Lenses on very curved zones of a singular line field of C^2 or of a singular plane field of C^3

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February 21, 2020

Abstract

We renormalize, using suitable lenses, small domains of a singular holomorphic line field of $\mathbb{C}^2$ or plane field of $\mathbb{C}^3$ where the curvature of a plane-field is concentrated. At a proper scale the field is almost invariant by translations. When the field is integrable, the leaves are locally almost translates of a surface that we will call *profile*. When the singular rays of the tangent cone (a generalization to a plane-field of the tangent cone of a singular surface is defined) are isolated, we obtain more precise results. We also generalize a result of Merle ([Me]) concerning the contact order of generic polar curves with the singular level $f = 0$ when $\omega = df$. On the way we obtain some classical results (Lê's carousels) on the knot $K = (\{f = 0\} \cap B_\varepsilon(0,0,0))$ in dimension 2 an a maybe less classical ones in dimension 3 .

key words
complex polynomial, complex one-form, isolated singularity, polar curve, profile
domains
[math.AG], [math.IT], [math.GT], [math.DG]

1 Introduction

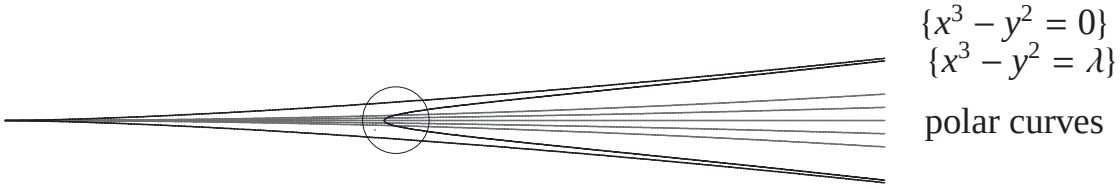


Figure 1: $\{x^3 - y^2 = 0\}$ and $\{x^3 - y^2 = \lambda\}$, a real picture.

In 1968 J. Milnor [Mil], published a book where he shows that levels of a complex polynomial $f : \mathbb{C}^n \to \mathbb{C}$ present some limit topology near an isolated singular point. Ness [Ne], Langevin [La1] studied the total curvature of the intersection of a level of f with a small ball centered at the singular point. Then, studying polynomials of two variables, Teissier [Tei3],

Garcia-Barroso [GarBar] and Garcia-Barroso and Teissier [GarBar-Te] got more and more precise results about where curvature is situated on a given level $\{f = \lambda\}$, λ close to 0. The first results, in dimension 2, about the pattern of the very curved zones of levels $\{f = \lambda\}$, λ close to 0, or of the leaves of a singular foliation defined by an integrable algebraic 1-form, were obtained by the author and J.C Sifre (see [La-Si]).

The results given here are also closely related to Merle's ([Me]) about *bouquets* (see Definition 3.2.2) of branches of polar curves in dimension 2.

The goal of this text is, given a 1-form in $\mathbb{C}^3$ with an isolated singular point at the origin, to renormalize small domains where the curvature is concentrated in order to find *profiles*, that is limit non-trivial plane field or foliation (see Definition 3.2.3). The choice of the dimension $n = 3$ of the ambient space is first intended to avoid heavy notations. It is also the last dimension where the author has still some geometrical intuition. When $\omega = df$, renormalizing simultaneously df and f, we prove that the profiles are in this case obtained translating the graph of a polynomial. We also observe the contact order of generic polar curves and the singular level $f = 0$, generalizing a result of Merle ([Me]). We obtain also a decomposition of the link $K = \{f = 0\} \cap S_\varepsilon$ in pieces which are either $\mathbb{S}^1$ fiber bundles or fiber bundles over $\mathbb{S}^1$. The pieces are glued along boundary tori.

The figures implying the use of a computer were made by J-C. Sifre and are taken from [La-Si] or [La2].

2 Preliminaries

2.1 Gauss map and polar curves (in dimension 2)

Given a 1-form ω defined on $\mathbb{R}^2$, we can define, at a point m where $\omega \neq 0$, a Gauss map $G_\omega(m) = ker\, \omega(m)$. It has values in $\mathbb{RP}^1$. The line field ker ω defines, where $\omega \neq 0$, a foliation $\mathcal{F}_\omega$. The Gauss map G_ω depends only on the foliation.

We can now define the *polar sets* of a foliation $\mathcal{F}_\omega$ of $\mathbb{R}^2$ by curves. They are inverse images of the points of $\mathbb{RP}^1$ by the Gauss map G_ω. These polar sets are in general curves, maybe with singularities. That is why we will abusively use the term *polar curves*.

Definition 2.1.1. - *The polar curve* $\Gamma_{[a,b]}$ *is the closure of the set (in general a curve, maybe with singularities) of points of the plane where the line tangent at a point m to the leaf* L_m *of the foliation which contains the point m is parallel to a direction* $[a, b]$.

- *When the foliation is defined by a* 1*-form* ω*, it is the closure of the set* $\{\omega(a, b) = 0\}$

This definition holds as well when the foliation $\mathcal{F}$ is a foliation of $\mathbb{C}^2$ by complex curves. The second part of this definition holds also when the 1-form in $\mathbb{C}^n$ or $\mathbb{R}^n$ defines only a hyperplane field.

2.2 Examples in dimension 2

Polar curves near an isolated singularity, the seminal example in dimension 2

Let ℓ be the line generated by the vector (a, b). The equation of the polar curve Γ_ℓ is then $df(a, b) = 0$. Here it writes $3ax^2 - 2by = 0$. The generic ($b \neq 0$) polar curves form a family of parabolas tangent to the x-axis and the x-axis.

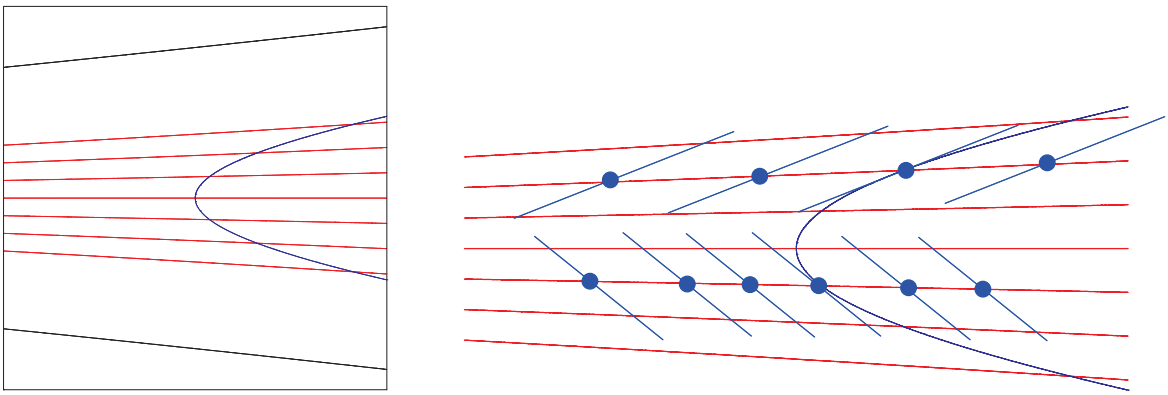


Figure 2: Polar curves of $x^3 - y^2 = \lambda$ using a suitable lens, the line-field tangent to $\mathcal{F}$ after enlargement

Notice that, when one observes a family of tangent generic polar curves $\Gamma_{[a,b]}$, they look like lines parallel to the common tangent direction after enlarging enough a small enough neighborhood of a point on the common tangent at the origin close to the origin (see Figure 2).

By definition of polar curves, the direction tangent to $\mathcal{F}$ at the points of a given segment of polar curve are parallel. If the direction $[a, b]$ is not tangent to $\Gamma_{[a,b]}$ at the origin, the integration of the line field will give pieces looking like parallel graphs (maybe with asymptotes parallel to the x-axis, the saddle-node of equation $\omega = ydx - x^2dy$ is already an example of this phenomenon in $\mathbb{R}^2$ or in $\mathbb{C}^2$.

Our tool providing a profile is a *Newton lens*.

Definition 2.2.1. *A Newton lens is a pair*

- an analytic curve $\gamma(t)$ ending at the origin and tangent at the origin to a privileged direction, say the x-axis,

- an enlargement rate $1/\rho = 1/\rho(x(\gamma(t)))$, $\lim_{|x(t)|\to 0} \rho(x(t)) = 0$

The analytic curve will first be the x-axis, then suitably chosen curves tangent to the x-axis at the singular point.

The definition will stay unchanged in $\mathbb{C}^3$.

2.3 Newton-lens cloud and Newton-lens polygon

The Newton cloud of a polynomial $f(x,y) = \sum a_{i,j}x^iy^j$ is the set $\{i, j\}$ of points of $\mathbb{N} \times \mathbb{N}$ such that $a_{i,j} \neq 0$ (see Figure 3).

The Newton polygon of f is the boundary of the convex hull of the union of the upper quadrants of $\mathbb{N} \times \mathbb{N}$ of vertices the points of the Newton cloud of f (see Figure 3).

Let Φ_1 be the change of variables

$$x = x_0 + \rho_1 X_1$$
$$y = \rho_1 Y_1$$

Then, we can see this change of variables as a moving lens settled to observe a neighborhood of the point $(x_0, 0)$ if ρ_1 is a function of $x_0 \to 0$.

