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Phase Transition of a Non-Linear Opinion Dynamics with Noisy Interactions

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Abstract

In several real *Multi-Agent Systems* (MAS), it has been observed that only weaker forms of *metastable consensus* are achieved, in which a large majority of agents agree on some opinion while other opinions continue to be supported by a (small) minority of agents. In this work, we take a step towards the investigation of metastable consensus for complex (non-linear) *opinion dynamics* by considering the popular UNDECIDED-STATE dynamics in the binary setting, which is known to reach consensus exponentially faster than the VOTER dynamics. We propose a simple form of uniform noise in which each message can change to another one with probability p and we prove that the persistence of a metastable consensus undergoes a phase transition for $p = \frac{1}{6}$. In detail, below this threshold, we prove the system reaches with high probability a metastable regime where a large majority of agents keeps supporting the same opinion for polynomial time. Moreover, this opinion turns out to be the initial majority opinion, whenever the initial bias is slightly larger than its standard deviation. On the contrary, above the threshold, we show that the information about the initial majority opinion is “lost” within logarithmic time even when the initial bias is maximum. Interestingly, we show our results have explicit connections to two different concrete frameworks. The first one concerns a specific setting of a well-studied value-sensitive decision mechanism inspired by cross-inhibition in house-hunting honeybee swarms. The second framework consists of a consensus process where a subset of agents behave in a stubborn way.

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1 Introduction

We consider a fully-decentralized Multi-Agent System (for short, MAS) formed by a set of n agents (i.e. nodes) which mutually interact by exchanging messages over an underlying communication graph. In this setting, opinion dynamics are mathematical models to investigate the way a fully-decentralized MAS is able to reach some form of consensus. Their study is a hot topic touching several research areas such as MAS [16, 25], Distributed Computing [7, 23, 34], Social Networks [1, 47], and System Biology [12, 13]. Typical examples of opinion dynamics are the Voter Model, the averaging rules, and the majority rules. Some of such dynamics share a surprising efficiency and resiliency that exploit common computational principles, as they rely on simple, lightweight, local, elementary rules [7, 23, 34]. A definition has been given by [7].

Within such framework, the tasks of (valid) consensus and majority consensus have attracted a lot of attention within different application domains in swarm robotics [46, 49, 50], in biological systems [31], in social networks [47], passively-mobile sensor networks [2] and chemical reaction networks [17]. In the consensus task, the system is required to converge to a stable configuration where all agents supports the same opinion

and this opinion must be valid, i.e. it must be supported by at least one agent in the initial configuration. While, in the majority consensus task, starting from an initial configuration where there is some positive bias towards one majority opinion, the system is required to converge to the configuration where all agents support the initial majority opinion. Here, the bias of a configuration is defined as the difference between the number of agents supporting the majority opinion (for short, we name this number as majority) and the number of agents supporting the second-largest opinion.

Different opinion dynamics have been studied in a variety of settings [18, 30], and then used as subroutine to solve more complex computational tasks [11, 22, 53]. As concrete instances of the latter scenario in fully-decentralized swarm of robots, we mention some applications of simple majority dynamics. In the presence of differential delays, dynamics have been used to coordinate the search for a shortest path [46]; they have been used in the collective decision making procedure for the Best-of- N problem, in the binary setting ($N = 2$), in presence of stubborn agents and spontaneous opinion switching (noise) [49]; finally, they have been considered in the study of the relation between the network topology (*social feedback*) and the presence of noise in order to perform coherent collective decision making [50].

In the aforementioned applicative scenarios, it has been nevertheless observed that only weaker forms of metastable consensus are achieved, in which the large majority of agents rapidly achieves a consensus (while other opinions continue to be supported by a small set of agents), and this setting is preserved for a relatively long regime. Models that have been considered to study such phenomena include MAS where: i) agents follow a linear dynamics, such as the VOTER model or the AVERAGING dynamics and ii) a small set of stubborn agents are present in the system [44, 45, 57], or the local interactions are affected by communication noise [55]. By linear dynamics, we refer to all those dynamics in which the local update rule is a linear function of the neighbor states.

We emphasize that the VOTER model has a slow (i.e. polynomial in the number n of agents) convergence time even in a fully-connected network (i.e. in the complete graph) and it does not guarantee a high probability to reach consensus on the initial majority opinion, even starting from a large initial bias (i.e. $\Theta(n)$, where n is the number of the agents of the system) [36]. On the other hand, the averaging dynamics requires agents to perform numerical operations and, very importantly, to have a large local memory (to guarantee a good-enough approximation of real numbers). For the reasons above, these aforementioned and known linear opinion dynamics cannot explain fast and reliable metastable consensus phenomena observed in some MAS [11, 17, 31].

The above discussion naturally leads us to investigate the behavior of other, non-linear dynamics in the presence of stubborn agents and/or communication noise. In a MAS, with n agents and working on the complete communication networks, we combine a simple model of communication noise with a popular dynamics, known as the UNDECIDED-STATE dynamics. Formerly introduced by [2] as an (asynchronous) population protocol for the majority consensus, in some previous papers [48] it has been also called the Third-State dynamics. We here prefer the term “undecided” since it well captures the role of this additional state.

The state of every agent can be either an opinion (chosen from a finite set Σ) or the undecided state. According to the parallel synchronous version of this simple dynamics, at every discrete-time step (i.e. round), every agent “pulls” the state of a random neighbor and updates its state according to the following rule: if a non-undecided agent pulls a different opinion from its current one, then it will get undecided, while in all other cases it keeps its opinion; moreover, if the node is undecided then it will get the state of the pulled neighbor.

The parallel UNDECIDED-STATE dynamics is known to compute consensus (and majority consensus) on the complete network within a logarithmic number of rounds [14] and, very importantly, it is optimal in terms of local memory since it requires just one extra state/opinion [42]. We remark that other, more complex non-linear dynamics for several versions of consensus problems have been studied via experiments and simulations in the context of swarm systems [33, 41].

While communication noise is a common feature of real-world systems and its effects have been thoroughly investigated in physics and information theory [19], its study has been mostly focused on settings in which communication happens over stable links where the use of error-correcting codes is feasible since message of large size are allowed; it has been otherwise noted that when interactions among the agents are random and opportunistic and consist of very-short messages, classical information-theoretic arguments do not carry on and new phenomena call for a theoretical understanding [12].

1.1 Our Contribution

In this work we show that, under a simple model of uniform noise, the UNDECIDED-STATE dynamics exhibits an interesting phase transition. We consider the binary case (i.e. $|\Sigma| = 2$) together with an oblivious and symmetric action of noise over messages: any sent message is changed upon being received to any value, independently and uniformly at random with probability p (where p is any fixed positive constant $< 1/3$).

We show the following. On one hand, if $p < 1/6$, starting from an arbitrary configuration of the complete network of n agents, we prove that the system with high probability¹ (w.h.p., for short) reaches, within logarithmic time, a metastable almost consensus regime where the bias towards one fixed valid opinion keeps large for an exponential number of rounds (Theorem 3). In particular, despite the presence of random communication noise, our result implies that the UNDECIDED-STATE dynamics is able to rapidly break the initial symmetry of any balanced configuration (e.g. the perfectly-balanced configuration with $n/2$ agents having one opinion and the other $n/2$ agents having the other opinion) and reach a metastable regime of almost consensus. Importantly enough, we also show that, for any $p < 1/6$, the system is able to “compute” the task of almost majority consensus, i.e. an almost consensus on the initial majority opinion, whenever the initial bias towards a majority community is of order $\Omega(\sqrt{n \log n})$.

On the other hand, if $p > 1/6$, even when the initial bias is maximum (i.e. when the system starts from any full-consensus configuration), after a logarithmic number of rounds, the information about the initial majority opinion is “lost”. We also provide some computer simulations that confirm our theoretical results (see Section 5 for further details).

Our work provides a theoretical analysis of MAS; however, we also establish clean connections with two popular concrete frameworks. The first framework consists of a value-sensitive decision mechanism inspired by cross-inhibition in house-hunting honeybee swarms [51] (formerly introduced by [52]). In Section 2.1, we show that a specific parameter range of this system can be fully described using our noise model and, thus, we can apply our results for this important task.

Moreover, in Section 2.3, we show that our noise model is equivalent to a noiseless setting in which stubborn agents are present in the system [57] (that is, agents that never change their state): we thus obtain an analogous phase transition in this setting. For more considerations about our results, we defer the reader to Section 6.

The obtained phase transition thus separates qualitatively the behavior of the UNDECIDED-STATE dynamics from that of the VOTER model which is, to the best of our knowledge, the only linear opinion dynamics (with a finite opinion set) which has been rigorously analyzed in the presence of communication noise or stubborn agents [45, 57]. For this dynamics, it has been shown that no form of consensus can be reached in the presence of uniform noise: this hints at a more general phenomenon for non-linear dynamics with fast convergence to some metastable consensus.

We believe this work contributes to the research endeavour of exploring the interplay between communication noise and stochastic interaction pattern in MAS. As we will discuss in the related work section, despite the fact that these two characteristics are quite common in real-world MAS, their combined effect is still far from being understood and poses novel mathematical challenges.

1.2 Further Related Work

The UNDECIDED-STATE dynamics has been originally studied as an efficient majority-consensus protocol by [2] and independently by [9] for the binary case (i.e. with two initial input values). They proved that w.h.p., within a logarithmic number of parallel rounds, all agents support the initial majority opinion. Some works have then extended the analysis of the UNDECIDED-STATE dynamics to non-complete topologies. In the Poisson-clock model (formally equivalent to the population protocol model), [27] derived an upper bound on the expected convergence time of the dynamics that holds for arbitrary connected graphs, which is based on the location of eigenvalues of some contact rate matrices. They also instantiate their bound for particular network topologies. Successively, [42] provided an analysis when the initial states of agents are assigned independently at random, and they also derive “bad” initial configurations on certain graph topologies such that the initial minority opinion eventually becomes the majority one. As for the use of UNDECIDED-STATE

¹An event E depending on a parameter n holds with high probability w.r.t. n if a constant $\gamma > 0$ exists such that $\mathbf{P}(E) \geq 1 - (1/n)^\gamma$.

as a generic consensus protocol, the recent work by [14] proves that, in the synchronous Uniform-PULL model, in which all agents update their state in parallel by observing the state of a random other node, the convergence time of the UNDECIDED-STATE dynamics is w.h.p. logarithmic.

The motivation to investigate opinion dynamics is twofold: they can be regarded as simplistic models of several real-world phenomena or as building blocks for more complex algorithms. While on the modelling side the UNDECIDED-STATE dynamics is an appealing model of opinion dynamics and it has also been considered as a model of some mechanism occurring in the biology of a cell [13], it has been employed as a sub-routine of efficient majority consensus protocols: [35], [10] and [29] consider majority consensus in the Uniform-PULL model, and design protocols (based on the UNDECIDED-STATE dynamics) which w.h.p. converge in poly-logarithmic time even if the number of initial opinions is very large.

Communication noise in random-interacting MAS area of investigation have been studied in several settings [5, 6, 16, 55]. However, most of the research studies focus on stable networks.² Among the investigations of communication noise in MAS, we note the Vicsek model [54], where agents are driven with a constant absolute velocity, and at each time step assume the average direction of motion of the agents in their neighborhood: this strategy is then combined with some random perturbation. The authors show that the average velocity of their model exhibits a phase transition around some critical value of the model parameters which include the noise.

Recently, [31] consider a settings in which agents interact uniformly at random by exchanged binary messages which are subject to noise. In detail, the authors provide simple and efficient protocols to solve the classical distributed-computing problems of broadcast (a.k.a rumor spreading) and majority consensus, in the Uniform-PUSH model with binary messages, in which each message can be changed upon being received with probability $1/2 - \epsilon$. We recall that in the Uniform-PUSH model nodes do not actively observe neighbor states but instead send their states to randomly chosen neighbors at each round. The state update occurs when a message is received. Their results have been generalized to the majority consensus problem for the multi-valued case by [32]. When the noise is constant, [31] prove that in their noisy version of the Uniform-PUSH model, the broadcast problem can be solved in logarithmic time. Rather surprisingly, [12] and [15] prove that solving the broadcast problem in the Uniform-PULL model takes linear time, while the time to perform majority consensus remains logarithmic in both models. In the context of swarm robots, some complex, non-linear dynamics have been employed to solve some versions of the best-of- n problem, in the presence of different forms of noise [20, 40].

The fact that real-world systems such as social networks fail to converge to consensus has been extensively studied in various disciplines; formal models developed to investigate the phenomenon include the multiple-state Axelrod model [4] and the bounded-compromise model by [56]; the failure to reach consensus in these models is due to the absence of interaction among agent opinions which are “too far apart”. A different perspective is offered by models which investigate the effect of stubborn agents (also known as zealotry in the literature), in which some stubborn/zealot agents never update their opinion.

Several works have been devoted to study such effects under linear models of opinion dynamics. Starting with [44], who proposes a statistical-physics method in order to study the VOTER model under the presence of a stubborn agent, followed by [45] who consider the case of several stubborn agents in the system. Later investigations analyzed various aspects of the stationary distribution of the systems, such as the works by [1] and [3], which investigate the relationship between the behavior of the opinion dynamics and the structure of the underlying interaction graph, or that by [57], which considers the VOTER dynamics and study the first and second moments of the number of the average agents’ opinion.

[21] perform an extensive experimental comparison between the weighted version of the VOTER model and a generalized version of the UNDECIDED-STATE dynamics where the state of the agents and the updating rules try to capture the strength of the specific belief each agent has on its current opinion by also using different activity periods. More in detail, each agent shows its own opinion A for a time window which is proportional to its current belief on A and this belief is, in turn, a function of its previous local interactions. The obtained results give an experimental evidence of the existence of some trade-off between the resilience and the convergence time. The resilience considered by [21] is w.r.t. some malfunctioning agents with the potential to disrupt this desired behavior by making decisions on the basis of random beliefs.

Very recently, [24] analyze another well-known non-linear opinion dynamics, the 3-MAJORITY [7], under

²By stable networks, we mean a network where communication between agents can be modeled as a classical channel the agents can use to exchange messages at will [19].

the same model of uniform communication noise we adopted in this work. The authors show an analogous phase-transition for the considered dynamics but at a lower noise threshold, thus establishing the higher resilience of the UNDECIDED-STATE dynamics w.r.t. the 3-MAJORITY dynamics. Notice that the definition of the noise adopted by [24] is different, but still equivalent to ours. The results by [24] corroborate our intuition that fast non-linear opinion dynamics may exhibit a common qualitative behavior under uniform communication noise.

1.3 Roadmap of the paper

In Section 2, we give some preliminaries, the equivalence result with a best-of- N selection process, and between communication noise and stubborn agents. In Section 3, we provide the probabilistic analysis of the UNDECIDED-STATE process when the initial bias is relatively large and its consequences on almost majority consensus. This analysis will be then combined with the analysis of the symmetry-breaking phase given in Section 4 to obtain our results on almost consensus. Some computer simulations validating experimentally our theoretical results are shown in Section 5. In Section 6 we discuss our findings and their applicability. At the end of the paper, there is an appendix which contains the probabilistic tools we used throughout the analysis (Appendix A) and the omitted proofs (Appendix B).

2 Preliminaries

We study the discrete-time, parallel version of the UNDECIDED-STATE dynamics on the complete graph in the binary setting [14]. In detail, there is an additional state/opinion, i.e. the undecided state, besides the two possible opinions (say, opinion *Alpha* and opinion *Beta*) an agent can support, and, in the absence of noise, the updating rule works as follows: at every round $t \in \mathbb{N}$, each agent u chooses a neighbor v (or, possibly, itself) independently and uniformly at random and, at the next round, it gets a new opinion according to the rule given in Table 1.³ The definition of noise we consider is the following.

$u \setminus v$	undecided	<i>Alpha</i>	<i>Beta</i>
undecided	undecided	<i>Alpha</i>	<i>Beta</i>
<i>Alpha</i>	<i>Alpha</i>	<i>Alpha</i>	undecided
<i>Beta</i>	<i>Beta</i>	undecided	<i>Beta</i>

Table 1: The update rule of the USD.

Definition 1 (Definition of noise). *Let p be a real number in the interval $(0, 1/3]$. When an agent u chooses a neighbor v and looks at (pulls) its opinion, it sees v 's opinion with probability $1 - 3p$, and, with probability $3p$, it sees one opinion chosen uniformly at random (u.a.r. for short).*

For instance, if v supports opinion *Alpha*, then u sees *Alpha* with probability $1 - 2p$, it sees *Beta* with probability p , and it sees the undecided state with probability p . In this work, the terms “agent” and “node” are interchangeable. In both its non-noisy and noisy versions, the UNDECIDED-STATE dynamics can be seen as a specific case of a best-of- N selection process in honeybees [51].

2.1 Best-of- N nest-site selection process in honeybees.

In this subsection, we describe an equation system that models the decision process of a bee swarm to select a new nest among N possible ones. This mathematical model has been introduced by [52] and also analysed by [51] (see Eq.s (1) in the latter work). In the next subsection, we show how to derive the UNDECIDED-STATE dynamics.

