lemma 3.1(i)). So from the representation form (3.2) and $\sup_{s > 0} |\phi^{(n)}| \leq \mathbb{E}Z^n < \infty$, we see that for $s \geq 0$,

$$t(s) = s + O(s^{1+\epsilon}) \quad \text{and} \quad t^{(n)}(s) = -\phi^{(n)}(s) + D + D's + O(s^{1+\epsilon}).$$

Hence from the definition of $T^{(n)}(s)$, we obtain that

$$\begin{aligned} e^{-T(s)}T^{(n)}(s) &= \sum_{i=1}^N A_i^n t^{(n)}(sA_i) - \sum_{i=1}^N t(sA_i) \cdot \sum_{i=1}^N A_i^n t^{(n)}(sA_i) + O(s^{1+\epsilon})Y_1^{1+\epsilon} \sum_{i=1}^N A_i^n t^{(n)}(sA_i) \\ &= -\sum_{i=1}^N A_i^n \phi^{(n)}(sA_i) + D \sum_{i=1}^N A_i^n + s \left[D' \sum_{i=1}^N A_i^{n+1} - Y_1 \sum_{i=1}^N A_i^n t^{(n)}(sA_i) \right] \\ &\quad + O(s^{1+\epsilon}) \left[\sum_{i=1}^N A_i^{n+1+\epsilon} + \left(\sum_{i=1}^N A_i^{1+\epsilon} + Y_1^{1+\epsilon} \right) \sum_{i=1}^N A_i^n t^{(n)}(sA_i) \right]. \end{aligned}$$

Since $\mathbb{E}A^{1+\epsilon}Y_1^n < \infty$ (by (5.7)) and $\sup_{s > 0} |t^{(n)}(s)| < \infty$ (by (3.7)), we see that

$$\mathbb{E} \left[\sum_{i=1}^N A_i^{n+1} + Y_1 \sum_{i=1}^N A_i^n |t^{(n)}(sA_i)| \right] \leq \left(1 + \sup_{s > 0} |t^{(n)}(s)| \right) \mathbb{E}A^{1+\epsilon}Y_1^{n-\epsilon} < \infty$$

and

$$\mathbb{E} \left[\sum_{i=1}^N A_i^{n+1+\epsilon} + \left(\sum_{i=1}^N A_i^{1+\epsilon} + Y_1^{1+\epsilon} \right) \sum_{i=1}^N A_i^n |t^{(n)}(sA_i)| \right] \leq \left[1 + 2 \sup_{s > 0} |t^{(n)}(s)| \right] \mathbb{E}A^{1+\epsilon}Y_1^n < \infty.$$

Therefore,

$$\begin{aligned} \mathbb{E}e^{-T(s)}T^{(n)}(s) &= -\mathbb{E} \sum_{i=1}^N A_i^n \phi^{(n)}(sA_i) + D_3 + D_4s + O(s^{1+\epsilon}) \\ &= -\mathbb{E}M^{n-1}\phi^{(n)}(sM) + D_3 + D_4s + O(s^{1+\epsilon}), \end{aligned}$$

where $D_3 = D\rho(n)$ and $D_4 = \mathbb{E} \left[D' \sum_{i=1}^N A_i^{n+1} - Y_1 \sum_{i=1}^N A_i^n t^{(n)}(sA_i) \right]$. This ends the proof of (5.19).

From (5.5), (5.10) and (5.19), together with (4.22), we obtain (5.4). This ends the proof of (2.12) according to the the analysis at the beginning of Part (b).

The proof of Theorem 2.1 for $\alpha \in \{2, 3, \dots\}$ is thus finished. \square

6 Proof of Theorem 1.1

Recall that without loss of generality, we can take ℓ as the form (1.10). Therefore, by Potter's theorem (cf. [11], p.25), for every $\epsilon > 0$, there is $C = C(\epsilon) > 0$ such that

$$\frac{\ell(y)}{\ell(x)} \leq C \max \left\{ \left(\frac{y}{x} \right)^\epsilon, \left(\frac{x}{y} \right)^\epsilon \right\}, \quad x, y > 0. \quad (6.1)$$

Proof of Theorem 1.1. We distinguish two cases according to $\alpha \in (1, \infty) \setminus \mathbb{N}$ or $\alpha \in \{2, 3, \dots\}$.

(i) Case $\alpha \in (1, \infty) \setminus \mathbb{N}$. By Proposition 2.1, it suffices to show that the following two assertions are equivalent:

$$h_n(s) \sim s^\alpha \ell(1/s) \quad (s \rightarrow 0), \quad (6.2)$$

$$\phi_n(s) \sim \frac{s^\alpha \ell(1/s)}{1 - \rho(\alpha)} \quad (s \rightarrow 0). \quad (6.3)$$

Step 1. We first show that (6.3) implies (6.2). By Theorem 2.2, we have

$$\phi_n(s) - \mathbb{E}M^{-1}\phi_n(sM) = h_n(s) + O(s^{\alpha+\epsilon}), \quad s \geq 0. \quad (6.4)$$

Notice that $\frac{\phi_n(s)}{s^\alpha}$ is bounded and $\ell(1/s) = 1$ for $s > 1$. Therefore (6.3) implies

$$|\phi_n(s)| \leq Cs^\alpha \ell(1/s), \quad s > 0, \quad (6.5)$$

and

$$\begin{aligned} M^{-1}\phi_n(sM) &\sim M^{-1}(sM)^\alpha \ell\left(\frac{1}{sM}\right) \cdot \frac{1}{1-\rho(\alpha)} \\ &\sim M^{\alpha-1}s^\alpha \ell\left(\frac{1}{s}\right) \cdot \frac{1}{1-\rho(\alpha)}. \end{aligned} \quad (6.6)$$

