

**ON L^p - ESTIMATES OF MILD SOLUTIONS FOR A CLASS OF
SEMILINEAR STOCHASTIC EVOLUTIONS EQUATIONS
DRIVEN BY LÉVY AND STABLE PROCESSES**

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ABSTRACT. We study existence and uniqueness of $L^p([0, T] \times \Omega)$ -bounded mild solutions for a class of semilinear stochastic evolutions equations driven by a general class of Lévy processes without Gaussian component including both the non square integrable (α -stable process) and the square integrable cases on a probability space. This is done using a stochastic analysis on the jumps of the Lévy process process with particular attention to the non square integrable case (for instance the α -stable process, $\alpha \in (0, 2)$) through a truncation method by separating the big and small jumps together with a classical fixed point theorem ; under local Lipschitz, Hölder, linear growth conditions on the coefficients. Finally, we give an example to show usefulness of the theoretical results that we obtain in this paper.

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1. INTRODUCTION

In this paper, we consider the following class of semilinear stochastic differential equation driven by a general class of Lévy processes Z defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\begin{cases} dX(t) = [AX(t) + F(t, X(t))]dt + G(t, X(t))dZ(t), & t \in [0, T] \\ X(0) = x_0 \in \mathbb{X}, \end{cases} \quad (1)$$

when Z is square integrable or non square integrable, for instance the stable process process with index $\alpha \in (0, 2)$) in an appropriate space \mathbb{X} with $T > 0$ a time horizon, whereas the operator $A : D(A) \subseteq \mathbb{X} \rightarrow \mathbb{X}$ is a linear operator, in general unbounded, defined on a certain dense linear subspace $D(A)$ which generates a strongly continuous one-parameter semigroup (also known as C_0 -semigroup) $S(t)$ of bounded operators on \mathbb{X} ; $F : [0, T] \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$, $G : [0, T] \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$ are two functions to be specied later and the initial value $x_0 \in \mathbb{X}$ is a random variable on Ω and \mathcal{F}_0 -adapted.

We first investigate the stochastic continuity and integrability of the solutions trajectories of the stochastic partial differential equation (1) and then the existence

and uniqueness of the mild solution in an appropriate space. We apply our abstract result to a stochastic partial differential equation (SPDE) namely the one dimensional stochastic heat equation driven by an α -stable process Z , $\alpha \in (0, 2)$, with Dirichlet boundary conditions:

$$\begin{cases} du(t, x) = \frac{\partial^2 u(t, x)}{\partial x^2} dt + F(t, u(t, x)) dt + \sin(t) \phi(u(t, x)) dZ(t) \\ u(t, 0) = u(t, 1) = 0, t \in [0, T], \\ u(0, x) = h(x) \in L^2[0, 1] \end{cases} \quad (2)$$

In the case where Z is a Wiener process, the theory of stochastic equations (1) is well understood as well as the case $A = 0$ where (1) is just a stochastic differential equation (SDE):

$$dX(t) = F(t, X(t)) dZ(t) + G(t, X(t)) dt, \quad X(0) = x_0. \quad (3)$$

Recall that, one of the most important property of Brownian motion, is that almost every path of Brownian motion is continuous, that is, has no jumps. In the case when Z is a Brownian motion (even in the infinite dimensional case), there is a rich literature on the existence and uniqueness of weak, strong solution and mild solutions under various conditions on the coefficient of the above SDE and SPDE, see [1, 2, 3] and references therein.

It is known that if we integrate with respect to a Brownian process, then it is enough to assume that the integrand is measurable, adapted and locally square integrable. Obviously if X is càdlàg and adapted, then $X(t-)$, $t \in [0, T]$ is predictable. Unfortunately, in some important cases, it is not known when solutions X of (1) have càdlàg trajectories and a possible lack of càdlàg regularity is pointed out for instance in [1], [Proposition 9.4.4]. The problem of predictability of integrands is also treated in several papers, see for instance [4], [5], [6], [7] and [8]. This motivates the authors in [10] to study the existence of adapted stochastic continuous solutions (this guarantees a predictable modification) of (1) when Z is a stable cylindrical process in a Hilbert space with $F(t, X(t)) := K(X(t))$ and $G = 1$ under the fact A generate a C_0 -semigroup of the form e^{tA} under classical Lipschitz condition on F, G . Again for such cylindrical stable process Z in Hilbert space, the authors in [9] investigate recently pathwise uniqueness and weak existence for the SPDE with Hölder continuous coefficient when $G = 1$ and $F(t, X(t)) = B(X(t))$ to generalise an infinite dimensional generalisation of the result of [10]. For a jump process Z for instance a Lévy process, there is also some significant existence results in both finite and infinite dimensional space; see for instance [11, 12, 13, 14] and references there in.

But if we consider a Lévy process Z with unbounded jumps and with infinite jump activity like α -stable processes, $\alpha \in (0, 2)$, in a finite time period, new phenomena (like the so called-heavy tailed phenomena) and difficulties appear so that one need to make a different analysis since the situation changes completely. For any $t \geq 0$, the expectation $\mathbb{E}|Z(t)|^p$ is finite when $0 \leq p < \alpha$, but when $p \geq \alpha$, it is infinite and in particular when $\alpha \leq 1$, even the expected value of X_t is not well-defined. The upshot of this is that an α -stable process may exhibit large-magnitude, low-intensity jumps which are very rare but whose size forces the expectation to be infinite.

The behavior of a stochastic integral driven by an α -stable driving process Z is perturbed by the regularly varying tails of the α -stable process Z . One cannot expect a stable stochastic integral to be square integrable, so that the tools often used in

stochastic calculus in a Hilbert space can't work directly in the case of an α -stable processes.

When $A = 0$ and Z being a purely discontinuous Lévy process including α -stable process, pathwise uniqueness and weak existence result of (3) was studied under various conditions on the coefficient such as Lipschitz continuity, boundness, Hölder continuity, see [15, 16, 17]. In the case $A \neq 1$, to the best of our knowledge there is no references dealing with existence, uniqueness, stochastically continuous and L^p - estimates of solutions of (1) in the non square integrable Lévy processes such as α -stable processes.

The novelty of our paper is based on the new model (1) considering a general Lévy processus including α -stable process Z and coefficient $G \neq 1$. For this stochastic model, we show that under an appropriate condition on F and G we can obtain the stochastic continuity of the mild solutions as well as some integrability properties using an original method based on a troncation procedure in the Lévy Itô decomposition of Z . For more references of this technique we refer the reader to our early paper [20]. In order to control and analyse the effect of the jumps, we separate big and small jumps and obtain some new stochastic convolutions integrals when the operator A generates a strongly continuous one-parameter semigroup.

This paper is organized as follows. Section 2 deals with some preliminaries intended to introduce briefly basic facts on general Lévy process including the non square integrable case, for instance the α -stable process and some of their important properties in order to clarify the computation of the tail behavior of the stochastic convolutions integrals equations. In section 3 and 4, we give some important results based on a square and non square integrable Lévy process namely the existence and uniqueness of adapted stochastic continuous $L^p([0, T] \times \Omega)$ -bounded solutions, see Theorem 3.3 and Theorem 4.6. We illustrate our conditions with the above one-dimensional stochastic heat equation driven by an α - stable process.

2. PRELIMINARIES

Let us recall some basics definitions and properties for Lévy processes. We follow the presentation in [13], [19]. Assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with some filtration $(\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions.

2.1. Basic facts about Lévy processes. Lévy processes are class of stochastic processes with discontinuous paths, simple enough to study and rich enough for applications or at least to be used in more realistic models.

Definition 2.1. *A stochastic process $(Z(t))_{t \in [0, T]}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it possesses the following properties:*

- i) *The paths of Z are \mathbb{P} -almost surely right continuous with left limits.*
- ii) $\mathbb{P}(Z_0 = 0) = 1$.
- iii) *For $0 \leq s \leq t$, $Z(t) - X(s)$ has is equal in distribution to $Z(t - s)$: its increments are stationary.*

- iv) For $0 \leq s \leq t$, $Z(t) - Z(s)$ is independent of $\{Z(u) : u \leq s\}$: its increments are independent.

From these properties it follows that X is continuous in probability :

$$\forall \epsilon > 0, \quad \lim_{s \rightarrow 0} \mathbb{P}(|Z(t+s) - Z(t)| > \epsilon) = 0.$$

Note that the properties of stationary and independent increments implies that a Lévy process is a (strong) Markov process. Lévy processes are essentially processes with jumps, because it can be shown that any Lévy process which has a.s. continuous trajectories is a Brownian motion with drift. There are plenty classical examples of Lévy processes. As a jump process, it can be described by its Poisson jump measure (jump measure of Z on interval $[0, t]$) defined as

$$\mu(t, A) = \sum_{0 \leq s \leq t} \mathbb{I}_A(Z(s) - Z(s-)),$$

the number of jumps of Z on the interval $[0, t]$ whose size lies in the set A bounded below. For such A , the process $\mu(\cdot, A)$ is a Poisson process with intensity $\nu(A) := \mathbb{E}(\mu(1, A))$.

We now examine the characteristic functions of Lévy processes. Denote by φ_{Z_t} the characteristic function of a Lévy process Z at time t .