In detail, consider any population of bees. Let x_i denote the proportion of bees agreeing on (which are *committed* to) option i , and x_u be the proportion of bees which are undecided (or *uncommitted*). The dynamics is described by the following system

³Notice that this dynamics requires no labeling of the agents, i.e. the network can be anonymous.

$$\begin{cases} \frac{dx_i}{dt} = \gamma_i x_u - \alpha_i x_i + \rho_i x_u x_i - x_i \sum_{1 \leq i \leq N} \beta_{ji} x_j; \\ x_u = 1 - \sum_{1 \leq i \leq N} x_i. \end{cases} \quad (1)$$

We now provide an intuitive explanation of the equation system, and the description of the parameters. Uncommitted bees explore the environment and discover possible options. While encountering option i , the uncommitted bee estimates its quality ν_i and may commit to that option at a rate γ_i , where $\gamma_i \propto \nu_i$ (more frequent commitments to better quality nests). Committed bees may spontaneously revert, through *abandonment*, to an uncommitted state at rate α_i . The abandonment rate is assumed to be inversely proportional to the option's quality $\alpha_i (\propto \nu_i^{-1})$. This abandonment feature allows bees to quickly discard bad options, and endows the swarm with a degree of flexibility since bees are not locked into their commitment state [51]. In addition to these two spontaneous individual transitions, bees interact with each other to achieve agreement in two forms: *recruitment* and *cross-inhibition*. The rate at which uncommitted bees are recruited to option i depends on the number of bees committed to i and on the strength of the recruitment process for i , that is $\rho_i (\propto \nu_i)$. Thus, the recruitment process is a form of positive feedback. On the other hand, cross-inhibition is a form of negative feedback: when a bee committed to option i encounters another bee committed to another option j , (with $j \neq i$), the first may deliver stop signals to the second which reverts to an uncommitted state at a rate $\beta_{ji} (\propto \nu_i)$.

We discretize Eq. (1) by using the forward Euler method [38] with $dt = 1$, obtaining

$$\begin{cases} x_i^{(t+1)} = \gamma_i x_u^{(t)} - (\alpha_i + 1)x_i^{(t)} + \rho_i x_u^{(t)} x_i^{(t)} - x_i^{(t)} \sum_{1 \leq i \leq N} \beta_{ji} x_j^{(t)}; \\ x_u^{(t+1)} = 1 - \sum_{1 \leq i \leq N} x_i^{(t+1)}. \end{cases} \quad (2)$$

As we will show, a specific parameter choice yields exactly the mean-field behavior of the UNDECIDED-STATE dynamics.

2.2 Notation, Characterization, and Mean-Field Behavior.

In this subsection, we characterize the mean-field behavior of the UNDECIDED-STATE dynamics. To do so, we introduce some notation and describe useful properties of the process. Let us name C the set of all possible configurations; notice that, since the graph is complete and its nodes are anonymous, a configuration $\mathbf{x} \in C$ is uniquely determined by giving the number of *Alpha* nodes, $a(\mathbf{x})$ and the number of *Beta* nodes, $b(\mathbf{x})$. Accordingly to this notation, we call $q(\mathbf{x})$ the number of undecided nodes in configuration \mathbf{x} , and $s(\mathbf{x}) = a(\mathbf{x}) - b(\mathbf{x})$ the *bias* of the configuration \mathbf{x} . When the configuration is clear from the context, we will omit \mathbf{x} and write just a, b, q , and s instead of $a(\mathbf{x}), b(\mathbf{x}), q(\mathbf{x})$, and $s(\mathbf{x})$. The UNDECIDED-STATE dynamics with uniform noise defines then a finite-state non reversible Markov chain $\{\mathbf{X}_t\}_{t \geq 0}$ with state space C and no absorbing states.

The stochastic process yielded by the UNDECIDED-STATE dynamics, starting from a given configuration, will be denoted as UNDECIDED-STATE process. Once a configuration \mathbf{x} at a round $t \geq 0$ is fixed, i.e. $\mathbf{X}_t = \mathbf{x}$, we use the capital letters A, B, Q , and S to refer to random variables $a(\mathbf{X}_{t+1}), b(\mathbf{X}_{t+1}), q(\mathbf{X}_{t+1})$, and $s(\mathbf{X}_{t+1})$. Notice that we consider the bias as $a(\mathbf{x}) - b(\mathbf{x})$ instead of $|a(\mathbf{x}) - b(\mathbf{x})|$ since the expectation of $|A - B|$ is much more difficult to evaluate than that of $A - B$.

The round-by-round expected behaviors of the above key random variables is described by the following equations:

$$\mathbb{E}[A \mid \mathbf{X}_t = \mathbf{x}] = \frac{a}{n}(1 - 3p)(a + 2q) + p(2a + q); \quad (3)$$

$$\mathbb{E}[B \mid \mathbf{X}_t = \mathbf{x}] = \frac{b}{n}(1 - 3p)(b + 2q) + p(2b + q); \quad (4)$$

$$\mathbb{E}[S \mid \mathbf{X}_t = \mathbf{x}] = s \left(1 - p + \frac{q}{n}(1 - 3p) \right); \quad (5)$$

$$\mathbb{E}[Q \mid \mathbf{X}_t = \mathbf{x}] = pn + \frac{1 - 3p}{2n} [2q^2 + (n - q)^2 - s^2]. \quad (6)$$

The proof of these equations can be found in Appendix B. In general, when conditioning on a particular configuration at time t , we will write just \mathbf{x} instead of $\mathbf{X}_t = \mathbf{x}$. Finally, we remark that, conditional on some

configuration \mathbf{x} , the r.v.s A , B , Q , and S can be written, respectively, as $\sum_{u \in V} \alpha_u$, $\sum_{u \in V} \beta_u$, $\sum_{u \in V} q_u$, and $\sum_{u \in V} (\alpha_u - \beta_u)$, where V is the set of all agents, α_u, β_u, q_u are random variables that yield 1 if u supports opinion α , β , or the undecided state at the next round, respectively, and 0 otherwise. Then, we can apply concentration inequalities such as the Chernoff bounds (Lemmas 16 and 17) to A, B, Q , and the Hoeffding bounds (Lemma 18) to S .

Particular case of the Best-of- N selection process. Consider the equation system in Eq. (2), and the following parameter choice⁴ (for $i, j = 1, \dots, N$)

$$\begin{aligned}\rho_i &= 1 - (N + 1)p; \\ \alpha_i &= p; \\ \gamma_i &= p; \\ \beta_{i,i} &= 0; \\ \beta_{i,j} &= 1 - (N + 1)p; \text{ if } i \neq j,\end{aligned}$$

which yields the followings equations:

$$\begin{aligned}x_i^{(t+1)} &= x_i^{(t)} + px_u^{(t)} - px_i^{(t)} + [1 - (N + 1)p] x_u^{(t)} x_i^{(t)} \\ &\quad - \sum_{\substack{1 \leq j \leq N \\ j \neq i}} [1 - (N + 1)p] x_i^{(t)} x_j^{(t)}; \end{aligned} \tag{7}$$

$$x_u^{(t+1)} = 1 - \sum_{1 \leq i \leq N} x_i^{(t+1)}. \tag{8}$$

We can now observe that Eqs. (7) and (8) are the mean-field equations of the UNDECIDED-STATE process we consider in this paper. Indeed, if y_i represents the number of agents supporting opinion i at some given configuration \mathbf{x} , Y_i is the r.v. yielding the number of agents supporting opinion i at the next round (analogously for y_u, Y_u), by substituting $x_i^{(t+1)} = \mathbb{E}[Y_i | \mathbf{x}]$, $x_i^{(t)} = y_i$ (the same for y_u, Y_u) we obtain the expected round-by-round behavior of the UNDECIDED-STATE process. If $N = 2$, we get exactly Eqs. (3), (4) and (6).

Significance of the parameters. We assume that the two opinion/options (nest-sites) *Alpha* and *Beta* have the same quality. The rate of the recruitment process for option i is, in our setting, $\rho_i = 1 - 3p$, which implies that only noise is responsible for the non-recruitment of uncommitted bees by committed bees. Since all nests have the same qualities, we choose the rate at which a bee committed to option i reverts to an uncommitted state when encountering a bee committed to option $j \neq i$ to be $B_{j,i} = 1 - 3p = \rho_i$. The abandonment rate is the spontaneous reversion to an uncommitted state from a committed state (say, committed to option i), and here it is assumed to be governed only by noise at a rate $\alpha_i = p$. Such choices are reasonable in the setting described in Section 2.1. The only difference lies in the choice of γ_i , the spontaneous commission rate to option i for an uncommitted bee. Usually, such parameter is proportional to the nest quality. In our setting, the nest quality plays no role in this a spontaneous phenomenon; instead, only noise is responsible, which corresponds to the choice $\gamma_i = \alpha_i = p$. Even though our setting is a strong restriction of the more general framework described by Eq. (2) [51], we believe it effectively describes a realistic scenario in which there is a somewhat equivalence between the option qualities, and the evolution of the system is dominated by another important factor (i.e. noise). In particular, based on our analysis on the convergence time, this setting represents a sort of worst-case scenario for the swarm system that, in general, requires a relatively longer symmetry-breaking phase but still achieves an almost-agreement.

Expected equilibria. Notice that, in expectation, the bias keeps constant iff $s = 0$ or

$$1 - p + \frac{q}{n}(1 - 3p) = 1$$

⁴We remark that these parameter choice is not the common one indicated by [51].

$$\Longleftrightarrow q = \frac{pn}{1-3p}.$$

If we substitute the latter value in (6) we get that q keeps constant, in expectation, iff

$$\begin{aligned} & pn + \frac{1-3p}{2n} \left[\frac{2p^2n^2}{(1-3p)^2} + \frac{(1-4p)^2n^2}{(1-3p)^2} - s^2 \right] = \frac{pn}{1-3p} \\ \Longleftrightarrow & s^2 \cdot \frac{1-3p}{2n} = \frac{2p^2n + (1-4p)^2n - 6p^2}{2(1-3p)} \\ \Longleftrightarrow & s^2 = \frac{n^2}{(1-3p)^2} \cdot (1-6p)(1-2p) \\ \Longleftrightarrow & s = \pm \frac{n}{1-3p} \sqrt{(1-6p)(1-2p)}. \end{aligned}$$

Let

$$s_{\text{eq}} = \frac{n}{1-3p} \sqrt{(1-6p)(1-2p)} \text{ and } q_{\text{eq}} = \frac{pn}{1-3p}.$$

The configuration with $s = \pm s_{\text{eq}}$ and $q = q_{\text{eq}}$ is in equilibrium, in expectation. We will show that, if $p < 1/6$, the latter is an “attractive” configuration, i.e. the process converges and oscillates around the above configuration w.h.p., no matter what the initial parameters are. Instead, if $p > 1/6$, the process converges to a configuration with $s = 0$ w.h.p.

For the sake of the analysis, we define the following quantities:

$$\begin{aligned} \check{s}_\epsilon &= \frac{n}{1-3p} \sqrt{(1-6p)(1-2p) - \epsilon}; \\ \hat{s}_\epsilon &= \frac{n}{1-3p} \sqrt{(1-6p)(1-2p) + \epsilon}; \\ \check{q}_\epsilon &= \frac{(p-\epsilon)n}{1-3p}; \\ \hat{q}_\epsilon &= \frac{(p+\epsilon)n}{1-3p}. \end{aligned}$$

2.3 Oblivious Noise and Stubborn Agents.

We can now consider the following more general message-oblivious model of noise.

Definition 2. *We say that the communication is affected by oblivious noise if the value of any sent message changes according to the following scheme:*

- (i) *with probability $1 - p_{\text{noise}}$ independent from the value of the sent message, the message remains unchanged;*
- (ii) *otherwise, the noise acts on the message and it changes its value according to a fixed distribution $\mathbf{p} = p_1, \dots, p_m$ over the possible message values $1, \dots, m$.*

In other words, according to the previous definition of noise (Definition 1), the probability that the noise changes any message to message i is $p_{\text{noise}} \cdot p_i$. It is immediate to verify that the definition of noise adopted in Theorems 1 and 2 corresponds to the aforementioned model of oblivious noise in the special case $m = 3$, $p_{\text{noise}} = p$, and $p_{\text{Alpha}} = p_{\text{Beta}} = p_{\text{undecided}} = \frac{1}{3}$.

Recalling that an agent is said to be stubborn if it never updates its state [57], we now observe that the above noise model is in fact equivalent to consider the behavior of the same dynamics in a noiseless setting with stubborn agents.

Lemma 1. *Consider the UNDECIDED-STATE dynamics on the complete graph with opinions (i.e. message values) in $\Sigma = \{1, \dots, m\}$. The following two processes are equivalent, i.e. their restrictions on the former graph of n nodes have the same transition probabilities.*

- (a) the UNDECIDED-STATE process with n agents in the presence of oblivious noise with parameters p_{noise} and $\mathbf{p} = p_1, \dots, p_m$;
- (b) the UNDECIDED-STATE process with n agents and $n_{stub} = \frac{p_{noise}}{1-p_{noise}}n$ additional stubborn agents present in the system, of which: $n_{stub} \cdot p_1$ are stubborn agents supporting opinion 1, $n_{stub} \cdot p_2$ are stubborn agents supporting opinion 2, and so on.

Proof of Lemma 1. Consider the complete graph of n nodes, K_n , over which the former process runs. Consider also the complete graph $K_{n+n_{stub}}$, which contains a sub-graph isomorphic to K_n we denote as \tilde{K}_n . Let $H = K_{n+n_{stub}} \setminus \tilde{K}_n$ be the subgraph of stubborn nodes. In the former model (a), the probability an agent pulls opinion $j \in \{1, \dots, m\}$ at any given round is

$$(1 - p_{noise}) \frac{c_j}{n} + p_{noise} \cdot p_j ,$$

where c_j is the size of the community of agents supporting opinion j ; in the model defined in (b), the probability a non-stubborn agent pulls opinion j at any given round is

$$\frac{c_j + n_{stub} \cdot p_j}{n + n_{stub}} = \frac{c_j + \frac{p_{noise}}{1-p_{noise}}n \cdot p_j}{n + \frac{p_{noise}}{1-p_{noise}}n} = (1 - p_{noise}) \cdot \frac{c_j}{n} + p_{noise} \cdot p_j .$$

Let C and \tilde{C} be the set of all possible configurations of, respectively, K_n and \tilde{K}_n . The latter result implies there exists a bijection $\phi : C \rightarrow \tilde{C}$ such that the probability to go from configuration \mathbf{x} to \mathbf{y} in C is the same as that of going from configuration $\phi(\mathbf{x})$ to $\phi(\mathbf{y})$ in \tilde{C} . \square

Basically, this equivalence implies that any result we state for the process defined in (a) has an analogous statement for the process defined in (b).

3 Process analysis for biased initial configurations

In this section, we analyze the UNDECIDED-STATE process when the system starts from biased configurations. The following two theorems show the phase transition exhibited by this process. We remind that our notion of noise is that of Definition 1.

Theorem 1 (Almost Majority Consensus). *Let $0 < p < 1/6$ be any constant denoting the noise probability. Furthermore, let $\epsilon_1 = (1 - 6p)^2/2$, $\epsilon_2 = 2p(1 - p)$. Assume the system starts from any configuration \mathbf{x} having bias $s(\mathbf{x}) \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$. Then, the UNDECIDED-STATE process reaches a configuration \mathbf{y} having bias $s(\mathbf{y}) \in \Delta = [\hat{s}_{\epsilon_1}(1 - 2p)^2, \hat{s}_{\epsilon_2}]$ in $\mathcal{O}_{p,\gamma}(\log n)$ rounds, w.h.p. Moreover, starting from \mathbf{y} , the UNDECIDED-STATE process enters a (metastable) phase of length $\exp(\Omega_p(n))$ where the bias remains in the range Δ , w.h.p.*

Observe that if the theorem is true, then it also holds analogously for the symmetrical case in which $s(\mathbf{x}) \leq -\gamma\sqrt{n \log n}$. We also remark that, for any given constant c , by $g(n) = \mathcal{O}_c(f(n))$ we mean that there exists a constant $h(c)$ depending on c such that $g(n) \leq h(c) \cdot (f(n))$ for n large enough. We will also write $\Omega_c(f(n))$ with an analogous meaning.

Theorem 2 (Victory of Noise). *Let $1/6 < p \leq 1/3$ be any constant denoting the noise probability. Assume the system starts from any configuration \mathbf{x} with $|s(\mathbf{x})| \geq \gamma\sqrt{n \log n}$, for some constant $\gamma > 0$. Then, the UNDECIDED-STATE process reaches a configuration \mathbf{y} having bias $|s(\mathbf{y})| \leq \gamma\sqrt{n \log n}$ in $\mathcal{O}_{p,\gamma}(\log n)$ rounds, w.h.p. Furthermore, starting from such a configuration, the UNDECIDED-STATE process enters a (metastable) phase of length $n^{\Omega_{p,\gamma}(1)}$ rounds where the absolute value of the bias keeps bounded by $\mathcal{O}_{p,\gamma}(\sqrt{n \log n})$ w.h.p.*

The next subsections are devoted to the proof of Theorem 1 (Section 3.1) and Theorem 2 (Section 3.2). We here just remark that the adopted arguments in the two proofs are similar.