By (6.5) and Potter's theorem, we see that

$$\begin{aligned} \left| \frac{M^{-1}\phi_n(sM)}{s^\alpha \ell(1/s)} \right| &\leq C \left(\frac{M^{-1}(sM)^\alpha \ell\left(\frac{1}{sM}\right)}{s^\alpha \ell(1/s)} \right) \\ &= C \left(M^{\alpha-1} \ell\left(\frac{1}{sM}\right) / \ell(1/s) \right) \\ &\leq C \max(M^{\alpha-1+\epsilon}, M^{\alpha-1-\epsilon}) \end{aligned}$$

(recall that $C > 0$ may differ from line to line). Therefore by the dominated convergence theorem, we have

$$\lim_{s \rightarrow 0} \frac{\mathbb{E}M^{-1}\phi_n(sM)}{s^\alpha \ell(1/s)} = \frac{\mathbb{E}M^{\alpha-1}}{1-\rho(\alpha)} = \frac{\rho(\alpha)}{1-\rho(\alpha)}. \quad (6.7)$$

As $\frac{O(s^\epsilon)}{\ell(1/s)} = o(1)$ ($s \rightarrow 0$), it follows from (6.4) that $\lim_{s \rightarrow 0} \frac{h_n(s)}{s^\alpha \ell(1/s)} = 1$. So (6.2) holds.

Step 2. We next show that (6.2) implies (6.3). Recall that by Theorem 2.2,

$$\phi_n(s) - \mathbb{E}M^{-1}\phi_n(sM) = h_n(s) + O(s^{\alpha+\epsilon}), \quad s \geq 0.$$

Therefore $\hat{\phi}_n(s) := \phi_n(s)/s^\alpha$ and $\hat{h}_n(s) := h_n(s)/s^\alpha$ satisfy

$$\hat{\phi}_n(s) - \mathbb{E}M^{\alpha-1}\hat{\phi}_n(sM) = \hat{h}_n(s) + O(s^\epsilon), \quad s \geq 0. \quad (6.8)$$

Let

$$\hat{\phi}(s) = \hat{\phi}_n(s) - \mathbb{E}M^{\alpha-1}\hat{\phi}_n(sM).$$

Then

$$\hat{\phi}(s) = \hat{\phi}_n(s) - \rho(\alpha)\mathbb{E}\hat{\phi}_n(sB), \quad (6.9)$$

where $B \geq 0$ is a random variable whose distribution is determined by

$$\mathbb{E}f(B) = \frac{1}{\rho(\alpha)}\mathbb{E}M^{\alpha-1}f(M), \quad (6.10)$$

for all measurable functions $f \geq 0$. Let B_i be independent copies of B . Then by (6.9), $\forall i \geq 1$,

$$\rho^{i-1}(\alpha)\hat{\phi}(sB_1 \cdots B_{i-1}) = \rho^{i-1}(\alpha)\hat{\phi}_n(sB_1 \cdots B_{i-1}) - \rho^i(\alpha)\mathbb{E}_i\hat{\phi}_n(sB_1 \cdots B_{i-1}B_i), \quad (6.11)$$

where \mathbb{E}_i denotes the expectation with respect to B_i , and $B_1 \cdots B_{i-1} = 1$ if $i = 1$. Taking expectation and then summing for i from 0 to $k-1$, we obtain

$$\sum_{i=0}^{k-1} \rho^i(\alpha)\mathbb{E}\hat{\phi}(sB_1 \cdots B_i) = \hat{\phi}_n(s) - \rho^k(\alpha)\mathbb{E}\hat{\phi}_n(sB_1 \cdots B_k). \quad (6.12)$$

Let us now prove that

$$\lim_{k \rightarrow \infty} \rho^k(\alpha)\mathbb{E}\hat{\phi}_n(sB_1 \cdots B_k) = 0. \quad (6.13)$$

As $\mathbb{E}Z^{\alpha-\epsilon} < \infty$, by (3.10),

$$|\hat{\phi}_n(s)| \leq Cs^{-\epsilon}, \quad (6.14)$$

so

$$\begin{aligned} \mathbb{E}\rho^k(\alpha)|\hat{\phi}_n(sB_1 \cdots B_k)| &\leq C\mathbb{E}\rho^k(\alpha)(s^{-\epsilon}B_1^{-\epsilon} \cdots B_k^{-\epsilon}) \\ &= Cs^{-\epsilon}\rho^k(\alpha)(\mathbb{E}B^{-\epsilon})^k \\ &= Cs^{-\epsilon}\rho^k(\alpha)\left(\frac{\mathbb{E}M^{\alpha-1-\epsilon}}{\rho(\alpha)}\right)^k \\ &= Cs^{-\epsilon}\rho^k(\alpha-\epsilon). \end{aligned} \quad (6.15)$$

As $\rho(\alpha-\epsilon) < 1$, this prove (6.13). So by (6.12), we obtain

$$\sum_{i=0}^{\infty} \rho^i(\alpha)\mathbb{E}\hat{\phi}(sB_1 \cdots B_i) = \hat{\phi}_n(s). \quad (6.16)$$

Notice that by the condition (6.2) and the equation (6.8), we have as $s \rightarrow 0$,

$$\hat{\phi}(s) \sim \ell(1/s)$$

and

$$\hat{\phi}(sB_1 \cdots B_i) \sim \ell\left(\frac{1}{sB_1 \cdots B_i}\right) \sim \ell(1/s).$$

By Potter's bound (cf. (6.1)) and the dominated convergence theorem, this yields

$$\lim_{s \rightarrow 0} \frac{\mathbb{E}\hat{\phi}(sB_1 \cdots B_i)}{\ell(1/s)} = 1. \quad (6.17)$$

From (6.16),(6.17) and the dominated convergence theorem, we obtain

$$\lim_{s \rightarrow 0} \frac{\hat{\phi}_n(s)}{\ell(1/s)} = \sum_{i=0}^{\infty} \rho^i(\alpha) = \frac{1}{1-\rho(\alpha)}, \quad (6.18)$$

which is just the desired result (6.3). Here the use of the dominated convergence theorem is justified by the fact that there is a sequence of non-negative numbers (a_i) with $\sum_{i=1}^{\infty} a_i < \infty$ such that for all $i \geq 1$ and all $s \in [0, 1]$,

$$\frac{\rho^i(\alpha)\mathbb{E}\hat{\phi}(sB_1 \cdots B_i)}{\ell(1/s)} \leq a_i. \quad (6.19)$$