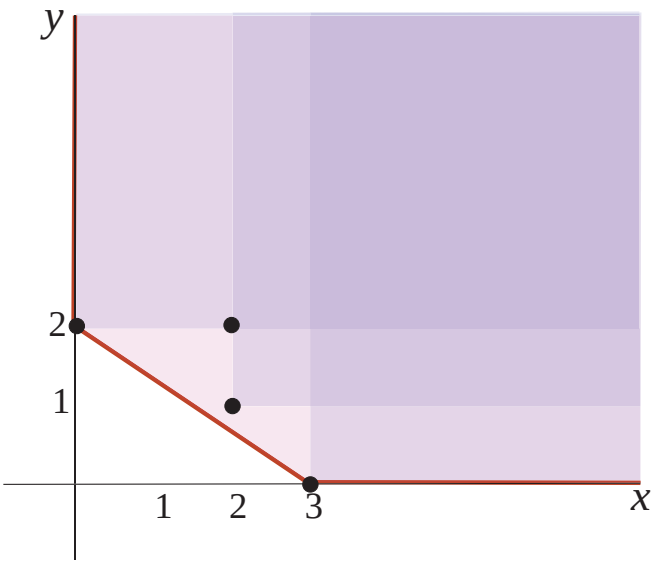


Figure 3: Newton cloud (black dots) and Newton polygon (in red) of $f(x,y) = x^3 - y^2 + x^2y^2 + x^2y$

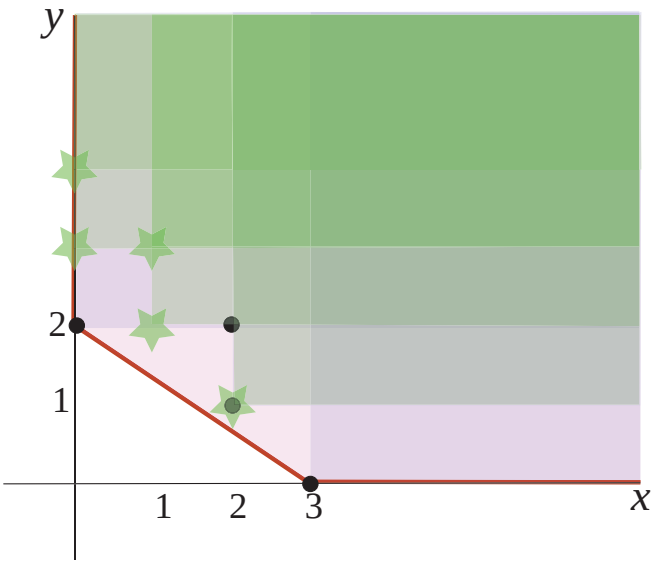


Figure 4: First Newton-lens cloud (black dots and green stars) and first Newton-lens polygon (in red) of $f(x,y) = x^3 - y^2 + x^2y^2 + x^2y$

Definition 2.3.1. *The first Newton-lens cloud of f is the Newton cloud associated to the polynomial $\Phi_1^*(f)$ of the variables x_0 and ρ_1.*

The first Newton-lens polygon of f is the boundary of the convex hull of the union of the upper quadrants of $\mathbb{N}\times\mathbb{N}$ of vertices the points of the first Newton cloud of f (see Figures 3 and 4).

We will first consider the exponents of the polynomial $\Phi_1^*(f)$ considered as a polynomial of the two variables x_0 and ρ_1.

Then the Newton-lens polygon allows us to chose the value $\rho_1 = x_0^{r_1}$. Let us denote by $\widetilde{\Phi_1}$

the change of variables

$$x = x_0 + x_0^{r_1} X_1$$
$$y = x_0^{r_1} Y_1$$

Then we will iterate the construction. After replacing ρ_{i-1} by a rational power of x_0, obtaining a change of variables $\widetilde{\Phi_{i-1}}$ we consider the change of variables Φ_i

$$X_{i-1} = \rho_i X_i$$
$$Y_{i-1} = y_i + \rho_i Y_i$$

We consider the exponents in x_0 and ρ_i of the polynomial $\Phi_i^* \widetilde{\Phi_{i-1}}^* \cdots \widetilde{\Phi_1}^*(f)$. The Newton-lens clouds are now in $\mathbb{Q}^+ \times \mathbb{N}$.

Then we will apply a similar construction to a polynomial one-form $\omega = Adx + Bdy$ plotting, after changes of variables, all the exponents of the monomials of A and of B.

We also apply this constructions to a polynomial $f(x,y,z)$ and a polynomial one-form $\omega = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz$. In this case, as it would be with more variables, the Newton-lens clouds are still planar (contained in $\mathbb{Q}^+ \times \mathbb{N}$) after the first step.

3 1-forms in dimension 3

The advantage of our method is to provide, in particular when ω is integrable, a precise description of the limit shapes of the leaves which appear near the origin. When $\omega = df$, we unfortunately loose the global structure of the levels of f, $f = \lambda$, $\lambda \to 0$.

Let us now consider a 1-form $\omega = Adx + Bdy + Cdz$.

Let Σ be the singular set of ω that is where ω is zero. The planes $\{ker(\omega)\}$ define a plane field $\mathcal{P}$ of $\mathbb{C}^3 \setminus \Sigma$.

In this text, we assume that the origin O is an isolated singular point of ω.

3.1 The tangent cone

Inspired by Euler's formula valid when $\omega = df$, we define the tangent cone of a 1-form ω. Let *low(ω)* be the homogeneous form selecting globally the lowest degree terms of the polynomials A, B and C coefficients of $\omega = Adx + Bdy + Cdz$; if this lowest degree is k, then $valA \geq k$; let $A_k(x,y) = low(A)$ if the valuation of A is k, zero if the valuation of A is larger than k; we use the same convention for B and C.

Definition 3.1.1. *The equation of the* tangent cone *of* $\omega = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz$ *at the origin is*

$$low(\omega(x,y,z))(x,y,z)) = A_k(x,y,z)x + B_k(x,y,z)y + C_k(x,y,z)z = 0.$$

Recall that, given in $\mathbb{C}^2$ the 1-form $\omega = A(x,y)dx + B(x,y)dy$, the polynomials $xA(x,y)$ and $yB(x,y)$ appear in [Ca-Li-Sa]. Notice that, when ω as an homogeneous 1-form, the plane field $ker\,\omega$ is constant along rays. In particular, along rays of the tangent cone, one has $\omega(x,y,z) = 0$, that is the plane $\ker\ \omega(x,y,z)$ contains the ray $\lambda \cdot (x,y,z)$, $\lambda \in \mathbb{C}$.

Notice also that, when $\omega = df$, $low(\omega(x,y,z))(x,y,z) = k \cdot low(f)$, where k is the degree of $low(f)$ (Euler's equality).

Example: the tangent cone of a linear forms

Let us suppose that $\omega = Adx + Bdy + Cdz$,

$$A = a_1x + b_1y + c_1z$$
$$B = a_2x + b_2y + c_2z$$
$$C = a_3x + b_3y + c_3z$$

Let us denote by $\mathcal{M}$ the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ we can write $\omega = \begin{pmatrix} dx & dy & dz \end{pmatrix} \cdot \mathcal{M} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The equation of the tangent cone is then $\begin{pmatrix} x & y & z \end{pmatrix} \cdot \mathcal{M} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

It is a quadratic equation which depends only on symmetrization $\frac{1}{2}({}^t\mathcal{M} + \mathcal{M})$.

When the matrix $\mathcal{M}$ is antisymmetric, say $\mathcal{M} = \begin{pmatrix} 0 & -a_2 & -a_3 \\ a_2 & 0 & -b_3 \\ a_3 & b_3 & 0 \end{pmatrix}$.

The form $\omega = \begin{pmatrix} dx & dy & dz \end{pmatrix} \cdot \mathcal{M} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is an example where the tangent cone is the whole space as

$${}^{\mathbf{t}}\left(\begin{pmatrix} dx & dy & dz \end{pmatrix} \mathcal{M} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-1)^3 \left(\begin{pmatrix} dx & dy & dz \end{pmatrix} \mathcal{M} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Notice that all rays are separatrices (see Definition 7.5.1).

In fact it is the general form of a linear 1-form such that the tangent cone is the whole space, as in that case $\left(\mathcal{M} + {}^{\mathbf{t}}\mathcal{M}\right)$ should be zero.

Remark 3.1.2. A parenthesis in $\mathbb{R}^3$ *When the matrix $\mathcal{M}$ is of rank 3, the integrability condition on the whole $\mathbb{R}^3$, $\omega \wedge d\omega = 0$ $\forall (x,y,z)$, writes $\mathcal{M} - {}^{\mathbf{t}}\mathcal{M} = 0$, in other words, the matrix is a symmetrical one.*

Remark When the image of the Gauss map is exactly a projective line $\mathcal{L}$, the foliation defined by $ker\,\omega$ is a family of planes rotating around an axis.

When ω is linear, we can represent the Gauss map using the matrix of a map $\mathcal{M} : \mathbb{C}^3 \to \mathbb{C}3$ as $ker\,\omega = \left(\mathcal{M} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)^{\perp}$.

When the Gauss map is of rank 1, and the linear map representing it of rank 2, the kernel of the matrix $\mathcal{M}$ is a line L. All the planes $ker\,\omega$ contain the line L. These planes form the projective line $\mathcal{L}$. The form ω defines a pencil of planes (see Figure **??**).

Remark Notice that, even if the tangent cone is a plane, the map $\gamma_{\omega_{low}}$ maybe locally surjective and guarantees, when ω is homogeneous, that all the polar curves are rays. This is the case of $\omega = \omega_{low} = xdx + zdy - ydz$ (the tangent cone has equation $x^2 = 0$, it is singular, see below Definition 3.1.3).

3.1.1 Singular rays of the tangent cone in dimension 3

Definition 3.1.3. *A ray* $\{\lambda \cdot (x,y,z),\ \lambda \in \mathbb{C}\}$ *of this cone is* singular for ω *if* $A_k(x,y,z) = B_k(x,y,z) = C_k(x,y,z) = 0$*, that is if* $low(\omega)(x,y,z) \equiv 0$.

Definition 3.1.4. *The equation of the tangent cone degenerates when all the lines of the tangent cone are singular (see Definition 3.1.3).*

This is the case when $\omega = df$ and if $f = g^2$.

The equation of the tangent cone provides a projective curve $C_\omega \subset \mathbb{CP}^2$.

Proposition 3.1.5. *When* $\omega = df$*, a line of the tangent cone is singular for* ω *if it is a singular point of* C_ω.