Theorem 2.1 (Lévy - Khintchine formula). *There exists (unique) $b \in \mathbb{R}$, $\sigma \geq 0$, and a measure ν (Lévy measure), with no atom at zero ($\nu(\{0\}) = 0$), satisfying*

$$\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$$

such that

$$\varphi_{Z_t}(u) = \exp t \left(i u b - \frac{1}{2} \sigma^2 u^2 + \int_{-\infty}^{+\infty} (e^{i u y} - 1 - i u y \mathbb{I}_{|y| \leq 1}) \nu(dy) \right), \quad t \in [0, T]. \quad (4)$$

Conversely, given any admissible choices (b, σ, ν) , there exists a Lévy process Z with characteristic exponent given by the above formula.

The first part of this theorem is rather technical, see [13], Theorem 8.1. The second part amounts to constructing a Lévy process out of its characteristics, and can be seen as part of the more detailed Lévy-Itô decomposition.

Theorem 2.2 (Lévy - Itô decomposition). *For a Lévy process Z , denote by*

$$\tilde{\mu}(t, A) = \mu(t, A) - t\nu(A)$$

the compensated random martingale measure of μ . Then, there exist $b \in \mathbb{R}$, $\sigma \geq 0$ and a standard Brownian motion B such that

$$Z(t) = bt + \sigma B(t) + \int_0^t \int_{|x| \leq 1} x \tilde{\mu}(ds, dx) + \int_0^t \int_{|x| > 1} x \mu(ds, dx). \quad (5)$$

This means that all Lévy processes are sum of a drift, a Brownian motion and a Poisson process. The Lévy measure is responsible for the richness of the class of Lévy processes and carries useful information about not only the path structure of the process but also on the finiteness of the moments of a Lévy process. In this paper we shall be concerned with purely discontinuous Lévy processes meaning without gaussian component. We will consider general class of Lévy processes including both the square integrable and the non square integrable cases; more

precisely we assume in the rest of this paper that

$(\mathbf{H}_\nu^{\beta, \mathbf{P}})$

$$\text{There exist } \beta \in (0, 2) \text{ such that } \forall p \in (0, \beta) : \\ \int_{|x| \geq 1} |x|^p \nu(dx) < \infty \quad \text{and} \quad \int_{|x| \geq 1} |x|^2 \nu(dx) = +\infty.$$

or

(\mathbf{H}_ν^2)

$$\int_{|x| \geq 1} |x|^2 \nu(dx) < \infty.$$

The following examples provide some concrete examples of (square intégrable) Lévy processes with condition (\mathbf{H}_ν^2) .

2.1.1. *Gamma Processes.* Let $\alpha, \lambda > 0$ and define the probability measure

$$\mu_{\alpha, \lambda}(dx) = \frac{\alpha^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-\alpha x} dx$$

supported in $[0, \infty)$: this is called *Gamma* (α, λ) distribution. Note that when $\lambda = 1$ this is the exponential distribution. The probability measure $\mu_{\alpha, \lambda}$ is infinitely divisible and we obtain the Lévy-Khintchine decomposition with $\sigma = 0$, $b = \int_0^1 x \nu(dx)$ where

$$\nu(dx) = \lambda x^{-1} e^{-\alpha x} dx$$

concentrated on $(0, \infty)$.

2.1.2. *Inverse Gaussian Processes.* Let $W = W(t)$, $t \geq 0$ be a standard Brownian motion. Define the first passage time

$$\tau_s = \inf\{t > 0 : W(t) + at \geq s\}$$

that is the first time a Brownian motion with linear drift $a > 0$ crosses above level s . It is shown that the process $\tau = \{\tau_t : t \geq 0\}$ has stationary independent increments. Continuity of the paths of W ensures that τ has right continuous paths. Further, it is clear that τ has almost surely non-decreasing paths, which guarantees its paths have left limits as well as being yet another example of a Subordinator.

According to the definition of τ as a sequence of first passage time which is a stopping time, τ is also the almost sure right inverse of the path of the graph of $\{W(t) + bt : t \geq 0\}$. From this τ earns its name as the inverse Gaussian process. The random variable τ_s is infinitely divisible for $s > 0$. The Lévy-Khintchine decomposition give

$$\sigma = 0; \quad b = -2sa^{-1} \int_0^a \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

and

$$\nu(dx) = \frac{s}{\sqrt{2\pi x^3}} e^{-a^2 x/2} dx$$

concentrated on $(0, \infty)$.

Now, let us introduce a particular class of Lvy processes which is not square integrable. An example of Lévy process which is not square integrable namely satisfying condition $(\mathbf{H}_\nu^{\beta, \mathbf{P}})$ is the following α -stable process (with $\beta = \alpha$).

2.1.3. *Stable process.* We follow the presentation in our paper [20] or [21], (see also [22]). For the sake of brevity, by an α -stable process Z where $\alpha \in (0, 2)$ with characteristics (b, c_+, c_-) , we will implicitly mean, in the remainder of this paper, an $(\mathcal{F}_t)_{t \geq 0}$ -adapted real càdlàg stable process with characteristic function given by

$$\varphi_{Z_t}(u) = \exp t \left(iub + \int_{-\infty}^{+\infty} (e^{iuy} - 1 - iuy \mathbb{I}_{|y| \leq 1}) \nu(dy) \right), \quad t \in [0, T], \quad (6)$$

where b stands for the drift parameter of Z and ν the Lévy measure defined on $\mathbb{R} \setminus \{0\}$ by

$$\nu(dx) := \frac{dx}{|x|^{\alpha+1}} (c_+ 1_{\{x>0\}} + c_- 1_{\{x<0\}}).$$

The parameters c_+, c_- above are non-negative with furthermore $c_+ + c_- > 0$ and $c_+ = c_-$ when $\alpha = 1$.

The process is said to be symmetric if $c_+ = c_- := c$. It is said to be strictly α -stable for $b = 0$.

In the case $\alpha \in (1, 2)$, the drift parameter is given by

$$b := - \int_{|y|>1} y \nu(dy) = - \frac{(c_+ - c_-)}{\alpha - 1} \quad \text{when } \alpha \neq 1.$$

However if $\alpha = 1$, we specify that $b = 0$ which is a restriction ensuring that the only (strictly) 1-stable process we consider is the symmetric Cauchy process.

Note that for $\alpha < 1$ the process Z is a finite variation process whereas when $\alpha \geq 1$, the process has unbounded jumps :

$$\sum_{s \leq t} \Delta Z_s = +\infty, \quad t > 0.$$

If $1 < \alpha < 2$ and Z symmetric (ν symmetric) note that the characteristic function becomes simply

$$\varphi_{Z_t}(u) = \exp \left(t \int_{-\infty}^{+\infty} (e^{iuy} - 1 - iuy) \nu(dy) \right), \quad t \in [0, T], \quad (7)$$

because

$$\int_{|y|>1} y \nu(dy) = 0.$$

An α -stable process is closely related to the notion of self-similar process. The process Z is said to be strictly α -stable if we have the self-similarity property

$$k^{-1/\alpha} (Z(kt))_{t \in [0, T]} \stackrel{d}{=} (Z(t))_{t \in [0, T]},$$

where $k > 0$ and the equality $\stackrel{d}{=}$ is understood in the sense of finite dimensional distributions. More generally an α -stable process Z is a process having the following self-similarity property : there exist $d : (0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ such that

$$k^{-1/\alpha} (Z(kt))_{t \in [0, T]} + d(k, t) \stackrel{d}{=} (Z(t))_{t \in [0, T]},$$

Note that α -stable processes are interesting due to the self-similarity property and the fact that the Lévy measure and the Lévy-Itô decomposition are almost totally explicit for the one dimensional case.

2.2. Basic definitions and notations. Recall that $L^p([0, T] \times \Omega)$ is the space of all measurable stochastic processes $X(t)$, $t \in [0, T]$ on $\Omega \times [0, T]$ such that $\|X(t)\|_p = (\mathbb{E}|X(t)|^p)^{1/p} < \infty$ when $p > 0$ and $t \in [0, T]$.

Definition 2.2. A stochastic process $X(t) : t \in [0, T]$ is said to be stochastically continuous (or continuous in probability) if for all $\epsilon > 0$

$$\lim_{h \rightarrow 0} \sup_{t \in [0, T]} \mathbb{P}(|X(t+h) - X(t)| > \epsilon) = 0.$$

Definition 2.3. A stochastic process $X(t) : t \in [0, T]$ is said to be bounded in probability or stochastically bounded if

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{P}(|X(t)| > N) = 0$$

Definition 2.4. A stochastic process $X(t) : t \in [0, T]$ is said to be continuous in the p -th moment if

$$\lim_{h \rightarrow 0} \sup_{t \in [0, T]} \mathbb{E}|X(t+h) - X(t)|^p = 0.$$

Denote by S a continuous semigroup on \mathbb{X} that is a map $S : \mathbb{R}_+ \rightarrow L(\mathbb{X})$ such that

1. $S(0) = I$ for all $t \geq 0$ where I is the identity operator on \mathbb{X} .
2. $S(t+s) = S(t)S(s)$ for all $s, t \geq 0$
3. $\|S(t)x - x\| \rightarrow 0$; $t \rightarrow 0$ for all $x \in \mathbb{X}$,

with $\|\cdot\|$ denoting the operator norm on $L(\mathbb{X})$.

Definition 2.5. [4] We say that S is a pseudo-contraction semigroup on \mathbb{X} if

$$\|S(t)\| \leq e^{at} \quad \forall t > 0,$$

for some constant $a > 0$.

If $a = 0$, $S(t)$ is called a contraction semigroup.