In fact, when $sB_1 \cdots B_i \leq 1$, as $\left|\frac{\hat{\phi}(s)}{\ell(1/s)}\right| \leq C$ for all $s \in [0, 1]$, we have by Potter's bound (cf. (6.1)) that

$$\begin{aligned} \frac{\rho^i(\alpha)\mathbb{E}|\hat{\phi}(sB_1 \cdots B_i)|}{\ell(1/s)} &= \rho^i(\alpha)\mathbb{E}\left[\frac{|\hat{\phi}(sB_1 \cdots B_i)|}{\ell((sB_1 \cdots B_i)^{-1})} \cdot \frac{\ell((sB_1 \cdots B_i)^{-1})}{\ell(s^{-1})}\right] \\ &\leq C\rho^i(\alpha)\mathbb{E}[(B_1 \cdots B_i)^\epsilon + (B_1 \cdots B_i)^{-\epsilon}] \\ &= C\rho^i(\alpha)[(\mathbb{E}B^\epsilon)^i + (\mathbb{E}B^{-\epsilon})^i], \end{aligned}$$

where $C > 0$ differs from line to line. When $sB_1 \cdots B_i > 1$, as $\frac{\ell(1)}{\ell((sB_1 \cdots B_i)^{-1})} \leq C(sB_1 \cdots B_i)^\epsilon$ (by (6.1)), and $|\hat{\phi}(sB_1 \cdots B_i)| \leq C[1 + \rho(\alpha-\epsilon)](sB_1 \cdots B_i)^{-\epsilon}$ (which is derived from (6.14) and the definition of $\hat{\phi}(\cdot)$), we have that

$$\begin{aligned} \frac{\rho^i(\alpha)\mathbb{E}|\hat{\phi}(sB_1 \cdots B_i)|}{\ell(1/s)} &= \rho^i(\alpha)\mathbb{E}\left[\frac{|\hat{\phi}(sB_1 \cdots B_i)|}{\ell(1)} \cdot \frac{\ell(1)}{\ell((sB_1 \cdots B_i)^{-1})} \cdot \frac{\ell((sB_1 \cdots B_i)^{-1})}{\ell(s^{-1})}\right] \\ &\leq C\rho^i(\alpha)\mathbb{E}\{(sB_1 \cdots B_i)^{-\epsilon} \cdot (sB_1 \cdots B_i)^\epsilon \cdot [(B_1 \cdots B_i)^\epsilon + (B_1 \cdots B_i)^{-\epsilon}]\} \\ &= C\rho^i(\alpha)[\mathbb{E}(B_1 \cdots B_i)^{-\epsilon} + \mathbb{E}(B_1 \cdots B_i)^\epsilon] \\ &= C\rho^i(\alpha)[(\mathbb{E}B^{-\epsilon})^i + (\mathbb{E}B^\epsilon)^i], \end{aligned}$$

where $C > 0$ differs from line to line. It should be pointed out that the $O(1)$ s above are not depending on i . From the distribution of B (cf. (6.11)), we have

$$\rho^i(\alpha) [(\mathbb{E}B^{-\epsilon})^i + (\mathbb{E}B^\epsilon)^i] = (\mathbb{E}M^{\alpha-1-\epsilon})^i + (\mathbb{E}M^{\alpha-1+\epsilon})^i = \rho^i(\alpha - \epsilon) + \rho^i(\alpha + \epsilon),$$

which is summable due to the facts that $\rho(\alpha - \epsilon) < 1$ and $\rho(\alpha + \epsilon) < 1$. Hence we have shown (6.19). This ends the proof of Step 2. Thus the proof for the case $\alpha \in (1, \infty) \setminus \mathbb{N}$ is finished.

(ii) Case $\alpha \in \{2, 3, \dots\}$. In this case we write

$$\hat{h}_{n-1}(s) = \frac{h_{n-1}(s)}{s^n}, \quad \hat{\phi}_{n-1}(s) = \frac{\phi_{n-1}(s)}{s^n}, \quad n = 2, 3, \dots \quad (6.20)$$

(this time we have $\alpha = (n-1) + 1$, which corresponds to the preceding case with n replaced by $n-1$ and $\beta = 1$). By Proposition 2.2, we only need to prove the equivalence between the following two statements: for any fixed $\lambda > 1$,

$$\hat{H}_{n-1}(s) := \hat{h}_{n-1}(s) - \hat{h}_{n-1}(\lambda s) \sim \frac{\ln \lambda}{(n-1)!} \ell(1/s) \quad (s \rightarrow 0), \quad (6.21)$$

$$\hat{\Phi}_{n-1}(s) := \hat{\phi}_{n-1}(s) - \hat{\phi}_{n-1}(\lambda s) \sim \frac{\ln \lambda}{(n-1)!} (1 - \rho(n))^{-1} \ell(1/s) \quad (s \rightarrow 0). \quad (6.22)$$

Recall that by Theorem 2.2,

$$\phi_{n-1}(s) - \mathbb{E}M^{-1}\phi_{n-1}(sM) = h_{n-1}(s) + \frac{C_{n-1}}{n!} s^n + O(s^{n+\epsilon}), \quad s \geq 0, \quad (6.23)$$

so that

$$\hat{\phi}_{n-1}(s) - \mathbb{E}M^{n-1}\hat{\phi}_{n-1}(sM) = \hat{h}_{n-1}(s) + \frac{C_{n-1}}{n!} + O(s^\epsilon), \quad s \geq 0. \quad (6.24)$$

Evaluating the equality at λs and subtracting, we obtain

$$\hat{\Phi}_{n-1}(s) - \mathbb{E}M^{n-1}\hat{\Phi}_{n-1}(sM) = \hat{H}_{n-1}(s) + O(s^\epsilon), \quad s \geq 0. \quad (6.25)$$

Using (6.25) instead of (6.8), the same argument as in Case (i) shows that (6.21) and (6.22) are equivalent. This ends the proof of Theorem 1.1. \square

7 Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we first show two lemmas. The first lemma states that the main term $\mathbb{E} [e^{-T(s)} - 1 + T(s)]$ of the right-hand-side of (2.15) behaves like h_1 .