Proof: Let us suppose that the x-axis is a singular ray of the tangent cone of $\omega = df$, and that f_{low} is homogeneous of degree $(k+1)$. It means that $\omega_{low}(1,0,0) = d(f_{low})(1,0,0) = 0$. The section of the tangent cone by the plane $x = 1$ has the equation $F(y,z) = A_k(1,y,z) + yB_k(1,y,z) + zC_k(1,y,z) = 0$. The gradient of the function F writes

$$\begin{array}{c} \frac{\partial A(1,y,z)}{\partial y} + B(1,y,z) + y\frac{\partial B}{\partial y} + z\frac{\partial C}{\partial y} \\ \frac{\partial A(1,y,z)}{\partial z} + y\frac{\partial B}{\partial z} + C(1,y,z) + z\frac{\partial C}{\partial z} \end{array}$$

The point $(1,0,0)$ is a singular point of the curve of equation $F = 0$ if

$$\frac{\partial A(1,y,z)}{\partial y}(1,0,0) + B(1,0,0) = 0 = \frac{\partial A(1,y,z)}{\partial z}(1,0,0) + C(1,0,0)$$

As the x-axis is singular we know already that $B(1,0,0) = C(1,0,0) = 0$.

The term of f which may contribute to $\frac{\partial A(1,y,z)}{\partial y}(1,0,0)$ and $\frac{A(1,y,z)}{\partial z}(1,0,0)$ are respectively of the form $x^k(ay)$ and $x^k(bz)$. But if $a \neq 0$ then $\frac{\partial f}{\partial y}(1,0,0) = B(1,0,0) \neq 0$ Similarly, if $b \neq 0$, $\frac{\partial f}{\partial z}(1,0,0) = C(1,0,0) \neq 0$. Therefore $a = b = 0$ and the point $(1,0,0)$ is a critical point of $F = 0$). □

In our search of profiles, the pertinent objects are the rays singular for ω.

Example of the type $\omega = df$: $x^4 + y^4 - x \cdot y^2 + z^3 = 0$.

The tangent cone has the equation $z^3 - xy^2 = 0$. As $low(\omega) = -y^2dx - 2xydy + 3z^2dz$ we see that the only degenerate ray is the x-axis.

The reader will find below a profile associated to this example.

3.2 Gauss map, polar surfaces and polar curves in dimension 3

Given a 1-form ω defined on $\mathbb{C}^3$, we can define, at a point m where $\omega \neq 0$, a Gauss map $G_\omega(m) = ker\,\omega(m)$. It has values in $\mathbb{CP}^2$. The map G_ω is defined on the complement of the singular locus of ω. It is therefore defined on a small enough neighborhood of the origin but for the origin itself.

Definition 3.2.1.

- The polar set $V_\ell = V_v$ *associated to a line* ℓ *generated by a vector v is the closure of the set of points where* $\omega(v) = 0$*. It is in general a surface.*

- The polar set Γ_h *associated to a plane h is the closure of the set of points where* $ker\,\omega = h$*. In general it is a curve.*

We will use only the terms “polar surface”, “polar curve” and deal with degenerate cases only if unavoidable.

In other terms the polar curve Γ_h is the inverse image of h by the Gauss map G_ω. A polar curve Γ_h, h a plane of equation $Ax+By+Cz=0$, is the intersection of two polar surfaces V_{a_i,b_i,c_i} such that the vectors (a_1,b_1,c_1) and (a_2,b_2,c_2) span the plane h.

An example of very degenerate case is the form $\omega = xdy - ydx$. It defines the foliation by planes containing the z-axis. The polar sets corresponding to planes transverse to the z-axis are empty and they coincide with the plane when it contains the z-axis.

Example

Suppose that $\omega = \omega_{low}$ is homogeneous. Then the plane field $\mathcal{P} = \{\ker \omega\}$ is invariant by homotheties of center the origin.

Given a plane h, in general the set of points where $ker\,\omega = h$ is a union of rays (lines containing the origin).

In this case, the polar curves are union of rays where the Gauss map $G_\omega : (\mathbb{C}^3 \setminus O) \to \mathbb{CP}^2;\ [x,y,z] \mapsto [A,B,C]$ is locally surjective.

In dimension 3, let us suppose that the plane h is transverse to the x-axis. Then h is spanned by the vectors $(a,b,0)$ and $(c,0,d)$, $b \neq 0, d \neq 0$ and the polar curve Γ_h is the intersection of the surface $S_{a,b,0}$ of equation $aA + bB = 0$ and the surface $S_{c,0,d}$ of equation $cA + dC = 0$.

Definition 3.2.2. *Given a subset $G \subset \mathbb{CP}^2$ of dimension* 1*, a neighborhood N_G of G, a family of branches of polar curves Γ_h, $h \notin N_G$ tangent to a line ℓ at the origin form a* bouquet.

A typical case, pertinent when the 1-form ω is the differential of a function f is $G_\ell = \{h$ such that $\ell \subset h\}$, ℓ a line of $\mathbb{CP}^2$.

Definition 3.2.3. *When a bouquet of polar curves is tangent to an isolated ray of the tangent cone of ω, say the x-axis, enlarging neighborhoods of a sequence of points of the x-axis converging to the origin may give rise to a limit plane field invariant by translations parallel to the x-axis that we will call* profile*. When ω is integrable we also call* profile *a typical leaf of the enlarged foliation, which is invariant by translations parallel to the x-axis.*

More systematically, we will use Newton lenses (see Definition 2.2.1) to find profiles.

The choice of the dimension $n = 3$ of the ambient space is intended to avoid heavy notations. The methods and most of the results extend straightforward to dimension n.

The induction leading to the construction of the profiles is similar to the Newton-Puiseux induction leading to a parametrization of an analytic curve of $\mathbb{C}^2$, but it relies on a different choice of variables and change of variables.

4 The Newton-lens algorithm, first step

We consider a complex polynomial differential form

$$\omega = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz$$

singular at the origin, that is such that $A(0,0,0) = B(0,0,0) = C(0,0,0) = 0$. We will suppose also that the origin is an isolated singular point of ω.

The *Newton-lens algorithms* provide Newton lenses, that is compute the suitable enlargement rates $\rho_i(x_0)$ along successive curves γ_i tangent to a singular ray of the tangent cone, that we will suppose to be the x-axis.

As in the dimension 2 examples (see Figure 2), the effect of a moving lens with adapted stronger and stronger strength when approaching the origin will be to straighten the polar curves tangent to the x-axis keeping the directions of the planes $ker(\omega)$.

4.1 The "quiet" regions

Let us first rule out the case where the x-axis is not in the tangent cone or is in the tangent cone but is not singular for ω. In other terms the three coefficients of $low(\omega)(x,0,0) = A_k(x,0,0)dx + B_k(x,0,0)dy + C_k(x,0,0)$ are not simultaneously zero.

Proposition 4.1.1. *When the x-axis is not in the tangent cone or is in the tangent cone but is not singular for ω, the limit of the enlargement of a ball of size $|x_0|^r, r > 0$ shows a plane field which is a family of parallel planes.*

Proof. The differential form $low(\omega) = A_k(x,y,z)dx + B_k(x,y,z)dy + C_k(x,y,z)dz$, with A_k, B_k and C_k homogeneous polynomials null or of degree k, verifies :

$$A_k(x_0,0,0) \neq 0 \quad \text{or} \quad B_k(x_0,0,0) \neq 0 \quad \text{or} \quad C_k(x_0,0,0) \neq 0.$$

For any exponent $r > 1$

$$\begin{aligned} &low(\omega)(x_0 + x_0^r X_1, x_0^r Y_1, x_0^r Z_1) \\ &\qquad = x_0^k(A_k(1,0,0)dx + B_k(1,0,0)dy + C_k(1,0,0)dz) + o(x_0^k), \end{aligned}$$

Notice that $dx = x_0^r dX_1$, $dy = x_0^r dY_1$, $dz = x_0^r dZ_1$. Factorizing in $low(\omega)(x_0 + x_0^r X_1, x_0^r Y_1, x_0^r Z_1)$ the maximal possible power of x_0, we get the differential form ω_1 which defines the profile (see Equation 1)

$$\omega_1(X_1,Y_1,Z_1) = A_k(1,0,0)dX_1 + B_k(1,0,0)dY_1 + C_k(1,0,0)dZ_1.$$

The form ω_1 is constant (and non-zero), therefore the plane field $ker\,\omega_1$ is just a family of parallel planes. □

In this case, after enlargement and when $x_0 \to 0$, when ω is integrable, the leaves of the foliation look more and more like parallel planes.

A simple example is the singularity $\omega = df$; $f(x,y,z) = xy + yz + zx = 0$. The leaves of the foliation are the levels $f(x,y,z) = \lambda$; they are surfaces, which, along any line through the origin $O = (0,0,0)$, give, near a point (t_0x_0, t_0y_0, t_0z_0), at any scale $x_0^r, r > 1$, a limit foliation by parallel planes. More generally, when ω is homogeneous and integrable, even if the x-axis is a singular ray of the tangent cone, near a point $(x_0,0,0)$, after enlargement with ratio $1/(x_0)^r$, $r > 1$, the leaves of the foliation $\mathcal{F}$ look like parallel planes.

4.2 The first change of variables

The focus of the first lens is again the point $(x_0,0,0)$ of the-axis, where $|x_0|$ is small. The first enlargement is of strength $(1/\rho_1) >> 1$. We will use values of ρ_1 of the form $\rho_1 = x_0^{r_1}, r_1 > 1, , r_1$rational. In our construction we can choose any determination of x_0^r. A choice of a complex enlargement ratio may introduce a rotation of the picture, but does not change the profile we want to observe. Now we suppose that the x-axis is a singular ray.