Let us now consider the equivalent model with stubborn agents according to Lemma 1, in which $p_{noise} = 3p$ and $p_{Alpha} = p_{Beta} = p_{undecided} = \frac{1}{3}$. We thus have $n_{stub} = \frac{3p}{1-3p}n$ additional stubborn nodes, of which $n_{stub} \cdot \frac{1}{3} = \frac{p}{1-3p}n$ support opinion *Alpha*, $n_{stub} \cdot \frac{1}{3} = \frac{p}{1-3p}n$ opinion *Beta*, and $n_{stub} \cdot \frac{1}{3} = \frac{p}{1-3p}n$ are undecided. On this new graph of $n + n_{stub}$ nodes, let the UNDECIDED-STATE dynamics run and call the resulting process the STUB process. The next result is an immediate corollary of the two previous theorems.

Corollary 1. *Let $0 < p < 1/3$ be any constant, and consider the STUB process with $n_{stub} = \frac{3p}{1-3p}n$ additional stubborn agents equally divided between the three opinions. Suppose the system starts from any configuration having bias $s \geq \gamma\sqrt{n \log n}$ for any constant $\gamma > 0$. If $p < 1/6$, then, in $\mathcal{O}_{p,\gamma}(\log n)$ rounds, the STUB process enters a metastable phase of almost consensus of length $\exp(\Omega_p(n))$ in which the bias is $\Theta(n)$, w.h.p. If $p \in (1/6, 1/3]$, then, in $\mathcal{O}(\log n)$ rounds, the STUB process enters a metastable phase of length $n^{\Omega_{p,\gamma}(1)}$ where the absolute value of the bias keeps bounded by $\mathcal{O}_{p,\gamma}(\sqrt{n \log n})$, w.h.p.*

Trivially, the corollary holds also in the symmetrical case in which $s \leq -\gamma\sqrt{n \log n}$.

3.1 Proof of Theorem 1

Informally, while the analysis is technically involved, it can be appreciated from it that the phase transition phenomenon at hand relies ultimately on the exponential drift of the UNDECIDED-STATE towards the majority opinion in the absence of noise: as long as the noise is kept under a certain threshold, the dynamics manages to quickly amplify and sustain the bias towards the majority opinion; as soon as the noise level reaches the threshold, the expected increase of the majority bias abruptly decreases below the standard deviation of the process and the ability of the dynamics to preserve a *signal* towards the initial majority rapidly vanishes.

We now proceed with the formal analysis. Wlog, in the remainder, for a given starting configuration \mathbf{x} , we will assume $a(\mathbf{x}) \geq b(\mathbf{x})$. Indeed, as it will be clear from the results, if $s(\mathbf{x}) \geq \gamma\sqrt{n \log n}$, then the plurality opinion does not change for $\exp(\Omega_p(n))$ rounds, w.h.p., and the argument for the case $b(\mathbf{x}) > a(\mathbf{x})$ is symmetric. Let $p < 1/6$. The key point to prove the first claim of the theorem is to show that, if the bias of the configuration is less than \tilde{s}_ϵ , and the number of undecided nodes is above \hat{q}_ϵ , then the bias at the next round increases by a constant factor and the number of undecided nodes remains greater than \hat{q}_ϵ , w.h.p.

Lemma 2. *Let \mathbf{x} be a configuration such that $q \geq \hat{q}_\epsilon$ and $s \geq 0$ for any arbitrarily small constant $\epsilon > 0$. Then, in the next round, $S \geq s(1 + \epsilon/2)$, with probability $1 - \exp(-\epsilon^2 s^2 / (2^3 n))$.*

Remark 1. If $s \geq \gamma\sqrt{n \log n}$ for any constant $\gamma > 0$, then the statement holds with probability $1 - n^{-\epsilon^2 \gamma^2 / 2^3}$.

Proof of Lemma 2. We first notice that $q \geq \hat{q}_\epsilon$ in eq. (5) implies

$$\mathbb{E}[S \mid \mathbf{x}] \geq s(1 + \epsilon).$$

Let $\lambda = (\epsilon/2) \cdot (s/n)$. Then, for the Hoeffding bound (Lemma 18).

$$\begin{aligned} \mathbb{P}\left[S \geq s\left(1 + \frac{\epsilon}{2}\right) \mid \mathbf{x}\right] &= \mathbb{P}[S \geq s(1 + \epsilon) - \lambda n \mid \mathbf{x}] \geq 1 - \exp\left(-\frac{2\lambda^2 n}{4}\right) \\ &= 1 - \exp\left(-\frac{\epsilon^2 s^2}{2^3 n}\right). \end{aligned}$$

□

Lemma 3. *Let ϵ be any constant with $(1 - 6p)^2/2 \leq \epsilon < (1 - 6p)(1 - 2p)$. Let \mathbf{x} be a configuration such that $s \leq \tilde{s}_\epsilon$. Then, in the next round, $Q \geq \hat{q}_{\epsilon/12}$ with probability $1 - \exp(-\epsilon^2 n / (2^3 3^2 (1 - 3p)^2))$.*

Proof of Lemma 3. Notice that eq. (6) reaches its minimum in $q = n/3$. Combined with $s \leq \tilde{s}_\epsilon$, we get

$$\begin{aligned} \mathbb{E}[Q \mid \mathbf{x}] &\geq pn + \frac{1 - 3p}{2n} \left[\frac{2n^2}{3} - \frac{(1 - 6p)(1 - 2p)n^2 - \epsilon n^2}{(1 - 3p)^2} \right] \\ &= pn + \frac{2(1 - 3p)^2 n - 3(1 - 6p)(1 - 2p)n + 3\epsilon n}{6(1 - 3p)} \\ &\geq \frac{6pn - 18p^2 n + 2n - 12pn + 18p^2 n - 3n + 24pn - 36p^2 n + 3\epsilon n}{6(1 - 3p)} \\ &= \frac{-36p^2 n + 18pn - n + 3\epsilon n}{6(1 - 3p)} = \frac{-(1 - 6p)^2 n + 6pn + 3\epsilon n}{6(1 - 3p)}. \end{aligned}$$

Now, for $(1 - 6p)^2/2 \leq \epsilon < (1 - 6p)(1 - 2p)$, we get

$$\mathbb{E}[Q \mid \mathbf{x}] \geq \frac{-(1 - 6p)^2 n + 6pn + 3\epsilon n}{6(1 - 3p)} \geq \frac{(6p + \epsilon)n}{6(1 - 3p)} = \frac{p + \epsilon/6}{1 - 3p} n = \hat{q}_{\frac{\epsilon}{6}}.$$

By the additive form of Chernoff bound (Lemma 17), we get

$$\mathbb{P}[Q \leq \hat{q}_{\frac{\epsilon}{12}} \mid \mathbf{x}] = \mathbb{P}\left[Q \leq \hat{q}_{\frac{\epsilon}{6}} - \frac{\epsilon}{12(1 - 3p)} n \mid \mathbf{x}\right] \leq \exp\left(-\frac{\epsilon^2 n}{2^3 3^2 (1 - 3p)^2}\right).$$

□

As we will see at the end of this section, the two lemmas above ensure that the system eventually reaches a configuration \mathbf{y} with bias $s(\mathbf{y}) > \check{s}_\epsilon$ within $\mathcal{O}(\log n)$ rounds, w.h.p. We now consider configurations in which $s > \check{s}_\epsilon$ and derive a useful bound on the possible decrease of s .

Lemma 4. *Let \mathbf{x} be any configuration such that $s \geq \gamma\sqrt{n \log n}$ for any constant $\gamma > 0$. Then, in the next round, it holds that $S \geq s(1 - 2p)$ with probability $1 - \exp(-p^2 s^2 / (2n))$.*

Remark 2. If $s \geq \gamma\sqrt{n \log n}$ for any constant $\gamma > 0$, then the statement holds with probability $1 - n^{-p^2 \gamma^2 / 2}$.

Proof of Lemma 4. From eq. (5), we get

$$\mathbb{E}[S \mid \mathbf{x}] \geq s(1 - p).$$

Let $\lambda = p \cdot (s/n)$. Then, for the Hoeffding bound (Lemma 18),

$$\mathbb{P}[S \geq s(1 - 2p) \mid \mathbf{x}] = \mathbb{P}[S \geq s(1 - p) - \lambda n \mid \mathbf{x}] \geq 1 - \exp\left(-\frac{p^2 s^2}{2n}\right).$$

□

Lemma 4 is used to control the decrease of the bias, showing that it keeps of magnitude $\Theta(n)$ for a polynomial number of rounds. The next two lemmas provide an upper bound on the bias during this phase.

Lemma 5. *Let \mathbf{x} be any configuration, and let $\epsilon = (p/2) \cdot (1 + 3p)$. Then, in the next round, $Q \geq \check{q}_\epsilon$, with probability $1 - \exp(-p^2 n / 2)$.*

Proof of Lemma 5. From Eq. (6), we get

$$\mathbb{E}[Q \mid \mathbf{x}] \geq pn + \frac{1 - 3p}{2n} [2q^2 + (n - q)^2 - (n - q)^2] = pn + \frac{(1 - 3p)q^2}{n} \geq pn.$$

where we used that $s \leq n - q$. Let $\lambda = pn/2$. For the additive form of Chernoff bound (Lemma 17), we get

$$\mathbb{P}\left[Q \geq \frac{pn}{2} \mid \mathbf{x}\right] = \mathbb{P}[Q \geq pn - 2\lambda \mid \mathbf{x}] \geq 1 - \exp\left(-\frac{2\lambda^2}{n}\right) = 1 - \exp\left(-\frac{p^2 n}{2}\right).$$

We conclude the proof by observing that

$$\frac{(p - \epsilon)n}{1 - 3p} = \frac{p(1 - 3p)n}{2(1 - 3p)} = \frac{pn}{2}.$$

□

Lemma 6. *Let \mathbf{x} be a configuration with $q \geq \check{q}_{\epsilon_1}$, for any constant $0 < \epsilon_1 < p$. Let $\epsilon_2 = 2p(1 - p)$. Then, in the next round, $S \leq \hat{s}_{\epsilon_2}$ with probability $1 - \exp(-(p^2/2) \cdot (p - \epsilon_1)^2 n / (1 - 3p)^2)$.*

Proof of Lemma 6. From Eq. (4), we get that

$$\mathbb{E}[B \mid \mathbf{x}] \geq pq \geq p\tilde{q}_{\epsilon_1} = \frac{p(p - \epsilon_1)n}{1 - 3p}.$$

Let $\lambda = (p/2) \cdot (p - \epsilon_1)n/(1 - 3p)$. Then, Lemma 17 implies that

$$\begin{aligned} \mathbb{P}\left[B \geq \frac{p(p - \epsilon_1)n}{2(1 - 3p)} \mid \mathbf{x}\right] &= \mathbb{P}\left[B \geq \frac{p(p - \epsilon_1)n}{1 - 3p} - \lambda \mid \mathbf{x}\right] \geq 1 - \exp\left(-\frac{2\lambda^2}{n}\right) \\ &= 1 - \exp\left(-\frac{p^2(p - \epsilon_1)^2n}{2(1 - 3p)^2}\right). \end{aligned}$$

Notice that $S \leq n - 2B$. Since

$$n - \frac{p(p - \epsilon_1)n}{1 - 3p} \leq \tilde{s}_{\epsilon_2},$$

for $\epsilon_2 = 2p(1 - p)$, then $S \leq \hat{s}_{\epsilon_2}$ with probability $1 - \exp(-(p^2/2) \cdot (p - \epsilon_1)^2n/(1 - 3p)^2)$. \square

Proof of Theorem 1. Suppose the initial configuration bias is outside the interval $I = [\tilde{s}_{\epsilon_1}, \hat{s}_{\epsilon_2}]$. By Lemmas 5 and 6, we have that in two rounds the bias is less than or equal to \hat{s}_{ϵ_2} with probability $1 - 2\exp(-p^4n/2^3)$ for the union bound, and keeps bounded by the same value for $T = \exp(-p^4n/2^4)$ rounds with probability $1 - \exp(-\Omega_p(n))$ for Lemmas 20 and 21.

Lemma 4 and Remark 2 imply that the bias at the second round is no less than $\gamma'\sqrt{n \log n}$ with probability $1 - n^{-p^2\gamma^2/2}$, with $\gamma' = \gamma(1 - 2p)$. Define the event

$$\begin{aligned} \mathcal{B}_k &= \left\{s(\mathbf{X}_k) \geq s(\mathbf{X}_{k-1}) \left(1 + \frac{\epsilon_1}{2 \cdot 12}\right)\right\}; \\ \mathcal{C}_k &= \left\{s(\mathbf{X}_k) \geq \gamma'\sqrt{n \log n}\right\} \cap \left\{q(\mathbf{X}_k) \geq \hat{q}_{\epsilon_1/12}\right\}; \\ \mathcal{D}_k &= \{s(\mathbf{X}_k) \geq \tilde{s}_{\epsilon_1}\} \\ \mathcal{E}_k &= \mathcal{B}_k \cap \mathcal{C}_k. \end{aligned}$$

Notice that, conditional on $\mathcal{E}_{k-1} \cap \mathcal{D}_{k-1}^C$, the probability of \mathcal{E}_{k+1} is $1 - 3n^{-(1/2^3)(\epsilon_1 ps)^2/n}$ for Lemmas 2 to 4, Remarks 1 and 2, and the union bound, where ϵ_1 depends only on p . Let $\tau^* = \min\{k \geq 0 : s(\mathbf{X}_k) \geq \tilde{s}_{\epsilon_1}\}$. Notice that $\cap_{k=1}^{T_2} (\mathcal{F}_k \cup \mathcal{D}_k) \subseteq \{\tau^* \leq T_2\}$, with $T_2 = \log n / \log(1 + \epsilon_1/24)$. Then, by Lemmas 20 and 21, we have that $\tau^* \leq T_2$ with probability $1 - n^{-\Omega_{p,\gamma}(1)}$. Let τ be the first time $s(\mathbf{X}_t)$ lies in the interval $I = [\tilde{s}_{\epsilon_1}, \hat{s}_{\epsilon_2}]$. For what we showed above, $\tau = \max\{2, \tau^*\}$, which is at most T_2 with high probability for the union bound.

Suppose now we start in a configuration inside the interval I . Then, the bias keeps bounded by \hat{s}_{ϵ_2} for time $T_1 - 2$ with probability $1 - \exp(-\Omega_p(n))$. Furthermore, it can decrease to, at most, $\tilde{s}_{\epsilon_1}(1 - 2p)^2$ for Lemma 4 with probability at least $1 - \exp(-\Omega_p(n))$ before starting to increase again towards interval I with probability at least $1 - \exp(-\Omega_p(n))$. Thus, the bias keeps inside the interval Δ for $\exp(\Omega_p(n))$ rounds, w.h.p. for the chain rule. \square

3.2 Proof of Theorem 2

This section provide all technical lemmas to prove Theorem 2. We assume the starting configuration \mathbf{x} to have bias $s(\mathbf{x}) = a(\mathbf{x}) - b(\mathbf{x}) \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$; the case in which $b(\mathbf{x}) > a(\mathbf{x})$ is analogous. Let $1/6 < p < 1/2$ be the probability of noise, with p being a constant.

We first show that if the bias is positive, it can become negative but its absolute value will be bounded by $\mathcal{O}(n \log n)$. The same holds if it is negative, by symmetry.

Lemma 7. *Let \mathbf{x} be any configuration such that $s \geq 0$. Then, for any $\gamma > 0$, $S \geq -\gamma\sqrt{n \log n}$ with probability $1 - n^{-\gamma^2/2}$.*

Proof of Lemma 7. From Eq. (5), we have that $\mathbb{E}[S \mid \mathbf{x}] \geq 0$. Indeed, if $p > 1/3$, we have $\mathbb{E}[S \mid \mathbf{x}] \geq s(2 - 4p) \geq 0$ since $q \leq n$ and $p < 1/2$. If, instead, $p \leq 1/3$, we have $\mathbb{E}[S \mid \mathbf{x}] \geq s(1 - p) \geq 0$. Let $\lambda = \gamma\sqrt{\log n/n}$. Then, by the Hoeffding bound (Lemma 18), we have that

$$\mathbb{P}\left[S \geq -\gamma\sqrt{n \log n} \mid \mathbf{x}\right] = \mathbb{P}[S \geq -\lambda n \mid \mathbf{x}] \leq \exp\left(-\frac{\gamma^2 \log n}{2}\right) = n^{-\frac{\gamma^2}{2}}.$$

□

We distinguish two cases: $p > 1/4$ and $1/6 < p \leq 1/4$.

3.2.1 First case: $p > \frac{1}{4}$ large noise.

Let $\epsilon > 0$ be a constant such that $1/4 + \epsilon < p < 1/2$. We first show a bound on the decrease of the bias.

Lemma 8. *Let \mathbf{x} be a configuration such that $s \geq \gamma\sqrt{n \log n}$ for any constant $\gamma > 0$. Then, in the next round, it holds that $S \leq s(1 - 2\epsilon)$ with probability $1 - n^{-2\epsilon^2\gamma^2}$.*

Proof of Lemma 8. Equation (5) implies that

$$\mathbb{E}[S \mid \mathbf{x}] \leq s \left[1 - \frac{1}{4} - \epsilon + 1 - \frac{3}{4} - 3\epsilon\right] = s(1 - 4\epsilon),$$

since $q/n \leq 1$. Let $\lambda = 2\epsilon(s/n)$. From the Hoeffding bound (Lemma 18), we have that

$$\mathbb{P}[S \leq s(1 - 2\epsilon) \mid \mathbf{x}] = \mathbb{P}[S \leq s(1 - 4\epsilon) + \lambda n \mid \mathbf{x}] \leq \exp\left(-\frac{2\epsilon^2 s^2}{n}\right) \leq n^{-2\epsilon^2\gamma^2}.$$

□

3.2.2 Second case: $p \leq 1/4$ small noise.

Let $\epsilon > 0$ be a constant such that $1/6 + \epsilon = p \leq 1/4$, which implies that $\epsilon < 1/12$. We remark that, in contrast to previous cases, here the parameter p for the noise is *exactly* $1/6 + \epsilon$ for the sake of the analysis.

