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Phase Transition of a Non-Linear Opinion Dynamics with Noisy Interactions

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Abstract

Consensus processes play a key role in self-organizing phenomena observed in several real *Multi-Agent Systems* (MAS). In models proposed for social and biological systems, consensus has been thoroughly investigated in the context of *opinion dynamics*. In such scenarios, it has been nevertheless observed that only weaker forms of *metastable* consensus are achieved, in which a large majority of agents agree on some opinion while other opinions continue to be supported by a (small) minority of agents.

In this work, we take a step towards the investigation of metastable consensus for complex (non-linear) dynamics by considering the famous UNDECIDED-STATE dynamics, which is known to reach consensus exponentially faster than the VOTER dynamics by making use of only one single additional agent state.

We consider a simple form of uniform noise in which each message can change to another one with probability p and we prove that the persistence of a *metastable consensus* undergoes a *phase transition* for $p = \frac{1}{6}$. In detail, below this threshold, we prove the system reaches with high probability a metastable regime where a large majority of agents keeps supporting the same opinion for polynomial time. Moreover, this opinion turns out to be the majority opinion in the initial configuration, whenever the initial bias is slightly larger than its standard deviation (the latter being computed under the random uniform configuration). On the contrary, above the threshold, we show that the information about the initial majority opinion is “lost” within logarithmic time even when the initial bias is maximum (i.e., even when the system starts from any full-consensus configuration).

Interestingly, using a simple coupling argument, we show the equivalence between our noisy model above and the model where a subset of agents behave in a *stubborn* way. This implies a similar phase transition in the latter model as well.

1 Introduction

Opinion dynamics are mathematical models to investigate the way a fully-decentralized Multi-Agent Systems (for short, MAS) is able to reach some form of *Consensus*. Their study is a hot topic touching several research areas such as MAS [18, 24], Distributed Computing [23, 33], Social Networks [1, 45], and System Biology [13, 14]. Typical examples of opinion dynamics are the Voter Model, the averaging rules, and the majority rules. Some of such dynamics share a surprising efficiency and resiliency that seem to exploit common *computational principles* [23, 33].

Within such framework, the tasks of (*valid*) *Consensus* and *Majority Consensus* have attracted a lot of attention within different application domains in social networks [45], in biological systems [31], passively-mobile sensor networks [2] and chemical reaction networks [19]. In the Consensus task, the system is required to converge to a stable configuration where all agents supports the same opinion and this opinion must be *valid*, i.e., it must be supported by at least one agent in the initial configuration. While, in the Majority Consensus task, starting from an initial configuration where there is some positive bias towards one *majority opinion*, the system is required to converge to the configuration where all agents support the initial majority opinion. Here, the *bias* of a configuration is defined as the difference between the number of agents supporting the majority opinion (for short, we name this number as *majority*) and the number of agents supporting the second-largest opinion.

Different opinion dynamics have been studied in a variety of settings [20, 30], and then used as subroutine to solve more complex computational tasks [22, 47, 12].

In the aforementioned applicative scenarios, it has been nevertheless observed that only weaker forms of *metastable* consensus are achieved, in which the large majority of agents rapidly achieves a consensus (while other opinions continue to be supported by a small set of agents), and this setting is preserved for a relatively-long regime. Models that have been considered to study such phenomenon include MAS where: i) agents follow a linear dynamics, such as the VOTER model or the AVERAGING dynamics and ii) a small set of *stubborn agents* are present in the system [42, 41, 50], or the local interactions are affected by *communication noise* [39, 15].

We emphasize that the VOTER model has a slow (i.e. polynomial in the number n of agents) convergence time even in a fully-connected network (i.e. in the complete graph) and it does not guarantee a high probability to reach consensus on the initial majority opinion, even starting from a large initial bias (i.e. $\Theta(n)$, where n is the number of the agents of the system) [35]. On the other hand, averaging dynamics requires agents to perform numerical operations and, very importantly, to have a large local

memory (to guarantee a good-enough approximation of real numbers). For the reasons above, linear opinion dynamics cannot explain fast and reliable metastable consensus phenomena observed in some MAS [12, 31, 19].

The above discussion naturally leads us to investigate the behaviour of other, non-linear dynamics in the presence of stubborn agents and/or communication noise. Over a MAS formed by a complete network of n agents, we introduce a simple model of *communication noise* in the stochastic process yielded by a popular dynamics, known as the UNDECIDED-STATE dynamics¹. According to this simple dynamics, the state of every agent can be either an opinion (chosen from a finite set Σ) or the *undecided state*. At every discrete-time step (i.e., round), every agent “pulls” the state of a random neighbor and updates its state according to the following rule: if a non-undecided agent pulls a different opinion from its current one, then it will get undecided, while in all other cases it keeps its opinion; moreover, if the node is undecided then it will get the state of the pulled neighbor.

This non-linear dynamics is known to compute Consensus (and Majority Consensus) on the complete network within a logarithmic number of rounds [2, 16] and, very importantly, it is optimal in terms of local memory since it requires just one extra state/opinion [40]. We provide an overview of the research on the UNDECIDED-STATE in Subsection 1.2.

While communication noise is a common feature of real-world systems and its effects have been thoroughly investigated in physics and information theory [21], its study has been mostly focused on settings in which communication happens over *stable links* where the use of error-correcting codes is feasible since message of large size are allowed; it has been otherwise noted that when interactions among the agents are random and opportunistic and consists of very-short messages, classical information-theoretic arguments do not carry on and new phenomena calls for a theoretical understanding [13].

1.1 Our Contribution

In this work, we show that, under a simple model of uniform noise, the UNDECIDED-STATE dynamics exhibits a rather surprising *phase transition*.

We consider the binary case (i.e., $|\Sigma| = 2$) together with an *oblivious* and *symmetric* action of noise over messages: any sent message is changed upon being received to any other value, independently and uniformly at random with probability p (where p is any fixed positive constant smaller than $1/2$).

On one hand, if $p < 1/6$, starting from an arbitrary configuration of the complete network of n agents, we prove that the system *with high probability*² (*w.h.p.*, for short) reaches, within $\mathcal{O}(\log n)$ rounds, a metastable almost

¹In some previous papers [46], this protocol has been called the *Third-State Dynamics*. We here prefer the term “undecided” since it well captures the role of this additional state.

²An event E holds *with high probability* if a constant $\gamma > 0$ exists such that $\mathbf{P}(E) \geq 1 - (1/n)^\gamma$.

consensus regime where the bias towards one fixed³ valid opinion keeps large, i.e. $\Theta(n)$, for at least a $\text{poly}(n)$ number of rounds (see Theorem 3). In particular, despite the presence of random communication noise, our result implies that the UNDECIDED-STATE dynamics is able to rapidly break the initial symmetry of any balanced configuration⁴ and reach a metastable regime of almost consensus.

Importantly enough, our probabilistic analysis also shows that, for any $p < 1/6$, the system is able to “compute” the task of almost Majority Consensus. Indeed, in Theorem 1, starting from an arbitrary configuration with bias $\Omega(\sqrt{n \log n})$,⁵ we prove the system w.h.p. reaches, within $\mathcal{O}(\log n)$ rounds, a metastable regime where the bias towards the initial majority opinion keeps large, i.e. $\Theta(n)$, for at least a $\text{poly}(n)$ number of rounds (see Theorem 1). For instance, our analysis for $p = 1/10$ implies that the process rapidly reaches a metastable regime where the bias keeps size larger than $n/3$.

On the other hand, if $p > 1/6$, even when the initial bias is maximum (i.e., when the system starts from any full-consensus configuration), after a logarithmic number of rounds, the information about the initial majority opinion is “lost”: we indeed show that the system w.h.p. enters into a regime where the bias keeps bounded by $\mathcal{O}(\sqrt{n \log n})$. We expect the majority opinion to switch continuously during this regime (see the simulation data and Table 2).

We also made computer simulations with values of n ranging from 5×10^3 to 15×10^4 , and for $p = 1/12$, $p = 1/8$, $p = 1/7$, and $p = 1/5$. The outcomes not only confirm the phase transition predicted by our theoretical analysis above, but they show this behaviour emerges even for reasonable sizes (i.e. n) of the system.

Indeed, we made the UNDECIDED-STATE dynamics run for 400 rounds for the above values of p . In the first three, we started from complete balanced configurations (i.e. in which opinion 1 and opinion 2 are supported by, respectively, $\frac{n}{2}$ agents) we found a fast convergence to the meta-stable regime of almost-consensus, which did not break for all the time we let the simulations run. Furthermore, we have noticed that the symmetry is always break when the bias is “roughly” $10\sqrt{n \log n}$.

While, for the case in which $p = 1/5$, we started from a configuration of complete consensus (namely, in which an opinion is supported by *all* the agents) and we observed that in a short time the system lost any information on the majority opinion (say, the bias becomes less than $10\sqrt{n \log n}$) and kept in this meta-stable phase in which no majority opinion emerged for

³This opinion never changes along the metastable regime.

⁴E.g., the perfectly-balanced configuration with $n/2$ agents having one opinion and the other $n/2$ agents having the other opinion.

⁵We remark that, when every agent chooses its initial binary opinion uniformly at random, the standard deviation of the bias is $\Theta(\sqrt{n})$.

all the remaining time. Actually, we observed many switches of majority opinion while the process is running. Here we report two table. In Table 1 we can see the average time (computed over 100 trials, and cut to the closest integer) in which the system enters the predicted meta-stable phase of almost-consensus for any value of $p = 1/12$, $p = 1/8$, and $p = 1/7$, for different input sizes. We also see that the more the probability p goes to $1/6$, the higher the number of nodes n has to be to observe the emergent behaviour (indeed, ours is an asymptotic result in n , which depends on ϵ), and some of the experiments fail.

Size n	Average times		
	$p = 1/12$	$p = 1/8$	$p = 1/7$
2^{10}	24	Failed	Failed
2^{11}	24	39	Failed
2^{12}	28	41	Failed
2^{13}	27	53	Failed
2^{14}	32	52	77
2^{15}	32	54	88
2^{16}	36	57	96
2^{17}	39	68	103

Table 1: Average time to reach a meta-stable almost-consensus.

In Table 2, we see the average times (computed over 100 trials, and cut to the closest integer) in which the bias of the system goes under $10\sqrt{n \log n}$ for different input sizes, and the corresponding number of switches of majority opinion during the remaining time.

Size n	$p = 1/5$	
	Average time	Number of switches
2^{10}	1	39
2^{11}	4	42
2^{12}	7	42
2^{13}	10	37
2^{14}	14	38
2^{15}	18	38
2^{16}	22	44
2^{17}	27	39

Table 2: Average times the bias goes under $10\sqrt{n \log n}$, and average number of times the maj. opinion switches.

Interestingly, in Subsection 2.0.2 we show that our noise model is in fact equivalent to a noiseless setting in which *stubborn* agents are present in the system [50] (that is, agents that never change their state), thus proving

an analogous phase transition in this setting. We can thus remark that the obtained phase transition separates qualitatively the behavior of the UNDECIDED-STATE dynamics from that of the VOTER model which is, to the best of our knowledge, the only opinion dynamics (with a finite opinion set) which has been rigorously analyzed in the presence of communication noise or stubborn agents [41, 50]: this hints at a more general phenomenon for dynamics with fast convergence to some metastable consensus.

Our proof of the above phase transition relies on a new tight probabilistic analysis of the dynamics process as a function of the initial bias and, in our new setting, of the noise parameter p . Informally, the key ideas of our analysis come out observing that the system admits only one global state of equilibrium in expectation, i.e. the one in which each opinion and the undecided state is supported by exactly $\frac{n}{3}$ agents: by barely perturbing this equilibrium, inserting a positive bias (e.g. a positive difference between the sizes of the communities of agents supporting the two opinions) and leaving the number of undecided agents to $\frac{n}{3}$, we prove that the expected bias in the next round turns out to grow exponentially fast when $p < \frac{1}{6}$, while it decreases exponentially fast when $p > \frac{1}{6}$. We believe this fact hints at an interesting phenomenon affecting a more general class of dynamics (e.g. those based on majority rules) in the presence of noise or stubborn agents.

We then use suitable Chernoff's bounds [28] to made our average-case analysis above a set of probability-concentration results that fully describe the behaviour of the UNDECIDED-STATE process. A special care is required to analyze the case of balanced initial configurations where there is no bias. For this setting, we need to show and exploit the presence of a large variance w.r.t. the expected behaviour and combine it with the emergence of any small bias observed along time windows of suitable length.

1.2 Related Work

Decentralized opinion dynamics and their ability to reach several forms of consensus is the subject of several research studies from different areas ranging from distributed computing [2, 8, 10, 11, 25, 35] to social networks [4, 43, 44, 45]. For brevity' sake, we here summarize those works having models and/or results which more directly related with ours.

The UNDECIDED-STATE dynamics has been originally studied as an efficient majority-consensus protocol by [2] and independently by [9] for the binary case (i.e. with two initial input values). They proved that w.h.p., within a logarithmic number of rounds, all agents support the initial majority opinion. Some works have then extended the analysis of the UNDECIDED-STATE dynamics to non-complete topologies. In the Poisson-clock model (formally equivalent to the Population Protocol model), [27] derive an upper bound on the expected convergence time of the dynamics that holds for arbitrary connected graphs, which is based on the location of eigenvalues of

some contact rate matrices. They also instantiate their bound for particular network topologies. Successively, [40] provided an analysis when the initial states of agents are assigned independently at random, and they also derive “bad” initial configurations on certain graph topologies such that the initial minority opinion eventually becomes the majority one. As for the use of UNDECIDED-STATE as a generic consensus protocol, [17] recently proved that, in the synchronous uniform PULL model in which all agents update their state in parallel by observing the state of a random other node the convergence time of the UNDECIDED-STATE dynamics is w.h.p. logarithmic.

The motivation to investigate opinion dynamics is twofold: they can be regarded as simplistic models of several real-world phenomena or as building blocks for more complex algorithms. While on the modelling side the UNDECIDED-STATE dynamics is an appealing model of opinion dynamics and it has also been considered as a model of some mechanism occurring in the biology of a cell [14], it has been employed as a sub-routine of efficient *Majority Consensus* protocols: [34], [11] and [29] consider Majority Consensus in the Uniform-PULL, and design protocols (based on the UNDECIDED-STATE dynamics) which w.h.p. converge in poly-logarithmic time even if the number of initial opinions is very large.