Lemma 7.1 *If $\rho(1 + \delta_0) < \infty$ for some $\delta_0 > 0$, then we have for $\epsilon \in (0, 1)$ and $s \geq 0$,*

$$h_1(s) \geq \mathbb{E} [e^{-T(s)} - 1 + T(s)] = h_1((1 - \epsilon)s) + O(s^{1+\delta_0}). \quad (7.1)$$

Proof. Let $H(x) = e^{-x} - 1 + x$ ($x \geq 0$). Then we have

$$\begin{aligned} \mathbb{E}H(sY_1) &= \mathbb{E} [e^{-sY_1} - 1 + sY_1] = h_1(s), \\ \mathbb{E}H(T(s)) &= \mathbb{E} [e^{-T(s)} - 1 + T(s)]. \end{aligned}$$

For $\epsilon \in (0, 1)$, there exists $s_0 = s_0(\epsilon) > 0$ such that $t(s) \geq (1 - \epsilon)s$ for all $s \in [0, s_0]$. Hence for $s \in [0, s_0]$, we have

$$T(s) \geq \sum_{i=1}^N (1 - \epsilon) A_i s \mathbf{1}_{\{A_i s \leq s_0\}} = (1 - \epsilon)s \sum_{i=1}^N A_i \mathbf{1}_{\{A_i s \leq s_0\}}. \quad (7.2)$$

As $H(x)$ is increasing on $(0, \infty)$, from (2.8) and (7.2) we see that

$$\mathbb{E}H(T(s)) \leq \mathbb{E}H(sY_1) = h_1(s), \quad (7.3)$$

$$\mathbb{E}H(T(s)) \geq \mathbb{E}H \left((1 - \epsilon)s \sum_{i=1}^n A_i \mathbf{1}_{\{A_i s \leq s_0\}} \right). \quad (7.4)$$

Notice that by the mean value theorem, $0 \leq H(b) - H(a) \leq b - a$ if $0 \leq a \leq b$. Hence

$$\begin{aligned} 0 &\leq H((1-\epsilon)sY_1) - H\left((1-\epsilon)s \sum_{i=1}^N A_i \mathbf{1}_{\{A_i s \leq s_0\}}\right) \\ &\leq (1-\epsilon)s \sum_{i=1}^N A_i \mathbf{1}_{\{A_i s > s_0\}} \\ &\leq (1-\epsilon)s^{1+\delta_0} \sum_{i=1}^N A_i^{1+\delta_0} / s_0^{\delta_0}. \end{aligned}$$

Taking expectation, we see that

$$\begin{aligned} 0 &\leq \mathbb{E} \left[H((1-\epsilon)sY_1) - H\left((1-\epsilon)s \sum_{i=1}^N A_i \mathbf{1}_{\{A_i s \leq s_0\}}\right) \right] \\ &\leq (1-\epsilon)s_0^{-\delta_0} \rho(1+\delta_0)s^{1+\delta_0}. \end{aligned} \quad (7.5)$$

As $\mathbb{E}H((1-\epsilon)sY_1) = h_1((1-\epsilon)s)$, it follows from (7.3), (7.4) and (7.5) that (7.1) holds. \square

The second lemma is a relation between $\tilde{\ell}$ and ℓ . It shows that we can present $\tilde{\ell}$ as a sum of ℓ composed with a random walk.

Lemma 7.2 *Let $(X_i)_{i \geq 0}$ be independent and identically distributed with $\mu = \mathbb{E}X_1 \in (0, \infty)$ and $\mathbb{E}e^{-\delta_0 X_1} < \infty$ for some $\delta_0 > 0$. Write $S_n = X_1 + \dots + X_n$. Let $\ell \in R_0$ be non-increasing with $\int_1^\infty \ell(t)dt/t < \infty$. Then as $x \rightarrow \infty$*

$$\frac{1}{\tilde{\ell}(x)} \sum_{n=0}^{\infty} \ell(xe^{S_n}) \rightarrow \frac{1}{\mu} \quad \text{a.s. and in } L^1, \quad (7.6)$$

where $\tilde{\ell}$ is defined in (1.17).

Proof. Without loss of generality, we assume that ℓ is given by (1.10). We first estimate the inferior limit of $\frac{\sum_{n=0}^{\infty} \mathbb{E}\ell(xe^{S_n})}{\tilde{\ell}(x)}$. As $\mathbb{E}X_1 = \mu \in (0, \infty)$, by the law of large numbers we have $S_n/n \rightarrow \mu$ a.s. For $\epsilon > 0$, there exists almost surely a finite N_ϵ such that S_n lies between $n(\mu \pm \epsilon)$ for $n \geq N_\epsilon$. Since ℓ is non-increasing and $\int_1^\infty \frac{\ell(x)}{x} dx < \infty$, we have $\lim_{x \rightarrow \infty} \ell(x) = 0$. Using this together with $\ell'(x) = \ell(x)\epsilon(x)/x$ (cf. (1.10)), we obtain that

$$\ell(x) = - \int_x^\infty \ell'(t)dt = \int_x^\infty \ell(t)(-\epsilon(t))dt/t.$$

Therefore

$$\frac{\ell(x)}{\tilde{\ell}(x)} = \frac{\int_x^\infty \ell(t)(-\epsilon(t))dt/t}{\int_x^\infty \ell(t)dt/t} \leq \sup_{t \in [x, \infty)} (-\epsilon(t)) \rightarrow 0 \quad (x \rightarrow \infty),$$

so that

$$\ell(x) = o(\tilde{\ell}(x)) \quad (x \rightarrow \infty). \quad (7.7)$$

As $\ell(xe^{S_n}) \sim \ell(x)$, it follows that a.s.