The following lemma states that, if $s \geq \frac{2}{3}n$, the bias decreases exponentially at the next round, w.h.p. On the other hand, if the bias is at most $\frac{2}{3}n$, it cannot grow over $\frac{2}{3}n$, w.h.p.

Lemma 9. *Let \mathbf{x} be any configuration. The following holds:*

- (1) *if $s \geq \frac{2}{3}n$, then $S \leq s(1 - \epsilon)$ with probability $1 - \exp(-2\epsilon^2 n/9)$;*
- (2) *if $s \leq \frac{2}{3}n$, then $S \leq \frac{2}{3}n$ with probability $1 - \exp(-16\epsilon^2 n/9)$.*

Proof of Lemma 9. We first address claim (1). Consider the first statement. If $s \geq \frac{2}{3}n$, then $q < \frac{1}{3}n$. Thus, from Eq. (5), we get

$$\mathbb{E}[S \mid \mathbf{x}] \leq s \left[1 - \frac{1}{6} - \epsilon + \frac{1}{3} - \frac{1}{6} - \epsilon\right] = s(1 - 2\epsilon).$$

Let $\lambda = \epsilon(s/n)$. By the Hoeffding bound (Lemma 18), we conclude that

$$\mathbb{P}[S \leq s(1 - \epsilon) \mid \mathbf{x}] = \mathbb{P}[S \leq s(1 - 2\epsilon) + \lambda n \mid \mathbf{x}] \leq \exp\left(-\frac{\epsilon^2 s^2}{2n}\right) \leq \exp\left(-\frac{2\epsilon^2 n}{9}\right),$$

since $s \geq 2n/3$.

Now we prove claim (2). Since $q \leq n - s$, from Eq. (5) we get that

$$\mathbb{E}[S \mid \mathbf{x}] \leq s \cdot \left[1 - \frac{1}{6} - \epsilon + \frac{n - s}{n} \cdot \left(1 - \frac{1}{2} - 3\epsilon\right)\right]$$

$$\begin{aligned}
&\leq \frac{2n}{3} \cdot \left[1 - \frac{1}{6} - \epsilon + 1 - \frac{1}{2} - 3\epsilon - \frac{2}{3} + \frac{1}{3} + 2 \right] \\
&= \frac{2n}{3} \cdot [1 - 4\epsilon].
\end{aligned}$$

Let $\lambda = 4\epsilon(2/3)$. For the Hoeffding bound (Lemma 18), it holds that

$$\mathbb{P} \left[S \leq \frac{2n}{3} \mid \mathbf{x} \right] = \mathbb{P} \left[S \leq \frac{2n}{3} (1 - 4\epsilon) + \lambda n \mid \mathbf{x} \right] \leq \exp \left(-\frac{16\epsilon^2 n}{9} \right).$$

□

Thus, we just have to take care of cases in which the bias is no more than $\frac{2}{3}n$. The key-point to show the decrease of the bias, as long as it is $\Omega(\sqrt{n \log n})$, it is the condition $q \leq \check{q}_{\epsilon/2} = (1 + 3\epsilon)(n/3)/(1 - 6\epsilon)$, as shown in the next lemma.

Lemma 10. *Let \mathbf{x} be a configuration such that $s \geq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$. If $q \leq \check{q}_{\epsilon/2}$, then in the next round it holds that $S \leq s(1 - \frac{\epsilon}{4})$ with probability $1 - n^{-\epsilon^2 \gamma^2 / 2^6}$.*

Proof of Lemma 10. From Eq. (5) it follows that

$$\mathbb{E}[S \mid \mathbf{x}] \leq s \cdot \left[1 - \frac{1}{6} - \epsilon + \frac{1}{6} + \frac{\epsilon}{2} \right] = s \left(1 - \frac{\epsilon}{2} \right).$$

Let $\lambda = (\epsilon/4)(s/n)$. By the Hoeffding bound (Lemma 18), we get

$$\mathbb{P} \left[S \leq s \left(1 - \frac{\epsilon}{4} \right) \mid \mathbf{x} \right] = \mathbb{P} \left[S \leq s \left(1 - \frac{\epsilon}{2} \right) + \lambda n \mid \mathbf{x} \right] \leq \exp \left(-\frac{\epsilon^2 s^2}{2^6 n} \right) \leq n^{-\frac{\epsilon^2 \gamma^2}{2^6}}.$$

□

We now analyze the dynamics by partitioning the interval $(0, \frac{2}{3}n]$ and seeing what happens to the bias in each element of the partition. Let $\beta = \frac{2\sqrt{2}\epsilon}{\sqrt{(1+6\epsilon)(1-6\epsilon)}}$ and define $S_{-1} := (0, \beta n]$, S_i the sequence of intervals

$$S_i := \left(\left(\frac{3}{2} \right)^i \beta n, \left(\frac{3}{2} \right)^{i+1} \beta n \right]$$

for $i = 0, 1, \dots, k-2$ where $k = \left\lceil \log_{\frac{3}{2}}(\beta) - 1 \right\rceil$, and $S_{k-1} := \left(\left(\frac{3}{2} \right)^{k-1} \beta n, \frac{2}{3}n \right]$. Furthermore, just for completeness, we define $S_k := (\frac{2}{3}n, n]$. In the next lemmas, we show that as long as $s \in S_i$ for $i = -1, \dots, k-1$, $s = \Omega(\sqrt{n \log n})$, and $\bar{q}_{i+1} \leq q \leq \check{q}_{\epsilon/2}$ for some decreasing sequence $\bar{q}_{-1}, \dots, \bar{q}_k$ accurately chosen, then, at the next round, the bias decreases exponentially w.h.p. and the number of undecided nodes moves to the interval $[\bar{q}_i, \check{q}_{\epsilon/2}]$ w.h.p. Note that since ϵ is a constant, so is k . The following lemma determines the sequence \bar{q}_i .

Lemma 11. *Let \mathbf{x} be any configuration.*

1. *If $-1 \leq i \leq k-1$ and $s \leq \left(\frac{3}{2} \right)^{i+1} \beta n$, it holds that $Q \geq \frac{n}{3} - \frac{2n\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+3}$ with probability $1 - \exp(-2\epsilon^2(3/2)^{4i+4}n/(1+6\epsilon)^2)$;*
2. *If $s \leq \frac{2}{3}n$, it holds that $Q \geq \frac{2}{9}n + \frac{\epsilon}{3}n$ with probability $1 - \exp(-2\epsilon^2 n/9)$;*
3. *Without any condition on s , it holds that $Q \geq \frac{n}{12} + \epsilon n$ with probability $1 - \exp(-\epsilon^2 n/2)$.*

Proof of Lemma 11. We start proving claim (1). From Equation (6), we have that

$$\mathbb{E}[Q \mid \mathbf{x}] \geq \frac{n}{3} - \frac{1-6\epsilon}{n} \left(\frac{s}{2} \right)^2 \geq \frac{n}{3} - \frac{1-6\epsilon}{4} \left(\frac{3}{2} \right)^{2i+2} \beta^2 n$$

$$= n \left[\frac{1}{3} - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+2} \right].$$

Thus, using the additive form of Chernoff bound (Lemma 17) with $\lambda = \frac{\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+2} n$, we have that $Q \geq n \left(\frac{1}{3} - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+3} \right)$ with probability $1 - \exp(-2\epsilon^2(3/2)^{4i+4}n/(1+6\epsilon)^2)$.

As for claim (2), we have that

$$\mathbb{E}[Q \mid \mathbf{x}] \geq \frac{n}{3} - \frac{1-6\epsilon}{n} \left(\frac{s}{2} \right)^2 \geq \frac{n}{3} - n \frac{1-6\epsilon}{9} = \frac{2+6\epsilon}{9}n$$

and we conclude by using the additive Chernoff bound with $\lambda = \epsilon n/3$, getting that $Q \geq \frac{2}{9}n + \frac{\epsilon}{3}n$ with probability $1 - \exp(-2\epsilon^2 n/9)$.

To prove claim (3), we use that $s \leq n$ and observe that

$$\mathbb{E}[Q \mid \mathbf{x}] \geq \frac{n}{3} - \frac{1-6\epsilon}{n} \left(\frac{s}{2} \right)^2 \geq \frac{n}{3} - n \frac{1-6\epsilon}{4} = \frac{1+18\epsilon}{12}n.$$

We conclude with the additive Chernoff bound (Lemma 17) with $\lambda = \epsilon n/2$, getting $Q \geq \frac{n}{12} + \epsilon n$ with probability $1 - \exp(-\epsilon^2 n/2)$. \square

Define $\bar{q}_i := n \left(\frac{1}{3} - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+3} \right)$ for $i = -1, \dots, k-2$, $\bar{q}_{k-1} := \frac{2}{9}n + \frac{\epsilon}{3}n$, and $\bar{q}_k := \frac{n}{12} + \epsilon n$, and notice that they form a decreasing sequence. With the next lemmas, we take care of controlling the behavior of the number of undecided nodes when $s > \inf(S_i)$ for $-1 \leq i \leq k-1$. The proof is highly technical and can be found in Appendix B.

Lemma 12. *Let $-1 \leq i \leq k-1$ and let \mathbf{x} be a configuration such that $\bar{q}_{i+1} \leq q \leq \check{q}_{\epsilon/2}$ and $s > \inf(S_i)$. Then, at the next round, $\bar{q}_i \leq Q \leq \check{q}_{\epsilon/2}$ with probability $1 - \exp(-2\epsilon^2 n/(1-6\epsilon)^2)$.*

At the same time, the following lemma implies that the possible decrease of the bias cannot move it from S_i beyond S_{i-1} .

Lemma 13. *Let \mathbf{x} be a configuration such that $s \geq \gamma \sqrt{n \log n}$ for some constant $\gamma > 0$. If $s > \left(\frac{3}{2} \right)^i \beta n$ for some $0 \leq i \leq k$, then $S > \inf(S_{i-1})$ with probability $1 - \exp(-\epsilon^2(3/2)^{2i} \beta^2 n/2)$.*

Proof of Lemma 13. We have

$$\mathbb{E}[S \mid \mathbf{x}] \geq s \left(\frac{5}{6} - \epsilon \right).$$

The Hoeffding bound (Lemma 18) implies that $S \geq s \left(\frac{5}{6} - 2\epsilon \right)$ with probability $1 - \exp(-\epsilon^2(3/2)^{2i} \beta^2 n/2)$. Observe that

$$s \left(\frac{5}{6} - 2\epsilon \right) > \left(\frac{3}{2} \right)^i \beta n \left(\frac{5}{6} - 2\epsilon \right) \geq \left(\frac{3}{2} \right)^{i-1} \beta n = S_{i-1}$$

since $\frac{3}{2} \left(\frac{5}{6} - 2\epsilon \right) \geq 1$, which concludes the proof. \square

We still need to control the process when $q > \check{q}_{\epsilon/2}$. The next lemma addresses this issue by showing that there is a decrease of the number of undecided nodes when they are more than $\check{q}_{\epsilon} = (n/3)/(1-6\epsilon)$, and provides a lower bound on the decrease, depending on the bias. Again, the proof is highly technical and is thus deferred to Appendix B.

Lemma 14. *Let \mathbf{x} be a configuration such that $q \geq \check{q}_{\epsilon}$. Then, it holds that*

- (1) $Q \leq q(1-\epsilon)$ with probability $1 - \exp(-2\epsilon^2(n/3)/(1-6\epsilon)^2)$;
- (2) if $s \leq \sup(S_i)$ for any $-1 \leq i \leq k-1$, then $Q \geq \bar{q}_{i+1}$ with probability $1 - \exp(-\epsilon^2 n/18)$.

The next and last lemma guarantees that once the process reaches a configuration having bias $\mathcal{O}(\sqrt{n \log n})$, then it “enters” a metastable phase that lasts $\Omega(n^{\lambda'})$ rounds w.h.p. in which the absolute value of the bias remains $\mathcal{O}(\sqrt{n \log n})$, since it can be used symmetrically when the bias is negative.

Lemma 15. *Let \mathbf{x} be any configuration. If $s \leq \gamma\sqrt{n \log n}$ for some constant $\gamma > 0$ and $q \leq \frac{n}{3} \left(\frac{1+3\epsilon}{1-6\epsilon} \right)$, then $S \leq 2\gamma\sqrt{n \log n}$ with probability $1 - n^{-\gamma^2/2}$.*

Proof of Lemma 15. From Eq. (5), we have

$$\mathbb{E}[S \mid \mathbf{x}] \leq \gamma\sqrt{n \log n} \left(\frac{5}{6} - \epsilon + \frac{1}{6} + \frac{\epsilon}{2} \right) \leq \gamma\sqrt{n \log n}.$$

We conclude applying the Hoeffding bound with $\lambda = \gamma\sqrt{\log n/n}$, obtaining that $S \leq 2\gamma\sqrt{n \log n}$ with probability $1 - n^{-\gamma^2/2}$. \square

3.2.3 Proof of Theorem 2

If $p > 1/4 + \epsilon$, Lemmas 7 and 8, and Lemmas 20 and 21 imply that the process reaches a configuration \mathbf{y} with bias $|s(\mathbf{y})| \leq (\gamma/2) \cdot \sqrt{n \log n}$ within time $\mathcal{O}_{p,\gamma}(\log n)$ w.h.p., since ϵ can be chosen as $(4p-1)/8$. Furthermore, the bias remains bounded by the same value for $n^{\Omega_{p,\gamma}(1)}$ rounds, w.h.p.

Let us now assume $1/6 + \epsilon = p \leq 1/4$. The proof is divided into different cases. We remark that ϵ depends only on p . Let $\gamma > 0$ be any constant.

(1) $s \in S_i$ for some $-1 \leq i \leq k-1$ and $s \geq (\gamma/2) \cdot \sqrt{n \log n}$,

(1.1) $\bar{q}_{i+1} \leq q \leq \frac{n(1+3\epsilon)}{3(1-6\epsilon)}$: the bias decreases exponentially fast each round, w.h.p., until $|s| \leq (\gamma/2) \cdot \sqrt{n \log n}$ due to the combination of Lemmas 20 and 21, Lemma 7, and Lemmas 10 to 13. This phase lasts $\mathcal{O}_{p,\gamma}(\log n)$ rounds;

(1.2) $\frac{n(1+3\epsilon)}{3(1-6\epsilon)} < q$: Lemmas 20 and 21, and Lemmas 11 and 14 imply that in $\mathcal{O}_p(\log n)$ rounds the number of undecided nodes reaches the interval $\left[\bar{q}_{i+1}, \frac{n}{3(1-6\epsilon)} \right]$ where i is such that the round before the undecided nodes become less than $\frac{n}{3(1-6\epsilon)}$, the bias is in S_i : recall that during this whole process (which lasts $\mathcal{O}_p(\log n)$ rounds) the bias never goes over $\frac{2}{3}n$ thanks to Lemma 9. At the same time, the value of the bias will belong to one set between S_{i-1}, \dots, S_{k-1} due to Lemma 13. Since \bar{q}_i is a decreasing sequence, we are in Case 1.1, and we conclude.

(1.3) $q < \bar{q}_{i+1}$: in this case, Lemma 11 implies $Q \geq \bar{q}_{i+1}$ in the next round, w.h.p. Since Lemma 9 guarantees that the bias remains under the value $\frac{2}{3}n$ w.h.p., either we are in Case 1.1 or in Case 1.2, and we conclude.

(2) $s > \frac{2}{3}n$: Lemma 9 implies that the bias gets less than or equal to $\frac{2}{3}n$ in $\mathcal{O}_p(\log n)$ rounds, w.h.p.; then we are in Case 1 and we conclude.

Now, we can suppose the process starts from a configuration \mathbf{y} having bias $0 \leq s(\mathbf{y}) \leq (\gamma/2) \cdot \sqrt{n \log n}$, and such that $\bar{q}_0 \leq q(\mathbf{y}) \leq \frac{n}{3} \left(\frac{1+3\epsilon}{1-6\epsilon} \right)$, as Case 1.1 or 1.3 leaves it. In the next round, it holds that the number of undecided nodes is $\bar{q}_{-1} \leq Q \leq \frac{n}{3} \left(\frac{1+3\epsilon}{1-6\epsilon} \right)$, w.h.p., due to Lemma 12; at the same time, w.h.p., $|S| \leq \gamma\sqrt{n \log n}$ for Lemmas 7 and 15 (which can be used symmetrically on $A - B$ and $B - A$). Thus, the absolute value of the bias is either still less than $(\gamma/2) \cdot \sqrt{n \log n}$ or has become greater than or equal to $|S| \geq (\gamma/2) \cdot \sqrt{n \log n}$, in which case it starts decreasing exponentially fast each round, w.h.p. for Lemma 10, until becoming smaller than $(\gamma/2) \cdot \sqrt{n \log n}$ (as explained in Case 1.1, which works analogously if the bias is negative, because of symmetry). This phase lasts $n^{\Omega_{p,\gamma}(1)}$ rounds w.h.p. for Lemmas 20 and 21.