Notably, communication noise in random-interacting MAS appears to be a neglected area of investigation [39, 6, 7, 18]. Such shortage of studies contrasts with the vast literature on communication noise over *stable* networks⁶. Among the few investigations of communication noise in MAS, we note the Vicsek model [48], where agents are driven with a constant absolute velocity, and at each time step assume the average direction of motion of the agents in their neighborhood: this strategy is combined with some random perturbation following a uniform distribution with parameter η , which represents a form of *temperature* of the system. The authors show that the average velocity of their model exhibits a phase transition around some critical value of the model parameters which include the noise. [37] later provided a theoretical analysis of the convergence to consensus in a *noise-less* version of the Vicsek model. [39] and [15] then reconsidered generalized version of the original Vicsek model and proved conditions under which the system converges to robust form of consensus despite the presence of noise. More recently, in [31], the authors consider a settings in which agents interact uniformly at random by exchanged binary messages which are subject to noise. In detail, the authors provide simple and efficient protocols to solve the classical distributed-computing problems of Broadcast (a.k.a Rumor Spreading) and Majority Consensus, in the Uniform-PUSH model with binary messages, in which each message can be changed upon being received

⁶For stable networks, we here mean a network where communication between agents can be modeled as a classical *channel* the agents can use to exchange messages at will [21].

with probability $1/2 - \epsilon$. Their results have been generalized to the Majority Consensus Problem for the multi-valued case in [32]. When the noise is constant, [31] proves that in their noisy version of the Uniform-PUSH model, the Broadcast Problem can be solved in *logarithmic* time. Rather surprisingly, [13] and [16] prove that solving the Broadcast Problem in the Uniform-PULL model takes linear time, while the time to perform Majority Consensus remains logarithmic in both models.

The fact that real-world systems such as social networks fail to converge to consensus has been extensively studied in various disciplines; formal models developed to investigate the phenomenon include the multiple-state Axelrod model [5] and the bounded-compromise model by Weisbuch et al [49]; the failure to reach consensus in these models is due to the absence of interaction among agent opinions which are “*too far apart*”. A different perspective is offered by models which investigate the effect of stubborn agents (also known as *zealotry* in the literature), in which some *stubborn/zealot* agents never update their opinion.

Several works have been devoted to study such effect under linear models of opinion dynamics, starting with [42] which proposed a statistical-physics method in order to study the VOTER model under the presence of a stubborn agent, followed by [41] which considers the case of several stubborn agents in the system. Later investigations analyzed various aspects of the stationary distribution of the systems, such as [1] which investigates the relationship between the behavior of the opinion dynamics and the structure of the underlying interaction graph, or [50], in which the authors consider the VOTER dynamics and study the first and second moments of the number of the average agents’ opinion.

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Roadmap of the paper. In Section 2, we give some preliminaries and the equivalence result between communication noise and stubborn agents. In Section 3, we provide the probabilistic analysis of the UNDECIDED-STATE process when the initial bias is relatively large and its consequences on almost Majority Consensus. This analysis will be then combined with the analysis of the symmetry-breaking phase given in Section 4 to obtain our results on almost Consensus. For the sake of readability, all the technical

proofs are given in Appendix B (Proofs).

2 Preliminaries

We study the discrete-time, parallel version of the UNDECIDED-STATE dynamics on the complete graph [17]. There is an additional state/opinion, i.e. the *undecided state*, besides the $k \geq 2$ possible opinions an agent can support, and, in the absence of noise, the updating rule works as follows: at every round $t \geq 0, t \in \mathbb{N}$, each agent u chooses a neighbor v (or, possibly, itself) independently and uniformly at random and, at the next round, it gets a new opinion according to the rule given in Table 3⁷. In this work

$u \setminus v$	undecided	opinion i	opinion j
undecided	undecided	i	j
opinion i	i	i	undecided
opinion j	j	undecided	j

Table 3: The update rule of the USD, where $i, j \in [k]$ and $i \neq j$.

we consider the binary case: there are two possible opinions (say opinion *Alpha* and opinion *Beta*) and the undecided state. The definition of noise we consider is the following.

Definition 1 (Definition of noise). Let p be a real number in the interval $(0, 1/2]$. When an agent u chooses a neighbor v and looks at (pulls) its opinion, it sees v 's opinion with probability $1 - 2p$, and, with probability p , it sees one of the two other opinions.

For instance, if v supports opinion *Alpha*, then u sees *Alpha* with probability $1 - 2p$, it sees *Beta* with probability p , and it sees the undecided state with probability p .

2.0.1 Notation, Characterization, and Expected Values.

Let us name C the set of all possible configurations; notice that, since the graph is complete, a configuration $\mathbf{x} \in C$ is uniquely determined by giving the number of *Alpha* nodes, $a(\mathbf{x})$ and the number of *Beta* nodes, $b(\mathbf{x})$. Accordingly to this notation, we call $q(\mathbf{x})$ the number of undecided nodes in configuration \mathbf{x} , and $s(\mathbf{x}) = a(\mathbf{x}) - b(\mathbf{x})$ the *bias* of the configuration \mathbf{x} . When the configuration is clear from the context, we will omit \mathbf{x} and write just a, b, q , and s instead of $a(\mathbf{x}), b(\mathbf{x}), q(\mathbf{x})$, and $s(\mathbf{x})$. The UNDECIDED-STATE dynamics defines a finite-state non reversible Markov chain $\{\mathbf{X}_t\}_{t \geq 0}$ with state space C and no absorbing states.

⁷Notice that this dynamics requires no labeling of the agents, i.e., the network can be anonymous.

We denote the stochastic process that results applying the UNDECIDED-STATE dynamics starting from a given configuration as the UNDECIDED-STATE process. Once a configuration \mathbf{x} at a round $t \geq 0$ is fixed, i.e. $\mathbf{X}_t = \mathbf{x}_t$, we use the capital letters A, B, Q, and S to refer to random variables $a(\mathbf{X}_{t+1})$, $b(\mathbf{X}_{t+1})$, $q(\mathbf{X}_{t+1})$, and $s(\mathbf{X}_{t+1})$.

We will make explicit use of the following expected values:

$$\begin{aligned} \mathbb{E}[A \mid \mathbf{x}] &= \frac{a}{n}(a + 2q)(1 - 2p) \\ &\quad + [a(a + b) + (a + q)(b + q)] \frac{p}{n}, \end{aligned} \quad (1)$$

$$\begin{aligned} \mathbb{E}[B \mid \mathbf{x}] &= \frac{b}{n}(b + 2q)(1 - 2p) \\ &\quad + [b(a + b) + (a + q)(b + q)] \frac{p}{n}, \end{aligned} \quad (2)$$

$$\mathbb{E}[S \mid \mathbf{x}] = s \left(1 - p + (1 - 3p) \frac{q}{n} \right), \quad (3)$$

$$\mathbb{E}[Q \mid \mathbf{x}] = pn + \frac{1 - 3p}{2n} [2q^2 + (n - q)^2 - s^2]. \quad (4)$$

The proof of such expressions can be found in Appendix B.1.

2.0.2 Oblivious Noise and Stubborn Agents.

We can now consider the following more general message-*oblivious* model of noise.

Definition 2. We say that the communication is affected by *oblivious noise* if the value of any sent message changes according to the following scheme:

- (i) with probability $1 - p_{noise}$ independent from the value of the sent message, the message remains unchanged;
- (ii) otherwise, the noise acts on the message and it changes its value according to a fixed distribution $\mathbf{p} = p_1, \dots, p_m$ over the possible message values $1, \dots, m$.

In other words, according to the previous definition of noise (Definition 1), the probability that the noise changes any message to message i is $p_{noise} \cdot p_i$. It is immediate to verify that the definition of noise adopted in our Theorems 1 and 2 corresponds to the aforementioned model of oblivious noise in the special case $p_{noise} = 3p$ and $p_{Alpha} = p_{Beta} = p_{undecided} = \frac{1}{3}$.

Recalling that an agent is said to be *stubborn* if it never updates its state [50], we now observe that the above noise model is in fact equivalent to consider the behavior of the same dynamics *in a noiseless setting with stubborn agents*.

Lemma 1. Consider any discrete-time, synchronous dynamics \mathcal{P} on the complete graph of n agents with opinions (i.e. message values) in $\Sigma = \{1, \dots, m\}$. The following two processes are equivalent⁸:

- (a) the process defined by \mathcal{P} in the presence of oblivious noise with parameters p_{noise} and $\mathbf{p} = p_1, \dots, p_m$;
- (b) the process defined by \mathcal{P} in the presence of $n_{\text{stub}} = \frac{p_{\text{noise}}}{1-p_{\text{noise}}}n$ stubborn agents present in the system, of which $n_{\text{stub}} \cdot p_1$ are stubborn agents supporting opinion 1, $n_{\text{stub}} \cdot p_2$ are stubborn agents supporting opinion 2, and so on.

Proof idea. The equivalence follows by considering a natural coupling between the two processes, in which the events in which v interacts with a stubborn agent with state γ in process (a) correspond to the event in which the noise changes the value of the message to γ in process (b). The full proof is given in Appendix B.1. \square

Basically, this equivalence implies that any result we state for the process defined in (a) has an analogous statement for the process defined in (b).

3 Process analysis for biased initial configurations

In this section, we analyze the UNDECIDED-STATE process when the system starts from biased configurations. The following two theorems show the phase transition exhibited by this process. We remind that our notion of noise is that of Definition 1.

Theorem 1 (Almost Majority Consensus). *Let \mathbf{x} be any initial configuration having bias $s(\mathbf{x}) \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$, and let $\epsilon \in (0, 1/6)$ be some absolute constant. If $p = 1/6 - \epsilon$ is the noise probability, then the UNDECIDED-STATE process reaches a configuration \mathbf{y} having bias $s(\mathbf{y}) \in \Delta = \left[\frac{2\sqrt{\epsilon}}{1+6\epsilon}n, \left(1 - \left(\frac{1-6\epsilon}{12}\right)^3\right)n \right]$ within $\mathcal{O}(\log n)$ rounds, w.h.p. Moreover, starting from such a configuration, within two rounds, the UNDECIDED-STATE process enters a (metastable) phase of length $\Omega(n^\lambda)$ rounds (for some constant $\lambda > 0$)⁹ where the bias remains in the range Δ , w.h.p.*

Observe that if the theorem is true, then it also holds analogously for the symmetrical case in which $s(\mathbf{x}) \leq -\gamma\sqrt{n \log n}$

⁸Formally, by *equivalent* here we mean that there exists a *coupling* between the two processes such that each agent's state is the same in both processes; if some agents (in our case, the stubborn ones) are present in only one process, their state does not play any role in the coupling (in our case this is obvious as stubborn agents never change their state).

⁹The constant λ depends only on the values of ϵ and γ .

Theorem 2 (Victory of Noise). *Let $p = 1/6 + \epsilon$ be the noise probability for some absolute constant $\epsilon \in (0, 1/3]$. Assume the system starts from any configuration \mathbf{x} with bias $s(\mathbf{x})$ such $|s(\mathbf{x})| \geq \gamma\sqrt{n \log n}$ ¹⁰, for some constant $\gamma > 0$. Then, the UNDECIDED-STATE process reaches a configuration \mathbf{y} having bias $|s(\mathbf{y})| = \mathcal{O}(\sqrt{n \log n})$ in $\mathcal{O}(\log n)$ rounds, w.h.p. Furthermore, starting from such a configuration, the UNDECIDED-STATE process enters a (metastable) phase of length $\Omega(n^{\lambda'})$ rounds (for some constant $\lambda' > 0$)¹¹ where the absolute value of the bias keeps bounded by $\mathcal{O}(\sqrt{n \log n})$, w.h.p.*

Let us now consider the equivalent model with stubborn agents according to Lemma 1, in which $p_{noise} = 3p$ and $p_{Alpha} = p_{Beta} = p_{undecided} = \frac{1}{3}$. We thus have $n_{stub} = \frac{3p}{1-3p}n$ additional stubborn nodes, of which $n_{stub} \cdot \frac{1}{3} = \frac{p}{1-3p}n$ support opinion *Alpha*, $n_{stub} \cdot \frac{1}{3} = \frac{p}{1-3p}n$ opinion *Beta*, and $n_{stub} \cdot \frac{1}{3} = \frac{p}{1-3p}n$ are undecided. On this new graph of $n + n_{stub}$ nodes¹², let the UNDECIDED-STATE dynamics run and call the resulting process the STUB process. The next result is an immediate corollary of the two previous theorems.

Corollary 1. *Let $\frac{1}{2} > p > 0$ be a constant, and let the STUB process start from any configuration having bias $s \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$. If $p < \frac{1}{6}$, then the STUB process enters in $\mathcal{O}(\log n)$ rounds a metastable phase of almost consensus of length $\Omega(n^\lambda)$ for some constant $\lambda > 0$, w.h.p., in which the bias is $\Theta(n)$. If $p \in (\frac{1}{6}, \frac{1}{2}]$, then the STUB process enters in $\mathcal{O}(\log n)$ rounds a metastable phase of length $\Omega(n^{\lambda'})$ for some constant $\lambda' > 0$ where the absolute value of the bias keeps bounded by $\mathcal{O}(\sqrt{n \log n})$, w.h.p.*

Trivially, the corollary holds also in the symmetrical case in which $s \leq -\gamma\sqrt{n \log n}$.

3.1 Proof of Theorem 1

Mostly, the proofs consist in estimations on the expectations of the random variables involved in the process, and probability concentration arguments to derive events which hold w.h.p. Throughout the analysis, for a given configuration \mathbf{x} , we will suppose $a(\mathbf{x}) \geq b(\mathbf{x})$ for the sake of simplicity. Indeed, as it will be clear from the results, if $s(\mathbf{x}) \geq \gamma\sqrt{n \log n}$, then the plurality opinion does not change for $\Omega(n^\lambda)$ rounds, w.h.p., and the argument for the case $b(\mathbf{x}) > a(\mathbf{x})$ is symmetric.

Let $\epsilon \in (0, 1/6)$ and $p = 1/6 - \epsilon$. Observe that Equations (3) and (4) become

$$\mathbb{E}[S \mid \mathbf{x}] = s \left(\frac{5}{6} + \epsilon + \frac{1}{2}(1 + 6\epsilon)\frac{q}{n} \right), \quad (5)$$

¹⁰So, even with large bias, say $|s(\mathbf{x})| = \Theta(n)$.

¹¹The constant λ' depends only on the values of ϵ and γ .

¹²See the proof of Lemma 1 for more details on the new graph.

$$\begin{aligned}\mathbb{E}[Q \mid \mathbf{x}] &= \frac{3}{4} \left(\frac{1+6\epsilon}{n} \right) q^2 - \frac{1+6\epsilon}{2} q + \frac{5+6\epsilon}{12} n \\ &\quad - \frac{1+6\epsilon}{n} \left(\frac{s}{2} \right)^2.\end{aligned}\tag{6}$$

The key-point to prove the first part of the theorem is showing that, if the bias of the configuration is under some quantity βn (for some accurately chosen constant β), and the number of undecided nodes is some other constant factor of n , then the bias grows exponentially fast each round, w.h.p. At the same time, as long as the bias is below βn , the number of undecided nodes in the next round is sufficiently large, w.h.p.

Lemma 2. *Let \mathbf{x} be a configuration such that $q \geq \frac{1-4\epsilon}{3(1+6\epsilon)}n$ and $s \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$. Then, in the next round, $S \geq s \left(1 + \frac{\epsilon}{6}\right)$, w.h.p.*

Let now $\beta = \frac{2\sqrt{3\epsilon}}{1+6\epsilon}$ be a constant. Then the following holds.