For any exponent $r > 1$, the differential form ω writes, after enlargement using a lens centered at the point $(x_0, 0, 0)$ ($|x_0|$ small) of the x-axis (i.e. change of variables Φ_1), as above :

$$\begin{cases} \Phi_1^*(\omega)(X_1, Y_1, Z_1) = A(x_0 + x_0^r X_1, x_0^r Y_1, x_0^r Z_1)dX_1 + B(x_0 + x_0^r X_1, x_0^r Y_1, x_0^r Z_1)dY_1 \\ \qquad\qquad + C(x_0 + x_0^r X_1, x_0^r Y_1, x_0^r Z_1)dZ_1 \\ \qquad\qquad = x_0^\theta \widetilde{\omega_1} = x_0^\theta \omega_1(X_1, Y_1, Z_1) + o(x_0^\theta), \qquad \text{with} \\ \omega_1(X_1, Y_1, Z_1) = A_1(X_1, Y_1, Z_1)dX_1 + B_1(X_1, Y_1, Z_1)dY_1 + C_1(X_1, Y_1, Z_1)dZ_1 \end{cases} \tag{1}$$

where x_0^θ is the highest power of x_0 which is a factor of $\Phi_1^*(\omega)(X_1, Y_1, Z_1)$; $x_0^\theta \omega_1(X_1, Y_1, Z_1)$ is the sum of terms of lowest degree in x_0 of $\Phi_1^*(\omega)$ and θ the valuation (in x_0) of $\Phi_1^*(\omega)$. This defines the differential form

$$\omega_1(X_1, Y_1, Z_1) = A_1(X_1, Y_1, Z_1)dX_1 + B_1(X_1, Y_1, Z_1)dY_1 + C_1(X_1, Y_1, Z_1)dZ_1.$$

If r is not strictly superior to 1, when x_0 tends to 0, the origin $(0, 0, 0)$ will stay at a finite distance from the new origin, the eventual renormalized plane field contains the origin and will stay singular.

When $[A_1(X_1, Y_1, Z_1), B_1(X_1, Y_1, Z_1), C_1(X_1, Y_1, Z_1)]$ define a point of $\mathbb{CP}^2$ independent of (X_1, Y_1, Z_1), the planes of the plane-field $ker\,\omega$ defined by

$\omega_1 = A_1(X_1, Y_1, Z_1)dX_1 + B_1(X_1, Y_1, Z_1)dY_1 + C_1(X_1, Y_1, Z_1)dZ_1$ are parallel planes.

From now on, we suppose that the x-axis is an isolated singular ray of the tangent cone.

We shall give necessary conditions on r_1 to obtain a profile where the planes $ker\,\omega_1$ are not all vertical.

The exponent r_1 of $\rho_1 = x_0^{r_1}$ is determined using a vertex or a side of the *first Newton-lens polygon* of ω.

Recall that the *Newton-lens cloud* of the polynomial

$\mathcal{A}(x_0, \rho_1) = A(x_0 + \rho_1 X_1, \rho_1 Y_1, \rho Z_1)$

is the set of pairs (i, j) of exponents of a non zero monomial of $\mathcal{A}$ (in the variables (x_0, ρ_1)). We obtain the same way Newton-lens clouds for

$\mathcal{B}(x_0, \rho_1) = B(x_0 + \rho_1 X_1, \rho_1 Y_1, \rho_1 Z_1)$ and $C(x_0, \rho_1) = C(x_0 + \rho_1 X_1, \rho_1 Y_1, \rho_1 Z_1)$.

Definition 4.2.1. *The* first Newton-lens cloud *of ω is the union of the Newton-lens clouds of the three polynomials $\mathcal{A}$, $\mathcal{B}$ and C. The* first Newton-lens polygon *of ω is the lower convex hull of the union of the upper quadrants of vertices the points of the first Newton-lens cloud of ω.*

An example in dimension 2**,** $f(x, y) = ay^3 + bx^2y^2 - cx^4y + x^5$**,** $\omega = df = (2bxy^2 - 4cx^3y + 5x^4)dx + (3ay^2 + 2bx^2y - cx^4)dy$ We get, setting $\rho_1 \widetilde{\omega_1}(x_0, \rho_1) = \omega(x_0 + \rho_1 X_1, \rho_1 Y_1)$ (we forget

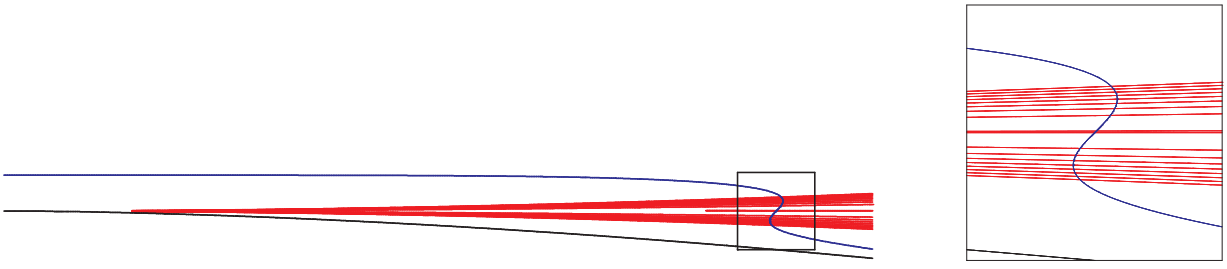


Figure 5: A cubic profile.

about the ρ_1-factor coming from the differentials).

$\widetilde{\omega_1}(x_0,\rho_1) = [2b(x_0+\rho_1 X_1)\rho_1^2 Y_1^2 - 4c(x_0+\rho_1 X_1)^3\rho_1 Y_1 + 5(x_0+\rho_1 X_1)^4]\rho_1 dX_1 + [3a\rho_1^2 Y_1^2 + 2b(x_0+\rho_1 X_1)^2\rho_1 Y_1 - c(x_0+\rho_1 X_1)^4]\rho_1 dY_1$.

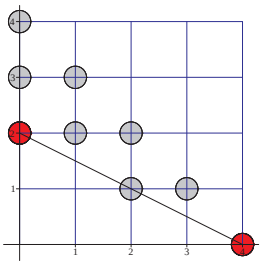


Figure 6: Newton-lens polygon for a cubic profile.

Taking, as the Newton-lens polygon suggests, $\rho_1 = x_0^2$, the lowest power of x_0 is x_0^4. Factorizing x_0^4 provides the differential equation $10\,dX_1 = 3aY_1^2 + 2bY_1$ and therefore a cubic profile $X_1 = \frac{a}{10}Y_1^3 + \frac{b}{10}Y_1^2$.

Examples in dimension 3

•**Example 1** $f(x,y,z) = x^3 - y^2 - z^2$ We get $df = 3x^2dx - 2ydy - 2zdz$ and $low(df) = 2ydy - 2zdz$. The tangent cone is the cylinder of equation $y^2 + z^2 = 0$; the x-axis is degenerate. It is the only singular line of the tangent cone.

Its Newton-lens polygon has, as the Newton-lens polygon of $f(x,y) = x^3 - y^2$, just one side of slope $-1/2$, therefore take $\rho_0 = x_0^2$.

Therefore we chose $M \cdot x_0^2$, M as large as we want, as radius of the ball $B = B(x_0, R = M \cdot x_0^2)$ where we will look at the enlarged plane-field.

Performing the change of variable

$$\begin{array}{l}(x - x_0) = x_0^2 \cdot X_1 \\ y = x_0^2 \cdot Y_1\end{array} \quad ,$$

we see that the level $C_{x_0^2} = \{f = x_0^2\}$ have in the ball B a shape similar to the solution of the differential equation

$$\begin{array}{l} a \simeq \partial X_1/\partial Y_1 \simeq (2/3)Y_1 \\ b \simeq \partial X_1/\partial Z_1 \simeq (2/3)Z_1 \end{array}$$

which admits the solution $X_1 = -(1/3)(Y_1^2 + Z_1^2)$

• **Example 2,** $f(x,y,z) = x^4 + y^4 - xy^2 + z^3$ (see Figure **??**).

$\omega = df = (4x^3 - y^2)dx + (4y^3 - 2xy)dy + 3z^2dz$

$\omega_{low} = -y^2dx - 2xydy + 3z^2dz$

The tangent cone at the origin has the equation $z^3 - xy^2 = 0$,

$\omega_{low} = -y^2dx - 2xydy + 3z^2dz$. The unique singular ray has equations $z = y = 0$, so it is the x-axis.

After the change of variables

$$x = x_0 + \rho_1 X_1,\ y = \rho_1 Y_1,\ z = \rho_1 Z_1,\ ,$$

we get, setting $\rho_1\widetilde{\omega_1}(x_0,\rho_1) = \omega(x_0 + \rho_1 X_1, \rho_1 Y_1, \rho_1 Z_1)$ (we forget about the ρ_1-factor coming from the differentials).

$$\widetilde{\omega_1} = [4(x_0+\rho_1 X_1)^3 - \rho_1^2 Y_1^2]dX_1 + [4\rho_1^3 Y_1^3 - 2\rho_1(x_0+\rho_1 X_1)Y_1]dY_1 + 3\rho_1^2 Z_1^2 dZ_1$$

In our construction, we will choose $\rho_1 = x_0^{r_1}$, $r_1 > 1$, r_1 rational (we can choose any determination of $x_0^{r_1}$) using the slope of a side of the Newton-lens polygon.

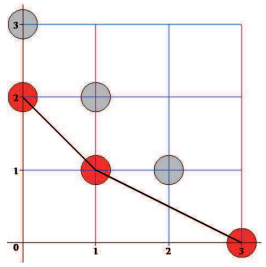


Figure 7: Newton-lens cloud and polygon of $\widetilde{\omega_1} = [4(x_0+\rho_1 X_1)^3 - \rho_1^2 Y_1^2]dX_1 + [4\rho_1^3 Y_1^3 - 2\rho_1(x_0+\rho_1 X_1)Y_1]dY_1 + 3\rho_1^2 Z_1^2 dZ_1$

Here (see Figure 7) we take $\rho_1 = x_0^2$, then $\omega_1 = 4dX_1 - 2Y_1 dY_1$, so

$$\frac{dX_1}{dY_1} = \frac{1}{2}Y_1 \text{ and}$$
$$X_1 = \frac{Y_1^2}{4} + c$$

If we want an estimation in terms of the level λ of $f = \lambda$, we see that the intersection point x_0 of Ox and the level $f = \lambda$ satisfies $|x_0| = |\lambda|^{1/4}$, therefore $|\rho_1| = |\lambda|^{1/2}$.

Notice that we obtained interesting profiles when the bottom-left point of the Newton-lens polygon, end of a side we considered, was coming from the "$x_0^k dX_1$" term of ω_1 itself "coming" from the a monomial of f which is a power of x only. This computation takes care of half of what we can observe on Figure **??**; the conical shape should be attributed to the homogeneous part.