Recall that for any C_0 -semigroup S there are constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t} \quad \forall t > 0 \tag{8}$$

If (8) holds with $M = 1$ then S is a pseudo-contraction (generalized contraction) semigroup. If moreover, $\omega \leq 0$ that is there exist $a > 0$ such that

$$\|S(t)\| \leq e^{-at} \quad \forall t > 0$$

then S is a contraction semigroup. In this case we'll say that S is exponentially stable.

Throughout this paper, we will need first the following assumption :

(H1): A generates a C_0 -semigroup process $(S(t))_{t \geq 0}$ on \mathbb{X} which is a generalized contraction semigroup.

Note that if A generates a semi group $(S(t))_{t \geq 0}$ then the process Y defined by $Y(s) = S(t-s)X(s)$ verify for $s < t$ the following relation

$$\begin{aligned} dY(s) &= -AS(t-s)X(s) + S(t-s)dX(s) \\ &= -AS(t-s)X(s) + S(t-s)AX(s) + S(t-s)F(s, X(s))ds + S(t-s)G(s, X(s))dZ(s) \\ &= S(t-s)F(s, X(s))ds + S(t-s)G(s, X(s))dZ(s). \end{aligned}$$

Thus

$$dY(s) = S(t-s)F(s, X(s))ds + S(t-s)G(s, X(s))dZ(s). \quad (9)$$

Integrating (9) on $[0, t]$ we obtain that

$$X(t) - S(t)x_0 = \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)G(s, X(s))dZ(s).$$

Therefore,

Definition 2.6. *By a mild solution of equation (1) with initial condition $X_0 = x_0$, we mean a predictable stochastic process $(X(t))_{t \in [0, T]}$ with respect to the natural filtration of Z that satisfies the following corresponding stochastic convolution integral equation :*

$$X(t) = S(t)x_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)G(s, X(s))dZ(s).$$

Now let come back to a general Lévy process Z without Gaussian component. Let $\mathcal{F}_t^Z := \sigma(Z_s : s \in [0, t])$, $t \in [0, T]$ its natural filtration that satisfies the usual hypothesis, that is completeness and right-continuity.

As a Lévy process, Z is a semimartingale whose Lévy-Itô decomposition is given by

$$Z_t = bt + \int_0^t \int_{|x| \leq 1} x(\mu - \sigma)(ds, dx) + \int_0^t \int_{|x| > 1} x\mu(ds, dx), \quad t \in [0, T],$$

Note that for all $t \in [0, T]$ we have :

$$\begin{aligned} \int_0^t S(t-s)G(s, X(s))dZ(s) &= \int_0^t bS(t-s)G(s, X(s))ds \\ &\quad + \int_0^t \int_{|y| \leq 1} yS(t-s)G(s, X(s))\tilde{\mu}(ds, dy) \\ &\quad + \int_0^t \int_{|y| > 1} yS(t-s)G(s, X(s))\mu(ds, dy) \end{aligned}$$

Note that the last two Lévy processes defined by the integrals above are independent. The first one has a compactly supported Lévy measure and is a square-integrable martingale with infinitely many jumps bounded by R (small jumps) on each compact time interval, whereas the second one is an integrable compound Poisson process with jumps larger than R (big jumps).

3. THE SQUARE INTEGRABLE CASE

We consider here Lévy process which satisfies condition (\mathbf{H}_ν^2) and we make the following additional assumptions :

(H2) $F, G : [0, T] \times \mathbb{X} \rightarrow \mathbb{X}$ are jointly measurable functions such that

$$|F(t, y)|^2 \vee |G(t, y)|^2 \leq C(1 + |y|^2), \quad C > 0 \quad \forall y \in \mathbb{X}, \quad t \in [0, T].$$

3.1. L^2 -boundedness property. Now let us established an L^2 -boundedness property of the stochastic convolution integral process X .

Lemma 3.1. *Assume that $\mathbb{E}|x_0|^2 < \infty$. Set*

$$K_\nu := b^2 + a \int_{\mathbb{R}^*} y^2 \nu(dy) + a\nu(\{|y| \geq 1\}) \int_{|y| \geq 1} y^2 \nu(dy)$$

*Under Hypothesis **(H1)** and **(H2)** we have uniformly in $t \in [0, T]$:*

$$\mathbb{E}|X(t)|^2 \leq \left(3\mathbb{E}|x_0|^2 + \frac{3C^2}{a}(1 + 3K_\nu) \right) e^{K_\nu}.$$

Proof. We have

$$\begin{aligned} \mathbb{E}|X(t)|^2 &\leq 3\mathbb{E}|S(t)x_0|^2 + 3^p \mathbb{E} \left| \int_0^t S(t-s)F(s, X(s))ds \right|^2 \\ &\quad + 3\mathbb{E} \left| \int_0^t S(t-s)G(s, X(s))dZ(s) \right|^2 \\ &\leq 3\|S(t)\|\mathbb{E}|x_0|^2 + 3\mathbb{E} \left(\int_0^t |S(t-s)F(s, X(s))|ds \right)^2 \\ &\quad + 3\mathbb{E} \left(\int_0^t |S(t-s)G(s, X(s))|dZ(s) \right)^2 \\ &= I_1(t, p) + I_2(t, p) + I_3(t, p) \end{aligned}$$

It is clear that $I_1(t, p) \leq 3\mathbb{E}|x_0|^2$.

Using Cauchy Schwarz inequality, Assumption **(H2)** on F and Assumption **(H1)**, we have the following estimate

$$\begin{aligned} I_2(t, p) &\leq 3\mathbb{E} \left(\int_0^t \|S(t-s)\| \times |F(s, X(s))|ds \right)^2 \\ &\leq 3\mathbb{E} \left(\int_0^t e^{-a(t-s)} |F(s, X(s))|ds \right)^2 \\ &\leq 3 \int_0^t e^{-a(t-s)} ds \int_0^t e^{-a(t-s)} \mathbb{E}|F(s, X(s))|^2 ds \\ &\leq 3 \frac{C}{a} \int_0^t e^{-a(t-s)} (1 + \mathbb{E}|X(s)|^2) ds \\ &\leq \frac{3C}{a^2} + \frac{3C}{a} \int_0^t e^{-a(t-s)} \mathbb{E}|X(s)|^2 ds. \end{aligned}$$

Now, let us deal with the estimation of $I_3(t, p)$. Note that

$$\begin{aligned}
\int_0^t S(t-s)G(s, X(s))dZ(s) &= \int_0^t bS(t-s)G(s, X(s))ds + \int_0^t \int_{|y|\leq 1} yS(t-s)G(s, X(s))\tilde{\mu}(ds, dy) \\
&\quad + \int_0^t \int_{|y|>1} yS(t-s)G(s, X(s))\mu(ds, dy) \\
&= \int_0^t bS(t-s)G(s, X(s))ds + \int_0^t \int_{\mathbb{R}^*} yS(t-s)G(s, X(s))\tilde{\mu}(ds, dy) \\
&\quad + \int_0^t \int_{|y|>1} yS(t-s)G(s, X(s))ds\nu(dy).
\end{aligned}$$

We have

$$\begin{aligned}
\mathbb{E} \left| \int_0^t S(t-s)G(s, X(s))dZ(s) \right|^2 &\leq 3\mathbb{E} \left| \int_0^t bS(t-s)G(s, X(s))ds \right|^2 \\
&\quad + 3\mathbb{E} \left| \int_0^t \int_{\mathbb{R}^*} yS(t-s)G(s, X(s))\tilde{\mu}(ds, dy) \right|^2 \\
&\quad + 3\mathbb{E} \left(\int_0^t \int_{|y|>1} yS(t-s)G(s, X(s))ds\nu(dy) \right)^2.
\end{aligned}$$

By Assumption **(H1)** we have :

$$\mathbb{E} \left| \int_0^t bS(t-s)G(s, X(s))ds \right|^2 \leq b^2\mathbb{E} \left(\int_0^t e^{-a(t-s)}|G(s, X(s))|ds \right)^2. \quad (10)$$

Using Cauchy-Schwarz' inequality, and Assumption **(H2)** on G we have obtain

$$\begin{aligned}
\mathbb{E} \left| \int_0^t bS(t-s)G(s, X(s))ds \right|^2 &\leq \frac{b^2}{a} \int_0^t e^{-a(t-s)}\mathbb{E}G(s, X(s))^2 ds \\
&\quad + \frac{b^2C}{a} \int_0^t e^{-a(t-s)}(1 + \mathbb{E}(X(s))^2)ds
\end{aligned}$$

By the isometry formula for Poisson stochastic integrals, Assumption **(H1)** and Assumption **(H2)** on G we have :

$$\begin{aligned}
\mathbb{E} \left(\int_0^t \int_{\mathbb{R}^*} |yS(t-s)G(s, X(s))|\tilde{\mu}(ds, dy) \right)^2 &\leq C \int_{\mathbb{R}^*} y^2\nu(dy) \int_0^t e^{-2a(t-s)}(1 + \mathbb{E}X(s)^2)ds \\
&\quad + C \int_{\mathbb{R}^*} y^2\nu(dy) \int_0^t e^{-a(t-s)}(1 + \mathbb{E}X^2(s))ds
\end{aligned}$$

Again by Chebychev's inequality, Assumption **(H1)**, Cauchy-Schwarz inequality and Assumption **(H2)** on G we have :

$$\begin{aligned} \mathbb{E}\left(\int_0^t \int_{|y|>1} |yS(t-s)G(s, X(s))|ds\nu(dy)\right)^2 &\leq C \left(\int_{|y|\geq 1} |y|\nu(dy)\right)^2 \times \\ &\mathbb{E}\left(\int_0^t e^{-a(t-s)} |G(s, X(s))|ds\right)^2 \\ &\leq C\nu(\{|y| \geq 1\}) \int_{|y|\geq 1} y^2\nu(dy) \times \\ &\int_0^t e^{-a(t-s)} (1 + \mathbb{E}|X(s)|^2)ds \end{aligned}$$