Lemma 3. *Let \mathbf{x} be a configuration such that $s \leq \beta n$. Then, in the next round, $Q \geq \frac{1-4\epsilon}{3(1+6\epsilon)}n$ w.h.p.*

It is easy to see that, if the system lies in any configuration with $q \geq \frac{1-4\epsilon}{3(1+6\epsilon)}n$ and $s \in [\gamma\sqrt{n \log n}, \beta n]$ for some $\gamma > 0$, then Lemmas 2 and 3 ensure that it reaches a configuration \mathbf{y} with bias $s(\mathbf{y}) > \beta n$ within $\mathcal{O}(\log n)$ rounds, w.h.p.¹³

We now consider configurations in which $q < \frac{1-4\epsilon}{3(1+6\epsilon)}n$ or $s > \beta n$, and we derive a useful bound on the possible decrease of s .

Lemma 4. *Let \mathbf{x} be any configuration such that $s \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$. Then, in the next round, it holds that $S \geq s \left(\frac{5}{6} + \frac{\epsilon}{2}\right)$ w.h.p.*

Observe that

$$\left(\frac{5}{6} + \frac{\epsilon}{2}\right)^2 \cdot \beta n \geq \frac{2\sqrt{\epsilon}}{1+6\epsilon}n$$

which means that, if at some round the bias is such that $s \geq \beta n$, and in the next round it moves to the interval $\left[\frac{2\sqrt{\epsilon}}{1+6\epsilon}n, \beta n\right]$ (it cannot go below that interval), then it remains in that range for at least another round w.h.p., thus guaranteeing that the number of undecided nodes becomes at least $\frac{1-4\epsilon}{3(1+6\epsilon)}n$ w.h.p. Then, in another round, the bias increases again w.h.p. (thanks to Lemmas 2 and 3).

If the initial configuration is such that $s \geq \gamma\sqrt{n \log n}$ but $q < \frac{1-4\epsilon}{3(1+6\epsilon)}n$, then in the next round $S \geq \gamma'\sqrt{n \log n}$ for some constant $\gamma' > 0$ (Lemma 4), and $Q \geq \frac{1-4\epsilon}{3(1+6\epsilon)}n$ (Lemma 3), w.h.p. Hence, the bias increases each round w.h.p. as long as it is still less than βn (Lemmas 2 and 3).

¹³As follows from Footnote 14, the intersection of $\mathcal{O}(\log n)$ events which hold w.h.p. is still an event which holds w.h.p.

The previous four lemmas are sufficient to show that starting from any initial configuration having bias $s \geq \gamma\sqrt{n \log n}$, for some positive constant $\gamma > 0$, within $\mathcal{O}(\log n)$ rounds the process “enters” w.h.p. a metastable almost consensus phase which lasts $\Omega(n^\lambda)$ rounds w.h.p. (see the proof of Theorem 1) for some $\lambda > 0$, in which the bias remains at least $\frac{2\sqrt{\epsilon}}{1+6\epsilon}n$. Actually, this value is a lower bound of the range which the bias “lives in” during this metastable phase. Through the next two lemmas, we provide the upper bound of this range.

Lemma 5. *Let \mathbf{x} be any configuration. It holds that, in the next round, $Q \geq \frac{n}{12}(1 - 6\epsilon)$ w.h.p.*

Now we simply have to notice that the number of nodes supporting the minority opinion is at least the number of undecided nodes who receive it, after having pulled some opinion, either because of the noise or not. Next lemma shows that the minority opinion will be supported by at least $\Theta(n)$ nodes w.h.p., after two rounds from the start of the process; hence, the bias of the system cannot reach the value $n - o(n)$.

Lemma 6. *Let \mathbf{x} be a configuration with $q \geq \frac{n}{12}(1 - 6\epsilon)$. Then, in the next round, $B \geq \left(\frac{1-6\epsilon}{12}\right)^3 n$ w.h.p.*

Proof of Theorem 1. The proof follows from a simple combination of the previous lemmas, observing that the intersection of n^λ events $\{\xi_i\}_{i \in I}$ which hold w.h.p. (say, $1 - n^{-\eta}$, with $\lambda < \eta$) is still an event which holds w.h.p.¹⁴ Furthermore, since η depends only on ϵ and γ , so does λ . \square

3.2 Proof of Theorem 2

We now assume the starting configuration \mathbf{x} to have bias $s(\mathbf{x}) = a(\mathbf{x}) - b(\mathbf{x}) \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$; the case in which $b(\mathbf{x}) > a(\mathbf{x})$ is analogous. Let $\epsilon \in (0, \frac{1}{3}]$ be a constant, and $p = 1/6 + \epsilon$ be the probability of noise. Equations (3) and (4) become

$$\mathbb{E}[S \mid \mathbf{x}] = s \left(\frac{5}{6} - \epsilon + \frac{1}{2}(1 - 6\epsilon)\frac{q}{n} \right), \quad (7)$$

$$\begin{aligned} \mathbb{E}[Q \mid \mathbf{x}] &= \frac{3}{4} \left(\frac{1 - 6\epsilon}{n} \right) q^2 - \frac{1 - 6\epsilon}{2} q + \frac{5 - 6\epsilon}{12} n \\ &\quad - \frac{1 - 6\epsilon}{n} \left(\frac{s}{2} \right)^2. \end{aligned} \quad (8)$$

From Equation (7) it is clear that the bias decreases in expectation exponentially fast each round as long as $\epsilon \geq 1/6$ (actually, $\epsilon > 1/12$ is enough) or $q/n < 1/3 \cdot (1 + 6\epsilon)/(1 - 6\epsilon)$. We analyze two cases: $\epsilon > 1/12$ and $\epsilon \leq 1/12$.

¹⁴By the union bound, $\Pr(\cap_{i \in I} \xi_i) = 1 - \Pr(\cup_{i \in I} \bar{\xi}_i) \geq 1 - \sum_{i \in I} n^{-\eta} = 1 - n^{\lambda - \eta} \geq 1 - n^{-\delta}$, where $\bar{\xi}_i$ denotes the negation of ξ_i and $\delta = \frac{\eta - \lambda}{2}$.

3.2.1 First Case: $\epsilon > \frac{1}{12}$ Large Epsilon

We first show a bound on the decrease of the bias.

Lemma 7. *Let \mathbf{x} be a configuration such that $s \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$. Then, in the next round, it holds that $S \leq s(1 - 2\epsilon + \frac{1}{6})$ w.h.p.*

We then obtain the first part of Theorem 2 by using a simple symmetric argument for configurations such that $b > a$, and by observing that the intersection of $n^{\lambda'}$ events $\{\xi_i\}_{i \in I}$ which hold w.h.p. (say, $1 - n^{-\eta}$, with $\lambda' < \eta$) is still an event which holds w.h.p. (see Footnote 14).

3.2.2 Second Case: $\epsilon \in (0, \frac{1}{12}]$ Small Epsilon

The following lemma states that, if $s \geq \frac{2}{3}n$, the bias decreases exponentially at the next round, w.h.p. On the other hand, if the bias is at most $\frac{2}{3}n$, it cannot grow over $\frac{2}{3}n$, w.h.p.

Lemma 8. *Let \mathbf{x} be any configuration. The followings hold:*

- (1) *if $s \geq \frac{2}{3}n$, then $S \leq s(1 - \epsilon)$ w.h.p.;*
- (2) *if $s \leq \frac{2}{3}n$, then $S \leq \frac{2}{3}n$ w.h.p.*

Thus, we just have to take care of cases in which the bias is no more than $\frac{2}{3}n$. The key-point to show the decrease of the bias, as long as it is $\Omega(\sqrt{n \log n})$, is the condition $q \leq \frac{n(1+3\epsilon)}{3(1-6\epsilon)}$, as shown in the next lemma.

Lemma 9. *Let \mathbf{x} be a configuration such that $s \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$. If $q \leq \frac{n}{3} \left(\frac{1+3\epsilon}{1-6\epsilon} \right)$, then in the next round it holds that $S \leq s(1 - \frac{\epsilon}{4})$ w.h.p.*

We now analyze the dynamics by partitioning the interval $(0, \frac{2}{3}n]$ and seeing what happens to the bias in each element of the partition. Let $\beta = \frac{2\sqrt{2\epsilon}}{\sqrt{(1+6\epsilon)(1-6\epsilon)}}$ and define $S_{-1} := (0, \beta n]$, S_i the sequence of intervals

$$S_i := \left(\left(\frac{3}{2} \right)^i \beta n, \left(\frac{3}{2} \right)^{i+1} \beta n \right]$$

for $i = 0, 1, \dots, k-2$ where $k = \left\lceil \log_{\frac{3}{2}}(\beta) - 1 \right\rceil$, and $S_{k-1} := \left(\left(\frac{3}{2} \right)^{k-1} \beta n, \frac{2}{3}n \right]$. Furthermore, just for completeness, we define $S_k := \left(\frac{2}{3}n, n \right]$. In the next lemmas, we show that as long as $s \in S_i$ for $i = -1, \dots, k-1$, $s = \Omega(\sqrt{n \log n})$, and $\bar{q}_{i+1} \leq q \leq \frac{n}{3} \left(\frac{1+3\epsilon}{1-6\epsilon} \right)$ for some decreasing sequence $\bar{q}_{-1}, \dots, \bar{q}_k$ accurately chosen, then, at the next round, the bias decreases exponentially w.h.p. and

the number of undecided nodes moves to the interval $\left[\bar{q}_i, \frac{n(1+3\epsilon)}{3(1-6\epsilon)}\right]$ w.h.p. Note that since ϵ is a constant, so is k . The following lemma determines the sequence \bar{q}_i .

Lemma 10. *Let \mathbf{x} be any configuration.*

1. *If $-1 \leq i \leq k-1$ and $s \leq \left(\frac{3}{2}\right)^{i+1} \beta n$, it holds that $Q \geq \frac{n}{3} - \frac{2n\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2i+3}$ w.h.p.;*
2. *If $s \leq \frac{2}{3}n$, it holds that $Q \geq \frac{2}{9}n + \frac{\epsilon}{3}n$ w.h.p.;*
3. *Without any condition on s , it holds that $Q \geq \frac{n}{12} + \epsilon n$ w.h.p.*

Define $\bar{q}_i := n \left(\frac{1}{3} - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2i+3}\right)$ for $i = -1, \dots, k-2$, $\bar{q}_{k-1} := \frac{2}{9}n + \frac{\epsilon}{3}n$, and $\bar{q}_k := \frac{n}{12} + \epsilon n$, and notice that they form a decreasing sequence. Next, with few lemmas, we take care of controlling the behaviour of the number of undecided nodes when $s > \inf(S_i)$ for $-1 \leq i \leq k-1$.

Lemma 11. *Let $-1 \leq i \leq k-1$ and let \mathbf{x} be a configuration such that $\bar{q}_{i+1} \leq q \leq \frac{n(1+3\epsilon)}{3(1-6\epsilon)}$ and $s > \inf(S_i)$. Then, at the next round, $\bar{q}_i \leq Q \leq \frac{n(1+3\epsilon)}{3(1-6\epsilon)}$ w.h.p.*

At the same time, the following lemma implies that the possible decrease of the bias cannot move it from S_i beyond S_{i-1} .

Lemma 12. *Let \mathbf{x} be a configuration such that $s \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$. If $s > \left(\frac{3}{2}\right)^i \beta n$ for some $0 \leq i \leq k$, then $S > \inf(S_{i-1})$ w.h.p.*

We still need to “control” the dynamics, in particular the case in which $q > \frac{n(1+3\epsilon)}{3(1-6\epsilon)}$. The next lemma fulfill this need. It shows that there is a decrease of the number of undecided nodes when they are more than $\frac{n(1+3\epsilon)}{3(1-6\epsilon)}$, and provides a lower bound on the decrease, depending on the bias.

Lemma 13. *Let \mathbf{x} be a configuration such that $q \geq \frac{n}{3(1-6\epsilon)}$. Then, it holds that*

- (1) $Q \leq q(1 - \epsilon)$ w.h.p.;
- (2) *if $s \leq \sup(S_i)$ for some $-1 \leq i \leq k-1$, then $Q \geq \bar{q}_{i+1}$ w.h.p.*

Next and last lemma guarantees that once the process reaches a configuration having bias $\mathcal{O}(\sqrt{n \log n})$, then it “enters” a metastable phase that lasts $\Omega(n^{\lambda'})$ rounds w.h.p. in which the absolute value of the bias remains $\mathcal{O}(\sqrt{n \log n})$, since it can be used symmetrically when the bias is negative.

Lemma 14. *Let \mathbf{x} be any configuration. If $s \leq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$ and $q \leq \frac{n}{3} \left(\frac{1+3\epsilon}{1-6\epsilon}\right)$, then $S \leq 2\gamma\sqrt{n \log n}$ w.h.p.*

Proof of Theorem 2. If $\frac{1}{12} < \epsilon$ the theorem is true due to Lemma 7. Let us assume $0 < \epsilon \leq \frac{1}{12}$. The proof is divided into different cases¹⁵. Let $\gamma > 0$ be any constant.

(1) $s \in S_i$ for some $-1 \leq i \leq k-1$ and $s \geq \gamma\sqrt{n \log n}$,

(1.1) $\bar{q}_{i+1} \leq q \leq \frac{n(1+3\epsilon)}{3(1-6\epsilon)}$: the bias decreases exponentially fast each round, w.h.p., until $0 \leq s \leq \gamma\sqrt{n \log n}$ due to the combination of Lemmas 9, 10, 11, and 12. This phase lasts $O(\log n)$ rounds;

(1.2) $\frac{n(1+3\epsilon)}{3(1-6\epsilon)} < q$: Lemmas 13 and 10 imply that in $O(\log n)$ rounds the number of undecided nodes reaches the interval $\left[\bar{q}_{i+1}, \frac{n}{3(1-6\epsilon)}\right]$ where i is such that the round before the undecided nodes become less than $\frac{n}{3(1-6\epsilon)}$, the bias is in S_i : remind that during the whole process (which lasts $O(\log n)$ rounds) the bias never goes over $\frac{2}{3}n$ thanks to Lemma 8. At the same time, the bias will be in one set between S_{i-1}, \dots, S_{k-1} due to Lemma 12. Since \bar{q}_i is a decreasing sequence, we are in Case 1.1, and we conclude.

(1.3) $q < \bar{q}_{i+1}$: in this case, Lemma 10 implies $Q \geq \bar{q}_{i+1}$ in the next round, w.h.p. Since Lemma 8 guarantees that the bias remains under the value $\frac{2}{3}n$ w.h.p., either we are in Case 1.1 or in Case 1.2, and we conclude.

(2) $s > \frac{2}{3}n$: Lemma 8 implies that the bias gets less than or equal to $\frac{2}{3}n$ in $O(\log n)$ rounds, w.h.p.; then we are in Case 1 and we conclude.