$$\sum_{n=0}^{N_\epsilon-1} \ell(xe^{S_n}) = o(\tilde{\ell}(x)) \quad (x \rightarrow \infty). \quad (7.8)$$

On the other hand, for $n \geq N_\epsilon$, as ℓ is non-increasing, we have a.s.

$$\sum_{n=N_\epsilon}^{\infty} \ell(xe^{n(\mu+\epsilon)}) \leq \sum_{n=N_\epsilon}^{\infty} \ell(xe^{S_n}) \leq \sum_{n=N_\epsilon}^{\infty} \ell(xe^{n(\mu-\epsilon)}). \quad (7.9)$$

Notice that for any $c > 0$,

$$\sum_{n=N_\epsilon+1}^{\infty} \ell(xe^{nc}) \leq \int_{N_\epsilon}^\infty \ell(xe^{ct})dt \leq \sum_{n=N_\epsilon}^{\infty} \ell(xe^{nc}),$$

and that the integral above (the middle term) is

$$\frac{1}{c} \int_{xe^{cN_\epsilon}}^{\infty} \ell(u) du / u = \frac{1}{c} \tilde{\ell}(xe^{cN_\epsilon}) \sim \frac{1}{c} \tilde{\ell}(x) \quad (x \rightarrow \infty).$$

Therefore

$$\sum_{n=N_\epsilon}^{\infty} \ell(xe^{nc}) \sim \frac{1}{c} \tilde{\ell}(x) \quad (x \rightarrow \infty). \quad (7.10)$$

It follows from (7.8), (7.9) and (7.10) that a.s.

$$\frac{\sum_{n=0}^{\infty} \ell(xe^{S_n})}{\tilde{\ell}(x)} \rightarrow \frac{1}{\mu} \quad (x \rightarrow \infty). \quad (7.11)$$

So by Fatou's lemma,

$$\liminf_{x \rightarrow \infty} \frac{\sum_{n=0}^{\infty} \mathbb{E} \ell(xe^{S_n})}{\tilde{\ell}(x)} \geq \mathbb{E} \liminf_{x \rightarrow \infty} \frac{\sum_{n=0}^{\infty} \ell(xe^{S_n})}{\tilde{\ell}(x)} = \frac{1}{\mu}. \quad (7.12)$$

We next consider the superior limit of $\frac{\sum_{n=0}^{\infty} \mathbb{E} \ell(xe^{S_n})}{\tilde{\ell}(x)}$. We divide S_n into two parts according to $\{S_n \leq n(\mu - \epsilon)\}$ or $\{S_n > n(\mu - \epsilon)\}$, so that

$$\sum_{n=0}^{\infty} \mathbb{E} \ell(xe^{S_n}) = \sum_{n=0}^{\infty} \mathbb{E} \ell(xe^{S_n}) \mathbf{1}_{\{S_n \leq n(\mu - \epsilon)\}} + \sum_{n=0}^{\infty} \mathbb{E} \ell(xe^{S_n}) \mathbf{1}_{\{S_n > n(\mu - \epsilon)\}}. \quad (7.13)$$

As ℓ is non-increasing, the second term is bounded by

$$\sum_{n=0}^{\infty} \ell(xe^{n(\mu - \epsilon)}) \sim \frac{1}{\mu - \epsilon} \tilde{\ell}(x) \quad (x \rightarrow \infty) \quad (7.14)$$

(cf. (7.10) with $c = \mu - \epsilon$). We now deal with the first term in the right hand side of (7.13). Write $\bar{X}_i = X_i - \mu$ and $\bar{S}_n = \bar{X}_1 + \dots + \bar{X}_n = S_n - n\mu$. By Potter's theorem, for any $\delta > 0$, we have

$$\begin{aligned} \mathbb{E} \ell(xe^{S_n}) \mathbf{1}_{\{S_n \leq n(\mu - \epsilon)\}} &\leq A_\delta \ell(x) \mathbb{E} (e^{\delta S_n} + e^{-\delta S_n}) \mathbf{1}_{\{S_n \leq n(\mu - \epsilon)\}} \\ &\leq A_\delta \ell(x) \mathbb{E} (e^{n\mu\delta} + e^{-\delta S_n}) \mathbf{1}_{\{\bar{S}_n \leq -n\epsilon\}}, \end{aligned} \quad (7.15)$$

where $A_\delta > 0$ is a constant depending on δ .

We shall prove that thanks to the condition $\mathbb{E} e^{-\delta_0 X_1} < \infty$, we have for $\epsilon > 0$,

$$\mathbb{P}(\bar{S}_n \leq -n\epsilon) \leq e^{-cn} \quad (7.16)$$

for some $c = c(\epsilon) > 0$ and all $n \geq 1$. In fact, as (X_i) are independent and identically distributed, we have for all $\delta > 0$

$$\begin{aligned} \mathbb{P}(\bar{S}_n \leq -n\epsilon) &= \mathbb{P}(e^{-\delta \bar{S}_n} \geq e^{n\delta\epsilon}) \\ &\leq e^{-n\delta\epsilon} \mathbb{E} e^{-\delta \bar{S}_n} \\ &= \left[e^{-\delta\epsilon} \mathbb{E} e^{-\delta \bar{X}_1} \right]^n. \end{aligned} \quad (7.17)$$

Let $\Lambda(\delta) = \ln \mathbb{E} e^{-\delta \bar{X}_1}$. By the dominated convergence theorem, it can be easily seen that $\frac{d}{d\delta} \mathbb{E} e^{-\delta \bar{X}_1} = \mathbb{E} [-\bar{X}_1 e^{-\delta \bar{X}_1}]$, so that

$$\Lambda'(\delta) = \frac{\mathbb{E} [-\bar{X}_1 e^{-\delta \bar{X}_1}]}{\mathbb{E} e^{-\delta \bar{X}_1}} \rightarrow -\mathbb{E}[\bar{X}_1] = 0 \quad (\delta \rightarrow 0).$$

Hence

$$\lim_{\delta \rightarrow 0} \frac{\Lambda(\delta)}{\delta} = \lim_{\delta \rightarrow 0} \Lambda'(\delta) = 0. \quad (7.18)$$

Therefore, for $\delta > 0$ small enough, we have $\Lambda(\delta) < \delta\epsilon$, which is equivalent to

$$\mathbb{E}e^{-\delta X_1} < e^{\delta\epsilon}. \quad (7.19)$$

So from (7.17), we obtain (7.16) with $c = -\ln \left[\frac{\mathbb{E}e^{-\delta X_1}}{e^{\delta\epsilon}} \right] > 0$.