4 Symmetry Breaking from Balanced Configurations

In this section, we consider the UNDECIDED-STATE process starting from arbitrary initial configurations: in particular, from configurations having no bias. Interestingly enough, we show a transition phase similar to that proved in the previous section. Informally, the next theorem states that when $p < 1/6$, the UNDECIDED-STATE process is able to break the symmetry of any perfectly-balanced initial configuration and to compute almost consensus within $O(\log n)$ rounds, w.h.p.

Theorem 3. *Let \mathbf{x} be any initial configuration, and let $p \in (0, 1/6)$ be some absolute constant denoting the noise probability. Furthermore, let $\epsilon_1 = (1 - 6p)^2/2$, $\epsilon_2 = 2p(1 - p)$. Then, the UNDECIDED-STATE process reaches a configuration \mathbf{y} having bias s towards some opinion $j \in \{\text{Alpha}, \text{Beta}\}$ such that $s(\mathbf{y}) \in \Delta = [\hat{s}_{\epsilon_1}(1 - 2p)^2, \hat{s}_{\epsilon_2}]$ within $\mathcal{O}_p(\log n)$ rounds, w.h.p. Moreover, once reached configuration \mathbf{y} , the UNDECIDED-STATE process enters a (metastable) phase of length $\exp(\Omega_p(n))$ rounds where the majority opinion is j and the bias keeps within the range Δ , w.h.p.*

What follows is an outline of the proof of the theorem, while more details are given in Appendix B.

Outline of Proof of Theorem 3. If the initial configuration \mathbf{x} has bias $s = \Omega(\sqrt{n \log n})$ then the claim of the theorem is equivalent to that of Theorem 1, so we are done. Hence, we next assume the initial bias s be $o(\sqrt{n \log n})$: for this case, our proof proceeds along the following main steps.

Step I. Whenever the bias s is small, i.e. $o(n)$, we prove that, within the next $\mathcal{O}(\log n)$ rounds, the number of undecided nodes turns out to keep always in a suitable linear range: roughly speaking, we get that this number lies in $(n/3, n/2]$, w.h.p.

Step II. Whenever s is very small, i.e. $s = o(\sqrt{n})$, there is no effective drift towards any opinion. However, we can prove that, thanks to Step I, the random variable S , representing the bias in the next round, has *high variance*, i.e. $\Theta(n)$. The latter holds since S can be written as a suitable sum whose addends include some random variables having binomial distribution of expectation 0: so, we can apply the Berry-Esseen Theorem (Lemma 19) to get a lower bound on the variance of S . Then, thanks to this large variance, standard arguments for the standard deviation imply that, in this parameter range, there is a positive constant probability that S will get some value of magnitude $\Omega(\sqrt{n})$ (see Claim 1 of Lemma 24). Not surprisingly, in this phase, we find out that the variance of S is not decreased by the communication noise. We can thus claim that the process, at every round, has positive constant probability to reach a configuration having bias $s = \omega(\sqrt{n})$ and $q \in (n/3, n/2]$. Then, after $\mathcal{O}(\log n)$ rounds, this event will happen w.h.p.

Step III. Once the process reaches a configuration with $s = \omega(\sqrt{n})$ and $q \in (n/3, n/2]$, we then prove that the expected bias increases by a constant factor. Observe that we cannot use here the same round-by-round concentration argument that works for bias over $\sqrt{n \log n}$: this is in fact the minimal magnitude required to apply the Chernoff's bounds [28]. We instead exploit a useful general tool [14] that bounds the stopping time of some class of Markov chains having rather mild conditions on the drift towards their absorbing states (see Lemma 22). This tool in fact allows us to consider the two phases described, respectively, in Step II and Step III as a unique *symmetry-breaking* phase of the process. Our final technical contribution here is to show that the conditions required to apply this tool hold whenever the communication noise parameter is such that $p \in (0, 1/6)$. This allows us to prove that, within $\mathcal{O}(\log n)$ rounds, the process reaches a configuration with bias $s = \Omega(\sqrt{n \log n})$, w.h.p. \square

Large communication noise (the case $p > 1/6$). When $p > 1/6$, Theorem 2 a fortiori holds when the initial bias is small, i.e. $s = o(\sqrt{n \log n})$: thus, we get that, in this case, the system enters into a long regime of non consensus, starting from any initial configuration. Then, by combining the results for biased configurations in Section 3 with those in this section, we can observe the phase transition of the UNDECIDED-STATE process starting from any possible initial configuration.

Theorem 4. *Let \mathbf{x} be any initial configuration, and let $p \in (1/6, 1/3]$ be some absolute constant denoting the noise probability. The UNDECIDED-STATE process reaches a configuration \mathbf{y} having bias $|s(\mathbf{y})| = \mathcal{O}_p(\sqrt{n \log n})$ within $\mathcal{O}_p(\log n)$ rounds, w.h.p. Furthermore, starting from such a configuration, the UNDECIDED-STATE process enters a (metastable) phase of length $n^{\Omega_p(1)}$ rounds where the absolute value of the bias keeps bounded by $\mathcal{O}_p(\sqrt{n \log n})$, w.h.p.*

Stubborn agents. We conclude this section by observing that the equivalence result shown in Lemma 1 holds independently of the choices of the noise parameter $p \in (0, 1/2]$, and of the initial bias: the phase transition of the UNDECIDED-STATE process in the presence of stubborn agents thus holds even in the case of unbiased configurations.

Corollary 2. *Let $0 < p \leq 1/3$ be a constant, and let the STUB process start from any initial configuration. If $p < 1/6$, then, in $\mathcal{O}_p(\log n)$ rounds, the STUB process enters a metastable phase of almost-consensus towards some opinion $j \in \{\text{Alpha}, \text{Beta}\}$ of length $\exp_p(\Omega(n))$, in which the absolute value of the bias is $\Theta(n)$, w.h.p.*

If $p \in (1/6, 1/3]$, then, in $\mathcal{O}_p(\log n)$ rounds, the STUB process enters a metastable phase of length $n^{\Omega_p(1)}$ where the absolute value of the bias keeps bounded by $\mathcal{O}_p(\sqrt{n \log n})$, w.h.p.

5 Simulations

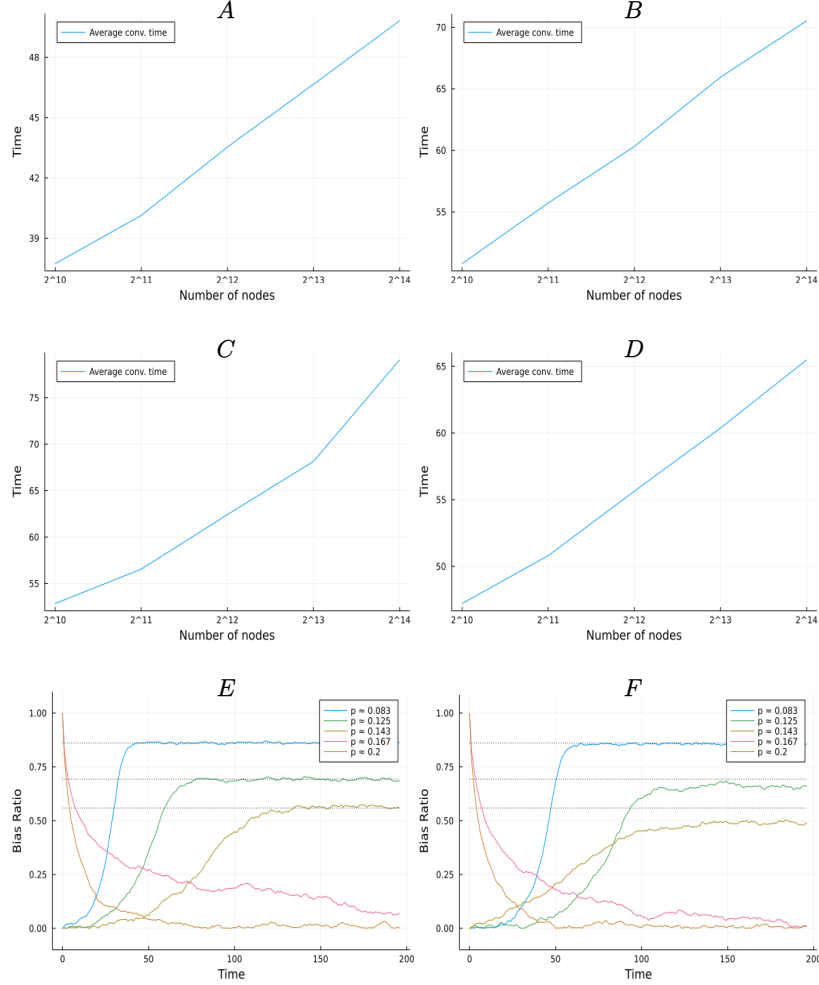


Figure 1: (A) Clique, victory of majority: average convergence time to almost consensus for a clique with noise parameter $p = 1/12$. (B) Clique, victory of noise: average convergence time to a symmetric configuration for a clique with noise parameter $p = 1/5$. (C) Erdős-Rényi graph, victory of majority: same as in (A) but for an Erdős-Rényi graph. (D) Erdős-Rényi graph, victory of noise: same as in (B) but for an Erdős-Rényi graph. (E) Clique, bias behavior: evolution of the bias for different values of the noise parameter $p \in \{1/12, 1/8, 1/7, 1/6, 1/5\}$ in the clique of size 2^{14} . (F) Erdős-Rényi graph, bias behavior: same as in (E) but for an Erdős-Rényi graph of size 2^{14} . In (A), (B), (C), and (D) the x -axis shows the size of the graph in logarithmic scale and the y -axis the average time of convergence over 1000 trials. In (E) and (F), the x -axis shows the time instant of the process, while the y -axis indicates the ratio of the bias, i.e. $|s(\mathbf{x})|/n$ for a given configuration \mathbf{x} . The dotted lines represent the expected equilibrium values for the clique. In all figures, the starting configuration is symmetric and random for $p < 1/6$, and the monochromatic one for $p \geq 1/6$.

We simulated the UNDECIDED-STATE dynamics with values of the input size in the range $n \in \{2^{10}, 2^{11}, \dots, 2^{14}\}$, and for noise probabilities in the range $p \in \{1/12, 1/8, 1/7, 1/6, 1/5\}$. Besides confirming the phase transition

predicted by our theoretical analysis, the outcomes show this behavior emerges even for reasonable sizes of the system. Fig. 1.(A) shows a convergence time to almost-consensus in the complete graph (equivalently, clique) for $p = 1/12$ which is perfectly approximated by a logarithmic function. The convergence time has been calculated as the average over 1000 trials. In Fig. 1.(B), the same process is shown for a “victory of noise” case, i.e., for $p = 1/5$. The behavior of the bias for the different values of p over the clique with 2^{14} nodes is shown in Fig. 1.(E). Our intuition suggests that our results carry on for sparser topologies that exhibit good expansion properties. To establish experimentally this, we ran the same simulations on the Erdős-Rényi graph model of size n , with edge probability $\log_2(n)/n$: it is well-known that this setting generate, w.h.p., a connected graph with logarithmic node degree. The results in Fig. 1.(C) and Fig. 1.(D) seem to show that even such a topology yields logarithmic convergence time; moreover, Fig. 1.(F) suggests that the phase-transition threshold either is very close or coincides with the same value $p = 1/6$.

6 Conclusions

While our mathematical analysis for the UNDECIDED-STATE dynamics does not directly apply to other opinion dynamics, it might suggest the emergence of a general phase-transition phenomenon for a large class of dynamics characterized by an exponential drift towards consensus configurations. Our work naturally poses the general question of whether it is possible to provide a characterization of opinion dynamics with stochastic interactions, in terms of their critical behavior with respect to uniform communication noise. To support this possible interesting scenario, we remark the analytical study in [24] showing that at least another well-known non-linear opinion dynamics, the 3-MAJORITY dynamics, exhibits a similar behavior. In Section 2.2, we described the corresponding specific setting of the general best-of- N nest site selection process that corresponds to UNDECIDED-STATE dynamics we considered: roughly speaking, the latter somewhat describes the case in which the options supported by the agents have the same quality and, thus, have the same impact on convincing undecided agents, while the dominant factor is the communication noise. This concrete setting may represent a worst-case scenario to adopt in order to “stress” the system in terms of convergence time and resilience. Following the line of research proposed in [51], an interesting future analytical study is to include a further parameter in the UNDECIDED-STATE local updating rule representing the strength of each possible options: then the goal would be finding clean mathematical relationship among the almost-consensus time, the rate of noise, and the presence of initial options having high/low strength.

As for the specific mathematical questions that follow from our results, our assumption of a complete topology as underlying graph is, for several real MAS, a rather strong condition. However, two remarks on this issue follow. On one hand, we observe that, according to the adopted communication model, at every round, every agent can pull information from just one other agent: the *dynamic* communication pattern is thus random and sparse. This setting may model opportunistic MAS where mobile agents use to meet randomly, at a relatively-high rate. On the other hand, we believe that a similar transition phase does hold even for sparse topologies having good expansion/conductance [37]. Our conjecture is supported by the set of experimental tests we performed on the classic Erdős-Rényi model: the results confirm the presence of a threshold behavior of the UNDECIDED-STATE dynamics equivalent to that we proved for the complete graph. We instead believe that the behavior of the dynamics over non-expanding graphs cannot be directly exploited to get fast and reliable almost-consensus since it strongly depends on the specific “geometric” shape of the initial configuration: hence, this study requires to introduce further, more refined parameters.

Finally, the properties we show for the UNDECIDED-STATE dynamics in the case of the complete graphs might also result useful to perform another fundamental task in swarm computing (which is not the subject of this work): that of distributed community detection [8, 23]. Informally, if the swarm system has a hidden community structure, the evolution of the UNDECIDED-STATE process might have a long metastable phase where the opinion/state of each agent is consistent with the hidden community it belongs to.

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Appendix A Useful tools

We recall the concentration results that we use in the analysis. For an overview on the forms of Chernoff bounds see the works by [\[28\]](#) or [\[26\]](#).

Lemma 16 (Multiplicative forms of Chernoff bounds). *Let X_1, X_2, \dots, X_n be independent $\{0, 1\}$ random variables. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then:*

(i) *for any $\delta \in (0, 1)$ and $\mu \leq \mu_+ \leq n$, it holds that*

$$P(X \geq (1 + \delta)\mu_+) \leq e^{-\frac{1}{3}\delta^2\mu_+}, \quad (9)$$

(ii) *for any $\delta \in (0, 1)$ and $0 \leq \mu_- \leq \mu$, it holds that*

$$P(X \leq (1 - \delta)\mu_-) \leq e^{-\frac{1}{2}\delta^2\mu_-}. \quad (10)$$

Lemma 17 (Additive forms of Chernoff bounds). *Let X_1, X_2, \dots, X_n be independent $\{0, 1\}$ random variables. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$. Then:*

(i) *for any $0 < \lambda < n$ and $\mu \leq \mu_+ \leq n$, it holds that*

$$P(X \geq \mu_+ + \lambda) \leq e^{-\frac{2}{n}\lambda^2}, \quad (11)$$

(ii) *for any $0 < \lambda < \mu_-$ and $0 \leq \mu_- \leq \mu$, it holds that*

$$P(X \leq \mu_- - \lambda) \leq e^{-\frac{2}{n}\lambda^2}. \quad (12)$$

We also make use of the Hoeffding bounds [\[43\]](#).

Lemma 18 (Hoeffding bounds). *Let $0 < a < b$ be two constants. Let X_1, X_2, \dots, X_n be independent random variables such that $\mathbb{P}[a \leq X_i \leq b] = 1$ and $\mathbb{E}[X_i] = \mu/n$ for all $i \leq n$, and let $X = \sum_{i=1}^n X_i$. Then:*

(i) *for any $\lambda > 0$ and $\mu \leq \mu_+$, it holds that*

$$\mathbb{P}[X \geq \mu_+ + \lambda n] \leq \exp\left(-\frac{2\lambda^2 n}{(b - a)^2}\right); \quad (13)$$

(ii) *for any $\lambda > 0$ and $0 \leq \mu_- \leq \mu$, it holds that*

$$\mathbb{P}[X \leq \mu_- - \lambda n] \leq 2 \exp\left(-\frac{2\lambda^2 n}{(b - a)^2}\right). \quad (14)$$

The Berry-Esseen theorem is well treated by [\[39\]](#), and it gives an estimation on “how far” is the distribution of the normalized sum of i.i.d. random variables to the standard normal distribution.

Lemma 19 (Berry-Esseen). *Let X_1, \dots, X_n be n i.i.d. (either discrete or continuous) random variables with zero mean, variance $\sigma^2 > 0$, and finite third moment. Let Z the standard normal random variable, with zero mean and variance equal to 1. Let $F_n(x)$ be the cumulative function of $\frac{S_n}{\sigma\sqrt{n}}$, where $S_n = \sum_{i=1}^n X_i$, and $\Phi(x)$ that of Z . Then, there exists a positive constant $C > 0$ such that*

$$\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \leq \frac{C}{\sqrt{n}}$$

for all $n \geq 1$.