Now, we can suppose the process starts from a configuration \mathbf{y} having bias $0 \leq s(\mathbf{y}) \leq \gamma\sqrt{n \log n}$, and such that $\bar{q}_0 \leq q(\mathbf{y}) \leq \frac{n}{3} \left(\frac{1+3\epsilon}{1-6\epsilon}\right)$, as Case 1.1 or 1.3 leaves it. In the next round, it holds that the number of undecided nodes is $\bar{q}_{-1} \leq Q \leq \frac{n}{3} \left(\frac{1+3\epsilon}{1-6\epsilon}\right)$, w.h.p., due to Lemma 11; at the same time, w.h.p., $|S| \leq 2\gamma\sqrt{n \log n}$ for Lemma 14 (which can be used symmetrically on $A - B$ and $B - A$). Thus, the absolute value of the bias is either still less than $\gamma\sqrt{n \log n}$ or has become greater than or equal to $|S| \geq \gamma\sqrt{n \log n}$, in which case it starts decreasing exponentially fast each round, w.h.p., for Lemma 9 until becoming again less than $\gamma\sqrt{n \log n}$ (as explained in Case 1.1, which works analogously if the bias is negative, because of symmetry). This phase lasts $\Omega(n^{\lambda'})$ rounds, for some sufficiently small constant $\lambda' > 0$ ¹⁶. \square

¹⁵Recall that the intersection of $n^{\lambda'}$ events $\{\xi_i\}_{i \in I}$ which hold w.h.p. (say, $1 - n^{-\eta}$, with $\eta > \lambda'$), is still an event which holds w.h.p. (see Footnote 14).

¹⁶We refer to the proof of Theorem 3.1 and to Footnote 14 for this statement. As in the proof of Theorem 1, λ' depends only on γ and ϵ .

4 Symmetry Breaking from Balanced Configurations

In the previous sections, we considered the UNDECIDED-STATE process starting from biased configurations and we established a phase-transition on its ability to compute almost Majority Consensus which depends on the communication noise parameter p . In this section, we consider the UNDECIDED-STATE process starting from arbitrary initial configurations: in particular, from configurations having no bias. Interestingly enough, we will show that a similar transition phase holds in this setting as well. Informally, the next theorem states that when $p < 1/6$, the UNDECIDED-STATE process is able to break the symmetry of any perfectly-balanced initial configuration and to compute almost Consensus within $O(\log n)$ rounds, w.h.p.

Theorem 3. *Let \mathbf{x} be any initial configuration, and let $\epsilon \in (0, 1/6)$ be some absolute constant. If $p = 1/6 - \epsilon$ is the noise probability, then the UNDECIDED-STATE process reaches a configuration \mathbf{y} having bias s toward some opinion $j \in \{\text{Alpha}, \text{Beta}\}$ such that $|s(\mathbf{y})| \in \Delta = \left[\frac{2\sqrt{\epsilon}}{1+6\epsilon}n, \left(1 - \left(\frac{1-6\epsilon}{12}\right)^3\right)n \right]$ within $O(\log n)$ rounds, w.h.p. Moreover, once reached configuration \mathbf{y} , within two rounds, the UNDECIDED-STATE process enters a (metastable) phase of length $\Omega(n^\lambda)$ rounds (for some constant $\lambda > 0$) where the majority opinion is j and the bias keeps within the range Δ , w.h.p.*

What follows is an outline of the proof of the theorem, while more details are given in the next subsection.

Outline of Proof of Theorem 3. If the initial configuration \mathbf{x} has bias $s = \Omega(\sqrt{n \log n})$ then the claim of the theorem is equivalent to that of Theorem 1, so we are done. Hence, we next assume the initial bias s be $o(\sqrt{n \log n})$: for this case, our proof proceeds along the following main steps.

Step I. Whenever the bias s is small, i.e. $o(n)$, we prove that, within the next $O(\log n)$ rounds, the number of undecided nodes turns out to keep always in a suitable linear range: roughly speaking, we get that this number lies in $(n/3, n/2]$, w.h.p.

Step II. Whenever s is very small, i.e. $s = o(\sqrt{n})$, there is no effective drift towards any opinion. However, we can prove that, thanks to Step I, the random variable S , representing the bias in the next round, has high variance, i.e. $\Theta(n)$. This is essentially due to the fact that S can be written as a suitable sum whose addends include some random variables having binomial distribution of expectation 0: so, we can apply the Berry-Essen Theorem (Theorem 6 in Appendix A) to get a lower bound on the variance of S . Then, using the above fact, standard arguments for the standard deviation imply that, in this parameter range, there is a positive constant probability that S will get some value of magnitude $\Omega(\sqrt{n})$ (see Claim 1 of Lemma 17).

Not surprisingly, in this phase, we find out that the variance of S is not decreased by the communication noise. We can thus claim that the process, at every round, has positive constant probability to reach a configuration having bias $s = \omega(\sqrt{n})$ and $q \in (n/3, n/2]$. Then, after $O(\log n)$ rounds, this event will happen w.h.p.

Step III. Once the process reaches a configuration with $s = \omega(\sqrt{n})$ and $q \in (n/3, n/2]$, we then prove that the expected bias increases by a constant factor (which depends on ϵ). Observe that we cannot use here the same round-by-round concentration argument that works for bias over $\sqrt{n \log n}$ (this is in fact the minimal magnitude required to apply the Chernoff's bounds [28]). We instead exploit a useful general tool [17] that bounds the stopping time of some class of Markov chains having rather mild conditions on the drift towards their absorbing states (see Lemma 15). This tool in fact allows us to consider the two phases described, respectively, in Step II and Step III as a unique *symmetry-breaking* phase of the process. Our main technical contribution here is to show that the conditions required to apply this tool hold whenever the communication noise parameter is such that $p \in (0, 1/6)$. This allows us to prove that, within $O(\log n)$ rounds, the process reaches a configuration with bias $s = \Omega(\sqrt{n \log n})$, w.h.p. □

Large communication noise (the case $p > 1/6 + \epsilon$). On the other hand, when $p > 1/6 + \epsilon$, Theorem 2 a fortiori holds when the initial bias is small, i.e. $s = o(\sqrt{n \log n})$: thus, we get that, in this case, the system enters into a long regime of non consensus, starting from any initial configuration. Then, by combining the results for biased configurations in Section 3 with those in this section, we can observe the phase transition of the UNDECIDED-STATE process starting from any possible initial configuration.

Stubborn agents. We conclude this section by observing that the equivalence result shown in Lemma 1 holds independently of the choices of the noise parameter $p \in (0, 1/2)$, and of the initial bias: the phase transition of the UNDECIDED-STATE process in the presence of stubborn agents thus holds even in the case of unbiased configurations.

4.1 Proof of Theorem 3: More details

The proof of Theorem 3 essentially relies on the following lemma which has been proved in [17], whose proof we report in Appendix B.4.

Lemma 15. *Let $\{X_t\}_{t \in \mathbb{N}}$ be a Markov Chain with finite-state space Ω and let $f : \Omega \mapsto [0, n]$ be a function that maps states to integer values. Let c_3 be any positive constant and let $m = c_3 \sqrt{n \log n}$ be a target value. Assume the following properties hold:*

(1) for any positive constant h , a positive constant $c_1 < 1$ exists such that for any $x \in \Omega : f(x) < m$,

$$\mathbb{P}\{f(X_{t+1}) < h\sqrt{n} \mid X_t = x\} < c_1;$$

(2) there exist two positive constants δ and c_2 such that for any $x \in \Omega : h\sqrt{n} \leq f(x) < m$,

$$\mathbb{P}\{f(X_{t+1}) < (1 + \delta)f(X_t) \mid X_t = x\} < e^{-c_2 f(x)^2/n}.$$

Then the process reaches a state x such that $f(x) \geq m$ within $\mathcal{O}(\log n)$ rounds, w.h.p.

Our goal is to apply the above lemma to the UNDECIDED-STATE process (which defines a finite-state Markov chain) starting with bias of size $o(\sqrt{n \log n})$ where we set $f(\mathbf{X}_t) = s(\mathbf{X}_t)$, $c_3 = \gamma > 0$ for some constant $\gamma > 0$, and $m = \gamma\sqrt{n} \log n$: this would imply the upper bound $\mathcal{O}(\log n)$ on the number of rounds needed to reach a configuration having bias $\Omega(\sqrt{n \log n})$, w.h.p., breaking the symmetry because Theorem 1 then holds. To this aim, with the next two lemmas we show that the UNDECIDED-STATE process satisfies the hypotheses of Lemma 15 in this setting, w.h.p.

Lemma 16. *Let \mathbf{x} be any configuration in which $s \leq \beta n$ and $\frac{n}{3} \left(\frac{1-4\epsilon}{1+6\epsilon} \right) \leq q \leq \frac{n}{2}$. Then, in the next round, it holds that $\frac{n}{3} \left(\frac{1-4\epsilon}{1+6\epsilon} \right) \leq Q \leq \frac{n}{2}$ w.h.p.*

Lemma 17. *Let \mathbf{x} be any configuration such that $q(\mathbf{x}) \in \left[\frac{n}{3} \left(\frac{1-4\epsilon}{1+6\epsilon} \right), \frac{n}{2} \right]$. Then, it holds that*

(1) for any constant $h > 0$ there exists a constant $c_1 > 0$ such that

$$\mathbb{P}(|S| < h\sqrt{n} \mid \mathbf{X}_t = \mathbf{x}) < c_1;$$

(2) there exist two positive constants δ and c_2 such that

$$\mathbb{P}(|S| \geq (1 + \delta)s \mid \mathbf{X}_t = \mathbf{x}) \geq 1 - e^{-c_2 \frac{s^2}{n}}.$$

The reader may notice that Lemma 16 requires the number of undecided nodes to be inside the interval $\left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2} \right]$. We will later take care of this issue with Lemmas 20 and 21, showing that whenever this number is not within the above interval, in at most $\mathcal{O}(\log n)$ rounds it will lie in it. Furthermore, Lemma 16 guarantees that the condition on the undecided nodes holds “only” w.h.p., while Lemma 15 requires this condition to hold with probability 1. We show this issue can be solved using a coupling argument similar to that in [17]. The key point is that, starting from any configuration

\mathbf{x} with $q(\mathbf{x}) \in \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2}\right]$, the probability that the process goes in one of those “bad” configurations with q outside the above interval is negligible. Intuitively speaking, the configurations *actually visited* by the process before breaking symmetry do satisfy the hypothesis of Lemma 15. In order to make this argument rigorous, we define a *pruned* process, by removing all the *unwanted* transitions.

Let $\bar{s} \in \{0, 1, \dots, n\}$, and $\mathbf{z}(\bar{s})$ the configuration such that $s(\mathbf{z}(\bar{s})) = \bar{s}$, and $q(\mathbf{z}(\bar{s})) = \frac{n}{2}$. Let $p_{\mathbf{x}, \mathbf{y}}$ be the probability of a transition from the configuration \mathbf{x} to the configuration \mathbf{y} in the UNDECIDED-STATE process. The PRUNED process behaves exactly as the original process but every transition from a configuration \mathbf{x} such that $q(\mathbf{x}) \in \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2}\right]$ and $s(\mathbf{x}) = \mathcal{O}(\sqrt{n \log n})$ to a configuration \mathbf{y} such that $q(\mathbf{y}) < \frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}$ or $q(\mathbf{y}) > \frac{n}{2}$ has probability $p'_{\mathbf{x}, \mathbf{y}} = 0$. Moreover, for any $\bar{s} \in [n]$, starting from the configuration \mathbf{x} , the probability of reaching the configuration $\mathbf{z}(\bar{s})$ is

$$p'_{\mathbf{x}, \mathbf{z}(\bar{s})} = p_{\mathbf{x}, \mathbf{z}(\bar{s})} + \sum_{\substack{\mathbf{y}: s(\mathbf{y})=\bar{s} \text{ and} \\ q(\mathbf{y}) \notin \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2}\right]}} p_{\mathbf{x}, \mathbf{y}}.$$

All the other transition probabilities remain the same. Observe that the UNDECIDED-STATE process is defined in such a way that it has exactly the same marginal probability of the original process w.r.t. the random variable $s(\mathbf{X}_t)$; thus, Lemma 17 holds for the PRUNED process as well and we can apply Lemma 15. Then, the PRUNED process reaches a configuration having bias $\Omega(\sqrt{n \log n})$ within $\mathcal{O}(\log n)$ rounds, w.h.p., as shown in the following lemma.

Lemma 18. *Starting from any configuration \mathbf{x} such that $q(\mathbf{x}) \in \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2}\right]$ and $s(\mathbf{x}) = \mathcal{O}(\sqrt{n \log n})$, the PRUNED process reaches a configuration having bias $\Omega(\sqrt{n \log n})$ within $\mathcal{O}(\log n)$ rounds, w.h.p.*

We now want to go back to the original process. The definition of the PRUNED process suggests a natural coupling between it and the original one. If the two process are in different states, then they act independently, while, if they are in the same state \mathbf{x} , they move together unless the UNDECIDED-STATE process goes in a configuration \mathbf{y} such that $q(\mathbf{y}) \notin \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2}\right]$. In that case, the PRUNED process goes in $\mathbf{z}(s(\mathbf{y}))$. In the proof of the next lemma, we show that the time the PRUNED process takes to reach bias $\Omega(\sqrt{n \log n})$ stochastically dominates the one of the original process, giving the result.

Lemma 19. *Starting from any configuration \mathbf{x} such that $q(\mathbf{x}) \in \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2}\right]$ and $s(\mathbf{x}) = \mathcal{O}(\sqrt{n \log n})$, the UNDECIDED-STATE process reaches a configuration having bias $\Omega(\sqrt{n \log n})$ within $\mathcal{O}(\log n)$ rounds, w.h.p.*

Now, we take care of those cases in which the starting configuration is such that $q \notin \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2} \right]$. Indeed, if $q < \frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}$, the following holds.

Lemma 20. *Let \mathbf{x} be any starting configuration such that $q(\mathbf{x}) \leq \frac{n}{2}$, and $s(\mathbf{x}) \leq \frac{2\epsilon}{(1+6\epsilon)^2}n$. Then, at the next round, it holds that $q(\mathbf{x}) \in \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2} \right]$, w.h.p.*

On the other hand, the number of undecided nodes decreases as long as it is more than $n/2$, w.h.p. The following lemma shows this behaviour.

Lemma 21. *Let \mathbf{x} be any starting configuration such that $q(\mathbf{x}) > \frac{n}{2}$. Then, at the next round, it holds that $Q \leq q\left(\frac{5}{6} + \epsilon\right)$, w.h.p.*

Proof of Theorem 3: Wrap-Up. Let $\gamma > 0$ be a constant and $m = \gamma\sqrt{n} \log n$ be the target value of the bias in Lemma 15. Let \mathbf{x} be any initial configuration having bias $|s| < m$. We have two cases.

- (i) If the number of undecided nodes is such that $q(\mathbf{x}) \in \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2} \right]$, then Lemma 19 implies that the UNDECIDED-STATE process reaches a configuration having bias $\Omega(\sqrt{n \log n})$ in $\mathcal{O}(\log n)$ rounds, w.h.p.;
- (ii) else, if the starting configuration is such that $q(\mathbf{x}) \notin \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2} \right]$, then, for Lemmas 20 and 21, the UNDECIDED-STATE process reaches within $\mathcal{O}(\log n)$ rounds a configuration \mathbf{y} having $q(\mathbf{y}) \in \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2} \right]$, w.h.p. Then, either $s(\mathbf{y}) = \Omega(\sqrt{n \log n})$, or we are in case (i)¹⁷.