4.3 Why the variables X_1 does not appear after the first step

We now proceed to understand the effect of zooming near a point $(x_0, 0, 0)$ close to the origin in the x-axis. Let us first consider slopes $-1 < (-1/r) < 0$ which are not slope of a side of the first Newton-lens polygon.

We perform the change of variables

$$\begin{cases} x = x_0 + x_0^r X_1 \\ y = x_0^r Y_1 \\ z = x_0^r Z_1 \end{cases} \tag{2}$$

Then

$$\begin{cases} \Phi_1^*(\omega)(X_1,Y_1,Z_1) = A(x_0+x_0^rX_1,x_0^rY_1,x_0^rZ_1)dX_1 \\ \qquad\qquad + B(x_0+x_0^rX_1,x_0^rY_1,x_0^rZ_1)dY_1 \\ \qquad\qquad + C(x_0+x_0^rX_1,x_0^rY_1,x_0^rZ_1)dZ_1 \\ \qquad\qquad = x_0^\theta\widetilde{\omega_1}(X_1,Y_1,Z_1) \quad \text{with} \\ \tilde{\omega_1}(X_1,Y_1,Z_1)\widetilde{A_1}(X_1,Y_1,Z_1) + \widetilde{B_1}(X_1,Y_1,Z_1) + \widetilde{C_1}(X_1,Y_1,Z_1) \\ \qquad\qquad = x_0^\theta\omega_1(X_1,Y_1,Z_1) + o(x_0^\theta), \qquad \text{with} \\ \omega_1(X_1,Y_1,Z_1) = A_1(X_1,Y_1,Z_1)dX_1 + B_1(X_1,Y_1,Z_1)dY_1 \\ \qquad\qquad + C_1(X_1,Y_1,Z_1) \end{cases} \tag{3}$$

Proposition 4.3.1. *The polynomials $A_1(X_1,Y_1,Z_1)$, $B_1(X_1,Y_1,Z_1)$ and $C_1(X_1,Y_1,Z_1)$ (see relations (3) above) do not contain the variable X_1.*

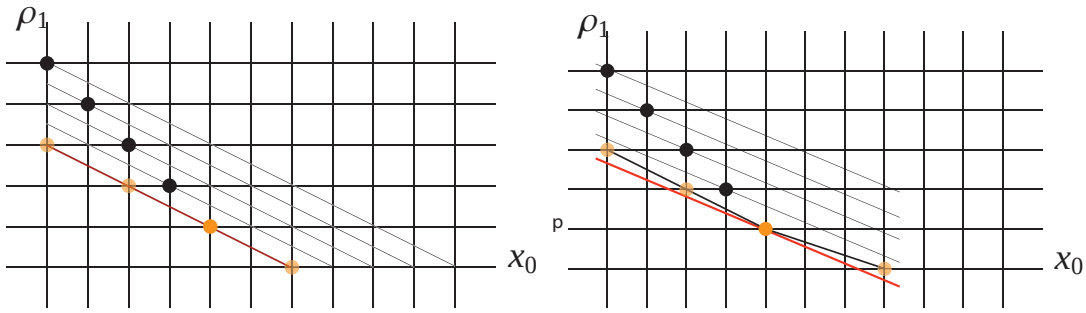


Figure 8: A tail coming from a point of the Newton-lens polygon of ω_1, left: from a point on a side, right: from a vertex

Proof. We prove separately that $A_1(X_1,Y_1,Z_1)$, $B_1(X_1,Y_1,Z_1), C_1(X_1,Y_1,Z_1)$ do not contain X_1.

Writing $A(x,y,z) = \sum a_{p,q,\ell}x^py^qz^\ell$, a non zero monomial $a_{p,q,\ell}x^py^qz^\ell$ gives a polynomial $a_{i,j,\ell}(x_0+\rho_1X_1)^p\rho_1^qY_1^q\rho_1^\ell Z_1^\ell$ in $A(x_0+\rho_1X_1,\rho_1Y_1,\rho Z_1)$.

One point comes from the monomial $x_0^p\rho_1^qY_1^q\rho_1^\ell Z_1^\ell$.

The other points introduced by the terms of $a_{p,q,\ell}(x_0+\rho_1X_1)^i\rho_1^jY_1^j\rho_1^kZ_1^k$ containing X_1 form a *tail* of slope -1 on the left of (p,q,ℓ) (in black in Figure 8).

Then a line in $\mathbb{R}^2$ of slope $-1/r$, $r>1$ below the Newton-lens cloud of $\mathcal{A}(x_0,\rho_1)$ cannot contain a point in this tail.

We proceed in the same way with $\mathcal{B}(x_0,\rho_1) = B(x_0+\rho_1X_1,\rho_1Y_1,\rho_1Z_1)$ and $C(x_0,\rho_1) = C(x_0+\rho_1X_1,\rho_1Y_1,\rho_1Z_1)$. □

A support line of the Newton-lens polygon, contains either a single pair (i,j), or an edge of the Newton-lens polygon of ω. All the corresponding monomials $a_{i,j,k}x_0^i\rho_1^jY_1^j\rho_1^kZ_1^k$, $b_{i,j,k}x_0^i\rho_1^jY_1^j\rho_1^kZ_1^k$ or $c_{i,j,k}x_0^i\rho_1^jY_1^j\rho_1^kZ_1^k$ give terms which do not contain X_1. They are followed by a tail of slope -1 the points of which cannot belong to any side of slope $-1<r<0$ of the Newton polygon.

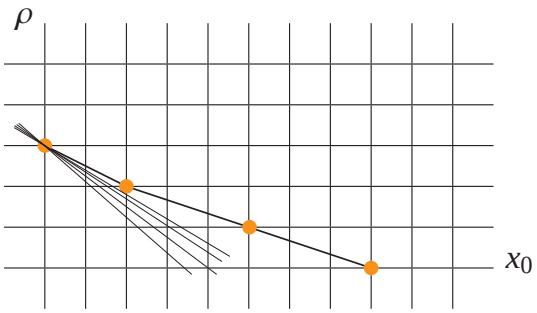


Figure 9: Support lines of slopes between $-1/a_1$ and $-1/b_1$ used to build a basic safety annulus

4.4 Vertices of the first Newton-lens polygon

First let $\rho = x_0^{r_1}, r_1 > 1$ an enlargement rate such that $-1/r_1$ is not a slope of a side of the first Newton-lens polygon. Performing the change of variables (Formula 1) we get:

$$\begin{cases} \Phi_1^*(\omega)(X_1,Y_1,Z_1) = A(x_0 + x_0^r X_1, x_0^r Y_1, x_0^r Z_1)dX_1 \\ \qquad\qquad + B(x_0 + x_0^r X_1, x_0^r Y_1, x_0^r Z_1)dY_1 \\ \qquad\qquad + C(x_0 + x_0^r X_1, x_0^r Y_1, x_0^r Z_1)dZ_1 \\ \qquad\qquad = x_0^\theta \omega_1(X_1,Y_1,Z_1) + o(x_0^\theta), \quad \text{with} \\ \omega_1(X_1,Y_1,Z_1) = A_1(X_1,Y_1,Z_1)dX_1 + B_1(X_1,Y_1,Z_1)dY_1 \\ \qquad\qquad + C_1(X_1,Y_1,Z_1) \end{cases} \tag{4}$$

where A_1, B_1 and C_1 are homogeneous polynomials in the variables Y_1 and Z_1 of the same degree which depend only on the vertex of the Newton-lens polygon "first" touched by an affine line of slope $-1/r$. Indeed, the degrees in ρ of the monomials of $A(x_0, \rho Y_1, \rho Z_1)$, $B(x_0 + \rho X_1, \rho Y_1)$, $C(x_0 + \rho X_1, \rho Y_1, \rho Z_1)$ are the same as the degrees of the homogeneous polynomials in Y_1, Z_1.

Then the plane field $ker\,\omega_1$ is constant on planes projecting on lines containing the origin of the (Y_1, Z_1)-plane. It defines a Gauss map $G_{\omega_1}(X_1,Y_1,Z_1) \mapsto ker\,\omega_1(X_1,Y_1,Z_1)$ In this case image of G_{ω_1} is of dimension 1 in $\mathbb{CP}^2$.

Definition 4.4.1. - *Given $x_0 \neq 0 \in \mathbb{C}$, a positive constants $1 < a$, the zone $Z_a(x_0) \subset \{x = x_0\} \subset \mathbb{C}^3$ is the set of points $|(y,z)| < x_0^a$. The zone $Z_{Ox,a}$ is the union $\bigcup_{x_0 \in \mathbb{C}\setminus 0} Z_a(x_0)$*

- Given $x_0 \neq 0 \in \mathbb{C}$, positive constants $1 < a < b$, the zone $Z_{a,b}(x_0) \subset \{x = x_0\} \subset \mathbb{C}^3$ is the set of points $x_0^b < |(y,z)| < x_0^a$. The zone $Z_{Ox,a,b}$ is the union $\bigcup_{x_0 \in \mathbb{C}\setminus 0} Z_{a,b}(x_0)$

The zone $Z_{Ox,a}$ looks like a thinned cone.
The zone $Z_{Ox,a,b}$ looks like a thinned cone on an annulus.
Both admit a tangent line at the origin: the x-axis.

Observe that any vertex, but the upper left one, of the Newton-lens polygon is the boundary of two sides of slopes $-(1/(r_{low})) > -(1/(r_{sup}))$. Lemma 4.4.3 describes the limit, when $|x_0| \to 0$ of the plane-field $ker\,\omega$ in the region $Z_{Ox,r_{sup}-\delta,r_{low}+\delta}$, δ is supposed to be very small compared to r_{sup} and r_{low}. We will call the region $Z_{Ox,r_{sup}-\delta,r_{low}+\delta}$, δ, a *safety funnel*; this funnel has a very thick side and a very thin hole.

Let us concentrate on the upper vertex of the union of sides of slope strictly between -1 and 0 (see Figure 9). The support lines touching the Newton-lens polygon at this vertex define a

zone Z_{0x,a_1,b_1}. The form ω_1 has coefficients A, B, C which are homogeneous polynomials in X_1 and Z_1 and define a Gauss map of image G of dimension 1. Therefore the image of the zone Z_{0x,a_1,b_1} by the Gauss map of the form ω is contained in a neighborhood of G. Getting closer to the origin makes the neighborhood of G as thin as desired.