Combining the above calculations, we obtain that

$$\begin{aligned} \mathbb{E}\left(\int_0^t S(t-s)G(s, X(s))dZ(s)\right)^2 &\leq 3\frac{b^2C}{a} \int_0^t e^{-a(t-s)} (1 + \mathbb{E}(X(s))^2)ds \\ &+ 3C \int_{\mathbb{R}^*} y^2\nu(dy) \int_0^t e^{-a(t-s)} (1 + \mathbb{E}X^2(s))ds \\ &+ 3C\nu(\{|y| \geq 1\}) \int_{|y|\geq 1} y^2\nu(dy) \int_0^t e^{-a(t-s)} (1 + \mathbb{E}|X(s)|^2)ds \end{aligned}$$

Finally, gathering all this estimates, we have

$$\begin{aligned} \mathbb{E}|X(t)|^2 &\leq 3\mathbb{E}|x_0|^2 \\ &+ \frac{3C}{a^2} + \frac{3C}{a} \int_0^t e^{-a(t-s)} \mathbb{E}|X(s)|^2 ds. \\ &\leq 9\frac{b^2C}{a} \int_0^t e^{-a(t-s)} (1 + \mathbb{E}(X(s))^2)ds \\ &+ 9C \int_{\mathbb{R}^*} y^2\nu(dy) \int_0^t e^{-a(t-s)} (1 + \mathbb{E}X^2(s))ds \\ &+ 9C\nu(\{|y| \geq 1\}) \int_{|y|\geq 1} y^2\nu(dy) \int_0^t e^{-a(t-s)} (1 + \mathbb{E}|X(s)|^2)ds \end{aligned}$$

This implies by the well-known Gronwall inequality the desired result. \square

3.2. Stochastic continuity. As mentioned in the introduction, it is known that when integrating with respect to a Brownian process, it is enough to assume that the integrand is measurable, adapted and locally square integrable with respect to time with probability 1. But if X is càdlàg and adapted, unfortunately, in important cases X does not have a predictable modification. For instance this is the case for stochastic convolution processes :

$$\int_0^t S(t-s)H(s, X(s))dZ(s)$$

where the integrand depends on t and Z is a discontinuous Lévy process. Since any adapted stochastically continuous process has a predictable modification ([1], Prop. 3.21), it suffices to check if the stochastic continuity holds.

Lemma 3.2. *Assume that $\mathbb{E}|x_0|^2 < \infty$. Under Hypothesis **(H1)** and **(H2)** we have :*

$$\lim_{h \rightarrow 0} \sup_{t \in [0, T]} \mathbb{E}|X(t+h) - X(t)|^2 = 0.$$

Hence X is stochastically continuous.

Proof. We have

$$\begin{aligned} X(t+h) - X(t) &= (S(t+h) - T(t))x_0 \\ &+ \int_0^{t+h} S(t+h-s)F(s, X(s))ds - \int_0^t C(t-s)F(s, X(s))ds \\ &+ \int_0^{t+h} S(t+h-s)G(s, X(s))dZ(s) - \int_0^t S(t-s)G(s, X(s))dZ(s). \end{aligned}$$

Then we have

$$\mathbb{E}(|X(t+h) - X(t)|^2) \leq 3\mathbb{E}A(t, h)^2 + 3\mathbb{E}B(t, h)^2 + 3\mathbb{E}C(t, h)^2.$$

where

$$\begin{aligned} A(t, h) &= |S(t+h) - S(t)|x_0 = |S(t)(S(h) - I)x_0| \\ B(t, h) &= \left| \int_0^{t+h} S(t+h-s)F(s, X(s))ds - \int_0^t S(t-s)F(s, X(s))ds \right| \\ \text{and } C(t, h) &= \left| \int_0^{t+h} S(t+h-s)g(s)dZ(s) - \int_0^t S(t-s)g(s)dZ(s) \right|. \end{aligned}$$

We have,

$$\begin{aligned} \mathbb{E}A(t, h)^2 &\leq \|S(t)(S(h) - I)\|^2 \mathbb{E}|x_0|^2 \\ &\leq e^{-at} \|S(h) - I\| \mathbb{E}|x_0|^2. \end{aligned}$$

Since $S(t)$ is a uniform strongly continuous semi group and $\mathbb{E}|x_0|^2 < \infty$ then

$$\lim_{h \rightarrow 0} \sup_{t \geq 0} \mathbb{E}A(t, h)^2 = 0.$$

To estimate $\mathbb{E}B(t, h)^2$, we will start from the following identity :

$$\int_0^{t+h} S(t+h-s)F(s, X(s))ds - S(h) \int_0^t S(t-s)F(s, X(s))ds = S(h) \int_t^{t+h} S(t-s)F(s, X(s))ds. \quad (EM1)$$

Then we get

$$\begin{aligned} \mathbb{E}B(t, h)^2 &\leq 2\|S(h) - I\|^2 \mathbb{E} \left(\int_0^t \|S(t-s)\| \|F(s, X(s))\| ds \right)^2 \\ &+ 2\|S(h)\|^2 \mathbb{E} \left(\int_t^{t+h} \|S(t-s)\| \|F(s, X(s))\| ds \right)^2 \\ &\leq J_1(t, h) + J_2(t, h). \end{aligned}$$

where

$$J_1(t, h) = 2\|S(h) - I\|^2 \mathbb{E} \left(\int_0^t \|S(t-s)\| \|F(s, X(s))\| ds \right)^2$$

and

$$J_2(t, h) = 2\|S(h)\|^2 \mathbb{E} \left(\int_t^{t+h} \|S(t-s)\| \|F(s, X(s))\| ds \right)^2.$$

Similarly to the estimation of $I_2(t, p)$ we obtain that

$$\begin{aligned} J_1(t, h) &\leq \|S(h) - I\|^2 \frac{6C}{a^2} \left(1 + a \int_0^t e^{-a(t-s)} \mathbb{E}|X(s)|^2 ds \right) \\ &\leq \|S(h) - I\|^2 \frac{6C}{a^2} (1 + \sup_{t \geq 0} \mathbb{E}|X(t)|^2) \end{aligned}$$

and

$$J_2(t, h) \leq \frac{6C}{a^2} \|S(h)\|^2 (1 + (e^{ah} - 1) \sup_{t \geq 0} \mathbb{E}|X(t)|^2)$$

From the above computation, it is clear that

$$\lim_{h \rightarrow 0} \sup_{t \geq 0} \mathbb{E} B(t, h)^2 = 0.$$

Now, let come back to the estimation of $\mathbb{E} C(t, h)^2$. We start from the following identity :

$$\begin{aligned} &\int_0^{t+h} S(t+h-s) G(s, X(s)) dZ(s) - S(h) \int_0^t S(t-s) G(s, X(s)) dZ(s) \\ &= \int_t^{t+h} S(t+h-s) G(s, X(s)) dZ(s). \quad (EM2) \end{aligned}$$

We have

$$3\mathbb{E} C(t, h)^2 \leq 6\mathbb{E} U(t, h)^2 + 6\mathbb{E} V(t, h)^2$$

where

$$U(t, h) = (S(h) - I) \int_0^t S(t-s) G(s, X(s)) dZ(s)$$

and

$$V(t, h) = \int_t^{t+h} S(t+h-s) G(s, X(s)) dZ(s)$$

Note that

$$\mathbb{E} U(t, h)^2 \leq \|S(h) - I\|^2 \mathbb{E} \left(\int_0^t S(t-s) G(s, X(s)) dZ(s) \right)^2$$

Using the computation of $I_3(t, p)$ we have

$$\mathbb{E} U(t, h)^2 \leq \|S(h) - I\|^2 \frac{3C}{a^2} K_\nu \left(1 + \sup_{t \geq 0} \mathbb{E}|X(t)|^2 \right)$$

$$\mathbb{E}V(t, h)^2 \leq \|S(h)\|^2 \frac{3C}{a^2} K_\nu (e^{ah} - 1) \left(1 + \sup_{t \geq 0} \mathbb{E}|X(t)|^2 \right)$$

Hence , we conclude that

$$\lim_{h \rightarrow 0} \sup_{t \geq 0} \mathbb{E}C(t, h)^2 = 0;$$

so that the trajectories of $X(t)$ are stochastically continuous. □

3.3. Existence of predictable and bounded solutions of the SPDE. Now we can establish the main result of this section. We make the following additional assumptions :

(H3) $F, G : [0, T] \times \mathbb{X} \rightarrow \mathbb{X}$ are continuous L -Lipchitz functions that is :

$$|F(t, y) - F(t, z)|^2 \leq L|y - z|^2 \quad L_F > 0 \quad y, z \in \mathbb{X}$$

Theorem 3.3. *Under the hypothesis (H1), (H2) and (H3), if $\mathbb{E}|x_0|^2 < \infty$ then for any $T > 0$, there exists a unique mild solution of (1) which is*

- (1) *uniformly $L^2(\Omega \times [0, T])$ -bounded that is $t \rightarrow \mathbb{E}|X(t)|^p$ is bounded for all $t \in [0, T]$.*
- (2) *\mathcal{F}_t -adapted and stochastically continuous, hence has a predictable version.*

Remark 3.1. *For any C_0 - semigroup S such that*

$$\|S(t)\| \leq Me^{\omega t}, \quad M > 0, \quad \omega \geq 0,$$

the statement of the previous theorem will always remain true, it suffices to replace for instance $-a$ by ω .