Then, Theorem 1 gives the desired result. □

5 Conclusions

This work contributes to the research endeavour of exploring the interplay between communication noise and stochastic interaction pattern in MAS. As discussed in the Related Work, despite the fact that these two characteristics are quite common in real-world MAS, their combined effect is still far from being understood and poses novel mathematical challenges. Within such framework, in this we have identified and rigorously analyzed a phase transition behaviour of the famous UNDECIDED-STATE process in the presence of communication noise (or, of stubborn agents) on the complete graph. While our mathematical analysis doesn't directly apply to other opinion dynamics, it suggests that a general phase transition phenomenon may hold for a large class of dynamics characterized by an *exponential drift* towards consensus configurations. Our work thus naturally poses the general question

¹⁷As Footnote 14 implies, the intersection of $\mathcal{O}(\log n)$ events that hold w.h.p. is an event which holds w.h.p.

of whether it is possible to provide a characterization of opinion dynamics with stochastic interactions, in terms of their critical behavior with respect to uniform communication noise.

As for the specific mathematical questions that follows from our result, our assumption of a complete topology as underlying communication graph is, for several real MAS, a rather strong condition. However, two remarks on this issue follows. On one hand, we observe that, according to the adopted communication model, at every round, every agent can pull information from just one other agent: the *dynamic* communication pattern is thus random and sparse. This setting may model opportunistic MAS where mobile agent use to meet randomly, at a relatively-high rate. On the other hand, we believe that a similar transition phase does hold even for sparse topologies having good expansion/conductance [36]: this is an interesting open issue.

A Useful Tools

Here we present the concentration results we have used all over the analysis. For an overview on the forms of Chernoff bounds see [28] or [26].

Theorem 4 (Multiplicative forms of Chernoff bounds). *Let X_1, X_2, \dots, X_n be independent $\{0, 1\}$ random variables. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then:*

(i) *for any $\delta \in (0, 1)$ and $\mu \leq \mu_+ \leq n$, it holds that*

$$P(X \geq (1 + \delta)\mu_+) \leq e^{-\frac{1}{3}\delta^2\mu_+}, \quad (9)$$

(ii) *for any $\delta \in (0, 1)$ and $0 \leq \mu_- \leq \mu$, it holds that*

$$P(X \leq (1 - \delta)\mu_-) \leq e^{-\frac{1}{2}\delta^2\mu_-}. \quad (10)$$

Theorem 5 (Additive forms of Chernoff bounds). *Let X_1, X_2, \dots, X_n be independent $\{0, 1\}$ random variables. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then:*

(i) *for any $0 < \lambda < n$ and $\mu \leq \mu_+ \leq n$, it holds that*

$$P(X \geq \mu_+ + \lambda) \leq e^{-\frac{2}{n}\lambda^2}, \quad (11)$$

(ii) *for any $0 < \lambda < \mu_-$ and $0 \leq \mu_- \leq \mu$, it holds that*

$$P(X \leq \mu_- - \lambda) \leq e^{-\frac{2}{n}\lambda^2}. \quad (12)$$

The Berry-Eseen theorem is well treated in [38], and it gives an estimation on “how far” is the distribution of the normalized sum of i.i.d. random variables to the standard normal distribution.

Theorem 6 (Berry-Eseen). *Let X_1, \dots, X_n be n i.i.d. (either discrete or continuous) random variables with zero mean, variance $\sigma^2 > 0$, and finite third moment. Let Z the standard normal random variable, with zero mean and variance equal to 1. Let $F_n(x)$ be the cumulative function of $\frac{S_n}{\sigma\sqrt{n}}$, where $S_n = \sum_{i=1}^n X_i$, and $\Phi(x)$ that of Z . Then, there exists a positive constant $C > 0$ such that*

$$\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \leq \frac{C}{\sqrt{n}}$$

for all $n \geq 1$.

B Proofs

B.1 Proofs: Preliminaries

Equations 3 and 4.

$$\begin{aligned} \mathbb{E}[S \mid \mathbf{x}] &= \mathbb{E}[A \mid \mathbf{x}] - \mathbb{E}[B \mid \mathbf{x}] \\ &= \frac{1-2p}{n} [s(a+b) + 2qs] + \frac{p}{n} [s(a+b)] \\ &= s \frac{1-2p}{n} [n+q] + s \frac{p}{n} [n-q] \\ &= s \left(1-p + (1-3p) \frac{q}{n} \right), \\ \mathbb{E}[Q \mid \mathbf{x}] &= + \frac{a}{n} [p(a+q) + (1-2p)b] \\ &\quad + \frac{b}{n} [p(b+q) + (1-2p)a] \\ &\quad + \frac{q}{n} [p(a+b) + (1-2p)q] \\ &= \frac{p}{n} [a^2 + b^2 + 2q(a+b)] \\ &\quad + \frac{1-2p}{n} [2ab + q^2] \\ &= pn + \frac{1-3p}{n} [2ab + q^2] \\ &= pn + \frac{1-3p}{2n} [2q^2 + (n-q)^2 - s^2]. \end{aligned}$$

□

Proof of Lemma 1. The equivalence between the two processes is showed through a coupling. Formally, consider the complete graph of n nodes, namely K_n , over which the former process runs. We define another graph G_n , which contains a sub-graph isomorphic to K_n in the following way. Let K'_n be a copy of K_n , and let H be a graph of $n_{stub} = \frac{p_{noise}}{1-p_{noise}}n$ isolated nodes (which will be the *stubborn agents*). Then, each node $u \in H$ is

connected by edges to all nodes of K'_n , namely, each node of K'_n has as its neighborhood the whole set of nodes of $K'_n \cup H$, while each node of H has as its neighborhood only the set of nodes of K'_n . The nodes of H are such that $n_{stub} \cdot p_1$ are stubborn agents supporting opinion 1, $n_{stub} \cdot p_2$ are stubborn agents supporting opinion 2, and so on. Observe that $\sum_{i=1}^m p_i = 1$, so this partition is well defined.

The UNDECIDED-STATE dynamics behaves in exactly the same way over G_n , with the exception that the stubborn agents never change their opinion and that there is no noise perturbing communications between agents. The coupling is any bijective function $f : K_n \rightarrow K'_n$ such that, for each $v \in K_n$, v and $f(v)$ support the same opinion at the beginning of the process. Consider the two resulting Markov processes $\{\mathbf{X}_t\}_{t \geq 0}$ over K_n and $\{\mathbf{X}'_t\}_{t \geq 0}$ over K'_n , denoting the opinion configuration at time t in K_n and in K'_n , respectively. It is easy to see that the two transition matrix are exactly the same (this is the meaning of *equivalence* between the two processes). Indeed, in the former model (a), the probability an agent pulls opinion j at any given round is

$$(1 - p_{noise}) \frac{c_j}{n} + p_{noise} \cdot p_j ,$$

where c_j is the size of the community of agents supporting opinion j ; in the model defined in (b), the probability a non-stubborn agent pulls opinion j at any given round is

$$\frac{c_j + n_{stub} \cdot p_j}{n + n_{stub}} = \frac{c_j + \frac{p_{noise}}{1-p_{noise}} n \cdot p_j}{n + \frac{p_{noise}}{1-p_{noise}} n} = (1 - p_{noise}) \cdot \frac{c_j}{n} + p_{noise} \cdot p_j .$$

□

B.2 Proofs: Subsection 3.1

Proof of Lemma 2. From Equation (5) we obtain

$$\mathbb{E} [S \mid \mathbf{x}] \geq s \left(1 + \frac{\epsilon}{3}\right) .$$

Now, for the additive form of Chernoff bound (Theorem 5 in Appendix A) it holds that

$$\begin{aligned} \mathbb{P} \left(A \leq \mathbb{E} [A \mid \mathbf{x}] - \frac{\epsilon}{12} \gamma \sqrt{n \log n} \mid \mathbf{x} \right) &\leq e^{-\frac{2n \log n}{144n}} \\ &= n^{-\frac{1}{77}} ; \\ \mathbb{P} \left(B \geq \mathbb{E} [B \mid \mathbf{x}] + \frac{\epsilon}{12} \gamma \sqrt{n \log n} \mid \mathbf{x} \right) &\leq e^{-\frac{2n \log n}{144n}} \\ &= n^{-\frac{1}{77}} . \end{aligned}$$

Let $E_1 = \{A \leq \mathbb{E}[A | \mathbf{x}] - \frac{\epsilon}{12}\gamma\sqrt{n \log n}\}^C$ and $E_2 = \{B \geq \mathbb{E}[B | \mathbf{x}] + \frac{\epsilon}{12}\gamma\sqrt{n \log n}\}^C$ be the complementaries of the two events above. It follows that

$$\begin{aligned}
& \mathbb{P}\left(S \geq s\left(1 + \frac{\epsilon}{6}\right) \mid \mathbf{x}\right) \\
&= \mathbb{P}\left(S \geq s\left(1 + \frac{\epsilon}{3}\right) - \frac{\epsilon}{6}s \mid \mathbf{x}\right) \\
&\geq \mathbb{P}\left(S \geq \mathbb{E}[S | \mathbf{x}] - \frac{\epsilon}{6}\gamma\sqrt{n \log n} \mid \mathbf{x}\right) \\
&= \mathbb{P}\left(A - B \geq \mathbb{E}[A - B | \mathbf{x}] - 2\frac{\epsilon}{12}\gamma\sqrt{n \log n} \mid \mathbf{x}\right) \\
&\geq \mathbb{P}(E_1 \cap E_2 \mid \mathbf{x}) \\
&= \mathbb{P}(E_1 \mid \mathbf{x}) + \mathbb{P}(E_2 \mid \mathbf{x}) - \mathbb{P}(E_1 \cup E_2 \mid \mathbf{x}) \\
&\geq 1 - 2n^{-\frac{1}{7}}.
\end{aligned}$$

Where in the last inequality we bounded the probability of the union with 1. \square

Proof of Lemma 3. Equation (6) has its minimum in $\bar{q} = \frac{n}{3}$; thus, we have

$$\begin{aligned}
\mathbb{E}[Q | \mathbf{x}] &\geq (1 + 6\epsilon)\frac{n}{12} - (1 + 6\epsilon)\frac{n}{6} + (5 + 6\epsilon)\frac{n}{12} \\
&\quad - (1 + 6\epsilon)\left(\frac{\beta}{2}\right)^2 n \\
&= \frac{n}{12} \left(1 + 6\epsilon - 2 - 12\epsilon + 5 + 6\epsilon - \frac{36\epsilon}{1 + 6\epsilon}\right) \\
&= \frac{1 - 3\epsilon}{3(1 + 6\epsilon)}n.
\end{aligned}$$

Hence, with a simple application of the additive form of Chernoff bound (Theorem 5 in Appendix A), we can see that $Q \geq \frac{1-4\epsilon}{3(1+6\epsilon)}n$, w.h.p. (actually, with probability $1 - \exp(-\Theta(n))$, which we will use in a later lemma). Indeed

$$\begin{aligned}
& \mathbb{P}\left(Q \leq \frac{1 - 4\epsilon}{3(1 + 6\epsilon)}n \mid \mathbf{x}\right) \\
&= \mathbb{P}\left(Q \leq \frac{1 - 3\epsilon}{3(1 + 6\epsilon)}n - \frac{\epsilon}{3(1 + 6\epsilon)}n \mid \mathbf{x}\right) \\
&\leq \mathbb{P}\left(Q \leq \mathbb{E}[Q | \mathbf{x}] - \frac{\epsilon}{3(1 + 6\epsilon)}n \mid \mathbf{x}\right) \\
&\leq e^{-\frac{2}{n}\frac{\epsilon^2}{9(1+6\epsilon)^2}n^2} = e^{-\frac{2\epsilon^2}{9(1+6\epsilon)^2}n}.
\end{aligned}$$

\square

Proof of Lemma 4. From Equation (5), we have that

$$\mathbb{E}[S | \mathbf{x}] \geq s\left(\frac{5}{6} + \epsilon\right).$$

For the additive form of Chernoff bound (as we did in Lemma 2), and the union bound, it holds that $S \geq s \left(\frac{5}{6} + \frac{\epsilon}{2} \right)$ w.h.p. (we do as in the proof of Lemma 2). \square

Proof of Lemma 5. From Equation (6)

$$\begin{aligned} \mathbb{E}[Q \mid \mathbf{x}] &\geq \frac{3}{4} \left(\frac{1+6\epsilon}{n} \right) q^2 - \frac{1+6\epsilon}{2} q + \frac{5+6\epsilon}{12} n \\ &\quad - \frac{1+6\epsilon}{n} \left(\frac{n-q}{2} \right)^2 \\ &\geq \frac{1}{2} \left(\frac{1+6\epsilon}{n} \right) q^2 + \frac{5+6\epsilon-3-18\epsilon}{12} n \\ &\geq \frac{1}{2} \left(\frac{1+6\epsilon}{n} \right) q^2 + \frac{1-6\epsilon}{6} n \\ &\geq \frac{1-6\epsilon}{6} n \end{aligned}$$

since $s \leq n - q$. For the additive form of Chernoff bound (Theorem 5 in Appendix A) we obtain that $Q \geq \frac{1-6\epsilon}{12} n$ w.h.p. \square

Proof of Lemma 6. From the last term of Equation (2), we have

$$\mathbb{E}[B \mid \mathbf{x}] \geq \frac{1}{6} \left(\frac{1-6\epsilon}{n} \right) (q^2) \geq \frac{n}{6 \cdot 12^2} (1-6\epsilon)^3.$$

The additive form of Chernoff bound (Theorem 5 in Appendix A) implies that $B \geq \left(\frac{1-6\epsilon}{12} \right)^3 n$, w.h.p. \square

B.3 Proofs: Subsection 3.2

Proof of Lemma 7. From Equation (7), we have

$$\begin{aligned} \mathbb{E}[S \mid \mathbf{x}] &\leq s \left(\frac{5}{6} - \epsilon + \frac{1}{2}(1-6\epsilon) \right) = s \left(\frac{4}{3} - 4\epsilon \right) \\ &\leq s \left(1 - 4\epsilon + \frac{1}{3} \right). \end{aligned}$$

Observe that $4\epsilon - \frac{1}{3} > 0$ if and only if $\epsilon > \frac{1}{12}$. Now, let $\lambda = \gamma \left(\epsilon - \frac{1}{12} \right) \sqrt{n \log n}$, and define the events

$$\begin{aligned} E_1 &= \{A \geq \mathbb{E}[A \mid \mathbf{x}] + \lambda\}, \quad \text{and} \\ E_2 &= \{B \leq \mathbb{E}[B \mid \mathbf{x}] - \lambda\}. \end{aligned}$$

Then, it for the additive form of Chernoff bound (Theorem 5 in Appendix A) it holds that

$$\mathbb{P}(E_1 \mid \mathbf{x}) \leq e^{-\frac{\gamma(\epsilon-1/12)^2 \log n}{2}} = n^{-\frac{\gamma(\epsilon-1/12)^2}{2}}, \quad \text{and}$$

$$\mathbb{P}(E_2 \mid \mathbf{x}) \leq e^{-\frac{\gamma(\epsilon-1/12)^2 \log n}{2}} = n^{-\frac{\gamma(\epsilon-1/12)^2}{2}}.$$

Then, for the union bound, we have that

$$\begin{aligned} \mathbb{P}\left(S \leq s \left(1 - 2\epsilon + \frac{1}{6}\right) \mid \mathbf{x}\right) &= \mathbb{P}\left(S \leq s \left(1 - 4\epsilon + \frac{1}{3}\right) + 2\lambda \mid \mathbf{x}\right) \\ &\geq \mathbb{P}(S \leq \mathbb{E}[S \mid \mathbf{x}] + 2\lambda \mid \mathbf{x}) \\ &= \mathbb{P}(A - B \leq \mathbb{E}[A - B \mid \mathbf{x}] + 2\lambda \mid \mathbf{x}) \\ &\geq \mathbb{P}(E_1^C, E_2^C \mid \mathbf{x}) \\ &\geq \mathbb{P}(E_1^C \mid \mathbf{x}) + \mathbb{P}(E_2^C \mid \mathbf{x}) - \mathbb{P}(E_1^C \cup E_2^C \mid \mathbf{x}) \\ &\geq 1 - 2n^{-\frac{\gamma(\epsilon-1/12)^2}{2}}, \end{aligned}$$

where in the last inequality we used that the probability of the union of two events is at most 1. \square

Proof of Lemma 8. Consider the first statement. If $s \geq \frac{2}{3}n$, then $q < \frac{1}{3}n$. Thus

$$\mathbb{E}[S \mid \mathbf{x}] \leq s \left(\frac{5}{6} - \epsilon + \frac{1}{6} - \epsilon \right) \leq s(1 - 2\epsilon).$$

We conclude using the additive form of Chernoff bound (Theorem 5 in Appendix A), as we have done in the proof of Lemma 7, getting that $S \leq s(1 - \epsilon)$, w.h.p.