Our analysis makes use of the following probabilistic results, which hold for n large enough.

Lemma 20. *Let $\eta > \lambda > 0$ be two constants, and $0 < p \leq 1$ be a probability. Consider any family of events $\{\xi_i\}_{i \leq M}$ with $M > 1$ being some integer. Suppose $\mathbb{P}[\xi_1] \geq p$, and, for $i \geq 2$, that $\mathbb{P}[\xi_i \mid \xi_1, \dots, \xi_{i-1}] = \mathbb{P}[\xi_i \mid \xi_{i-1}] \geq p$. The following holds.*

(i) *If $p = 1 - n^{-\eta}$ and $M \leq n^\lambda$, then $\cap_{i \leq M} \xi_i$ holds with probability $1 - \mathcal{O}(n^{-(\eta-\lambda)})$.*

(ii) *If $p = 1 - \exp(-\eta n)$ and $M \leq e^\lambda$. Then $\cap_{i \leq M} \xi_i$ holds with probability $1 - \mathcal{O}(e^{-(\eta-\lambda)n})$.*

Proof of Lemma 20. We have that

$$\begin{aligned} \mathbb{P}[\cap_{i \leq M} \xi_i] &= \mathbb{P}[\xi_M \mid \cap_{i \leq M-1} \xi_i] \cdot \mathbb{P}[\cap_{i \leq M-1} \xi_i] \\ &= \mathbb{P}[\xi_M \mid \xi_{M-1}] \cdot \mathbb{P}[\cap_{i \leq M-1} \xi_i] \\ &= \prod_{i=2}^M \mathbb{P}[\xi_i \mid \xi_{i-1}] \cdot \mathbb{P}[\xi_1] \\ &\geq p^M \end{aligned}$$

Now, let $f(n) = 1 - p = o(1)$. Notice that $1 - x \geq e^{-x/(1-x)}$ for $|x| < 1$. Then,

$$\begin{aligned} p^M &= [1 - f(n)]^M \\ &\geq \exp\left(-\frac{Mf(n)}{1 - f(n)}\right) \\ &\geq \exp(-2Mf(n)) \\ &\stackrel{(a)}{\geq} 1 - 4Mf(n), \end{aligned}$$

where (a) holds of the exponential function Taylor's expansion, since $Mf(n) = o(1)$ by the hypotheses. As for (i), we get

$$p^M \geq 1 - \mathcal{O}\left(\frac{n^\lambda}{n^\eta}\right) = 1 - \mathcal{O}\left(\frac{1}{n^{\eta-\lambda}}\right).$$

As for (ii), we get

$$p^M \geq 1 - \mathcal{O}\left(\frac{\exp(\lambda n)}{\exp(\eta n)}\right) = 1 - \mathcal{O}\left(e^{-(\eta-\lambda)n}\right).$$

□

More easily, the union bound implies the following.

Lemma 21. *Let $\eta > \lambda > 0$ be two constants, and $0 < p \leq 1$ be a probability. Consider any family of events $\{\xi_i\}_{i \leq M}$ with $M > 1$ being some integer. Suppose $\mathbb{P}[\xi_i] \geq p$ for all i . The following holds.*

(i) *If $p = 1 - n^{-\eta}$ and $M \leq n^\lambda$, then $\cap_{i \leq M} \xi_i$ holds with probability $1 - n^{-(\eta-\lambda)}$.*

(ii) *If $p = 1 - \exp(-\eta n)$ and $M \leq e^\lambda$. Then $\cap_{i \leq M} \xi_i$ holds with probability $1 - e^{-(\eta-\lambda)n}$.*

Appendix B Omitted proofs

Proofs: Preliminaries

Proof of Equations (3) to (6). Conditional on any configuration at time t , the probability an agent pulls opinion *Alpha* at the next round is

$$(1 - 3p)\frac{a}{n} + p,$$

and symmetrical expressions hold for opinion *Beta* and the undecided state. An agent updates its opinion to *Alpha* at time $t+1$ if it is undecided at time t and pulls opinion *Alpha*, or if it supports opinion *Alpha* at time t and pulls either opinion *Alpha* or the undecided state. If V_q and V_a denote the sets of agents supporting the undecided state and opinion *Alpha*, respectively, at time t , we have

$$\begin{aligned}\mathbb{E}[A \mid \mathbf{X}_t = \mathbf{x}] &= \sum_{u \in V_a} \left((1 - 3p)\frac{a}{n} + p + (1 - 3p)\frac{q}{n} + p \right) + \sum_{u \in V_q} \left((1 - 3p)\frac{a}{n} + p \right) \\ &= a \left((1 - 3p)\frac{a}{n} + p + (1 - 3p)\frac{q}{n} + p \right) + q \left((1 - 3p)\frac{a}{n} + p \right) \\ &= (1 - 3p)\frac{a}{n}(a + 2q) + p(2a + q).\end{aligned}$$

Similarly, we get the conditional expectation of B . Then

$$\begin{aligned}\mathbb{E}[S \mid \mathbf{X}_t = \mathbf{x}] &= \mathbb{E}[A \mid \mathbf{X}_t = \mathbf{x}] - \mathbb{E}[B \mid \mathbf{X}_t = \mathbf{x}] \\ &= \frac{1 - 2p}{n} [s(a + b) + 2qs] + \frac{p}{n} [s(a + b)] \\ &= s \frac{1 - 2p}{n} [n + q] + s \frac{p}{n} [n - q] = s \left(1 - p + (1 - 3p)\frac{q}{n} \right), \\ \mathbb{E}[Q \mid \mathbf{X}_t = \mathbf{x}] &= + \frac{a}{n} [p(a + q) + (1 - 2p)b] + \frac{b}{n} [p(b + q) + (1 - 2p)a] \\ &\quad + \frac{q}{n} [p(a + b) + (1 - 2p)q] \\ &= \frac{p}{n} [a^2 + b^2 + 2q(a + b)] + \frac{1 - 2p}{n} [2ab + q^2] \\ &= pn + \frac{1 - 3p}{n} [2ab + q^2] = pn + \frac{1 - 3p}{2n} [2q^2 + (n - q)^2 - s^2].\end{aligned}$$

□

Proofs: victory of noise

Proof of Lemma 12. Define $f(q)$ equal to $\mathbb{E}[Q \mid \mathbf{x}] = \frac{3}{4} \left(\frac{1 - 6\epsilon}{n} \right) q^2 - \frac{1 - 6\epsilon}{2} q + \frac{5 - 6\epsilon}{12} n - \frac{1 - 6\epsilon}{n} \left(\frac{s}{2} \right)^2$. We are going to evaluate $f(q)$ in \bar{q}_{i+1} and in $\bar{q} = \frac{n(1 + 3\epsilon)}{3(1 - 6\epsilon)}$. We take care of different cases: first, we assume $i = -1$, with the condition that $s > 0$. Thus

$$\begin{aligned}f(\bar{q}_0) &\leq \frac{3}{4} \left(\frac{1 - 6\epsilon}{n} \right) n^2 \left[\frac{1}{9} + \frac{4\epsilon^2}{(1 + 6\epsilon)^2} \left(\frac{3}{2} \right)^6 - \frac{4\epsilon}{3(1 + 6\epsilon)} \left(\frac{3}{2} \right)^3 \right] \\ &\quad - \frac{1 - 6\epsilon}{2} n \left[\frac{1}{3} - \frac{2\epsilon}{1 + 6\epsilon} \left(\frac{3}{2} \right)^3 \right] + \frac{5 - 6\epsilon}{12} n \\ &= \frac{n}{3} + n \frac{3\epsilon^2(1 - 6\epsilon)}{(1 + 6\epsilon)^2} \left(\frac{3}{2} \right)^6 = \frac{n}{3} \left[1 + \frac{3^8 \epsilon^2 (1 - 6\epsilon)}{2^6 (1 + 6\epsilon)^2} \right];\end{aligned}$$

now, we observe that

$$1 + \frac{3^8 \epsilon^2 (1 - 6\epsilon)}{2^6 (1 + 6\epsilon)^2} - \frac{1}{1 - 6\epsilon} < \frac{-6\epsilon}{1 - 6\epsilon} + \frac{3^8 \epsilon^2}{2^6 (1 + 6\epsilon)^2}$$

$$\begin{aligned}
&= \frac{-2^6 \cdot 6\epsilon(1+6\epsilon)^2 + 3^8\epsilon^2(1-6\epsilon)}{2^6(1-6\epsilon)(1+6\epsilon)^2} \\
&= \frac{3\epsilon(-2^7 - 3 \cdot 2^9\epsilon - 3^2 \cdot 2^9\epsilon^2 + 3^7\epsilon - 2 \cdot 3^8\epsilon^2)}{2^6(1-6\epsilon)(1+6\epsilon)^2} \\
&= \frac{3\epsilon(-2^7 + 3\epsilon(3^6 - 2^9) - 2 \cdot 3^2\epsilon^2(2^8 + 3^6))}{2^6(1-6\epsilon)(1+6\epsilon)^2} < 0
\end{aligned}$$

where in the last inequality we have used that $-2^7 + 3\epsilon(3^6 - 2^9) < 0$ for $\epsilon \leq \frac{1}{12}$. Thus, $f(\bar{q}_0) < \frac{n}{3(1-6\epsilon)}$.

Second, we assume $0 \leq i \leq k-3$, with the condition that $s > (\frac{3}{2})^i \beta n$.

$$\begin{aligned}
f(\bar{q}_{i+1}) &\leq \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) n^2 \left[\frac{1}{9} + \frac{4\epsilon^2}{(1+6\epsilon)^2} \left(\frac{3}{2} \right)^{4i+10} - \frac{4\epsilon}{3(1+6\epsilon)} \left(\frac{3}{2} \right)^{2i+5} \right] \\
&\quad - \frac{1-6\epsilon}{2} n \left[\frac{1}{3} - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+5} \right] + \frac{5-6\epsilon}{12} n - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i} n \\
&= \frac{n}{3} + n \frac{3\epsilon^2(1-6\epsilon)}{(1+6\epsilon)^2} \left(\frac{3}{2} \right)^{4i+10} - n \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i} \\
&= \frac{n}{3} \left\{ 1 + \frac{3\epsilon}{(1+6\epsilon)^2} \left(\frac{3}{2} \right)^{2i} \left[3\epsilon(1-6\epsilon) \left(\frac{3}{2} \right)^{2(i+5)} - 2(1+6\epsilon) \right] \right\};
\end{aligned}$$

for the evaluation of $f(\bar{q}_{i+1})$ we observe that β is a constant in $(0, 1)$ and that

$$3\epsilon(1-6\epsilon) \left(\frac{3}{2} \right)^{2(i+5)} - 2(1+6\epsilon) < 3\epsilon \left(\frac{2}{3} \frac{1}{\beta} \left(\frac{3}{2} \right)^3 \right)^2 - 2 < \frac{3^5}{2^7} - 2 < 0$$

because $(\frac{3}{2})^{i+2} < \frac{2}{3\beta}$ for $i+2 \leq k$; thus $f(\bar{q}_{i+1}) \leq \frac{n}{3}$.

Let now $i = k-2$; thus $s > (\frac{3}{2})^{k-2} \beta n$. We now evaluate $f(\bar{q}_{k-1})$.

$$\begin{aligned}
f(\bar{q}_{k-1}) &\leq \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) n^2 \left[\frac{4}{81} + \frac{\epsilon^2}{9} + \frac{4\epsilon}{27} \right] - \frac{1-6\epsilon}{2} n \left[\frac{2}{9} + \frac{\epsilon}{3} \right] + \frac{5-6\epsilon}{12} n \\
&\quad - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2k-4} n \\
&= \frac{-8(1-6\epsilon) + 45 - 54\epsilon}{108} n - \frac{\epsilon(1-6\epsilon)}{18} n - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2k-4} n \\
&= \frac{37 - 12\epsilon + 36\epsilon^2}{108} n - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2k-4} n \\
&= \frac{n}{3} \left[1 + \frac{(1-6\epsilon)^2}{36} - \frac{6\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2k-4} \right] \\
&= \frac{n}{3} \left[1 + \left(\frac{1-6\epsilon}{6} - \sqrt{\frac{6\epsilon}{1+6\epsilon}} \cdot \frac{3^{k-2}}{2^{k-2}} \right) \cdot \left(\frac{1-6\epsilon}{6} + \sqrt{\frac{6\epsilon}{1+6\epsilon}} \cdot \frac{3^{k-2}}{2^{k-2}} \right) \right].
\end{aligned}$$

Observe that, by definition of k , we have

$$\begin{aligned}
&\frac{1-6\epsilon}{6} - \sqrt{\frac{6\epsilon}{1+6\epsilon}} \cdot \frac{3^{k-2}}{2^{k-2}} \leq \frac{1-6\epsilon}{6} - \sqrt{\frac{6\epsilon}{1+6\epsilon}} \cdot \frac{2^3}{3^3\beta} \\
&= \frac{1-6\epsilon}{6} - \frac{\sqrt{3(1-6\epsilon)} \cdot 2^2}{3^3} = \frac{9 - 54\epsilon - 8\sqrt{3(1-6\epsilon)}}{54} \\
&< \frac{9 - 8\sqrt{3(1-6\epsilon)}}{54} < 0
\end{aligned}$$

for $\epsilon \leq \frac{1}{12}$. Thus, $f(\bar{q}_{k-1}) \leq \frac{n}{3}$.

Let now $i = k - 1$, which implies that $s > (\frac{3}{2})^{k-1} \beta n$. We evaluate $f(\bar{q}_k)$:

$$\begin{aligned}
f(\bar{q}_k) &\leq \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) n^2 \left[\frac{1}{144} + \epsilon^2 + \frac{\epsilon}{6} \right] - \frac{1-6\epsilon}{2} n \left[\frac{1}{12} + \epsilon \right] + \frac{5-6\epsilon}{12} n \\
&\quad - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2k-2} n \\
&= \frac{-7(1-6\epsilon) + 80 - 96\epsilon}{192} n - \frac{3\epsilon(1-6\epsilon)}{8} n - \frac{2\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2k-2} n \\
&= \frac{n}{3} \left[\frac{219 - 162\epsilon - 216\epsilon + 1296\epsilon^2}{192} - \frac{6\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2k-2} \right] \\
&= \frac{n}{3} \left[1 + \frac{27 - 378\epsilon + 1296\epsilon^2}{192} - \frac{6\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2k-2} \right].
\end{aligned}$$

By definition of k , we have that

$$\frac{27 - 378\epsilon + 1296\epsilon^2}{192} - \frac{6\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2k-2} \leq \frac{27 - 378\epsilon + 1296\epsilon^2}{192} - \frac{1-6\epsilon}{2} \left(\frac{2}{3} \right)^3 < 0,$$

where the first and the second inequalities hold for $\epsilon \leq \frac{1}{12}$. Thus, $f(\bar{q}_k) \leq \frac{n}{3}$.

We finally evaluate $f(\bar{q})$:

$$\begin{aligned}
f(\bar{q}) &\leq \frac{n}{12} \left(\frac{(1+3\epsilon)^2}{(1-6\epsilon)} \right) - \frac{n(1+3\epsilon)}{6} + \frac{n(5-6\epsilon)}{12} \\
&= \frac{n}{12} \left(\frac{1-6\epsilon+9\epsilon^2}{1-6\epsilon} - 2 - 6\epsilon + 5 - 6\epsilon \right) \\
&= \frac{n}{12(1-6\epsilon)} \left(1 - 6\epsilon + 9\epsilon^2 - 2 + 6\epsilon + 36\epsilon^2 + 5 - 36\epsilon + 36\epsilon^2 \right) \\
&= \frac{n}{12} \left(\frac{4 - 36\epsilon + 81\epsilon^2}{1-6\epsilon} \right) = \frac{n}{12} \left(\frac{(2-9\epsilon)^2}{1-6\epsilon} \right).
\end{aligned}$$

It holds that $f(\bar{q}) \leq \frac{n}{3(1-6\epsilon)}$ (remember that $\epsilon \leq 1/12$); in all cases, f is no more than $\frac{n}{3(1-6\epsilon)}$, and, from an immediate application of the additive Chernoff bound (Lemma 17) with $\lambda = \frac{\epsilon n}{1-6\epsilon}$, and by observing that $\bar{q}_i \geq \bar{q}_{i+1}$, we get that

$$\bar{q}_i \leq Q \leq \check{q}_{\epsilon/2} = \frac{n(1+3\epsilon)}{3(1-6\epsilon)}$$

with probability $1 - \exp(-2\epsilon^2 n / (1-6\epsilon)^2)$. □

Proof of Lemma 14. Consider the first claim. We define $f(q) = \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) q^2 - \frac{1-6\epsilon}{2} q + \frac{5-6\epsilon}{12} n$. We now show that $f(q) \leq q(1-2\epsilon)$. Indeed, $f(q) - q(1-2\epsilon)$ is equal to

$$\begin{aligned}
&\frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) q^2 - \left(\frac{1-6\epsilon}{2} + 1 - 2\epsilon \right) q + \frac{5-6\epsilon}{12} n \\
&= \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) q^2 - \left(\frac{3-10\epsilon}{2} \right) q + \frac{5-6\epsilon}{12} n.
\end{aligned}$$

This expression is a convex parabola which has its maximum in either $\bar{q}_1 = \frac{n}{3(1-6\epsilon)}$ or $\bar{q}_2 = n$. We calculate $f(q) - q(1-2\epsilon)$ in these two points

$$\frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) \bar{q}_1^2 - \left(\frac{3-10\epsilon}{2} \right) \bar{q}_1 + \frac{5-6\epsilon}{12} n$$

$$= \frac{n^2}{12(1-6\epsilon)} (1-6+20\epsilon+5-36\epsilon+36\epsilon^2) = \frac{n^2}{12(1-6\epsilon)} [-4\epsilon(4-9\epsilon)] < 0$$

for all $0 < \epsilon \leq \frac{1}{12}$. At the same time it holds that

$$\begin{aligned} & \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) q_2^2 - \left(\frac{3-10\epsilon}{2} \right) q_2 + \frac{5-6\epsilon}{12} n = \frac{n}{12} [9-54\epsilon-18+60\epsilon+5-6\epsilon] \\ & = -\frac{n}{3} < 0. \end{aligned}$$

Thus, $\mathbb{E}[Q \mid \mathbf{x}] \leq q(1-2\epsilon)$. The additive form of Chernoff bound (Lemma 17) with $\lambda = \frac{\epsilon n}{3(1-6\epsilon)}$ implies that $Q < q(1-\epsilon)$ with probability $1 - \exp(-2\epsilon^2(n/3)/(1-6\epsilon)^2)$.