Proof. Denote by \mathbb{X} the collection of all stochastically continuous and $L^2([0, T] \times \Omega)$ -bounded stochastic processes $X(t)$, $t \in [0, T]$ such that $\mathbb{E}|X(t)|^2 < \infty$ for all $T > 0$. It is well-known that for the space \mathbb{X} is a Banach space when it is equipped with the norm

$$\|X\|_T = \left(\mathbb{E} \sup_{t \in [0, T]} |X(t)|^2 \right)^{1/2}.$$

We consider the family of equivalent norms for some $\gamma > 0$ to be choose later.

$$\|X\|_\gamma := \sup_{t \in [0, T]} e^{-\gamma t} \left(\mathbb{E}|X(t)|^2 \right)^{1/2} \quad \text{for some } \gamma > 0.$$

Then \mathbb{X} equipped with $\|\cdot\|_\gamma$ is a Banach space. Define the following operator Γ on the Banach space $(\mathbb{X}, \|\cdot\|)$ by

$$(\Gamma X)(t) = S(t)x_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)g(s)\phi(X(s))dZ(s).$$

Note that $S(t)x_0$, $\int_0^t S(t-s)F(s, X(s))ds$ and $\int_0^t S(t-s)G(s, X(s))dZ(s)$ are in \mathbb{X} according to the previous sections. Then for any $X \in \mathbb{X}$, Γ maps the space \mathbb{X} into itself.

Finally for any $X, Y \in \mathbb{X}$ we have :

$$\begin{aligned}
e^{-2\gamma t} \mathbb{E} |\Gamma X(t) - \Gamma Y(t)|^2 &\leq 2 \left(\int_0^t e^{-2a(t-s)} ds \right) \mathbb{E} \int_0^t e^{-2\gamma(t-s)} e^{-2\gamma s} |F(s, X(s)) - F(s, Y(s))|^2 ds \\
&\quad + 2\mathbb{E} \left(\int_0^t e^{-a(t-s)} e^{-\gamma t} |G(s, X(s)) - G(s, Y(s))| dZ(s) \right)^2 \\
&\leq \frac{L}{a} \sup_{t \in [0, T]} e^{-2\gamma t} \mathbb{E} |X(t) - Y(t)|^2 \int_0^t e^{-2\gamma(t-s)} ds \\
&\quad + 2\mathbb{E} \left(\int_0^t e^{-(\gamma+a)(t-s)} e^{-\gamma s} |G(s, X(s)) - G(s, Y(s))| dZ(s) \right)^2 \\
&\leq \frac{L}{a\gamma} \sup_{t \in [0, T]} e^{-2\gamma t} E |X(t) - Y(t)|^2 \\
&\quad + \frac{6L}{a\gamma} K_\nu \sup_{t \in [0, T]} e^{-2\gamma t} E |X(t) - Y(t)|^2
\end{aligned}$$

So that

$$\|\Gamma X - \Gamma Y\|_\gamma \leq \sqrt{\frac{L}{a\gamma} + \frac{6L}{a\gamma} K_\nu} \|X - Y\|_\gamma$$

Choosing γ such that

$$\sqrt{\frac{L}{a\gamma} + \frac{6L}{a\gamma} K_\nu} < 1$$

allows us to complete the proof by the well-known Banach fixed-point theorem. \square

4. THE NON SQUARE INTEGRABLE CASE

We consider a Lévy process Z satisfying condition $(\mathbf{H}_\nu^{\beta, \mathbf{P}})$.

In order to control the jump size of Z and the moment behavior of the stochastic convolution integral, let us introduce a truncation method like in our paper [20], [21]; see also [22].

As a Lévy process, Z is a semimartingale whose Lévy-Itô decomposition is given by

$$Z_t = b_R t + \int_0^t \int_{|x| \leq R} x (\mu - \sigma)(ds, dx) + \int_0^t \int_{|x| > R} x \mu(ds, dx), \quad t \in [0, T],$$

where R is some arbitrary positive truncation level (classically chosen to be 1) and μ is a Poisson random measure on $[0, T] \times \mathbb{R}$ with intensity $\sigma(dt, dx) = dt \otimes \nu(dx)$. Here b_R is the drift parameter given by

$$b_R := b + \int_{1 < |x| \leq R} x \nu(dx)$$

We make the following additional assumptions :

(A1) We assume that the Lévy measure ν of Z is supported on \mathbb{R}^* with $\nu(\{0\}) = 0$ such that for all $R \geq 1$

$$\nu(\{y \in \mathbb{R} : |y| > R\}) \leq C_1(\nu) R^{-\beta} \quad \text{and} \quad \nu(\{y \in \mathbb{R} : 0 < |y| \leq R\}) \leq C_2(\nu) R^{-\beta}$$

where $C_1(\nu)$ and $C_2(\nu)$ are non negative constants depending on the parameters of the Lévy measure. Note that this latter assumption is sufficient to ensure that the integral in the Lévy-Khintchine formula converges.

(A2) $F, G : [0, T] \times \mathbb{X} \rightarrow \mathbb{X}$ are measurable functions such that

$$|F(t, y)|^p \leq C(1 + |y|^p) \quad C > 0 \quad y \in \mathbb{X}$$

and

$$G(t, z) = g(t) \phi(z), \quad t \in [0, T]; \quad z \in \mathbb{X}$$

where $\phi : \mathbb{X} \rightarrow \mathbb{X}$ is a bounded function and $g : [0, T] \times \mathbb{R}$ a measurable function such that

$$\eta(a, T, g) := \left(\sup_{0 \leq t \leq T} \int_0^t e^{-a(t-s)} g^2(s) ds \right)^{1/2} < \infty.$$

(A3) $F : [0, T] \times \mathbb{X} \rightarrow \mathbb{X}$ is a continuous L_F -Lipchitz function that is :

$$|F(t, y) - F(t, z)|^p \leq L_F |y - z|^p \quad L_F > 0 \quad y, z \in \mathbb{X}$$

The function $\phi : \mathbb{X} \rightarrow \mathbb{X}$ is Hölder continuous with exponent $\frac{p}{2}$ that is

$$|\phi(y) - \phi(z)| \leq |y - z|^{p/2} \quad y, z \in \mathbb{X}$$

In the sequel, we'll frequently make use of the following constant :

$$\begin{aligned} \eta(a, T, g, h) &:= \left(\sup_{0 \leq t \leq T} \int_t^{t+h} e^{-a(t-s)} g^2(s) ds \right)^{1/2} \\ K_\nu(a, b, \phi) &= \frac{8b^2 \|\phi\|_\infty^2}{a} + \frac{8 \|\phi\|_\infty^2 C_1(\nu) C_2(\nu)}{a} + 4 \|\phi\|_\infty^2 C_2(\nu) \\ K_\nu(a, b, \phi, h) &= \frac{8b^2 \|\phi\|_\infty^2}{a} + \frac{8 \|\phi\|_\infty^2 C_1(\nu) C_2(\nu)}{a} + 4 \|\phi\|_\infty^2 C_2(\nu) \end{aligned}$$

4.1. L^p -boundedness property. We analyse the tail and moment behavior of the stochastic convolution integrals obtained by convolution of the contractive semi-group $T(t)$ with respect to the non square integrable Lévy process Z .

Now let us first established an explicit bound on the tail behavior of

$$\int_0^t T(t-s) G(s, X(s)) dZ(s)$$

In view of this, we have

Lemma 4.1. *Let Z be a non square integrable Lévy process satisfying Condition (\mathbf{H}_ν^2) . Under Hypothesis **(A1)** and **(A2)** we have for some $\beta \in (0, 2)$ and all $x \geq \eta(a, T, g)$*

$$\mathbb{P} \left(\left| \int_0^t S(t-s) G(s, X(s)) dZ(s) \right| \geq x \right) \leq \frac{\eta^\beta(a, T, g)}{x^\beta} (K_\nu(a, b, \phi) + TC_1(\nu)).$$

Corollary 4.2. *Let Z be a stable process with index $\alpha \in (0, 2)$. Under Hypothesis **(A2)** we have for all $x \geq \eta(a, T, g)$*

$$\mathbb{P} \left(\left| \int_0^t S(t-s)G(s, X(s))dZ(s) \right| \geq x \right) \leq \frac{\eta^\alpha(a, T, g)}{x^\alpha} \left(\frac{8b^2 \|\phi\|_\infty^2}{a} + \frac{8\|\phi\|_\infty^2(c_+ + c_-)^2}{a\alpha(2-\alpha)} \right. \\ \left. + \frac{4\|\phi\|_\infty^2(c_+ + c_-)}{2-\alpha} + \frac{T(c_+ + c_-)}{\alpha} \right).$$

Indeed for a stable process Z with index $\alpha \in (0, 2)$ Condition **(A1)** and **(H $_\nu^{\beta, \mathbf{P}}$)** holds for $\beta = \alpha$ and

$$C_1(\nu) = \frac{c_+ + c_-}{\alpha} \quad \text{and} \quad C_2(\nu) = \frac{c_+ + c_-}{2-\alpha}.$$

Now, let us start the proof of Lemma 4.1.