As for the second statement, we take the expectation of S from Equation (7) and we observe that

$$\begin{aligned} \mathbb{E}[S \mid \mathbf{x}] &= s \left(\frac{5}{6} - \epsilon + \frac{1}{2}(1 - 6\epsilon)\frac{q}{n} \right) \\ &\leq s \left(\frac{5}{6} - \epsilon + \frac{1}{2}(1 - 6\epsilon)\frac{n-s}{n} \right) \\ &= s \left(\frac{4}{3} - 4\epsilon - \frac{s}{2n}(1 - 6\epsilon) \right) \\ &\leq \frac{2}{3}n(1 - 2\epsilon). \end{aligned}$$

We conclude applying the additive form of Chernoff bound (Theorem 5 in Appendix A) on A and B , and the union bound, as we have done in the proof of Lemma 7, getting that $S \leq \frac{2}{3}n$, w.h.p. \square

Proof of Lemma 9. From Equation (7) it follows that

$$\mathbb{E}[S \mid \mathbf{x}] \leq s \left(\frac{5}{6} - \epsilon + \frac{1}{6} + \frac{\epsilon}{2} \right) \leq s \left(1 - \frac{\epsilon}{2} \right).$$

The thesis follows from an easy application of the additive form of Chernoff bound (Theorem 5 in Appendix A) and the union bound, as we have done in the proof of Lemma 7, getting that $S \leq s(1 - \epsilon/4)$, w.h.p. \square

Proof of Lemma 10. We start proving Item 1. From Equation (8), we have that

$$\begin{aligned}\mathbb{E}[Q \mid \mathbf{x}] &\geq \frac{n}{3} - \frac{1-6\epsilon}{n} \left(\frac{s}{2}\right)^2 \\ &\geq \frac{n}{3} - \frac{1-6\epsilon}{4} \left(\frac{3}{2}\right)^{2i+2} \beta^2 n \\ &= n \left[\frac{1}{3} - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3^{2i+2}}{2}\right) \right].\end{aligned}$$

Thus, using the additive form of Chernoff bound (Theorem 5 in Appendix A) with $\lambda = \frac{\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2i+2} n$, we have that $Q \geq n \left(\frac{1}{3} - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2i+3}\right)$, w.h.p.

As for Item 2, we have that

$$\begin{aligned}\mathbb{E}[Q \mid \mathbf{x}] &\geq \frac{n}{3} - \frac{1-6\epsilon}{n} \left(\frac{s}{2}\right)^2 \geq \frac{n}{3} - n \frac{1-6\epsilon}{9} \\ &= \frac{2+6\epsilon}{9} n\end{aligned}$$

and we conclude by using the additive Chernoff bound (Theorem 5 in Appendix A) with $\lambda = \frac{\epsilon}{3}n$, getting that $Q \geq \frac{2}{9}n + \frac{\epsilon}{3}n$, w.h.p.

To prove Item 3 we use that $s \leq n$ and observe that

$$\begin{aligned}\mathbb{E}[Q \mid \mathbf{x}] &\geq \frac{n}{3} - \frac{1-6\epsilon}{n} \left(\frac{s}{2}\right)^2 \geq \frac{n}{3} - n \frac{1-6\epsilon}{4} \\ &= \frac{1+18\epsilon}{12} n.\end{aligned}$$

We conclude with the additive Chernoff bound (Theorem 5 in Appendix A) with $\lambda = \frac{\epsilon}{2}n$, getting $Q \geq \frac{n}{12} + \epsilon n$, w.h.p. \square

Proof of Lemma 11. Define $f(q)$ equal to $\mathbb{E}[Q \mid \mathbf{x}] = \frac{3}{4} \left(\frac{1-6\epsilon}{n}\right) q^2 - \frac{1-6\epsilon}{2} q + \frac{5-6\epsilon}{12} n - \frac{1-6\epsilon}{n} \left(\frac{s}{2}\right)^2$. We are going to evaluate $f(q)$ in \bar{q}_{i+1} and in $\bar{q} = \frac{n(1+3\epsilon)}{3(1-6\epsilon)}$. We take care of different cases: first, we assume $i = -1$, with the condition that $s > 0$. Thus

$$\begin{aligned}f(\bar{q}_0) &\leq \frac{3}{4} \left(\frac{1-6\epsilon}{n}\right) n^2 \left[\frac{1}{9} + \frac{4\epsilon^2}{(1+6\epsilon)^2} \left(\frac{3}{2}\right)^6 - \frac{4\epsilon}{3(1+6\epsilon)} \left(\frac{3}{2}\right)^3 \right] \\ &\quad - \frac{1-6\epsilon}{2} n \left[\frac{1}{3} - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^3 \right] + \frac{5-6\epsilon}{12} n \\ &= \frac{n}{3} + n \frac{3\epsilon^2(1-6\epsilon)}{(1+6\epsilon)^2} \left(\frac{3}{2}\right)^6 \\ &= \frac{n}{3} \left[1 + \frac{3^8 \epsilon^2 (1-6\epsilon)}{2^6 (1+6\epsilon)^2} \right];\end{aligned}$$

now, we observe that

$$\begin{aligned}
& 1 + \frac{3^8 \epsilon^2 (1 - 6\epsilon)}{2^6 (1 + 6\epsilon)^2} - \frac{1}{1 - 6\epsilon} \\
& < \frac{-6\epsilon}{1 - 6\epsilon} + \frac{3^8 \epsilon^2}{2^6 (1 + 6\epsilon)^2} \\
& = \frac{-2^6 \cdot 6\epsilon (1 + 6\epsilon)^2 + 3^8 \epsilon^2 (1 - 6\epsilon)}{2^6 (1 - 6\epsilon) (1 + 6\epsilon)^2} \\
& = \frac{3\epsilon (-2^7 - 3 \cdot 2^9 \epsilon - 3^2 \cdot 2^9 \epsilon^2 + 3^7 \epsilon - 2 \cdot 3^8 \epsilon^2)}{2^6 (1 - 6\epsilon) (1 + 6\epsilon)^2} \\
& = \frac{3\epsilon (-2^7 + 3\epsilon(3^6 - 2^9) - 2 \cdot 3^2 \epsilon^2 (2^8 + 3^6))}{2^6 (1 - 6\epsilon) (1 + 6\epsilon)^2} < 0
\end{aligned}$$

where in the last inequality we have used that $-2^7 + 3\epsilon(3^6 - 2^9) < 0$ for $\epsilon \leq \frac{1}{12}$. Thus, $f(\bar{q}_0) < \frac{n}{3(1-6\epsilon)}$.

Second, we assume $0 \leq i \leq k - 3$, with the condition that $s > (\frac{3}{2})^i \beta n$.

$$\begin{aligned}
f(\bar{q}_{i+1}) & \leq \frac{3}{4} \left(\frac{1 - 6\epsilon}{n} \right) n^2 \left[\frac{1}{9} + \frac{4\epsilon^2}{(1 + 6\epsilon)^2} \left(\frac{3}{2} \right)^{4i+10} - \frac{4\epsilon}{3(1 + 6\epsilon)} \left(\frac{3}{2} \right)^{2i+5} \right] \\
& \quad - \frac{1 - 6\epsilon}{2} n \left[\frac{1}{3} - \frac{2\epsilon}{1 + 6\epsilon} \left(\frac{3}{2} \right)^{2i+5} \right] + \frac{5 - 6\epsilon}{12} n - \frac{2\epsilon}{1 + 6\epsilon} \left(\frac{3}{2} \right)^{2i} n \\
& = \frac{n}{3} + n \frac{3\epsilon^2 (1 - 6\epsilon)}{(1 + 6\epsilon)^2} \left(\frac{3}{2} \right)^{4i+10} - n \frac{2\epsilon}{1 + 6\epsilon} \left(\frac{3}{2} \right)^{2i} \\
& = \frac{n}{3} \left\{ 1 + \frac{3\epsilon}{(1 + 6\epsilon^2)} \left(\frac{3}{2} \right)^{2i} \left[3\epsilon(1 - 6\epsilon) \left(\frac{3}{2} \right)^{2(i+5)} - 2(1 + 6\epsilon) \right] \right\};
\end{aligned}$$

for the evaluation of $f(\bar{q}_{i+1})$ we observe that β is a constant in $(0, 1)$ and that

$$\begin{aligned}
& 3\epsilon(1 - 6\epsilon) \left(\frac{3}{2} \right)^{2(i+5)} - 2(1 + 6\epsilon) \\
& < 3\epsilon \left(\frac{2}{3} \frac{1}{\beta} \left(\frac{3}{2} \right)^3 \right)^2 - 2 \\
& < \frac{3^5}{2^7} - 2 < 0
\end{aligned}$$

because $(\frac{3}{2})^{i+2} < \frac{2}{3\beta}$ for $i + 2 \leq k$; thus $f(\bar{q}_{i+1}) \leq \frac{n}{3}$.

Let now $i = k - 2$; thus $s > (\frac{3}{2})^{k-2} \beta n$. We now evaluate $f(\bar{q}_{k-1})$.

$$f(\bar{q}_{k-1}) \leq \frac{3}{4} \left(\frac{1 - 6\epsilon}{n} \right) n^2 \left[\frac{4}{81} + \frac{\epsilon^2}{9} + \frac{4\epsilon}{27} \right] - \frac{1 - 6\epsilon}{2} n \left[\frac{2}{9} + \frac{\epsilon}{3} \right] + \frac{5 - 6\epsilon}{12} n$$

$$\begin{aligned}
& -\frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2k-4} n \\
&= \frac{-8(1-6\epsilon) + 45 - 54\epsilon}{108} n - \frac{\epsilon(1-6\epsilon)}{18} n - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2k-4} n \\
&= \frac{37-12\epsilon+36\epsilon^2}{108} n - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2k-4} n \\
&= \frac{n}{3} \left[1 + \frac{(1-6\epsilon)^2}{36} - \frac{6\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2k-4} \right] \\
&= \frac{n}{3} \left[1 + \left(\frac{1-6\epsilon}{6} - \sqrt{\frac{6\epsilon}{1+6\epsilon}} \cdot \frac{3^{k-2}}{2^{k-2}} \right) \cdot \left(\frac{1-6\epsilon}{6} + \sqrt{\frac{6\epsilon}{1+6\epsilon}} \cdot \frac{3^{k-2}}{2^{k-2}} \right) \right].
\end{aligned}$$

Observe that, by definition of k , we have

$$\begin{aligned}
& \frac{1-6\epsilon}{6} - \sqrt{\frac{6\epsilon}{1+6\epsilon}} \cdot \frac{3^{k-2}}{2^{k-2}} \leq \frac{1-6\epsilon}{6} - \sqrt{\frac{6\epsilon}{1+6\epsilon}} \cdot \frac{2^3}{3^3\beta} \\
&= \frac{1-6\epsilon}{6} - \frac{\sqrt{3(1-6\epsilon)} \cdot 2^2}{3^3} = \frac{9-54\epsilon-8\sqrt{3(1-6\epsilon)}}{54} \\
&< \frac{9-8\sqrt{3(1-6\epsilon)}}{54} < 0
\end{aligned}$$

for $\epsilon \leq \frac{1}{12}$. Thus, $f(\bar{q}_{k-1}) \leq \frac{n}{3}$.

Let now $i = k-1$, which implies that $s > \left(\frac{3}{2}\right)^{k-1} \beta n$. We evaluate $f(\bar{q}_k)$:

$$\begin{aligned}
f(\bar{q}_k) &\leq \frac{3}{4} \left(\frac{1-6\epsilon}{n}\right) n^2 \left[\frac{1}{144} + \epsilon^2 + \frac{\epsilon}{6} \right] - \frac{1-6\epsilon}{2} n \left[\frac{1}{12} + \epsilon \right] + \frac{5-6\epsilon}{12} n \\
&\quad - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2k-2} n \\
&= \frac{-7(1-6\epsilon) + 80 - 96\epsilon}{192} n - \frac{3\epsilon(1-6\epsilon)}{8} n - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2k-2} n \\
&= \frac{n}{3} \left[\frac{219-162\epsilon-216\epsilon+1296\epsilon^2}{192} - \frac{6\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2k-2} \right] \\
&= \frac{n}{3} \left[1 + \frac{27-378\epsilon+1296\epsilon^2}{192} - \frac{6\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2k-2} \right].
\end{aligned}$$

By definition of k , we have that

$$\begin{aligned}
& \frac{27-378\epsilon+1296\epsilon^2}{192} - \frac{6\epsilon}{1+6\epsilon} \left(\frac{3}{2}\right)^{2k-2} \\
&\leq \frac{27-378\epsilon+1296\epsilon^2}{192} - \frac{1-6\epsilon}{2} \left(\frac{2}{3}\right)^3 < 0,
\end{aligned}$$

where the first and the second inequalities hold for $\epsilon \leq \frac{1}{12}$. Thus, $f(\bar{q}_k) \leq \frac{n}{3}$.