By the Cauchy-Schwartz inequality, it follows from (7.15) and (7.16) that

$$\begin{aligned} \mathbb{E}\ell(xe^{S_n})\mathbf{1}_{\{S_n \leq n(\mu-\epsilon)\}} &\leq A_\delta \ell(x) \left\{ e^{n\mu\delta} \mathbb{P}(\bar{S}_n \leq -n\epsilon) + [\mathbb{E}e^{-2\delta S_n}]^{1/2} [\mathbb{P}(\bar{S}_n \leq -n\epsilon)]^{1/2} \right\} \\ &\leq A_\delta \ell(x) \left\{ [e^{\mu\delta} \cdot e^{-c}]^n + [\mathbb{E}e^{-2\delta X_1} \cdot e^{-c}]^{n/2} \right\}. \end{aligned} \quad (7.20)$$

As $\mathbb{E}e^{-2\delta X_1} < \infty$ for $\delta \in [0, \delta_0/2]$, we can choose $\delta > 0$ small enough such that

$$e^{\mu\delta} \cdot e^{-c} < 1 \quad \text{and} \quad \mathbb{E}e^{-2\delta X_1} \cdot e^{-c} < 1. \quad (7.21)$$

Hence from (7.20) and (7.7), we see that

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{E}\ell(xe^{S_n})\mathbf{1}_{\{S_n \leq n(\mu-\epsilon)\}} &\leq A_\delta \ell(x) \sum_{n=1}^{\infty} \left\{ [e^{\mu\delta} \cdot e^{-c}]^n + [\mathbb{E}e^{-2\delta X_1} \cdot e^{-c}]^{n/2} \right\} \\ &\leq C\ell(x) = o(\tilde{\ell}(x)). \end{aligned} \quad (7.22)$$

It follows from (7.13), (7.14) and (7.22) that

$$\sum_{n=0}^{\infty} \mathbb{E}\ell(xe^{S_n}) \leq o(\tilde{\ell}(x)) + \frac{1}{\mu - \epsilon} \tilde{\ell}(x),$$

which implies that

$$\limsup_{x \rightarrow \infty} \frac{\sum_{n=0}^{\infty} \mathbb{E}\ell(xe^{S_n})}{\tilde{\ell}(x)} \leq \frac{1}{\mu - \epsilon}.$$

Letting $\epsilon \rightarrow 0$, we have

$$\limsup_{x \rightarrow \infty} \frac{\sum_{n=0}^{\infty} \mathbb{E}\ell(xe^{S_n})}{\tilde{\ell}(x)} \leq \frac{1}{\mu}. \quad (7.23)$$

Combining this with (7.12), we obtain

$$\lim_{x \rightarrow \infty} \frac{1}{\tilde{\ell}(x)} \sum_{n=0}^{\infty} \mathbb{E}\ell(xe^{S_n}) = \frac{1}{\mu}. \quad (7.24)$$

By Scheffé's theorem, (7.11) and (7.24) give (7.6). \square

Proof of Theorem 1.2. By Proposition 2.1 (with $n = 1$ and $\beta = 0$), we just need to show that

$$h_1(s) \sim s\ell(1/s) \quad (s \rightarrow 0) \quad (7.25)$$

implies

$$\phi_1(s) \sim \mu^{-1} s \tilde{\ell}(1/s) \quad (s \rightarrow 0), \quad (7.26)$$

where $\tilde{\ell}$ is defined in (1.17). Notice that if $\ell_1(x) \sim \ell_2(x)$ ($x \rightarrow \infty$), then $\int_x^\infty \frac{\ell_1(t)}{t} dt \sim \int_x^\infty \frac{\ell_2(t)}{t} dt$ ($x \rightarrow \infty$) by l'Hôpital's rule. Therefore we can suppose that $\ell(1/s) = h_1(s)/s$, $s > 0$, which is non-decreasing, so that $\ell(x)$ is non-increasing.

Recall the definitions of $\hat{\phi}_1(s)$ and $\hat{\phi}(s)$ in the proof of Theorem 1.1(i) (with $\alpha = 1$):

$$\hat{\phi}_1(s) = \phi_1(s)/s \quad \text{and} \quad \hat{\phi}(s) = \hat{\phi}_1(s) - \mathbb{E}\hat{\phi}_1(sM).$$

Corresponding to (6.12), we get

$$\sum_{i=0}^{k-1} \mathbb{E}\hat{\phi}(sM_1 \cdots M_i) = \hat{\phi}_1(s) - \mathbb{E}\hat{\phi}_1(sM_1 \cdots M_k), \quad (7.27)$$

where $\{M_i\}$ are independent copies of M . Notice that $0 < 1 - e^{-x} < x$ ($\forall x > 0$), we see that $0 \leq \phi_1(s) = \mathbb{E}(e^{-sZ} - 1 + sZ) \leq 2s\mathbb{E}Z = 2s$, and hence

$$\sup_{s>0} |\hat{\phi}_1(s)| < \infty.$$

Therefore, by the dominated convergence theorem and the fact that $M_1 \cdots M_k \rightarrow 0$ (which follows from the law of large numbers), we conclude that

$$\lim_{k \rightarrow \infty} \mathbb{E} \hat{\phi}_1(sM_1 \cdots M_k) = 0.$$

So letting $k \rightarrow \infty$ in (7.27), we obtain

$$\sum_{n=0}^{\infty} \mathbb{E} \hat{\phi}_1(sM_1 \cdots M_n) = \hat{\phi}_1(s). \quad (7.28)$$