Proof. Let $x > 0$ be fixed. We have

$$\mathbb{P} \left(\left| \int_0^t S(t-s)G(s, X(s))dZ(s) \right| \geq x \right) \leq \mathbb{P} \left(\int_0^t |b_R S(t-s)G(s, X(s))| ds \geq \frac{x}{2} \right) \\ + \mathbb{P} \left(\int_0^t \int_{|y| \leq R} |yS(t-s)G(s, X(s))| \tilde{\mu}(ds, dy) \geq \frac{x}{2} \right) \\ + \mathbb{P} \left(\int_0^t \int_{|y| > R} |yS(t-s)G(s, X(s))| \mu(ds, dy) > 0 \right).$$

Let us, firstly, investigate the drift integral part

$$\int_0^t b_R |S(t-s)G(s, X(s))| ds.$$

By Chebychev inequality, Cauchy-Schwarz inequality, Assumption **(A1)** and **(A2)** we have :

$$\mathbb{P} \left(\int_0^t b_R |S(t-s)G(s, X(s))| ds \geq \frac{x}{2} \right) \leq \frac{4b_R^2}{x^2} \mathbb{E} \left(\int_0^t e^{-a(t-s)} |g(s)| |\phi(X(s))| ds \right)^2 \\ \leq \frac{4b_R^2}{ax^2} \int_0^t e^{-a(t-s)} g^2(s) \mathbb{E} \phi(X(s))^2 ds.$$

Using the fact that ϕ is bounded, the condition on g , Assumption **(H1)** and **(A1)**, we get for $\beta \in (0, 2)$ and all $x \geq \eta(a, T, g)$:

$$\mathbb{P} \left(\int_0^t b_R |S(t-s)G(s, X(s))| ds \geq \frac{x}{2} \right) \leq \frac{4b_R^2 \|\phi\|_\infty^2}{x^2} \eta^2(a, T, g) \\ \leq \frac{8\|\phi\|_\infty^2}{ax^2} (b^2 + \nu(\{y \in \mathbb{R} : |y| > 1\})R^{2-\beta}) \eta^2(a, T, g). \\ \leq \frac{8b^2 \|\phi\|_\infty^2}{ax^\beta} \eta^\beta(a, T, g) \\ + \frac{8\|\phi\|_\infty^2}{ax^2} C_1(\nu)C_2(\nu)R^{2-\beta} \eta^2(a, T, g)$$

because

$$\begin{aligned}
b_R^2 &\leq 2b^2 + 2\nu\{y \in \mathbb{R} : 1 < |y| \leq R\} \int_{1 < |y| \leq R} y^2 \nu(dy) \\
&\leq 2b^2 + 2\nu\{y \in \mathbb{R} : |y| > 1\} \int_{|y| \leq R} y^2 \nu(dy) \\
&\leq 2b^2 + 2C_1(\nu)R^2 \int_{|y| \leq R} \nu(dy) \\
&\leq 2b^2 + 2C_1(\nu)C_2(\nu)R^{2-\beta}.
\end{aligned}$$

From Chebychev's inequality, isometry formula for Poisson stochastic integrals, assumptions **(A1)** and **(A2)**, we have

$$\begin{aligned}
\mathbb{P}\left(\int_0^t \int_{|y| \leq R} |yS(t-s)g(s)\phi(X(s))| \tilde{\mu}(ds, dy) \geq \frac{x}{2}\right) &\leq \frac{4}{x^2} \mathbb{E}\left(\int_0^t \int_{|y| \leq R} |yS(t-s)g(s)\phi(X(s))| \tilde{\mu}(ds, dy)\right)^2 \\
&\leq \frac{4}{x^2} \int_{|y| \leq R} y^2 \nu(dy) \int_0^t e^{-2a(t-s)} g^2(s) \mathbb{E}\phi(X(s))^2 ds \\
&\leq \frac{4C_2(\nu)R^{2-\beta}}{x^2} \int_0^t e^{-a(t-s)} g^2(s) \mathbb{E}\phi(X(s))^2 ds \quad (*) \\
&\leq \frac{4C_2(\nu)R^{2-\beta}}{x^2} \|\phi\|_\infty^2 \eta^2(a, T, g).
\end{aligned}$$

Now, we proceed with the study of the compound Poisson stochastic integral

$$N_t = \int_0^t \int_{|y| > R} y S(t-s)g(s)\phi(X(s)) \mu(ds, dy).$$

Now, denote by T_1^R , the first jump time of the Poisson process $\mu(\{y \in \mathbb{R} : |y| > R\} \times [0, t])$ on the set $\{y \in \mathbb{R} : |y| > R\}$ which is exponentially distributed with parameter $\nu(\{y \in \mathbb{R} : |y| > R\})$, see e.g. [13], [Theorem 21.3].

If a.s. T_1^R occurs after time t , then the compound Poisson stochastic integral N_t is identically 0 on the interval $[0, t]$. Thus we have

$$\begin{aligned}
&\mathbb{P}(N_t > 0) \\
&= 1 - \mathbb{P}(N_t = 0) \\
&\leq 1 - \mathbb{P}(T_1^R > t) \\
&= 1 - \exp -t\nu(\{y \in \mathbb{R} : |y| > R\}) \\
&\leq t\nu(\{y \in \mathbb{R} : |y| > R\}) \\
&= TC_1(\nu)R^{-\beta}
\end{aligned}$$

Therefore choosing the truncation level

$$R = \frac{x}{\eta(a, T, g)} \geq 1$$

and rearranging the terms, we obtain for all $x \geq \eta(a, T, g)$:

$$\mathbb{P}\left(\left|\int_0^t S(t-s)g(s)\phi(X(s))dZ(s)\right| \geq x\right) \leq \frac{\eta^\beta(a, T, g)}{x^\beta} (K_\nu(a, b, \phi) + TC_1(\nu)).$$

□

Now, we will established an L^p -boundedness property of the stochastic convolution integral process X .

Lemma 4.3. *Assume that $\mathbb{E}|x_0|^p < \infty$ for $p \in (0, \beta)$. Under Hypothesis **(A1)** and **(A2)** we have uniformly in $t \in [0, T]$:*

$$\mathbb{E}|X(t)|^p \leq e^{3^p a^{-p} C} 3^p \left(\mathbb{E}|x_0|^p + C \left(\frac{1}{a}\right)^{p/q} + \eta^p(a, T, g) \left(1 + (K_\nu(a, b, \phi) + TC_1(\nu)) \frac{p}{\beta - p}\right) \right)$$

where q stands for the conjuguate of p .

According to the same reason as in ... we deduce that

Corollary 4.4. *Let Z be a stable process with index $\alpha \in (0, 2)$. Under Hypothesis **(A2)** we have for we have uniformly in $t \in [0, T]$:*

$$\mathbb{E}|X(t)|^p \leq e^{3^p a^{-p} C} 3^p \left(\mathbb{E}|x_0|^p + C \left(\frac{1}{a}\right)^{p/q} + \eta^p(a, T, g) \left(1 + K(T, a, b, \phi, \alpha, c_+, c_-) \frac{p}{\alpha - p}\right) \right)$$

where

$$K(T, a, b, \phi, \alpha, c_+, c_-) = \frac{8b^2 \|\phi\|_\infty^2}{a} + \frac{8\|\phi\|_\infty^2 (c_+ + c_-)^2}{a\alpha(2-\alpha)} + \frac{4(c_+ + c_-)\|\phi\|_\infty^2}{2-\alpha} + \frac{T(c_+ + c_-)}{\alpha}.$$

Now, let us start the proof of Lemma 4.3.

Proof. Let $p > 0$. We have

$$\begin{aligned} \mathbb{E}|X(t)|^p &\leq 3^p \mathbb{E}|S(t)x_0|^p + 3^p \mathbb{E} \left| \int_0^t S(t-s)F(s, X(s))ds \right|^p \\ &\quad + 3^p \mathbb{E} \left| \int_0^t S(t-s)G(s, X(s))dZ(s) \right|^p \\ &\leq 3^p \|S(t)\| \mathbb{E}|x_0|^p + 3^p \mathbb{E} \left(\int_0^t |S(t-s)F(s, X(s))| ds \right)^p \\ &\quad + 3^p \mathbb{E} \left(\int_0^t |S(t-s)G(s, X(s))| dZ(s) \right)^p \\ &= I_1(t, p) + I_2(t, p) + I_3(t, p) \end{aligned}$$

It is clear that $I_1(t, p) \leq 3^p \mathbb{E}|x_0|^p < \infty$ for all $p > 0$.

Using Holder inequality and the linear growth assumption, we have the following estimate

$$\begin{aligned}
I_2(t, p) &\leq 3^p \mathbb{E} \left(\int_0^t \|S(t-s)\| \times |F(s, X(s))| ds \right)^p \\
&\leq 3^p \mathbb{E} \left(\int_0^t e^{-a(t-s)} |F(s, X(s))| ds \right)^p \\
&\leq 3^p \left(\int_0^t e^{-a(t-s)} ds \right)^{p/q} \int_0^t e^{-a(t-s)} \mathbb{E} \|F(s, X(s))\|^p ds \\
&\leq 3^p C \left(\int_0^t e^{-a(t-s)} ds \right)^{p/q} \int_0^t e^{-a(t-s)} (1 + \mathbb{E}|X(s)|^p) ds \\
&\leq 3^p \frac{C}{a^p} + 3^p \left(\frac{1}{a} \right)^{p/q} C \int_0^t e^{-a(t-s)} \mathbb{E}|X(s)|^p ds
\end{aligned}$$

where q stands for the conjuguate of p .