We finally evaluate $f(\bar{q})$:

$$\begin{aligned}
f(\bar{q}) &\leq \frac{n}{12} \left(\frac{(1+3\epsilon)^2}{(1-6\epsilon)} \right) - \frac{n(1+3\epsilon)}{6} + \frac{n(5-6\epsilon)}{12} \\
&= \frac{n}{12} \left(\frac{1-6\epsilon+9\epsilon^2}{1-6\epsilon} - 2 - 6\epsilon + 5 - 6\epsilon \right) \\
&= \frac{n}{12(1-6\epsilon)} \left(1 - 6\epsilon + 9\epsilon^2 - 2 + 6\epsilon + 36\epsilon^2 + 5 - 36\epsilon + 36\epsilon^2 \right) \\
&= \frac{n}{12} \left(\frac{4 - 36\epsilon + 81\epsilon^2}{1-6\epsilon} \right) \\
&= \frac{n}{12} \left(\frac{(2-9\epsilon)^2}{1-6\epsilon} \right).
\end{aligned}$$

It holds that $f(\bar{q}) \leq \frac{n}{3(1-6\epsilon)}$ (remember that $\epsilon < \frac{1}{12}$); thus, all the evaluations are no more than $\frac{n}{3(1-6\epsilon)}$, and, from an immediate application of the additive Chernoff bound (Theorem 5 in Appendix A) with $\lambda = \frac{\epsilon n}{1-6\epsilon}$, and by observing that $\bar{q}_i \geq \bar{q}_{i+1}$, we get that

$$\bar{q}_i \leq Q \leq \frac{n(1+3\epsilon)}{3(1-6\epsilon)},$$

w.h.p. □

Proof of Lemma 12. We have

$$\mathbb{E}[S \mid \mathbf{x}] \geq s \left(\frac{5}{6} - \epsilon \right).$$

The additive form of Chernoff bound (Theorem 5 in Appendix A) on A and B , together with the union bound, implies that $S \geq s \left(\frac{5}{6} - 2\epsilon \right)$ w.h.p. (we can proceed as in the proof of Lemma 2). Thus

$$s \left(\frac{5}{6} - 2\epsilon \right) > \left(\frac{3}{2} \right)^i \beta n \left(\frac{5}{6} - 2\epsilon \right) \geq \left(\frac{3}{2} \right)^{i-1} \beta n$$

since $\frac{3}{2} \left(\frac{5}{6} - 2\epsilon \right) \geq 1$. □

Proof of Lemma 13. Consider the first item. We define $f(q) = \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) q^2 - \frac{1-6\epsilon}{2} q + \frac{5-6\epsilon}{12} n$. We now show that $f(q) \leq q(1-2\epsilon)$. Indeed, $f(q) - q(1-2\epsilon)$ is equal to

$$\begin{aligned}
&\frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) q^2 - \left(\frac{1-6\epsilon}{2} + 1 - 2\epsilon \right) q + \frac{5-6\epsilon}{12} n \\
&= \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) q^2 - \left(\frac{3-10\epsilon}{2} \right) q + \frac{5-6\epsilon}{12} n.
\end{aligned}$$

This expression is a convex parabola which has its maximum in either $\bar{q}_1 = \frac{n}{3(1-6\epsilon)}$ or $\bar{q}_2 = n$. We calculate $f(q) - q(1-2\epsilon)$ in these two points

$$\begin{aligned} & \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) q_1^2 - \left(\frac{3-10\epsilon}{2} \right) q_1 + \frac{5-6\epsilon}{12} n \\ &= \frac{n^2}{12(1-6\epsilon)} (1-6+20\epsilon+5-36\epsilon+36\epsilon^2) \\ &= \frac{n^2}{12(1-6\epsilon)} [-4\epsilon(4-9\epsilon)] < 0 \end{aligned}$$

for all $0 < \epsilon \leq \frac{1}{12}$. At the same time it holds that

$$\begin{aligned} & \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) q_2^2 - \left(\frac{3-10\epsilon}{2} \right) q_2 + \frac{5-6\epsilon}{12} n \\ &= \frac{n}{12} [9-54\epsilon-18+60\epsilon+5-6\epsilon] \\ &= -\frac{n}{3} < 0. \end{aligned}$$

Thus, $\mathbb{E}[Q | \mathbf{x}] \leq q(1-2\epsilon)$. The additive form of Chernoff bound (Theorem 5 in Appendix A) with $\lambda = \frac{\epsilon n}{3(1-6\epsilon)}$ implies that $Q < q(1-\epsilon)$ w.h.p.

As for the second item, We consider two cases. First, assume $i < k-1$, thus $s \leq \left(\frac{3}{2}\right)^{i+1} \beta n$. Take the expectation (8) and observe that

$$\begin{aligned} \mathbb{E}[Q | \mathbf{x}] &\geq \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) \frac{n^2}{9(1-6\epsilon)^2} + \frac{5-6\epsilon}{12} n - \left(\frac{1-6\epsilon}{2} \right) \frac{n}{3(1-6\epsilon)} - \frac{1-6\epsilon}{n} \left(\frac{s}{2} \right)^2 \\ &\geq \frac{n}{12} \left(\frac{4-24\epsilon+36\epsilon^2}{1-6\epsilon} \right) - \frac{2n\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+2} \\ &\geq \frac{n}{3(1-6\epsilon)} (1-3\epsilon)^2 - \frac{2n\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+2} \\ &\geq \frac{n}{3} - \frac{2n\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+5} + \frac{19n\epsilon}{4} \left(\frac{3}{2} \right)^{2i+2} \\ &= \bar{q}_{i+1} + \frac{19n\epsilon}{4} \left(\frac{3}{2} \right)^{2i+2}. \end{aligned}$$

We conclude applying the additive Chernoff bound (Theorem 5 in Appendix A) with $\lambda = \frac{19n\epsilon}{4} \left(\frac{3}{2} \right)^{2i+2}$, obtaining $Q \geq \bar{q}_{i+1}$, w.h.p.

Second, let $i = k-1$; then $s \leq \frac{2}{3}n$. As before

$$\begin{aligned} \mathbb{E}[Q | \mathbf{x}] &\geq \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) \frac{n^2}{9(1-6\epsilon)^2} + \frac{5-6\epsilon}{12} n - \left(\frac{1-6\epsilon}{2} \right) \frac{n}{3(1-6\epsilon)} - \frac{1-6\epsilon}{n} \left(\frac{s}{2} \right)^2 \\ &\geq \frac{n}{12(1-6\epsilon)} (4-24\epsilon+36\epsilon^2) - \frac{n(1-6\epsilon)}{9} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{n(1-3\epsilon)^2}{3(1-6\epsilon)} - \frac{n(1-6\epsilon)}{9} \\
&= \frac{n}{9(1-6\epsilon)} (2-6\epsilon-9\epsilon^2) \\
&= \frac{n}{12} \left(\frac{4(2-6\epsilon-9\epsilon^2)}{3(1-6\epsilon)} \right) \\
&\geq \frac{n}{12} (1+14\epsilon) = \bar{q}_k + \frac{\epsilon n}{6}
\end{aligned}$$

since

$$\frac{4(2-6\epsilon-9\epsilon^2)}{3(1-6\epsilon)} - (1+14\epsilon) > \frac{5-48\epsilon}{3(1-6\epsilon)} > 0$$

for $\epsilon \leq \frac{1}{12}$. Thus, we conclude applying the additive form of Chernoff bound (Theorem 5 in Appendix A) with $\lambda = \frac{\epsilon}{6}n$, obtaining $Q \geq \bar{q}_k$, w.h.p. \square

Proof of Lemma 14. Consider the expectation of S from equation 7. We have

$$\begin{aligned}
\mathbb{E}[S \mid \mathbf{x}] &\leq \gamma \sqrt{n \log n} \left(\frac{5}{6} - \epsilon + \frac{1}{6} + \frac{\epsilon}{2} \right) \\
&\leq \gamma \sqrt{n \log n}.
\end{aligned}$$

We conclude applying the additive form of Chernoff bound (Theorem 5 in Appendix A), and the union bound, as we did in the proof of Theorem 7. \square

B.4 Proofs: Subsection 4.1

Proof of Lemma 15. Define a set of hitting times $T := \{\tau(i)\}_{i \in \mathbb{N}}$, where

$$\tau(i) = \inf_{i \in \mathbb{N}} \{t : t > \tau(i-1), f(X_t) \geq h\sqrt{n}\},$$

setting $\tau(0) = 0$. By the first hypothesis, for every $i \in \mathbb{N}$, the expectation of $\tau(i)$ is finite. Now, define the following stochastic process which is a subsequence of $\{X_t\}_{t \in \mathbb{N}}$:

$$\{R_i\}_{i \in \mathbb{N}} = \{X_{\tau(i)}\}_{i \in \mathbb{N}}.$$

Observe that $\{R_i\}_{i \in \mathbb{N}}$ is still a Markov chain. Indeed, if $\{x_1, \dots, x_{i-1}\}$ be a set of states in Ω , then

$$\begin{aligned}
&\mathbb{P}(R_i = x \mid R_{i-1} = x_{i-1}, \dots, R_1 = x_1) \\
&= \mathbb{P}(X_{\tau(i)} = x \mid X_{\tau(i-1)} = x_{i-1}, \dots, X_{\tau(1)} = x_1) \\
&= \sum_{t(i), \dots, t(0) \in \mathbb{N}} \mathbb{P}(X_{t(i)} = x \mid X_{t(i-1)} = x_{i-1}, \dots, X_{t(1)} = x_1) \\
&\quad \cdot \mathbb{P}(\tau(i) = t(i), \dots, \tau(1) = t(1))
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{P}(X_{\tau(i)} = x \mid X_{\tau(i-1)} = x_{i-1}) \\
&= \mathbb{P}(R_i = x \mid R_{i-1} = x_{i-1}).
\end{aligned}$$

By definition, the state space of R is $\{x \in \Omega : f(x) \geq h\sqrt{n}\}$. Moreover, the second hypothesis still holds for this new Markov chain. Indeed:

$$\begin{aligned}
&\mathbb{P}(f(R_{i+1}) < (1 + \epsilon)f(R_i) \mid R_i = x) \\
&= 1 - \mathbb{P}(f(R_{i+1}) \geq (1 + \epsilon)f(R_i) \mid R_i = x) \\
&= 1 - \mathbb{P}(f(X_{\tau(i+1)}) \geq (1 + \epsilon)f(X_{\tau(i)}) \mid X_{\tau(i)} = x) \\
&\leq 1 - \mathbb{P}(f(X_{\tau(i+1)}) \geq (1 + \epsilon)f(X_{\tau(i)}), \tau(i+1) = \tau(i) + 1 \mid X_{\tau(i)} = x) \\
&= 1 - \mathbb{P}(f(X_{\tau(i)+1}) \geq (1 + \epsilon)f(X_{\tau(i)}) \mid X_{\tau(i)} = x) \\
&= 1 - \mathbb{P}(f(X_{t+1}) \geq (1 + \epsilon)f(X_t) \mid X_t = x) \\
&< e^{-c_2 f(x)^2/n}.
\end{aligned}$$

These two properties are sufficient to study the number of rounds required by the new Markov chain $\{R_i\}_{i \in \mathbb{N}}$ to reach the target value m . Indeed, by defining the random variable $Z_i = \frac{f(R_i)}{\sqrt{n}}$, and considering the following “potential” function, $Y_i = \exp\left(\frac{m}{\sqrt{n}} - Z_i\right)$, we can compute its expectation at the next round as follows. Let us fix any state $x \in \Omega$ such that $h\sqrt{n} \leq f(x) < m$, and define $z = \frac{f(x)}{\sqrt{n}}$, $y = \exp\left(\frac{m}{\sqrt{n}} - z\right)$. We have

$$\begin{aligned}
\mathbb{E}[Y_{i+1} \mid R_i = x] &\leq \mathbb{P}(f(R_{i+1}) < (1 + \epsilon)f(x)) e^{m/\sqrt{n}} \\
&\quad + \mathbb{P}(f(R_{i+1}) \geq (1 + \epsilon)f(x)) e^{m/\sqrt{n} - (1 + \epsilon)z} \\
(\text{from Hypothesis (2)}) &\leq e^{-c_2 z^2} \cdot e^{m/\sqrt{n}} + 1 \cdot e^{m/\sqrt{n} - (1 + \epsilon)z} \\
&= e^{m/\sqrt{n} - c_2 z^2} + e^{m/\sqrt{n} - z - \epsilon z} \\
&= e^{m/\sqrt{n} - z} (e^{z - c_2 z^2} + e^{-\epsilon z}) \\
&\leq e^{m/\sqrt{n} - z} (e^{-2} + e^{-2}) \tag{13} \\
&< \frac{e^{m/\sqrt{n} - z}}{e} \\
&< \frac{y}{e},
\end{aligned}$$

where in (13) we used that z is always at least h and thanks to Hypothesis (1) we can choose a sufficiently large h .

By applying the Markov inequality and iterating the above bound, we get

$$\mathbb{P}(Y_i > 1) \leq \frac{\mathbb{E}[Y_i]}{1} \leq \frac{\mathbb{E}[Y_{i-1}]}{e} \leq \dots \leq \frac{\mathbb{E}[Y_0]}{e^i} e^{\tau R} \leq \frac{e^{m/\sqrt{n}}}{e^i}.$$

We observe that if $Y_i \leq 1$ then $R_i \geq m$, thus by setting $i = m/\sqrt{n} + \log n = (c_3 + 1) \log n$, we get:

$$\mathbb{P}(R_{(c_3+1)\log n} < m) = \mathbb{P}(Y_{(c_3+1)\log n} > 1) < \frac{1}{n}. \quad (14)$$

Our next goal is to give an upper bound on the hitting time $\tau_{(c_3+1)\log n}$. Note that the event “ $\tau_{(c_3+1)\log n} > c_4 \log n$ ” holds if and only if the number of rounds such that $f(X_t) \geq h\sqrt{n}$ (before round $c_4 \log n$) is less than $(c_3 + 1)\log n$. Thanks to Hypothesis (1), at each round t there is at least probability $1 - c_1$ that $f(X_t) \geq h\sqrt{n}$. This implies that, for any positive constant c_4 , the probability $\text{prob}(\tau_{(c_3+1)\log n} > c_4 \log n)$ is bounded by the probability that, within $c_4 \log n$ independent Bernoulli trials, we get less than $(c_3 + 1)\log n$ successes, where the success probability is at least $1 - c_1$. We can thus choose a sufficiently large c_4 and apply the multiplicative form of the Chernoff bound, obtaining

$$\mathbb{P}(\tau_{(c_3+1)\log n} > c_4 \log n) < \frac{1}{n}. \quad (15)$$

We are now ready to prove the Lemma using Inequalities (14) and (15), indeed

$$\begin{aligned} \mathbb{P}(X_{c_4 \log n} \geq m) &> \mathbb{P}(R_{(c_3+1)\log n} \geq m \wedge \tau_{(c_3+1)\log n} \leq c_4 \log n) \\ &= 1 - \text{prob}(R_{(c_3+1)\log n} < m \vee \tau_{(c_3+1)\log n} > c_4 \log n) \\ &\geq 1 - \mathbb{P}(R_{(c_3+1)\log n} < m) + \mathbb{P}(\tau_{(c_3+1)\log n} > c_4 \log n) \\ &> 1 - \frac{2}{n}. \end{aligned}$$

Hence, choosing a suitable big c_4 , we have shown that in $c_4 \log n$ rounds the process reaches the target value m , w.h.p. \square

Proof of Lemma 16. The fact that $Q \geq \frac{n}{3} \left(\frac{1-4\epsilon}{1+6\epsilon} \right)$ with probability $1 - \exp(-\Theta(n))$ (which is w.h.p.) comes from Lemma 3. At the same time, from Equation 4, it holds that

$$\mathbb{E}[Q \mid \mathbf{x}] \leq \frac{(1-6\epsilon)n}{6} + \frac{1+6\epsilon}{4n} (2q^2 + (n-q)^2).$$