By Lemma 7.1 and Theorem 2.2, we see that

$$\hat{\phi}_1(s) \leq \frac{h_1(s)}{s} + O(s^\epsilon), \quad s \geq 0, \quad (7.29)$$

$$\hat{\phi}_1(s) \geq \frac{h_1((1-\delta)s)}{s} + O(s^\epsilon), \quad s \geq 0, \quad (7.30)$$

for all $\delta \in (0, 1)$. From (7.28) and (7.29), we have for all $\epsilon > 0$,

$$\begin{aligned} \hat{\phi}_1(s) &\leq \sum_{n=0}^{\infty} \left[\mathbb{E} \frac{h_1(sM_1 \cdots M_n)}{sM_1 \cdots M_n} + O(\mathbb{E}(sM_1 \cdots M_n)^\epsilon) \right] \\ &= \sum_{n=0}^{\infty} \left[\mathbb{E} \ell \left(\frac{1}{sM_1 \cdots M_n} \right) + O(s^\epsilon) \mathbb{E}(M_1 \cdots M_n)^\epsilon \right]. \end{aligned} \quad (7.31)$$

Since (1.14) holds for $\alpha = 1$ and some $\delta_0 > 0$, we have $\rho(1 + \delta) < \infty$ for some $\delta > 0$ small enough (cf. (1.15)). Hence $\rho(x)$ is convex on $(1, 1 + \delta)$ with $\rho'(1) < 0$, and we can find some $\epsilon > 0$ small enough such that $\rho(1 + \epsilon) < 1$. By the definition of M (cf. (2.10)), we obtain that $\mathbb{E}(M_1 \cdots M_n)^\epsilon = \left(\mathbb{E} \sum_{i=1}^n A_i^{1+\epsilon} \right)^n = \rho^n(1 + \epsilon)$ and

$$\sum_{n=0}^{\infty} \mathbb{E}(M_1 \cdots M_n)^\epsilon = \sum_{n=0}^{\infty} \rho^n(1 + \epsilon) = \frac{1}{1 - \rho(1 + \epsilon)}. \quad (7.32)$$

Write $M_n = e^{-X_n}$ and $S_n = X_1 + \cdots + X_n$. So $\{X_n\}$ are independent and identically distributed, with $\mathbb{E}X_1 = -\mathbb{E} \ln M_1 = -\rho'(1) = \mu \in (0, \infty)$ and $\mathbb{E}e^{-\delta_0 X_1} = \mathbb{E}M_1^{\delta_0} = \rho(1 + \delta_0) < \infty$. By Lemma 7.2, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{E} \ell \left(\frac{1}{sM_1 \cdots M_n} \right) &= \sum_{n=0}^{\infty} \mathbb{E} \ell(e^{S_n}/s) \\ &\sim \mu^{-1} \tilde{\ell}(1/s) \quad (s \rightarrow 0). \end{aligned} \quad (7.33)$$

Notice that $\tilde{\ell}$ is a function slowly varying at ∞ , it follows from (7.31), (7.32) and (7.33) that

$$\limsup_{s \rightarrow 0} \frac{\hat{\phi}_1(s)}{\tilde{\ell}(1/s)} \leq \lim_{s \rightarrow 0} \left[\frac{1}{\mu} + \frac{O(s^\epsilon)}{\tilde{\ell}(1/s)} \right] = \mu^{-1}. \quad (7.34)$$

On the other hand, from (7.28) and (7.30), we have for $\delta \in (0, 1)$,

$$\begin{aligned} \hat{\phi}_1(s) &\geq \sum_{n=0}^{\infty} \left[\mathbb{E} \frac{h_1((1-\delta)sM_1 \cdots M_n)}{sM_1 \cdots M_n} + O(s^\epsilon) \mathbb{E}(M_1 \cdots M_n)^\epsilon \right] \\ &= \sum_{n=0}^{\infty} \left[(1-\delta) \mathbb{E} \ell \left(\frac{1}{(1-\delta)sM_1 \cdots M_n} \right) + O(s^\epsilon) \mathbb{E}(M_1 \cdots M_n)^\epsilon \right] \\ &= \sum_{n=0}^{\infty} \left[(1-\delta) \mathbb{E} \ell(e^{S_n}/[(1-\delta)s]) + O(s^\epsilon) \mathbb{E}(M_1 \cdots M_n)^\epsilon \right]. \end{aligned} \quad (7.35)$$

Using again Lemma 7.2, we have

$$\sum_{n=0}^{\infty} \mathbb{E} \ell(e^{S_n}/[(1-\delta)s]) \sim \mu^{-1} \tilde{\ell}(1/[(1-\delta)s]) \sim \mu^{-1} \tilde{\ell}(1/s) \quad (s \rightarrow 0), \quad (7.36)$$

where the last step holds since the slow variation of ℓ implies that of $\tilde{\ell}$. It follows from (7.32), (7.35) and (7.36) that

$$\liminf_{s \rightarrow 0} \frac{\hat{\phi}_1(s)}{\tilde{\ell}(1/s)} \geq \frac{1-\delta}{\mu}. \quad (7.37)$$

Letting $\delta \rightarrow 0$, we see that

$$\liminf_{s \rightarrow 0} \frac{\hat{\phi}_1(s)}{\tilde{\ell}(1/s)} \geq \mu^{-1}. \quad (7.38)$$

From (7.34) and (7.38), we obtain

$$\lim_{s \rightarrow 0} \frac{\hat{\phi}_1(s)}{\tilde{\ell}(1/s)} = \mu^{-1}, \quad (7.39)$$

which is equivalent to (7.26). This ends the proof of Theorem 1.2. \square

Remark 7.1 In Theorem 1.2, we fail to show the equivalence between (1.18) and (1.19), which can be interpreted to the equivalence between (7.25) and (7.26). In our approach, to show the implication (7.25) \Rightarrow (7.26), we use (7.28) (together with Lemma 7.2) to transfer information on h_1 (corresponding to $\hat{\phi}$) to that on ϕ_1 (corresponding to $\hat{\phi}_1$). The difficult for establishing the inverse implication (7.26) \Rightarrow (7.25) is that from (7.28), we cannot transfer the information from $\hat{\phi}_1$ to $\hat{\phi}$ (that is, from ϕ_1 to h_1).