As for the second claim, we consider two cases. First, assume $i < k-1$, thus $s \leq (\frac{3}{2})^{i+1} \beta n$. From Eq. (6), we observe that

$$\begin{aligned} \mathbb{E}[Q \mid \mathbf{x}] & \geq \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) \frac{n^2}{9(1-6\epsilon)^2} + \frac{5-6\epsilon}{12} n - \left(\frac{1-6\epsilon}{2} \right) \frac{n}{3(1-6\epsilon)} - \frac{1-6\epsilon}{n} \left(\frac{s}{2} \right)^2 \\ & \geq \frac{n}{12} \left(\frac{4-24\epsilon+36\epsilon^2}{1-6\epsilon} \right) - \frac{2n\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+2} \\ & \geq \frac{n}{3(1-6\epsilon)} (1-3\epsilon)^2 - \frac{2n\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+2} \\ & \geq \frac{n}{3} - \frac{2n\epsilon}{1+6\epsilon} \left(\frac{3}{2} \right)^{2i+5} + \frac{19n\epsilon}{4} \left(\frac{3}{2} \right)^{2i+2} = \bar{q}_{i+1} + \frac{19n\epsilon}{4} \left(\frac{3}{2} \right)^{2i+2}. \end{aligned}$$

We conclude applying the additive Chernoff bound with $\lambda = \frac{19\epsilon n}{4} \left(\frac{3}{2} \right)^{2i+2}$, obtaining $Q \geq \bar{q}_{i+1}$ with probability $1 - \exp(-19^2\epsilon^2(n/8)(3/2)^{4i+4})$.

Second, let $i = k-1$; then $s \leq \frac{2}{3}n$. As before

$$\begin{aligned} \mathbb{E}[Q \mid \mathbf{x}] & \geq \frac{3}{4} \left(\frac{1-6\epsilon}{n} \right) \frac{n^2}{9(1-6\epsilon)^2} + \frac{5-6\epsilon}{12} n - \left(\frac{1-6\epsilon}{2} \right) \frac{n}{3(1-6\epsilon)} - \frac{1-6\epsilon}{n} \left(\frac{s}{2} \right)^2 \\ & \geq \frac{n}{12(1-6\epsilon)} (4-24\epsilon+36\epsilon^2) - \frac{n(1-6\epsilon)}{9} \geq \frac{n(1-3\epsilon)^2}{3(1-6\epsilon)} - \frac{n(1-6\epsilon)}{9} \\ & = \frac{n}{9(1-6\epsilon)} (2-6\epsilon-9\epsilon^2) = \frac{n}{12} \left(\frac{4(2-6\epsilon-9\epsilon^2)}{3(1-6\epsilon)} \right) \\ & \geq \frac{n}{12} (1+14\epsilon) = \bar{q}_k + \frac{\epsilon n}{6} \end{aligned}$$

since

$$\frac{4(2-6\epsilon-9\epsilon^2)}{3(1-6\epsilon)} - (1+14\epsilon) > \frac{5-48\epsilon}{3(1-6\epsilon)} > 0$$

for $\epsilon \leq \frac{1}{12}$. Thus, we conclude applying the additive form of Chernoff bound with $\lambda = \frac{\epsilon}{6}n$, obtaining $Q \geq \bar{q}_k$ with probability $1 - \exp(-\epsilon^2 n/18)$. Hence, the second claim holds with probability at least $1 - \exp(-\epsilon^2 n/18)$, which is less than $1 - \exp(-19^2\epsilon^2(n/8)(3/2)^{4i+4})$. \square

Proofs: symmetry breaking

The proof of Theorem 3 essentially relies on the following lemma which has been proved by [14], for which we report a proof that corrects some minor mistakes.

Lemma 22. *Let $\{X_t\}_{t \in \mathbb{N}}$ be a Markov Chain with finite-state space Ω and let $f : \Omega \mapsto [0, n]$ be a function that maps states to integer values. Let c_3 be any positive constant and let $m = c_3 \sqrt{n} \log n$ be a target value. Assume the following properties hold:*

(1) for any positive constant h , a positive constant $c_1 < 1$ exists such that for any $x \in \Omega : f(x) < m$,

$$\mathbb{P}[f(X_{t+1}) < h\sqrt{n} \mid X_t = x] < c_1;$$

(2) there exist two positive constants δ and c_2 such that for any $x \in \Omega : h\sqrt{n} \leq f(x) < m$,

$$\mathbb{P}[f(X_{t+1}) < (1 + \delta)f(X_t) \mid X_t = x] < e^{-c_2 f(x)^2/n}.$$

Then the process reaches a state x such that $f(x) \geq m$ within $\mathcal{O}_{c_1, c_3}(\log n)$ rounds with probability at least $1 - 2/n$.

Proof. Define a set of hitting times $T := \{\tau(i)\}_{i \in \mathbb{N}}$, where

$$\tau(i) = \inf_{i \in \mathbb{N}} \{t : t > \tau(i-1), f(X_t) \geq h\sqrt{n}\},$$

setting $\tau(0) = 0$. By the first hypothesis, for every $i \in \mathbb{N}$, the expectation of $\tau(i)$ is finite. Now, define the following stochastic process which is a subsequence of $\{X_t\}_{t \in \mathbb{N}}$:

$$\{R_i\}_{i \in \mathbb{N}} = \{X_{\tau(i)}\}_{i \in \mathbb{N}}.$$

Observe that $\{R_i\}_{i \in \mathbb{N}}$ is still a Markov chain. Indeed, if $\{x_1, \dots, X_{i-1}\}$ is a set of states in Ω , then

$$\begin{aligned} & \mathbb{P}[R_i = x \mid R_{i-1} = x_{i-1}, \dots, R_1 = x_1] \\ &= \mathbb{P}[X_{\tau(i)} = x \mid X_{\tau(i-1)} = x_{i-1}, \dots, X_{\tau(1)} = x_1] \\ &= \sum_{t(i) > \dots > t(1) \in \mathbb{N}} \mathbb{P}[X_{t(i)} = x \mid X_{t(i-1)} = x_{i-1}, \dots, X_{t(1)} = x_1] \\ & \quad \cdot \mathbb{P}[\tau(i) = t(i), \dots, \tau(1) = t(1)] \\ &= \mathbb{P}[X_{\tau(i)} = x \mid X_{\tau(i-1)} = x_{i-1}] \\ &= \mathbb{P}[R_i = x \mid R_{i-1} = x_{i-1}]. \end{aligned}$$

By definition, the state space of R is $\{x \in \Omega : f(x) \geq h\sqrt{n}\}$. Moreover, the second hypothesis still holds for this new Markov chain. Indeed:

$$\begin{aligned} & \mathbb{P}[f(R_{i+1}) < (1 + \epsilon)f(R_i) \mid R_i = x] \\ &= 1 - \mathbb{P}[f(R_{i+1}) \geq (1 + \epsilon)f(R_i) \mid R_i = x] \\ &= 1 - \mathbb{P}[f(X_{\tau(i+1)}) \geq (1 + \epsilon)f(X_{\tau(i)}) \mid X_{\tau(i)} = x] \\ &\leq 1 - \mathbb{P}[f(X_{\tau(i+1)}) \geq (1 + \epsilon)f(X_{\tau(i)}), \tau(i+1) = \tau(i) + 1 \mid X_{\tau(i)} = x] \\ &= 1 - \mathbb{P}[f(X_{\tau(i)+1}) \geq (1 + \epsilon)f(X_{\tau(i)}) \mid X_{\tau(i)} = x] \\ &= 1 - \mathbb{P}[f(X_{t+1}) \geq (1 + \epsilon)f(X_t) \mid X_t = x] \\ &< e^{-c_2 f(x)^2/n}. \end{aligned}$$

These two properties are sufficient to study the number of rounds required by the new Markov chain $\{R_i\}_{i \in \mathbb{N}}$ to reach the target value m . Indeed, by defining the random variable $Z_i = \frac{f(R_i)}{\sqrt{n}}$, and considering the following “potential” function, $Y_i = \exp\left(\frac{m}{\sqrt{n}} - Z_i\right)$, we can compute its expectation at the next round as follows. Let us fix any state $x \in \Omega$ such that $h\sqrt{n} \leq f(x) < m$, and define $z = \frac{f(x)}{\sqrt{n}}$, $y = \exp\left(\frac{m}{\sqrt{n}} - z\right)$. We have

$$\begin{aligned} \mathbb{E}[Y_{i+1} \mid R_i = x] &\leq \mathbb{P}[f(R_{i+1}) < (1 + \epsilon)f(x)] e^{m/\sqrt{n}} \\ & \quad + \mathbb{P}[f(R_{i+1}) \geq (1 + \epsilon)f(x)] e^{m/\sqrt{n} - (1 + \epsilon)z} \\ (\text{from Hypothesis (2)}) &\leq e^{-c_2 z^2} \cdot e^{m/\sqrt{n}} + 1 \cdot e^{m/\sqrt{n} - (1 + \epsilon)z} \end{aligned}$$

$$\begin{aligned}
&= e^{m/\sqrt{n}-c_2 z^2} + e^{m/\sqrt{n}-z-\epsilon z} \\
&= e^{m/\sqrt{n}-z}(e^{z-c_2 z^2} + e^{-\epsilon z}) \\
&\leq e^{m/\sqrt{n}-z}(e^{-2} + e^{-2}) \\
&< \frac{e^{m/\sqrt{n}-z}}{e} \\
&= \frac{y}{e},
\end{aligned} \tag{15}$$

where in (15) we used that z is always at least h and thanks to Hypothesis (1) we can choose a sufficiently large h .

By applying the Markov inequality and iterating the above bound, we get

$$\mathbb{P}[Y_i > 1] \leq \frac{\mathbb{E}[Y_i]}{1} \leq \frac{\mathbb{E}[Y_{i-1}]}{e} \leq \dots \leq \frac{\mathbb{E}[Y_0]}{e^{\tau_R}} \leq \frac{e^{m/\sqrt{n}}}{e^i}.$$

We observe that if $Y_i \leq 1$ then $R_i \geq m$, thus by setting $i = m/\sqrt{n} + \log n = (c_3 + 1) \log n$, we get:

$$\mathbb{P}[R_{(c_3+1) \log n} < m] = \mathbb{P}[Y_{(c_3+1) \log n} > 1] < \frac{1}{n}. \tag{16}$$

Our next goal is to give an upper bound on the hitting time $\tau_{(c_3+1) \log n}$. Note that the event “ $\tau_{(c_3+1) \log n} > c_4 \log n$ ” holds if and only if the number of rounds such that $f(X_t) \geq h\sqrt{n}$ (before round $c_4 \log n$) is less than $(c_3 + 1) \log n$. Thanks to Hypothesis (1), at each round t there is at least probability $1 - c_1$ that $f(X_t) \geq h\sqrt{n}$. This implies that, for any positive constant c_4 , the probability $\mathbb{P}[\tau_{(c_3+1) \log n} > c_4 \log n]$ is bounded by the probability that, within $c_4 \log n$ independent Bernoulli trials, we get less than $(c_3 + 1) \log n$ successes, where the success probability is at least $1 - c_1$. We can thus choose a sufficiently large $c_4 = c_4(c_1, c_3)$ and apply the multiplicative form of the Chernoff bound (Lemma 16), obtaining

$$\mathbb{P}[\tau_{(c_3+1) \log n} > c_4 \log n] < \frac{1}{n}. \tag{17}$$

We are now ready to prove the Lemma using (16) and (17), indeed

$$\begin{aligned}
\mathbb{P}[X_{c_4 \log n} \geq m] &> \mathbb{P}[R_{(c_3+1) \log n} \geq m \wedge \tau_{(c_3+1) \log n} \leq c_4 \log n] \\
&= 1 - \mathbb{P}[R_{(c_3+1) \log n} < m \vee \tau_{(c_3+1) \log n} > c_4 \log n] \\
&\geq 1 - \mathbb{P}[R_{(c_3+1) \log n} < m] + \mathbb{P}[\tau_{(c_3+1) \log n} > c_4 \log n] \\
&> 1 - \frac{2}{n}.
\end{aligned}$$

Hence, choosing a suitable large c_4 , we have shown that in $c_4 \log n$ rounds the process reaches the target value m , w.h.p. \square

Our goal is to apply the above lemma to the UNDECIDED-STATE process (which defines a finite-state Markov chain) starting with bias of size $o(\sqrt{n} \log n)$ where we set $f(\mathbf{X}_t) = s(\mathbf{X}_t)$, $c_3 = \gamma > 0$ for some constant $\gamma > 0$, and $m = \gamma \sqrt{n} \log n$: this would imply the upper bound $\mathcal{O}(\log n)$ on the number of rounds needed to reach a configuration having bias $\Omega(\sqrt{n} \log n)$, w.h.p., breaking the symmetry because Theorem 1 then holds. To this aim, with the next two lemmas we show that the UNDECIDED-STATE process satisfies the hypotheses of Lemma 22 in this setting, w.h.p.

Lemma 23. *Let ϵ be any constant such that $(1 - 6p)^2/2 \leq \epsilon < (1 - 6p)(1 - 2p)$. Let \mathbf{x} be any configuration in which $s \leq \check{s}_\epsilon$, and $q \leq \frac{n}{2}$. Then, in the next round, it holds that $\hat{q}_{\epsilon/12} \leq Q \leq \frac{n}{2}$ with probability $1 - \exp(-\epsilon^2 p^2 n / (2^4 3^2))$.*

Proof of Lemma 23. Lemma 3 implies that $Q \geq \hat{q}_{\epsilon/12}$ with probability $1 - \exp(-\epsilon^2 n / (2^3 3^2 (1 - 3p)^2))$. At the same time, by Eq. (6) it holds that

$$\mathbb{E}[Q \mid \mathbf{x}] \leq pn + \frac{1 - 3p}{2n} [2q^2 + (n - q)^2] = f(q).$$

For $q \leq \frac{n}{2}$, the maximum of f is obtained either at $q_1 = 0$ or at $q_2 = \frac{n}{2}$. Then,

$$\begin{aligned} f(q_1) &= \left(p + \frac{1-3p}{2}\right)n = \frac{(1-p)n}{2}; \\ f(q_2) &= pn + \frac{1-3p}{2n} \left[\frac{n^2}{2} + n^2 - n^2 + \frac{n^2}{4}\right] = \left(p + \frac{3(1-3p)}{8}\right)n \\ &= \frac{(3-p)n}{8}. \end{aligned}$$

Therefore, we have $f(q) \leq (1-p)n/2$ for $q \leq \frac{n}{2}$, since $(1-p)/2 > (3-p)/8$ for $p < 1/2$. By the additive form of Chernoff bound (Lemma 17),

$$\begin{aligned} \mathbb{P}\left[Q \geq \frac{n}{2} \mid \mathbf{x}\right] &= \mathbb{P}\left[Q \geq \frac{(1-p)n}{2} + \frac{pn}{2} \mid \mathbf{x}\right] \leq \mathbb{P}\left[Q \geq \mathbb{E}[Q \mid \mathbf{x}] + \frac{pn}{2} \mid \mathbf{x}\right] \\ &\leq \exp(-p^2n/2). \end{aligned}$$

Hence, the joint probability that $Q \leq n/2$ and $Q \geq \hat{q}_{\epsilon/12}$ is at least $1 - \exp(-\epsilon^2 p^2 n / (2^4 3^2))$, since $\epsilon < 1$ and $p < 1$. \square

Lemma 24. *Let ϵ be any constant such that $(1-6p)^2/2 \leq \epsilon < (1-6p)(1-2p)$. Let \mathbf{x} be any configuration such that $q(\mathbf{x}) \in [\hat{q}_{\epsilon/12}, n/2]$. Then, it holds that*

(1) *for any constant $h > 0$ there exists a constant $c_1 > 0$ (depending only on p) such that*

$$\mathbb{P}[|S| < h\sqrt{n} \mid \mathbf{X}_t = \mathbf{x}] < c_1;$$

(2) *there exist two positive constants δ and c_2 (depending only on p) such that*

$$\mathbb{P}[|S| \geq (1+\delta)s \mid \mathbf{X}_t = \mathbf{x}] \geq 1 - e^{-c_2 \frac{s^2}{n}}.$$

Proof of Lemma 24. As for the first item, let \mathbf{x} and \mathbf{x}_0 be two states such that $|s(\mathbf{x})| < h\sqrt{n} \log n$, $|s(\mathbf{x}_0)| = 0$, $q(\mathbf{x}) = q(\mathbf{x}_0)$. A simple domination argument implies that

$$\mathbb{P}[|S| < h\sqrt{n} \mid \mathbf{X}_t = \mathbf{x}] \leq \mathbb{P}[|S| < h\sqrt{n} \mid \mathbf{X}_t = \mathbf{x}_0].$$

Thus, we can bound just the second probability, where the initial bias is zero, which implies that $a = b$.