Remark 4.4.2. *Given a neighborhood V_0 of $G \subset \mathbb{CP}^2$, any polar curve $\Gamma_h, h \notin V_0$ having a point in Z_{Ox,b_1} is trapped in Z_{Ox,b_1} when approaching the origin. In particular it is tangent to Ox at the origin. Therefore it will look closer and closer, in any C^k-topology, to a line parallel to Ox when x_0 goes to zero.*

As polar curves are analytic curves, the limits of the enlargement of such a polar curve trough a Newton-lens will therefore always be a line parallel to the x-axis.

The definition of polar curves implies already that the plane field $ker\,\omega_1$, that we will construct using the Newton-lens induction if it ends at the first step do not depend on the variable X_1. More precisely

Lemma 4.4.3. *- Through a lens of magnifying ratio x_0^{-r}, $(r_{sup} > r > r_{low})$ centered on the x-axis at a point close enough from the origin, where $-1/r_{low}$ and $-1/r_{sup}$ are the slopes of the adjacent sides to a vertex on the first Newton-lens polygon ($r_{sup} = 1$, if one adjacent side is of slope (-1) or if one extremity belongs to the ρ_1-axis (we do not consider the case $r_{low} = \infty$ corresponding to a point on the x_0-axis) we observe a region around the x-axis where, except along the x-axis, the plane-field looks, through the lens, invariant by translations parallel to the x-axis. The image of the Gauss map of the limit plane-field is of dimension* 1.

Proof: A vertex of the first Newton-lens polygon corresponds to terms of the form $x_0^p \rho_1^q$. The coefficients of the terms of the form $x_0^p \rho_1^q$ are homogeneous polynomials a priori in the new variables X_1, Y_1 and Z_1. The coefficients of the 1-form ω_1 depend in fact only in the variables Y_1 and Z_1 (see Proposition 4.3). In other terms the variable X_1 does not appear (see Figure 8, as the picture is the same for the Newton-lens polygon of a polynomial f or of the three coefficients (A, B, C) of a 1-form ω together. This proves again that the limit plane-field is invariant by translation in the X_1 direction. □

4.5 First step, $\omega = df$

Lemma 4.5.1. *When $\omega = df$, choosing an enlarging rate given by a support line of the first Newton-lens polygon containing only one vertex, the foliation looks through the lens more and more, when x_0 tends to zero, like the product of homogeneous foliation of the (Y_1, Z_1)-plane and the x-axis.*

Remark 4.5.2. *The separatrices (see definition 7.5.1) of the homogeneous foliation defined in the Y_1, Z_1-plane are lines containing the origin. Their products with the x-axis are pieces of the limit of the enlargement of $f = 0$ contained in the domain enlarged by the Newton lens.*

Proof: Proof of the lemma When $\omega = df$, a vertex of the first Newton-lens polygon not situated on the x_0-axis cannot correspond to a term $a_{p,q,\ell} x^{p-1} y^q z^\ell$ of $\partial f/\partial x$ as $\partial x^p y^q z^\ell/\partial y$ and/or $\partial x^p y^q z^\ell/\partial z$ would create a term below the support line containing the vertex. Then, the plane field defined by the form ω_1 is the product of a line field defined in the (Y_1, Z_1) plane by the x-axis (a priori, the planes of $ker\,\omega_1$ need not to be parallel to the x-axis when $\omega \neq df$.). □

Corollary 4.5.3. *When $\omega = df$ there is, associated to each vertex of the Newton-lens polynomial, for any $\delta < (r_{sup} - r_{low})/2$, a "security funnel" centered on the x-axis, crossing planes transverse to the x-axis in annuli of inner radius $x_0^{r_{low}+\delta}$ and outer radius $x_0^{r_{sup}-\delta}$ where the foliation defined by ω looks through a lens of strength $x_0^{r_{sup}-\delta}$ like a product of a transverse line-field and the x-axis.*

Remark 4.5.4. *When $\rho_1 = x_0^r, r > 1$ is an enlargement rate such that $-1/r$ is not a slope of a side of the first Newton-lens polygon, the limit Gauss map seen through a Newton lens is constant on the planes projecting on rays of the (Y_1, Z_1)-plane, and therefore is of rank less or equal to 1. Therefore a necessary condition in order to get a map*

$$(X_1, Y_1, Z_1) \to [A_1(X_1, Y_1, Z_1), B_1(X_1, Y_1, Z_1), C_1(X_1, Y_1, Z_1)] \in \mathbb{CP}^2$$

of rank 2 is that $\rho_1 = x_0^{r_1}$ where $-1/r_1$ is the slope of an edge of the first Newton-lens polygon of ω.

4.6 First step, $\omega = df$, polynomial profile

From now on, we suppose that $r_1 = p/q$, where $s = -q/p$ is a slope of an edge Δ of the Newton-lens polygon. It gives, after enlargement, a differential form (see Equation 4)

$$\omega_1 = A_1(Y_1, Z_1)dX_1 + B_1(Y_1, Z_1)dY_1 + C_1(Y_1, Z_1)dZ_1.$$

Notice that at least one of the coefficients $A(x, y, z)$, $B(x, y, z)$, $C(x, y, z)$ of dx, dy, dz in ω should contain a monomial of the form x^n, otherwise the whole x-axis would be a set of singular points.

The coefficients A_1, B_1, C_1 of dX_1, dY_1, dZ_1 in ω_1 need not a priori to contain a constant term. Let us suppose that a monomial $a_{n,0,0}x^n$ of $A(x, y, z)$ generates a monomial $a_{n,0,0}x_0^n$ of $A(x_0 + \rho X_1, \rho_1 Y_1, \rho_1 Z_1)$ such that the Newton-lens polygon of ω contains the point $(n, 0, 0)$. This point is the extremity of an edge of the Newton-lens polygon of slope $-(1/r_1), r_1 > 1$. Then A_1 contains a constant term. Therefore we get

$$dX_1 = \frac{B_1}{A_1}dY_1 + \frac{C_1}{A_1}dZ_1.$$

A solution of the system of partial differential equations

$$\frac{\partial X_1}{\partial Y_1} = -\frac{B_1(Y_1, Z_1)}{A_1(Y_1, Z_1)} \; \frac{\partial X_1}{\partial Z_1} = -\frac{C_1(Y_1, Z_1)}{A_1(Y_1, Z_1)} \tag{5}$$

is a profile (see Definition 3.2.3).

Otherwise the induction should be continued.

When ω if integrable, if $A_1(Y_1, Z_1)$ is non zero, then the partial differential equations 5 provide a foliation, as its solution is a limit of leaves of foliations. In particular this is the case when $\omega = df$.

Theorem 4.6.1. *When $\omega = df$, $f(x, y, z)$ a polynomial with an isolated singularity at the origin such that the x-axis is a singular ray of the tangent cone of $\omega = df$, when the first step of the induction provides a profile, lenses of strength $x_0^{-(r_1)}$ centered at $(x_0, 0, 0)$ provide, when $x_0 \to 0$, a polynomial profile, that is the leaves of $\mathcal{F}_{df}$ look as parallel graphs of the form $X_1 = P_1(Y_1, Z_1) + c$.*

We will use the notation $d_{y,z}f = \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$.

On Figure 10, left, the red bullets would correspond to a term of f providing a term of $d_{y,z}f$ on a side or slope $-(1/r)$, $r > 1$ of the Newton-lens polygon, and a term of $d_{yz}f$ under the side of the Newton-lens polygon.

On Figure 10, right, the red bullets correspond to a term of f providing a term of $d_{y,z}f$ on the x_0-axis of the Newton-lens polygon. The following lemma repeat with words the idea conveyed

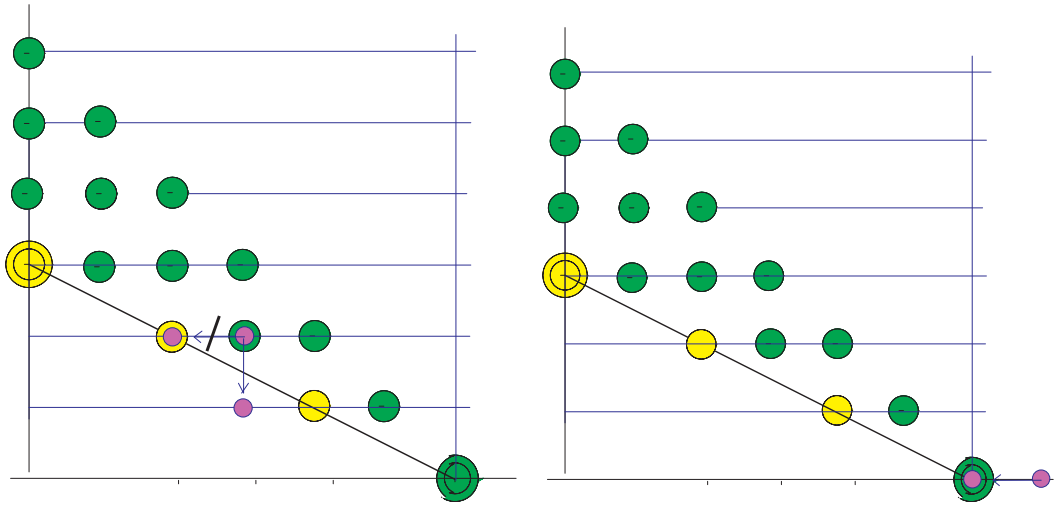


Figure 10: Impossible and possible contributions when $\omega = df$

by Figure 10 and proves Theorem 4.6.1.

Lemma 4.6.2. *If $\omega = df$, and if the induction ends at the first step, then one of the coefficients $B_1(Y_1, Z_1)$ of dY_1 or $C_1(Y_1, Z_1)$ of dZ_1 in ω_1 is not zero and A_1 is reduced to a x_0^n term.*

Proof. A monomial $ax^py^qz^\ell$ of f provides a monomial $pax^{p-1}y^qz^\ell$ of $\dfrac{\partial f}{\partial x}$ and monomials $qax^py^{q-1}z^\ell$ and $\ell ax^py^qz^{\ell-1}$ of $d_{yz}f$, except if $q = \ell = 0$. Then all the points of the edge Δ of the Newton-lens polygon but for the right extremity if it is on the Ox-axis come from a monomial of $\dfrac{\partial}{\partial y}$ or $\dfrac{\partial}{\partial z}$. Indeed, all the points corresponding to monomials of $\dfrac{\partial f}{\partial x}$ are strictly above the edge Δ of slope in $]-1, 0]$, except maybe the last point on the right if it is on the x_0 axis. □

Let us now prove a result analogous to the 2-dimensional one proved in Rouillé's thesis [Rou2].