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Appendix

Proof of Proposition 2.2. Write F for the distribution function of X . Define F_1, \dots, F_{n+1} by letting $F_1 = F$ and

$$1 - F_{k+1}(x) = \begin{cases} \int_x^\infty [1 - F_k(t)] dt, & x \geq 0, \\ 1 - F_{k+1}(0), & x < 0, \end{cases} \quad (\text{A.1})$$

if $1 \leq k \leq n$. By induction, it can be easily shown that when $\mu_n = \mathbb{E}X^n < \infty$, we have for $k = 0, \dots, n$,

$$1 - F_{k+1}(x) = \frac{\mathbb{E}[(X-x)^k \mathbf{1}_{\{X>x\}}]}{k!}, \quad x \geq 0;$$

in particular, $1 - F_{k+1}(x)$ is bounded by $1 - F_{k+1}(0) = \mu_k/k!$. By definition, for $1 \leq k \leq n$,

$$F_{k+1}(x) - F_{k+1}(0) = \int_0^x [1 - F_k(t)] dt, \quad x \geq 0.$$

In other words, for $k = 1, \dots, n$, $F_{k+1}(x) - F_{k+1}(0)$ is the distribution function of the measure concentrated on $(0, \infty)$ of density $1 - F_k(t)$ with respect to the Lebesgue measure, with mass $1 - F_{k+1}(0) = \mu_k/k!$. Define

$$F_{n+2}(x) = \int_0^x [1 - F_{n+1}(t)] dt, \quad x \geq 0$$

and $F_{n+2}(x) = 0$ for $x < 0$, namely, $F_{n+2}(x)$ is the distribution function of the measure concentrated on $(0, \infty)$ of density $1 - F_{n+1}(t)$ with respect to the Lebesgue measure. By induction, it can be easily checked that for $k = 0, \dots, n$,

$$\hat{f}_k(s) = \int_{[0, \infty)} e^{-st} d(F_{k+2}(x) - F_{k+2}(0)) = \int_{[0, \infty)} e^{-st} [1 - F_{k+1}(t)] dt. \quad (\text{A.2})$$

We first show that (2.3) implies (2.4). As $1 - F(x) \sim x^{-(n+1)}\ell(x)$ ($x \rightarrow \infty$), by induction we have

$$1 - F_{k+1}(x) \sim \frac{x^{-(n+1-k)}}{n \cdots (n-k+1)} \ell(x), \quad k = 0, \dots, n,$$

with the convention that $n \cdots (n-k+1) = 1$ when $k = 0$. Therefore, for any $\lambda > 0$, we have

$$\begin{aligned} F_{n+2}(\lambda x) - F_{n+2}(x) &= \int_x^{\lambda x} [1 - F_{n+1}(t)] dt \\ &\sim \int_x^{\lambda x} \frac{\ell(t)}{n!} \frac{dt}{t} \sim \frac{\ell(x)}{n!} \ln \lambda, \quad x \rightarrow \infty, \end{aligned}$$

where the last equivalence is due to the Uniform Convergence Theorem (see [11, Theorem 1.2.1]). Hence we obtain

$$\lim_{x \rightarrow \infty} \frac{F_{n+2}(\lambda x) - F_{n+2}(x)}{\ell(x)} = \frac{1}{n!} \ln \lambda, \quad \forall \lambda > 0. \quad (\text{A.3})$$

As \hat{f}_n is the Laplace-Stieltjes transform of F_{n+2} , it follows from (A.2) and de Haan's theorem (see [15] or [11, Theorem 3.9.1]) that

$$\lim_{s \rightarrow \infty} \frac{\hat{f}_n(\frac{1}{\lambda s}) - \hat{f}_n(\frac{1}{s})}{\ell(s)} = \frac{1}{n!} \ln \lambda, \quad \forall \lambda > 0, \quad (\text{A.4})$$

which gives (2.4).

We now show that (2.4) implies (2.3). Since $\hat{f}_n(\cdot)$ is the Laplace-Stieltjes transform of $F_{n+2}(\cdot)$, again by de Haan's theorem, we see that (A.3) holds true, which implies that

$$\lim_{x \rightarrow \infty} \frac{F_{n+2}(\lambda x) - F_{n+2}(x)}{F_{n+2}(e x) - F_{n+2}(x)} = \ln \lambda, \quad \forall \lambda > 0.$$

So by the lemma of [15], we see that $1 - F_{n+1}(x)$, which is the derivative of $F_{n+2}(x)$, is a regularly varying function of index -1 . Differentiating n times the function $1 - F_{n+1}(x)$, we know that $1 - F_1(x)$ varies regularly with index $-(n+1)$, that is,

$$1 - F_1(x) \sim x^{-(n+1)} \ell_1(x) \quad (x \rightarrow \infty), \quad (\text{A.5})$$

where $\ell_1(\cdot)$ is a function slowly varying at ∞ . Using this and the implication "(2.3) \Rightarrow (2.4)", we see that (2.4) holds also with ℓ replaced by ℓ_1 . Thus $\ell_1(x) \sim \ell(x)$ ($x \rightarrow \infty$), so that (A.5) implies (2.3). This ends the proof of the implication "(2.4) \Rightarrow (2.3)". \square

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