Denote this expression as $f(q)$. For $\frac{n}{3} \left(\frac{1-4\epsilon}{1+6\epsilon} \right) \leq q \leq \frac{n}{2}$, the maximum of f is obtained either at $q_1 = \frac{n}{3} \left(\frac{1-4\epsilon}{1+6\epsilon} \right)$ or at $q_2 = \frac{n}{2}$.

$$\begin{aligned} f(q_1) &= \frac{1-6\epsilon}{3}n + \frac{2(1-4\epsilon)^2 + 4(1+11\epsilon)^2}{3^2 \cdot 4(1+6\epsilon)}n \\ &= \frac{6(1-36\epsilon^2) + 3 + 36\epsilon + 258\epsilon^2}{2 \cdot 3^2(1+6\epsilon)}n \end{aligned}$$

$$\begin{aligned}
&= \frac{9 + 36\epsilon + 42\epsilon^2}{2 \cdot 9(1 + 6\epsilon)} n \\
&= \frac{1 + 4\epsilon + \frac{14}{3}\epsilon^2}{2(1 + 6\epsilon)} n \\
&\leq \left(\frac{1}{2} - c\right) n, \\
f(q_2) &= \frac{1 - 6\epsilon}{6} n + \frac{3(1 + 6\epsilon)}{16} n \\
&= \frac{17 + 6\epsilon}{48} n \\
&\leq \left(\frac{1}{2} - c\right) n,
\end{aligned}$$

for some suitable constant $c > 0$, and for $\epsilon < \frac{1}{6}$. Thus, $f(q) \leq (1/2 - c)n$ and we conclude by using the additive form of Chernoff bound (Theorem 5 in Appendix A), getting that $Q \leq \frac{n}{2}$, with probability $1 - \exp(-c^2 n)$. Then, the intersection of two events holding with probability $1 - \exp(-\Theta(n))$ is still an event holding with probability $1 - \exp(-\Theta(n))$. \square

Proof of Lemma 17. As for the first item, let \mathbf{x} and \mathbf{x}_0 be two states such that $|s(\mathbf{x})| < h\sqrt{n}$, $|s(\mathbf{x}_0)| = 0$, $q(\mathbf{x}) = q(\mathbf{x}_0)$. A simple domination argument implies that

$$\mathbb{P}(|S| < h\sqrt{n} \mid \mathbf{X}_t = \mathbf{x}) \leq \mathbb{P}(|S| < h\sqrt{n} \mid \mathbf{X}_t = \mathbf{x}_0).$$

Thus, we can bound just the second probability, where the initial bias is zero, which implies that $a = b$.

Define A^q , B^q , Q^q the random variables counting the nodes that were undecided in the configuration \mathbf{x}_0 and that, in the next round, get the opinion *Alpha*, *Beta*, and undecided, respectively. Similarly, A^a (B^b) counts the nodes that support opinion *Alpha* (*Beta*) in the configuration \mathbf{x}_0 and that, in the next round, still support the same opinion. Trivially, $A = A^q + A^a$ and $B = B^q + B^b$. Moreover, observe that, among these random variables, only A^q and B^q are mutually dependent. Thus, conditioned to the event $\{\mathbf{X}_t = \mathbf{x}_0\}$, if $\alpha = \mathbb{E}[A^a \mid \mathbf{X}_t = \mathbf{x}_0] = \mathbb{E}[B^b \mid \mathbf{X}_t = \mathbf{x}_0]$, it holds that

$$\begin{aligned}
&\mathbb{P}(|S| \geq h\sqrt{n}) \\
&\geq \mathbb{P}(A \geq B + h\sqrt{n}) \\
&\geq \mathbb{P}(A^q \geq B^q + h\sqrt{n}) \mathbb{P}(A^a \geq \alpha) \mathbb{P}(B^b \leq \alpha).
\end{aligned}$$

The random variables $A^a - \alpha$ and $B^b - \alpha$ happen to be binomial distribution with expectation 0 (recall that $a = b$), and finite second and third moment. Thus, the Berry-Essen Theorem (Theorem 6 in Appendix A) allows us to

approximate up to an arbitrary-small constant $\epsilon_1 > 0$ (as long as n is large enough) both the random variables with a normal distribution that has expectation 0. Thus,

$$\mathbb{P}(A^q \geq \alpha) = \mathbb{P}(B^q \leq \alpha) \geq \left(\frac{1}{2} - \epsilon_1\right).$$

As for the random variable $A^q - B^q$, notice that conditioned to the event $\{q - Q^q = k\}$, it is the sum of k Rademacher random variables. The hypothesis $q \leq \frac{n}{2}$ allows us to use the Chernoff bound on Q^q and show that $Q^q \leq \frac{3}{4}q$ w.h.p. Thus, since $q \geq \frac{n}{3} \left(\frac{1-4\epsilon}{1+6\epsilon}\right)$, it holds that $q - Q^q = \Theta(n)$ w.h.p. It follows that the conditional variance of $A^q - B^q$ given $q - Q^q$ yields $\Theta(n)$ w.h.p., and $A^q - B^q$ conditioned to the event $E = \{q - Q^q = \Theta(n)\}$ can be approximated by a normal distribution up to an arbitrary-small constant $\epsilon_2 > 0$. Then, we have that

$$\mathbb{P}(A^q \geq B^q + h\sqrt{n}) \geq \mathbb{P}(A^q \geq B^q + h\sqrt{n} \mid E)\mathbb{P}(E) \geq \epsilon_2.$$

Setting $c_1 = \epsilon_1 \cdot \epsilon_2$, we get property (1).

As for property (2), it is easy to see that the hypothesis on q implies that

$$\mathbb{E}[S \mid \mathbf{X}_t = \mathbf{x}] \geq s \left(1 + \frac{\epsilon}{3}\right).$$

We can get the property applying the additive Chernoff bound (Theorem 5 in Appendix A) separately on A and B , and then the union bound, as we did in the proof of Lemma 2, getting that $\mathbb{E}[S \mid \mathbf{x}] \geq s(1 + \epsilon/6)$, w.h.p. \square

Proof of Lemma 18. Let $\gamma > 0$ be a constant and $m = \gamma\sqrt{n}\log n$ be the target value of the bias in Lemma 15. Since $q(\mathbf{x}) \in \left[\frac{n}{3} \frac{1-4\epsilon}{1+6\epsilon}, \frac{n}{2}\right]$ and $s(\mathbf{x}) = \mathcal{O}(\sqrt{n}\log n)$, the PRUNED process satisfies Lemma 17 with probability 1, and thus we can apply Lemma 15 (setting the function $f(\mathbf{X}_t) = s(\mathbf{X}_t)$), which gives us that the PRUNED process reaches a configuration \mathbf{y} having bias $s(\mathbf{y}) \geq m = \Omega(\sqrt{n}\log n)$ within $\mathcal{O}(\log n)$ rounds, w.h.p. \square

Proof of Lemma 19. Let $\{\mathbf{X}_t\}$ and $\{\mathbf{Y}_t\}$ be the original process and the pruned one, respectively. Call H the set of possible initial configuration according to the hypothesis, and let $\mathbf{x} \in H$. Note that if $\mathbf{X}_t = \mathbf{Y}_t = \mathbf{x}$, then

$$\mathbf{Y}_{t+1} = \begin{cases} \mathbf{X}_{t+1} & \text{if } \mathbf{X}_{t+1} \in H \\ \mathbf{z}(s(\mathbf{X}_t)) & \text{otherwise} \end{cases}.$$

Let $\tau = \inf\{t : \mathbb{N} : |s(\mathbf{X}_t)| \geq \sqrt{n}\log n\}$, and let $\tau^* = \inf\{t \in \mathbb{N} : |s(\mathbf{Y}_t)| \geq \sqrt{n}\log n\}$. For any configuration $\mathbf{x} \in H$, define $\rho_{\mathbf{x}}^t$ the event that the two processes $\{\mathbf{X}_t\}$ and $\{\mathbf{Y}_t\}$ have separated at round $t + 1$, i.e. $\rho_{\mathbf{x}}^t = \{\mathbf{X}_t = \mathbf{Y}_t = \mathbf{x}\} \cap \{\mathbf{X}_{t+1} \neq \mathbf{Y}_{t+1}\}$. Observe that, if the two couple processes in

the same configuration $\mathbf{x}_0 \in H$ and $\tau > c \log n$, then either $\tau^* > c \log n$ or there exists a round $t \leq c \log n$ such that for some $\mathbf{x} \in H$ the event $\rho_{\mathbf{x}}^t$ has occurred. Hence, if $\mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}$ is the joint probability for the couple $(\mathbf{X}_t, \mathbf{Y}_t)$ which both start at \mathbf{x}_0 , we have

$$\begin{aligned} & \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}(\tau > c \log n) \\ & \leq \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}(\{\tau^* > c \log n\} \cup \{\exists t \leq c \log n, \exists \mathbf{x} \in H : \rho_{\mathbf{x}}^t\}) \\ & \leq \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}(\tau^* > c \log n) + \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}(\exists t \leq c \log n, \exists \mathbf{x} \in H : \rho_{\mathbf{x}}^t). \end{aligned}$$

As for the first item, since Lemma 15 holds for the PRUNED process, we have that it is upper bounded by $1/n^{-\Theta(1)}$. As for the second term, we get that

$$\begin{aligned} \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}(\exists t \leq c \log n, \exists \mathbf{x} \in H : \rho_{\mathbf{x}}^t) & \leq \sum_{t=1}^{c \log n} \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}(\exists \mathbf{x} \in H : \rho_{\mathbf{x}}^t) \\ & = \sum_{t=1}^{c \log n} \sum_{\mathbf{x} \in H} \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}(\rho_{\mathbf{x}}^t) \\ & \leq \sum_{t=1}^{c \log n} \frac{n^2}{e^{-\Theta(n)}} \\ & \leq \frac{1}{n}, \end{aligned}$$

where in the second inequality we used the probabilities computed in the proof of Lemma 16, and the fact that $|H|$ is at most all the combinations of parameters q and s . \square

Proof of Lemma 20. Let $f(q) = \frac{3}{4} \left(\frac{1+6\epsilon}{n}\right) q^2 - \frac{1+6\epsilon}{2} q + \frac{5+6\epsilon}{12} n$. By Equation (6) we have that

$$f(q) - \frac{\epsilon}{(1+6\epsilon)} n \leq \mathbb{E}[Q \mid \mathbf{x}] \leq f(q).$$

Then, $f(q)$ has its maximum in one of the two boundaries, namely $q = 0$ or $q = \frac{n}{2}$. Observe that

$$f(0) = \frac{n}{2} - \frac{1-6\epsilon}{12} n < \frac{n}{2}$$

since $\epsilon < \frac{1}{6}$. At the same time, we have that

$$f(n/2) = -\frac{1+6\epsilon}{16} n + \frac{5+6\epsilon}{12} n \leq \frac{n}{2} - \frac{1-6\epsilon}{12} n < \frac{n}{2}.$$

Thus, for the additive form of Chernoff bound (Theorem 5 in Appendix A), we have that

$$\mathbb{P}\left(Q \geq \frac{n}{2} \mid \mathbf{x}\right) \leq \mathbb{P}\left(Q \geq \mathbb{E}[Q \mid \mathbf{x}] + \frac{1-6\epsilon}{12} n\right) \leq \exp\left(-\frac{2(1-6\epsilon)}{144} n\right).$$

On the other hand, the function

$$f(q) - \frac{\epsilon}{(1+6\epsilon)}n = \frac{3}{4} \left(\frac{1+6\epsilon}{n} \right) q^2 - \frac{1+6\epsilon}{2}q + \frac{5+6\epsilon}{12}n - \frac{\epsilon}{(1+6\epsilon)}n$$

has its minimum in $\bar{q} = \frac{n}{3}$. Then

$$f(\bar{q}) - \frac{\epsilon}{(1+6\epsilon)}n = -\frac{1+6\epsilon}{12}n + \frac{5+6\epsilon}{12}n - \frac{\epsilon}{(1+6\epsilon)}n = \frac{n}{3} \left(\frac{1+3\epsilon}{1+6\epsilon} \right),$$

which is at most $\mathbb{E}[Q | \mathbf{x}] - \frac{6\epsilon n}{3(1+6\epsilon)}$. From the additive form of Chernoff bound (Theorem 5 in Appendix A), this implies the following.

$$\mathbb{P} \left(Q \leq \frac{1-3\epsilon}{1+6\epsilon} | \mathbf{x} \right) \leq \mathbb{P} \left(Q \leq \mathbb{E}[Q | \mathbf{x}] - \frac{2\epsilon n}{1+6\epsilon} | \mathbf{x} \right) \leq \exp \left(-\frac{8\epsilon^2 n}{(1+6\epsilon)^2} \right),$$

which, together with the previous result, gives the thesis. \square

Proof of Lemma 21. Consider $f(q) = \frac{3}{4} \left(\frac{1+6\epsilon}{n} \right) q^2 - \frac{1+6\epsilon}{2}q + \frac{5+6\epsilon}{12}n$, which no less than $\mathbb{E}[Q | \mathbf{x}]$. We see that, for $q > n/2$, the following is true:

$$\tilde{f}(q) = f(q) - q \left(\frac{2}{3} + 2\epsilon \right) \leq 0.$$

Indeed,

$$\tilde{f}(q) = \frac{3}{4} \left(\frac{1+6\epsilon}{n} \right) q^2 - \frac{7+30\epsilon}{6}q + \frac{5+6\epsilon}{12}n,$$

which has its maximum in one of the two boundaries, namely $q = n/2$ and $q = n$. We compute the expression in these quantities.

$$\tilde{f}(n/2) = \frac{(9+54\epsilon)n - (28+120\epsilon)n + (5+6\epsilon)n}{48} = -\frac{7+30\epsilon}{24}n < 0,$$

$$\tilde{f}(n) = \frac{(9+54\epsilon)n - (14+60\epsilon)n + (5+6\epsilon)n}{12} = 0.$$

Then, we have that

$$\mathbb{E}[Q | \mathbf{x}] \leq q \left(\frac{2}{3} + 2\epsilon \right).$$

The additive form of Chernoff bound (Theorem 5 in Appendix A) implies that

$$\begin{aligned} \mathbb{P} \left(Q \geq q \left(\frac{5}{6} + \epsilon \right) | \mathbf{x} \right) &= \mathbb{P} \left(Q \geq q \left(\frac{2}{3} + 2\epsilon \right) + q \left(\frac{1}{6} - \epsilon \right) | \mathbf{x} \right) \\ &\leq \mathbb{P} \left(Q \geq \mathbb{E}[Q | \mathbf{x}] + q \left(\frac{1}{6} - \epsilon \right) | \mathbf{x} \right) \\ &\leq \mathbb{P} \left(Q \geq \mathbb{E}[Q | \mathbf{x}] + \frac{n}{2} \left(\frac{1}{6} - \epsilon \right) | \mathbf{x} \right) \\ &\leq \exp \left(\frac{n}{2} \left(\frac{1}{6} - \epsilon \right)^2 \right). \end{aligned}$$

which gives the thesis. \square

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