Proposition 4.6.3. *The graph of any polynomial in the variables (Y_1, Z_1) can be a profile obtained at the first step of Newton-lens induction.*

Proof: It suffices to consider the polynomial

$$f(x, y, z) = a_n(y, z) + a_{n-1}(y, z)x^2 + \cdots + a_{n-k}(y, z)x^{2k} + \cdots a_1(y, z)x^{2(n-1)} + x^{2n-1}.$$

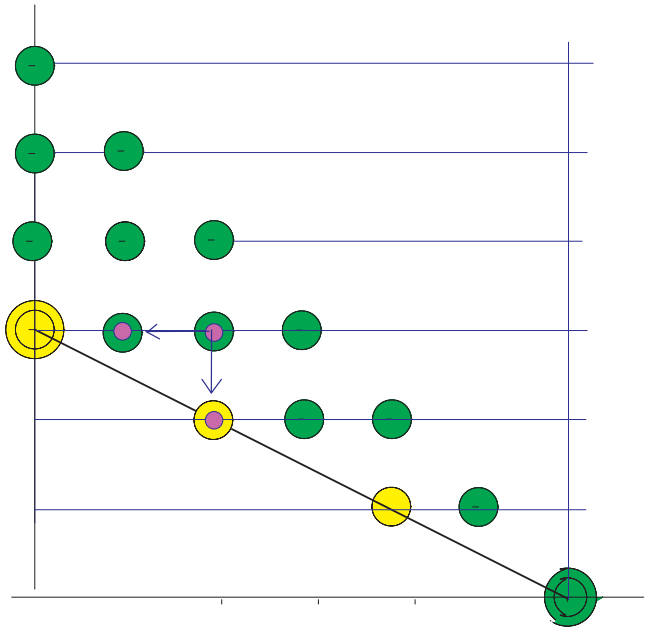


Figure 11: The contributions of a term of f.

where the a_i are homogeneous polynomials of degree i in the variables (y,z). We again use the notation $d_{y,z}$ which follows Theorem 4.6.1.

Then

$$\begin{aligned} df = {} & d_{y,z}a_n(y,z) + d_{y,z}a_{n-1}(y,z)\cdot x^2 + \cdots + d_{y,z}a_{n-k}(y,z)\cdot x^{2k} \\ & + \cdots + d_{y,z}a_1(y,z)x^{2(n-1)} \\ & + 2xa_{n-1}(x,y)dx + \cdots + 2kx^{2k-1}a_{n-k}(y,z)dx \\ & + \cdots + (2n-1)x^{2n-2}. \end{aligned}$$

The Newton-lens polygon is therefore analogous to Rouillé's example (see Figure 11). □

4.7 Merle-type results (1)

Proposition 4.7.1. *In $\mathbb{C}^3$, when $\omega = df$ and when only one step is needed to obtain a profile, a bouquet of generic polar curves containing the profile stays apart from the singular level $f = 0$.*

Remark 4.7.2. *Of course, Proposition 4.7.1 is valid in $\mathbb{C}^2$. In that case, it writes in terms of contact order: the contact order of two curves of a bouquet is larger than the contact order of a curve of the bouquet and the curve $f = 0$.*

Proof: For that, let us compare the Newton-lens polygons of $f(x_0+\rho_1X_1,\rho_1Y_1,\rho_1Z_1)$ and of $df(x_0+\rho_1X_1,\rho_1Y_1,\rho_1Z_1)$.

The slope of the bottom segment of the Newton-lens polygon of f ($3/2$ in Figure 12) is (in absolute value) bigger than ($1/2$ in Figure 12) the slope of the bottom side of the Newton-lens polygon of df.

This means that a renormalization ball centered at $(x_0,0,0)$ catching the profile (radius $\simeq x_0^2$) is much smaller that a renormalization ball centered at $(x_0,0,0)$ catching the singular level $f=0$ (radius $\simeq x_0^{3/2}$). In other terms, Merle's bouquet of polar curves corresponding to the tangent

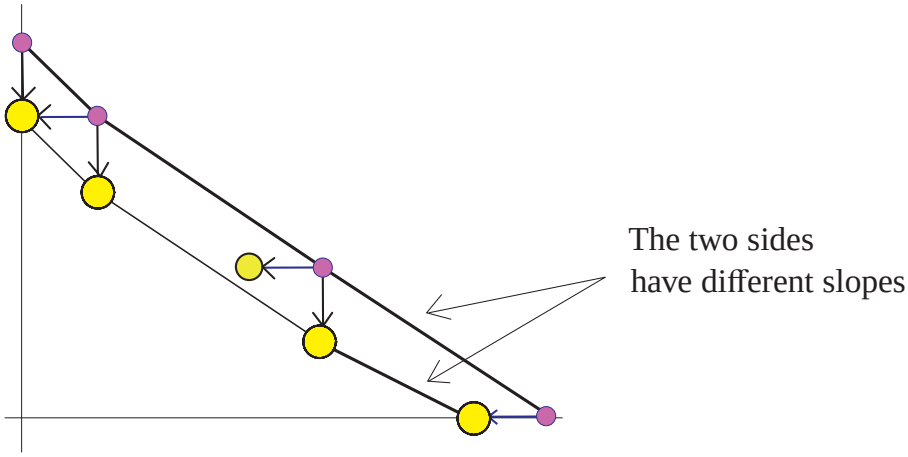


Figure 12: Comparison of Newton-lens polygons of f (purple) and df (yellow); the tails are not represented.

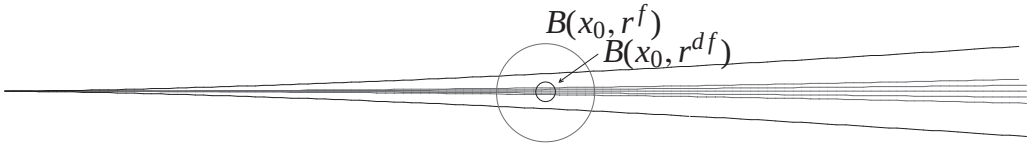


Figure 13: Comparison of $\{f = 0\}$ and a bouquet of polar curves.

directions at points of balls whose limits will provide the profile is "far" from the singular level $f = 0$. □

When $\omega = df$, the slope of the line joining terms coming from a term $x^p y^q z^\ell$ of $\partial f/\partial x$ as $\partial x^p y^q z^\ell/\partial y$ and/or $\partial x^p y^q z^\ell/\partial z$ would create a term below the support line containing the vertex. In other words, the bottom side of the Newton-lens polygon of df is the only one which is not parallel to a side of the Newton-lens polygon of f. The only way to get a side of $\mathcal{N}(df)$ non parallel to a side of $\mathcal{N}(f)$ is when the vertex on the x-axis of $\mathcal{N}(df)$ comes from $\frac{\partial}{\partial x}(f)$. When $\omega = df$ the Gauss image G of Z_{Ox,r_a,r_b} is a neighborhood of the set of h containing the x-axis which is a projective line in $\mathbb{CP}^2$. In that case, the bouquets of strands of polar curves Γ_h, h not too close from the line D_x of $\mathbb{CP}^2$ of planes containing Ox, look more and more like parallel lines, the leaves of $\mathcal{F}_{\omega_\ell}$ are graphs of polynomials.

5 Further steps

5.1 Example with more than one step

We give now one example, in $\mathbb{C}^2$, of a profile obtained with two steps of induction.

Example 5.1.1. *This example is certainly the shortest possible (three terms) :*

$$f(x,y) = y^2 + x^2y + x^4.$$

*Its first Newton-lens polygon (figure (**??**) on the left) has only one side. The differential form* $\omega_1 = A_1(Y_1)dX_1 + B_1(Y_1)dY_1$ *is* $\omega_1 = (2Y_1 + 1)dY_1$.

A second step is needed with the root $c = -1/2$. *The second Newton-lens polygon is shown*

*in Figure (**??**) on the right. It gives the differential equation :*

$$\frac{dX_2}{dY_2} = -\frac{2Y_2}{3}.$$

The integral curves of the profile are parabolas.

5.2 The general construction

We will detail the second step of the induction. We need to prove that the terms A_2, B_2, C_2 of $\omega_2 = A_2dX_2 + B_2dY_2 + C_2dZ_2$ do not depend on the variable X_2. The further steps, when needed, are analogous.

Let us now suppose that $A_1(Y_1, Z_1) = 0$. Let $\rho_1 = x_0^{r_1}$.

After dividing $\omega(x_0 + \rho_1 X_1, \rho_1 Y_1, \rho_1 Z_1)$ by the maximal power of x_0, we get a one-form in the variables X_1, Y_1, Z_1

$\tilde{\omega_1} = \tilde{A_1}dX_1 + \tilde{B_1}dY_1 + \tilde{C_1}dZ_1$

and, considering only the term ω_1 of $\tilde{\omega_1}$ which does not contain a factor x_0^s, as $A_1(Y_1, Z_1) = 0$,

$\omega_1 = B_1dY_1 + C_1dZ_1$

In the (Y_1, Z_1)-plane, ω_1 defines a one-dimensional foliation with singular points the roots (y_i^1, z_i^1) of $(B_1 = C_1 = 0)$

After choosing a root (y_i^1, z_i^1) of $B_1 + C_1 = 0$, let us perform the second change of variables

$$\begin{aligned} X_1 &= \rho_2 X_2 \\ Y_1 &= y_i^1 + \rho_2 Y_2 \\ Z_1 &= z_i^1 + \rho_2 Z_2 \end{aligned}$$

We get a Newton-lens cloud and a Newton-lens polygon, now with vertices in $\mathbb{Q}^+ \times \mathbb{N}$, plotting the coefficient of form $\tilde{\omega_1}(\rho_2 X_2, y_i^1 + \rho_2 Y_2, z_i^1 + \rho_2 Z_2)$; the horizontal axis corresponds to exponents of x_0 and the vertical axis to exponents of ρ_2.