Now, let us deal with the estimation of $I_3(t, p)$. Note that by integration, we have for any $p > 0$,

$$\begin{aligned}
\mathbb{E} \left(\int_0^t |S(t-s)g(s)\phi(X(s))| dZ(s) \right)^p &= p \int_0^{+\infty} px^{p-1} \mathbb{P} \left(\int_0^t |S(t-s)g(s)\phi(X(s))| dZ(s) \geq x \right) dx, \\
&= \eta(a, T, g)^p + \int_{\eta(a, T, g)}^{+\infty} \mathbb{P} \left(\int_0^t |S(t-s)g(s)\phi(X(s))| dZ(s) \geq x \right) dx.
\end{aligned}$$

Using Lemma 4.1 one obtain that :

$$\mathbb{E} \left(\int_0^t |S(t-s)g(s)\phi(X(s))| dZ(s) \right)^p \leq \eta^p(a, T, g) \left(1 + K_\nu(a, b, \phi) \frac{p}{\beta - p} \right) \quad \text{provided } p \in (0, \beta). \quad (11)$$

Finally, gathering all this estimates, we have

$$\begin{aligned}
\mathbb{E}|X(t)|^p &\leq 3^p \mathbb{E}|x_0|^p \\
&\quad + 3^p \frac{C}{a^p} + 3^p \left(\frac{1}{a} \right)^{p/q} C \int_0^t e^{-a(t-s)} \mathbb{E}|X(s)|^p ds \\
&\quad + 3^p \eta^p(a, T, g) \left(1 + K_\nu(a, b, \phi) \frac{p}{\beta - p} \right).
\end{aligned}$$

This implies by the well-known Gronwall inequality the desired result. \square

4.2. Stochastic continuity. For discontinuous and non square integrable Lévy process Z , we established the following stochastic continuity under condition $(\mathbf{H}_\nu^{\beta, \mathbf{P}})$

Lemma 4.5. *Assume that $\mathbb{E}|x_0|^p < \infty$ for $p \in (0, \beta)$. Under Hypothesis **(A1)** and **(A2)** we have :*

$$\lim_{h \rightarrow 0} \sup_{t \in [0, T]} \mathbb{E}|X(t+h) - X(t)|^p = 0.$$

We start the proof like in Lemma 3.2.

Proof. For any $\epsilon > 0$, we have

$$\begin{aligned}\mathbb{P}(|X(t+h) - X(t)| > \epsilon) &\leq \mathbb{P}(A(t, h) > \epsilon/3) \\ &\quad + \mathbb{P}(B(t, h) > \epsilon/3) \\ &\quad + \mathbb{P}(C(t, h) > \epsilon/3)\end{aligned}$$

where

$$\begin{aligned}A(t, h) &= |S(t+h) - S(t)|x_0 = |S(t)(S(h) - I)x_0| \\ B(t, h) &= \left| \int_0^{t+h} S(t+h-s)F(s, X(s))ds - \int_0^t S(t-s)F(s, X(s))ds \right| \\ \text{and } C(t, h) &= \left| \int_0^{t+h} S(t+h-s)g(s)\phi(X(s))dZ(s) - \int_0^t S(t-s)g(s)\phi(X(s))dZ(s) \right|.\end{aligned}$$

We have,

$$\mathbb{P}(A(t, h) > \epsilon/3) \leq \frac{3^p}{\epsilon^p} e^{-apt} \|S(h) - I\| \mathbb{E}|x_0|^p.$$

so that for all $\epsilon > 0$,

$$\limsup_{h \rightarrow 0} \sup_{t \geq 0} \mathbb{P}(A(t, h) > \epsilon/3) = 0.$$

To estimate $B(t, h)$, we start again from identity (EM1) :

$$\begin{aligned}\mathbb{P}(B(t, h) > \epsilon/3) &\leq \frac{3^p}{\epsilon^p} \mathbb{E}B(t, h)^p \leq \frac{6^p}{\epsilon^p} \|S(h) - I\|^p \mathbb{E} \left(\int_0^t \|S(t-s)\| \|F(s, X(s))\| ds \right)^p \\ &\quad + \frac{6^p}{\epsilon^p} \|S(h)\|^p \mathbb{E} \left(\int_t^{t+h} \|S(t-s)\| \|F(s, X(s))\| ds \right)^p \\ &\leq J_1(t, h) + J_2(t, h).\end{aligned}$$

where

$$J_1(t, h) = \frac{6^p}{\epsilon^p} \|S(h) - I\|^p \mathbb{E} \left(\int_0^t \|S(t-s)\| \|F(s, X(s))\| ds \right)^p$$

and

$$J_2(t, h) = \frac{6^p}{\epsilon^p} \|S(h)\|^p \mathbb{E} \left(\int_t^{t+h} \|S(t-s)\| \|F(s, X(s))\| ds \right)^p.$$

Similarly to the estimation of the previous $I_2(t, p)$ we obtain that

$$J_1(t, h) \leq \frac{6^p}{\epsilon^p} \|S(h) - I\|^p C \left(\frac{1}{a} \right)^p \left(1 + \sup_{t \in [0, T]} \mathbb{E}|X(t)|^p \right)$$

and

$$J_2(t, h) \leq \frac{6^p}{\epsilon^p} \|S(h)\|^p \left(\frac{e^{ah} - 1}{a} \right)^p C \left(1 + \sup_{t \in [0, T]} \mathbb{E}|X(t)|^p \right)$$

Thus, it is clear that

$$\limsup_{h \rightarrow 0} \sup_{t \geq 0} \mathbb{P}(B(t, h) > \epsilon/3) = 0.$$

Now, note also that from identity (EM2) with $G(s, X(s)) = g(s)\phi(X(s))$ we have :

$$\mathbb{P}(C(t, h) > \epsilon/3) \leq \mathbb{P}(U(t, h) > \epsilon/6) + \mathbb{P}(V(t, h) > \epsilon/6)$$

where

$$U(t, h) = (S(h) - I) \int_0^t |S(t-s)g(s)\phi(X(s))|dZ(s)$$

and

$$V(t, h) = \int_t^{t+h} |S(t+h-s)g(s)\phi(X(s))|dZ(s)$$

Note that

$$\mathbb{P}(U(t, h) > \epsilon/6) \leq \frac{6^p}{\epsilon^p} \|S(h) - I\|^p \mathbb{E} \left(\int_0^t |S(t-s)g(s)\phi(X(s))|dZ(s) \right)^p$$

Using the estimation in (11), we have

$$\mathbb{P}(U(t, h) > \epsilon/6) \leq \frac{6^p}{\epsilon^p} \|S(h) - I\|^p \eta^p(a, T, g) \left(1 + K_\nu(a, b, \phi) \frac{p}{\beta - p} \right) \quad \text{provided } p \in (0, \beta).$$

so that

$$\limsup_{h \rightarrow 0} \sup_{t \geq 0} \mathbb{P}(U(t, h) > \epsilon/6) = 0.$$

Finally, since $N_{t+h} - N_t$ has the same law as N_h ; using the same arguments of calculus in (11), we have

$$\mathbb{P} \left(\int_t^{t+h} |S(t+h-s)G(s, X(s)|dZ(s) \geq \frac{\epsilon}{6} \right) \leq \frac{6^p}{\epsilon^p} \|S(h)\|^p \eta^p(a, T, g, h) \left(hC_1(\nu) + K_\nu(a, b, \phi, h) \frac{p}{\beta - p} \right)$$

Hence, we conclude that

$$\limsup_{h \rightarrow 0} \sup_{t \geq 0} \mathbb{P}(C(t, h) > \epsilon/3) = 0;$$

and we conclude that the trajectories are stochastically continuous under the p -th moment. \square

4.3. Existence of stochastically continuous and bounded solutions of the SPDE. Now we can establish the main result of this section. We give a result, not only on the existence of the mild solution but also integrability of the solutions trajectories and predictable predictability).

Theorem 4.6. *Let $p \in (0, \beta)$ and $\beta \in (0, 2)$. Let Z be a non square integrable Lévy process more precisely satisfying Condition $(\mathbf{H}_\nu^{\beta, \mathbf{P}})$. Under hypothesis $(\mathbf{A1})$, $(\mathbf{A2})$ and $(\mathbf{A3})$, if $\mathbb{E}|x_0|^p < \infty$ and $T > 0$, there exist a unique mild solution of (1) which is $L^p(\Omega \times [0, T])$ -bounded that is $t \rightarrow \mathbb{E}|X(t)|^p$ is bounded for all $t \in [0, T]$ and predictable.*

In the case $\beta \in (0, 1]$ we need the following additional strong condition

$$2^p L_F^p \left(\frac{1}{a} \right)^{p/q} + 2^p \eta^p(a, T, g) \left(1 + K_\nu(a, b, \phi, T) \frac{p}{\beta - p} \right) < 1 \quad (**)$$

where q stand for the conjugate of p and

$$K_\nu(a, b, \phi, T) = \frac{8b^2 \|\phi\|_\infty^2}{a} + \frac{8 \|\phi\|_\infty^2 C_1(\nu) C_2(\nu)}{a} + 4 \|\phi\|_\infty^2 C_2(\nu) + C_1(\nu) T.$$

Remark 4.1. One can be disapointed to require the strong condition (**). But one can see that this latter condition holds for a symmetric α -stable process ($c_+ = c_- = c$) of order $\alpha \in (0, 1]$ and $g(t) = C_0 \sin(t)$ if we choose C_0 such that

$$2^p L_F^p \left(\frac{1}{a} \right)^{p/q} + (a2)^p C_0^p \left(1 + \frac{32c \|\phi\|_\infty^2}{a\alpha(2-\alpha)} + 8 \frac{c \|\phi\|_\infty^2}{2-\alpha} + \frac{2Tc}{\alpha} \right) < 1$$

Remark 4.2. Note also that the remark 3.1 formulated previously also applies for this result.