Define A^q, B^q, Q^q the random variables counting the nodes that were undecided in the configuration \mathbf{x}_0 and that, in the next round, get the opinion *Alpha*, *Beta*, and undecided, respectively. Similarly, $A^a (B^b)$ counts the nodes that support opinion *Alpha* (*Beta*) in the configuration \mathbf{x}_0 and that, in the next round, still support the same opinion. Trivially, $A = A^q + A^a$ and $B = B^q + B^b$. Moreover, observe that, among these random variables, only A^q and B^q are mutually dependent. Thus, conditioned to the event $\{\mathbf{X}_t = \mathbf{x}_0\}$, if $\alpha = \mathbb{E}[A^a \mid \mathbf{X}_t = \mathbf{x}_0] = \mathbb{E}[B^b \mid \mathbf{X}_t = \mathbf{x}_0]$, it holds that

$$\mathbb{P}[|S| \geq h\sqrt{n}] \geq \mathbb{P}[A \geq B + h\sqrt{n}] \geq \mathbb{P}[A^q \geq B^q + h\sqrt{n}] \mathbb{P}[A^a \geq \alpha] \mathbb{P}[B^b \leq \alpha].$$

The random variables $A^a - \alpha$ and $B^b - \alpha$ follow binomial distributions with expectation 0 (recall that $a = b$), and finite second and third moment. Thus, the Berry-Esseen theorem (Lemma 19 in Appendix A) allows us to approximate up to an arbitrary-small constant $\epsilon_1 > 0$ (as long as n is large enough) both the random variables with a normal distribution that has expectation 0. Thus,

$$\mathbb{P}[A^a \geq \alpha] = \mathbb{P}[B^b \leq \alpha] \geq \left(\frac{1}{2} - \epsilon_1\right).$$

As for the random variable $A^q - B^q$, notice that conditioned to the event $\{q - Q^q = k\}$, it is the sum of k Rademacher random variables. The hypothesis $q \leq \frac{n}{2}$ allows us to use the Chernoff bound on Q^q and show that $Q^q \leq \frac{3}{4}q$ w.h.p. Thus, since $q \geq \hat{q}_{\epsilon/12}$, it holds that $q - Q^q = \Theta(n)$ w.h.p. It follows that the

conditional variance of $A^q - B^q$ given $q - Q^q$ yields $\Theta(n)$ w.h.p., and $A^q - B^q$ conditional on the event $E = \{q - Q^q = \Theta(n)\}$ can be approximated by a normal distribution up to an arbitrary-small constant $\epsilon_2 > 0$. Then, we have that

$$\mathbb{P}[A^q \geq B^q + h\sqrt{n}] \geq \mathbb{P}[A^q \geq B^q + h\sqrt{n} \mid E] \mathbb{P}[E] \geq \epsilon_2.$$

Setting $c_1 = \epsilon_1 \cdot \epsilon_2$, we get property (1). We can choose ϵ_1, ϵ_2 to be equal to p , so that c_1 depends only on p . As for property (2), by Eq. (5) and the hypothesis on q we have that

$$\mathbb{E}[S \mid \mathbf{X}_t = \mathbf{x}] \geq s(1 + \epsilon/12).$$

We can get the property applying the Hoeffding bound (Lemma 18), getting that

$$\mathbb{P}\left[S \leq s\left(1 + \frac{\epsilon}{24}\right) \mid \mathbf{x}\right] = \mathbb{P}\left[S \leq s\left(1 + \frac{\epsilon}{12}\right) - \frac{\epsilon s}{24} \mid \mathbf{x}\right] \leq \exp\left(-\frac{\epsilon^2 s^2}{2^7 3^2 n}\right),$$

which is the thesis. \square

The reader may notice that Lemma 23 requires the number of undecided nodes to be inside the interval $[\hat{q}_{\epsilon/12}, n/2]$. We will later take care of this issue with Lemma 27, showing that whenever this number is not within the above interval, in at most $\mathcal{O}(\log n)$ rounds it will. Furthermore, Lemma 23 guarantees that the condition on the undecided nodes holds “only” w.h.p., while Lemma 22 requires this condition to hold with probability 1. We show this issue can be solved using a coupling argument similar to that by [14]. The key point is that, starting from any configuration \mathbf{x} with $q(\mathbf{x}) \in [\hat{q}_{\epsilon/12}, n/2]$, the probability that the process goes in one of those “bad” configurations with q outside the above interval is negligible. Intuitively speaking, the configurations *actually visited* by the process before breaking symmetry do satisfy the hypothesis of Lemma 22. In order to make this argument rigorous, we define a *pruned* process, by removing all the *unwanted* transitions.

Let $\bar{s} \in \{0, 1, \dots, n\}$, and $\mathbf{z}(\bar{s})$ the configuration such that $s(\mathbf{z}(\bar{s})) = \bar{s}$, and $q(\mathbf{z}(\bar{s})) = n/2$. Let $p_{\mathbf{x}, \mathbf{y}}$ be the probability of a transition from the configuration \mathbf{x} to the configuration \mathbf{y} in the UNDECIDED-STATE process. The PRUNED process behaves exactly as the original process but every transition from a configuration \mathbf{x} such that $q(\mathbf{x}) \in [\hat{q}_{\epsilon/12}, n/2]$ and $s(\mathbf{x}) = \mathcal{O}(\sqrt{n \log n})$ to a configuration \mathbf{y} such that $q(\mathbf{y}) < \hat{q}_{\epsilon/12}$ or $q(\mathbf{y}) > n/2$ has probability $p'_{\mathbf{x}, \mathbf{y}} = 0$. Moreover, for any $\bar{s} \in [n]$, starting from the configuration \mathbf{x} , the probability of reaching the configuration $\mathbf{z}(\bar{s})$ is

$$p'_{\mathbf{x}, \mathbf{z}(\bar{s})} = p_{\mathbf{x}, \mathbf{z}(\bar{s})} + \sum_{\substack{\mathbf{y}: s(\mathbf{y})=\bar{s} \text{ and} \\ q(\mathbf{y}) \notin [\hat{q}_{\epsilon/12}, \frac{n}{2}]}} p_{\mathbf{x}, \mathbf{y}}.$$

All the other transition probabilities remain the same. Observe that the PRUNED process is defined in such a way that it has exactly the same marginal probability of the original process w.r.t. the random variable $s(\mathbf{X}_t)$; thus, Lemma 24 holds for the PRUNED process as well and we can apply Lemma 22. Then, the PRUNED process reaches a configuration having bias $\Omega(\sqrt{n \log n})$ within $\mathcal{O}(\log n)$ rounds, w.h.p., as shown in the following lemma.

Lemma 25. *Let ϵ be any constant such that $(1-6p)^2/2 \leq \epsilon < (1-6p)(1-2p)$. Starting from any configuration \mathbf{x} such that $q(\mathbf{x}) \in [\hat{q}_{\epsilon/12}, n/2]$ and $s(\mathbf{x}) = \mathcal{O}(\sqrt{n \log n})$, the PRUNED process reaches a configuration having bias $\Omega(\sqrt{n \log n})$ within $\mathcal{O}_{p, \gamma}(\log n)$ rounds with probability $1 - 2/n$.*

Proof of Lemma 25. Let $\gamma > 0$ be a constant and $m = \gamma\sqrt{n \log n}$ be the target value of the bias in Lemma 22. Since $q(\mathbf{x}) \in [\hat{q}_{\epsilon/12}, n/2]$ and $s(\mathbf{x}) = \mathcal{O}(\sqrt{n \log n})$, the PRUNED process always satisfies Lemma 24, and thus we can apply Lemma 22 (setting the function $f(\mathbf{X}_t) = s(\mathbf{X}_t)$), which gives us that the PRUNED process reaches a configuration \mathbf{y} having bias $s(\mathbf{y}) \geq m = \Omega(\sqrt{n \log n})$ within $\mathcal{O}_{p, \gamma}(\log n)$ rounds with probability $1 - 2/n$. \square

We now want to go back to the original process. The definition of the PRUNED process suggests a natural coupling between it and the original one. If the two process are in different states, then they act independently,

while, if they are in the same state \mathbf{x} , they move together unless the UNDECIDED-STATE process goes in a configuration \mathbf{y} such that $q(\mathbf{y}) \notin [\hat{q}_{\epsilon/12}, n/2]$. In that case, the PRUNED process goes in $\mathbf{z}(s(\mathbf{y}))$. In the proof of the next lemma, we show that the time the PRUNED process takes to reach bias $\Omega(\sqrt{n \log n})$ stochastically dominates the one of the original process, giving the result.

Lemma 26. *Let γ, ϵ be any two positive constants, with $(1-6p)^2/2 \leq \epsilon < (1-6p)(1-2p)$. Starting from any configuration \mathbf{x} such that $q(\mathbf{x}) \in [\hat{q}_{\epsilon/12}, n/2]$ and $s(\mathbf{x}) \leq \gamma\sqrt{n \log n}$, the UNDECIDED-STATE process reaches a configuration having bias $s \geq \gamma\sqrt{n \log n}$ within $\mathcal{O}_{p,\gamma}(\log n)$ rounds with probability at least $1 - 3/n$.*

Proof of Lemma 26. Let $\{\mathbf{X}_t\}$ and $\{\mathbf{Y}_t\}$ be the original process and the pruned one, respectively. Denote the set of possible initial configuration according to the hypothesis by H , and let $\mathbf{x} \in H$. Note that if $\mathbf{X}_t = \mathbf{Y}_t = \mathbf{x}$, then

$$\mathbf{Y}_{t+1} = \begin{cases} \mathbf{X}_{t+1} & \text{if } \mathbf{X}_{t+1} \in H \\ \mathbf{z}(s(\mathbf{X}_t)) & \text{otherwise} \end{cases}.$$

Let $\tau = \inf\{t : \mathbb{N} : |s(\mathbf{X}_t)| \geq \sqrt{n \log n}\}$, and let $\tau^* = \inf\{t \in \mathbb{N} : |s(\mathbf{Y}_t)| \geq \sqrt{n \log n}\}$. For any configuration $\mathbf{x} \in H$, define $\rho_{\mathbf{x}}^t$ the event that the two processes $\{\mathbf{X}_t\}$ and $\{\mathbf{Y}_t\}$ have separated at round $t+1$, i.e. $\rho_{\mathbf{x}}^t = \{\mathbf{X}_t = \mathbf{Y}_t = \mathbf{x}_t\} \cap \{\mathbf{X}_{t+1} \neq \mathbf{Y}_{t+1}\}$. Observe that, if the two couple processes in the same configuration $\mathbf{x}_0 \in H$ and $\tau > c \log n$, then either $\tau^* > c \log n$ or there exists a round $t \leq c \log n$ such that for some $\mathbf{x} \in H$ the event $\rho_{\mathbf{x}}^t$ has occurred. Hence, if $\mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}[\cdot]$ is the joint probability for the couple $(\mathbf{X}_t, \mathbf{Y}_t)$ which both start at \mathbf{x}_0 , we have

$$\begin{aligned} & \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}[\tau > c \log n] \\ & \leq \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}[\{\tau^* > c \log n\} \cup \{\exists t \leq c \log n, \exists \mathbf{x} \in H : \rho_{\mathbf{x}}^t\}] \\ & \leq \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}[\tau^* > c \log n] + \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}[\exists t \leq c \log n, \exists \mathbf{x} \in H : \rho_{\mathbf{x}}^t]. \end{aligned}$$

We choose a suitable constant c which depends on p and γ , in order to apply Lemmas 22 and 23. As for the first item, since Lemma 22 holds for the PRUNED process, we have that it is upper bounded by $2/n$. As for the second term, we get that

$$\begin{aligned} & \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}[\exists t \leq c \log n, \exists \mathbf{x} \in H : \rho_{\mathbf{x}}^t] \leq \sum_{t=1}^{c \log n} \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}[\exists \mathbf{x} \in H : \rho_{\mathbf{x}}^t] \\ & = \sum_{t=1}^{c \log n} \sum_{\mathbf{x} \in H} \mathbb{P}'_{\mathbf{x}_0, \mathbf{x}_0}[\rho_{\mathbf{x}}^t] \stackrel{(a)}{\leq} \sum_{t=1}^{c \log n} \frac{n^2}{e^{-\frac{\epsilon^2 p^2 n}{2^4 3^2}}} \leq \frac{1}{n}, \end{aligned}$$

where in (a) the second inequality we used Lemma 23, and the fact that $|H|$ is at most all the combinations of parameters q and s . \square

Now, we take care of those cases in which the starting configuration is such that $q \notin [\hat{q}_{\epsilon/12}, n/2]$. If $q < \hat{q}_{\epsilon/12}$ then, for Lemma 23, $Q \in [\hat{q}_{\epsilon/12}, n/2]$ w.h.p. Next lemma takes care of the case $q > n/2$.

Lemma 27. *Let \mathbf{x} be any starting configuration such that $q(\mathbf{x}) > \frac{n}{2}$. Then, at the next round, it holds that $Q \leq q(1-p/2)$ with probability $1 - \exp(-p^2 n/2^3)$.*

Proof of Lemma 27. From Eq. (6), we have that

$$\begin{aligned} \mathbb{E}[Q \mid \mathbf{x}] - q(1-2p) & \leq pn + \frac{1-3p}{2n} [2q^2 + (n-q)^2] - q(1-p) \\ & = 3q^2 \left(\frac{1-3p}{2n} \right) - q(2-4p) + n \left(\frac{1-p}{2} \right) = f(q). \end{aligned}$$

Now, $f(q)$ takes its maximum either in $q_1 = n/2$, or in $q_2 = n$. Then,

$$f(q_1) = \frac{3n}{4} \left(\frac{1-3p}{2} \right) - \frac{n}{2}(2-4p) + n \left(\frac{1-p}{2} \right) = n \cdot \frac{3-9p-8+16p+4-4p}{8}$$

$$\begin{aligned}
&= n \cdot \frac{-1+3p}{8}; \\
f(q_2) &= 3n \left(\frac{1-3p}{2} \right) - n(2-4p) + n \left(\frac{1-p}{2} \right) = n \cdot \frac{3-9p-4+8p+1-p}{2} \\
&= -pn.
\end{aligned}$$

Since $p < 1/6$, we have that $f(q) < 0$ for $n/2 < q \leq n$. Thus,

$$\mathbb{E}[Q \mid \mathbf{x}] \leq q(1-p),$$

and we can use the Chernoff bound (Lemma 17).

$$\begin{aligned}
\mathbb{P} \left[Q \geq q \left(1 - \frac{p}{2} \right) \mid \mathbf{x} \right] &= \mathbb{P} \left[Q \geq q(1-p) + \frac{pq}{2} \mid \mathbf{x} \right] \leq \mathbb{P} \left[Q \geq \mathbb{E}[Q \mid \mathbf{x}] + \frac{pn}{4} \mid \mathbf{x} \right] \\
&\leq \exp \left(-\frac{p^2 n}{2^3} \right).
\end{aligned}$$

□

Finally, we are ready to prove Theorem 3.

Proof of Theorem 3: Wrap-Up. Let $\gamma = p$, and let $m = \gamma\sqrt{n} \log n$ be the target value of the bias in Lemma 22. Let \mathbf{x} be any initial configuration having bias $|s| < m$. Let ϵ be any constant such that $(1-6p)^2/2 \leq \epsilon < (1-6p)(1-2p)$. We have two cases.

- (i) If the number of undecided nodes is such that $q(\mathbf{x}) \in [\hat{q}_{\epsilon/12}, n/2]$, then Lemma 26 implies that the UNDECIDED-STATE process reaches a configuration having bias $s \geq \gamma\sqrt{n} \log n$ in $\mathcal{O}_p(\log n)$ rounds with probability $1 - 3/n$;
- (ii) else, if the starting configuration is such that $q(\mathbf{x}) \notin [\hat{q}_{\epsilon/12}, n/2]$, then, for Lemmas 23 and 27, the UNDECIDED-STATE process reaches within $\mathcal{O}_p(\log n)$ rounds a configuration having the number of undecided nodes $q \in [\hat{q}_{\epsilon/12}, n/2]$, with probability $1 - \exp(-p^2 n/2^4)$ for Lemmas 20 and 21. Then, either the bias is $s \geq \gamma\sqrt{n} \log n$, or we are in case (i). As Lemmas 20 and 21 in the preliminaries imply, the probability that case (ii) and then case (i) take place is a high probability (in this case, with probability at least $1 - 4/n$).

Then, Theorem 1 gives the desired result.

□