Proof. Denote by \mathbb{X} the collection of all adapted stochastically continous and $L^p([0, T] \times \Omega)$ -bounded stochastic processes $X(t)$, $t \in [0, T]$ such that $\mathbb{E}|X(t)|^p < \infty$ for $p > 0$. It is well-known that for $p \geq 1$, the space \mathbb{X} is a Banach space when it is equipped with the norm

$$\|X\|_T = \left(\mathbb{E} \sup_{t \in [0, T]} |X(t)|^p \right)^{1/p}.$$

Thus, we can consider the family of equivalent norms for some $\gamma > 0$ to be choose later :

$$\|X\|_\gamma := \sup_{t \in [0, T]} e^{-\gamma t} \left(\mathbb{E}|X(t)|^p \right)^{1/p}.$$

Note also that \mathbb{X} equipped with $\|\cdot\|_\gamma$ is again a Banach space.

Define the following operator Γ on the Banach space $(\mathbb{X}, \|\cdot\|)$ by

$$(\Gamma X)(t) = S(t)x_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)g(s)\phi(X(s))dZ(s).$$

Note that $S(t)x_0$, $\int_0^t S(t-s)F(s, X(s))ds$ and $\int_0^t S(t-s)g(s)\phi(X(s))dZ(s)$ are in \mathbb{X} according to previous sections.

For any $X \in \mathbb{X}$, Γ maps the space \mathbb{X} into itself. Finally for any $X, Y \in \mathbb{X}$ we have for all $p \geq 1$:

$$\begin{aligned} e^{-p\gamma t} \mathbb{E}|\Gamma X(t) - \Gamma Y(t)|^p &\leq 2^{p-1} \left(\int_0^t e^{-qa(t-s)} ds \right)^{p/q} \mathbb{E} \int_0^t e^{-\gamma(t-s)} e^{-\gamma s} |F(s, X(s)) - F(s, Y(s))|^p ds \\ &\quad + 2^{p-1} \mathbb{E} \left(\int_0^t e^{-\gamma t} e^{-a(t-s)} |g(s)| |\phi(X(s)) - \phi(Y(s))| dZ(s) \right)^p \\ &\leq 2^{p-1} L_F \left(\frac{1}{aq} \right)^{p/q} \sup_{t \in [0, T]} e^{-p\gamma t} \mathbb{E}|X(t) - Y(t)|^p \int_0^t e^{-\gamma(t-s)} ds \\ &\quad + 2^{p-1} \mathbb{E} \left(\int_0^t e^{-a(t-s)} |g(s)| |\phi(X(s)) - \phi(Y(s))| dZ(s) \right)^p \\ &\leq 2^{p-1} L_F \left(\frac{1}{aq\gamma} \right)^{p/q} \sup_{t \in [0, T]} e^{-p\gamma t} \mathbb{E}|X(t) - Y(t)|^p \\ &\quad + 2^{p-1} \mathbb{E} \left(\int_0^t e^{-(a+p\gamma)(t-s)} |g(s)| e^{-p\gamma s} |\phi(X(s)) - \phi(Y(s))| dZ(s) \right)^p \end{aligned}$$

Note that, from the previous calculus we have :

$$\mathbb{E} \left(\int_0^t e^{-(a+\gamma)(t-s)} |g(s)| e^{-\gamma s} |\phi(X(s)) - \phi(Y(s))| dZ(s) \right)^p \leq \sup_{t \in [0, T]} e^{-\gamma t} \mathbb{E} |X(t) - Y(t)|^p \eta^p(a, T, g, \gamma) \times \left(1 + K_\nu(a, b, \phi) \frac{p}{\beta - p} \right)$$

where

$$\eta^p(a, T, g, \gamma) = \left(\sup_{0 \leq t \leq T} \int_0^t e^{-(a+\gamma)(t-s)} g^2(s) ds \right)^{p/2}.$$

This implies that

$$\sup_{t \in [0, T]} e^{-p\gamma t} \mathbb{E} |\Gamma X(t) - \Gamma Y(t)|^p \leq \sup_{t \in [0, T]} e^{-p\gamma t} \mathbb{E} |X(t) - Y(t)|^p 2^{p-1} \times \left(L_F \left(\frac{1}{aq\gamma} \right)^{p/q} + \eta^p(a, T, g, \gamma) \left(1 + K_\nu(a, b, \phi, T) \frac{p}{\beta - p} \right) \right).$$

Choosing γ sufficiently large so that

$$\left(L_F \left(\frac{1}{aq\gamma} \right)^{p/q} + \eta^p(a, T, g, \gamma) \left(1 + K_\nu(a, b, \phi, T) \frac{p}{\beta - p} \right) \right) < 1$$

allows us to complete the proof by the well-known Banach fixed-point theorem.

For $p \in (0, 1)$, note that the space \mathbb{X} is a linear complete separable metric space when it is equipped with the following distance

$$d_p(X, Y) = \int_0^T \mathbb{E} |X(t) - Y(t)|^p dt.$$

In this case, it is easy to see that for any $X \in \mathbb{X}$, Γ maps the space \mathbb{X} into itself. Taking any $X, Y \in \mathbb{X}$ we have for all $p \in (0, 1)$:

$$\begin{aligned} \mathbb{E} |\Gamma X(t) - \Gamma Y(t)|^p &\leq 2^p M^p \left(\int_0^t e^{-a(t-s)} ds \right)^{p/q} \mathbb{E} \int_0^t e^{-a(t-s)} |F(s, X(s)) - F(s, Y(s))|^p ds \\ &\quad + 2^p \mathbb{E} \left(\int_0^t e^{-a(t-s)} |g(s)| |\phi(X(s)) - \phi(Y(s))| dZ(s) \right)^p \\ &\leq 2^p L_F^p \left(\int_0^t e^{-a(t-s)} ds \right)^{p/q} \int_0^t \mathbb{E} |X(s) - Y(s)|^p ds \\ &\quad + 2^p \mathbb{E} \left(\int_0^t e^{-a(t-s)} |g(s)| |\phi(X(s)) - \phi(Y(s))| dZ(s) \right)^p \\ &\leq 2^p L_F^p \left(\frac{1}{a} \right)^{p/q} d_p(X, Y) \\ &\quad + 2^p \mathbb{E} \left(\int_0^t e^{-a(t-s)} |g(s)| |\phi(X(s)) - \phi(Y(s))| dZ(s) \right)^p \end{aligned}$$

Note that the calculus above in previous section we have :

$$\mathbb{E} \left(\int_0^t e^{-a(t-s)} |g(s)| |\phi(X(s)) - \phi(Y(s))| dZ(s) \right)^p \leq \eta^p(a, T, g) \times \left(1 + K_\nu(a, b, \phi) \frac{p}{\beta - p} \right) \int_0^T \mathbb{E} |X(t) - Y(t)|^p$$

Thus,

$$d_p(\Gamma X, \Gamma Y) \leq \left(2^p L_F^p \left(\frac{1}{a} \right)^{p/q} + 2^p \eta^p(a, T, g) \left(1 + K_\nu(a, b, \phi) \frac{p}{\beta - p} \right) \right) d_p(X, Y).$$

The proof is completed by using the contraction mapping theorem. \square

4.4. An illustrative example. In order to illustrate usefulness of the theoretical results established, we consider the following one-dimensional stochastic heat equation driven by a strictly stable process Z with Dirichlet boundary conditions:

$$\begin{cases} du(t, x) = \frac{\partial^2 u(t, x)}{\partial x^2} dt + F(t, u(t, x)) dt + \sin(t) \phi(u(t, x)) dZ(t) \\ u(t, 0) = u(t, 1) = 0, t \in [0, T], \\ u(0, x) = h(x), x \in [0, 1]. \end{cases} \quad (12)$$

where $Z(t)$ is an α -stable process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$; $h \in L^p[0, 1]$. Denote by ϕ and F functions satisfying assumptions **(A3)** and **(A3)**. Define

$$\mathcal{D}(A) = \{v \text{ continuous}/v'(r) \text{ absolutely continuous on } [0, 1], v''(r) \in L^2[0, 1]$$

$$\text{and } v(0) = v(1) = 0\}$$

$$Av = v'' \text{ for all } v \in \mathcal{D}(A).$$

Let $\Phi_n(t) = \sqrt{2} \sin(n\pi t)$ for all $n \in \mathbb{N}$. Φ_n are eigenfunctions of the operator $(A, \mathcal{D}(A))$ with eigenvalues $\lambda_n = -n^2$. Then, A generates a C_0 semigroup $(S(t))$ of the form

$$S(t)\Phi = \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \langle \phi, \phi_n \rangle \Phi_n, \quad \forall \phi \in L^2[0, 1]$$

and

$$\|S(t)\| \leq e^{-\pi^2 t}, \text{ for all } t \in [0, T].$$

Thus $a = \pi^2$.

The equation (12) is of the form

$$\begin{cases} dy(t) = Ay(t) dt + F(t, y(t)) dt + G(t, y(t)) dZ(t), \\ y(0) = c_0. \end{cases}$$

On can apply Theorem 4.6 with $a = \pi^2$, $\beta(a, g, T) = \frac{1}{\pi^2}$ under Assumptions **(A2)** and **(A3)**.

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