Let $\rho_2 = x_0^{r_2}$, r_2 any positive rational number. In particular we will soon chose it using the slope of a side Δ of the second Newton-lens polygon.

We get

$$\left\{ \begin{array}{ll} \Phi_2^*(\widetilde{\omega_1})(X_2, Y_2, Z_2) & = x_0^{r_2}\widetilde{\omega_2} = x_0^{r_2}\omega_2(X_2, Y_2, Z_2) + o(x_0^{r_2}), \quad \text{with} \\ \omega_2(X_2, Y_2, Z_2) & = A_2(X_2, Y_2, Z_2)dX_2 + B_2(X_2, Y_2, Z_2)dY_2 \\ & +C_2(X_2, Y_2, Z_2)dZ_2 \end{array} \right. \tag{6}$$

Notice that $\widetilde{\omega_2}$ keeps terms of different orders in x_0. When dealing with the equation $f = 0$ we will use similarly the notations $\widetilde{f_2}$.

Let us first chose a vertex ν of the Newton-lens polygon. It corresponds to a term of the form $x_0^s Q_{A,2}$, $x_0^s Q_{B,2}$ or $x_0^s Q_{C,2}$ of $\widetilde{A}, \widetilde{B}$ or $\widetilde{C}$, where the polynomials $Q_{A,2}, Q_{B,2}$ or $Q_{C,2}$ are homogeneous of the same degree.

The plane-field is then invariant on rays. Therefore the rays are contained in the polar set at this scale.

The roots $(y_{i,1}z_{i,1})$ belong to the plane $x = x_0$

Moving the point x_0 we can follow the roots $(y_{i,1}z_{i,1})$. At the scale of the previous change of variable, and even more at the scale of the present one, they describe vertical lines.

Therefore the Gauss map associated to $\omega 2$ cannot contain an open set. Its image is therefore of dimension 1. The polar loci are then generically surfaces union of rays. The existence of two

different cones is also incompatible with a small change of x_0. The polar surfaces should then be planes containing the axis $0x$. This means that the direction $[A, B, C]$ depends only on its value on the Y_2, Z_2-plane where it is constant on rays.

We chose now an exponent r corresponding to the slope of a support line of the Newton-lens polygon at ν.

Choosing a side of the Newton-lens polygon ending on the x-axis if it exists, and if $A_2(x, y, z)$ contributes to this term, will provide a limit plane field. As before, we will prove that the limit plane-field is invariant by translations. Otherwise, we should continue the induction.

In both case we need to prove, first that the limit plane-field is invariant by translations along lines parallel to the X_2-axis, then that the polynomials A_2, B_2 and C_2 do not depend on the variable X_2.

Theorem 5.2.1. *The limit plane field is invariant by translations and therefore, when ω is integrable, the leaves of the foliation it defines are graphs, maybe with some vertical asymptotes, from the (Y_2, Z_2)-plane to the X_2-axis. We call* profile *one of these graphs.*

The reader can find in [La-Si] a less geometric proof in $\mathbb{C}^2$ of Theorem 5.2.1.

Remark 5.2.2. *In [GarBar-Te] Garcia- Barroso and Teissier compute the lowest exponent of the Puiseux serie of a strand of generic polar curve involving the slope of the line direction defining the polar curve in terms of the value λ defining the level $f = \lambda$. It gives both the size and the localization in terms of λ of a domain of our profile meeting the level $f = \lambda$.*

Proof: of theorem 5.2.1 Corollary 4.5.3 implies that there exists two values of $r_a > r_b > 1$ close to 1 which determine a zone Z_{Ox,r_a,r_b} (see Definition 4.4.1)where the planes $h = ker\,\omega$ define a Gauss map of very thin image G. Therefore a polar curve Γ_h, h not too close from G, which starts out of $Z_{Ox,(r_b-\delta)}$ (see also Definition 4.4.1) cannot cross Z_{Ox,r_a}. The polar curve Γ_h, if it starts at a point of $Z_{Ox,r_a}, r_a > r_c > r_b$ has to stay in Z_{Ox,r_b} and therefore is tangent to the axis Ox at the origin. Then the strands of polar curves Γ_h, h not too close from G, form a bouquet, and look through a lens of strength $x_0^r, r_a > r > r_b$, more and more like parallel lines when $|x_0| \to 0$. Therefore the plane field defined by $ker\,\omega_2$ is invariant by translation along the X_2-axis. Then, when ω is integrable, so is ω_2 and the leaves of ω_2 are graphs (we accept vertical asymptotes) of functions not depending on the variable X_2. □

Remark 5.2.3. *Remark 4.4.2 shows that the consideration of the larger zone Z_{Ox,a_1,b_1} is already sufficient to trap the polar branches.*

In order to continue the induction we need a more "algebraic" result.

Theorem 5.2.4. *The polynomials $A_2(X_2, Y_2, Z_2)$, $B_2(X_2, Y_2, Z_2)$ and $C_2(X_2, Y_2, Z_2)$ do not contain the variable X_2.*

Proof: We know that the plane field $ker\,\omega_2$ is invariant by translations parallel to the X_2-axis, therefore the point $[A_2(X_2, Y_2, Z_2), B_2(X_2, Y_2, Z_2), C_2(X_2, Y_2, Z_2)]$ does not depend of X_2 but A_2, B_2, C_2 may a priori depend on X_2.

Let us consider some (fixed) value $Y_{2,0}$ and $Z_{2,0}$ of Y_2 and Z_2. The polynomial $A_2(X_2, Y_{2,0}, Z_{2,0})$, if it is not constant, should have some root $X_{2,0}$. If one of the polynomials $B_2(X_{2,0}, Y_{2,0}, Z_{2,0})$ and $C_2(X_{2,0}, Y_{2,0}, Z_{2,0})$ is not zero, it is also not zero in a neighborhood of $(X_{2,0}, Y_{2,0}, Z_{2,0})$. Then the plane $ker\,\omega_2$ has a position which is different in the neighborhood of the point $X_2 \in$

the line $\{Y_2 = Y_{2,0}, Z_2 = Z_{2,0}\}$ and at the point $X_{2,0} \in$ the line $\{Y_2 = Y_{2,0}, Z_2 = Z_{2,0}\}$. Therefore the polynomial $A_2(X_2, Y_{2,0}, Z_{2,0})$ should be constant. Similarly B_2 and C_2 should be constant on the line $\{Y_2 = Y_{2,0}, Z_2 = Z_{2,0}\}$, unless the three polynomials A_2, B_2, C_2 are simultaneously zero on the line $\{Y_2 = Y_{2,0}, Z_2 = Z_{2,0}\}$.

Notice that one or two of the constant values of $A_2(X_2, Y_{2,0}, Z_{2,0})$, $B_2(X_{2,0}, Y_{2,0}, Z_{2,0})$ or $C_2(X_{2,0}, Y_{2,0}, Z_{2,0})$ maybe zero.

Let us now suppose that the three values
$A_{2,0}(X_{2,0}, Y_{2,0}, Z_{2,0})$, $B_{2,0}(X_{2,0}, Y_{2,0}, Z_{2,0})$ and $C_{2,0}(X_{2,0}, Y_{2,0}, Z_{2,0})$ are zero.

A priori the set $\Sigma_2 = \{A_2 = B_2 = C_2 = 0\}$ is an algebraic set.

If a zero $m_2 \in \Sigma_2$ of the map

$$(A_2, B_2, C_2) : \mathbb{C}^3 \to \mathbb{C}^3$$

is isolated, it is of non-zero index. Therefore, for x_0 small enough, the map $(\widetilde{A_2}, \widetilde{B_2}, \widetilde{C_2}) : \mathbb{C}^3 \to \mathbb{C}^3$ (see Equation 3 and the beginning of Subsection 5.2 for a definition of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$) has also a zero in a neighborhood of $(X_{2,0}, Y_{2,0}, Z_{2,0})$. This contradicts the fact that the origin in an isolated singular point of ω.

Otherwise the algebraic set Σ_2 may contain curves and surfaces. The curves cannot be only lines parallel to the X_2-axis as some common zeros of (A_2, B_2 and C_2) are isolated on the line parallel to the X_2-axis trough it.

Let us now consider a regular point m_2 of an arc α_2 contained in Σ_2, and a transverse holomorphic disc D_2 to the arc at the point. It should contain an isolated zero of the map $(B_2, C_2) : D_2 \to \mathbb{C}^2$. For the same reason as above the map $(\widetilde{B_2}, \widetilde{C_2}) : D_2 \to \mathbb{C}^2$ should have a zero close to m_2. Moving the point m_2 on a neighborhood of m_2 on α_2 we get a piece of the polar curve Γ_h, $h = (OY_2, OZ_2)$. It should have Ox as limit tangent at the origin. Therefore it should look more and more parallel through the second lens to the OX_2 axis.

Let us now consider a regular point m_2 of a surface S_2 contained in Σ_2 where the tangent plane does not contain OX_2. The polar curve trough m_2 cannot have a tangent having a limit in $T_{m_2}S_2$ providing a contradiction. □

Remark 5.2.5. *Using a Newton-lens providing a profile, we do not loose the polar branches going to the limit $x_0 \to 0$. When a branch of the Newton-lens induction ends at a vertex, the situation is different. Nevertheless the limit* 1*-form is still independent of the last X-variable say X_p. The conclusion of Remark 4.4.2 are still valid. Therefore, in any C^k topology the dependence of the form ω_p on the variable X_p, considering derivatives of order up to k, is going to zero. The limit form, which is polynomial, therefore does not depend on the variable X_p. This is true although the Gauss map of the limit form ω_p is of rank strictly less that* 2 *as the limit polar loci are planes containing the X_p-axis.*

If needed, the following steps of the induction provide forms $\omega_p = A_p dX_p + B_p dY_p + C_p dZ_p$ where the polynomial A_p, B_p and C_p do not depend on the variable X_p.

The induction stops when $A_p \neq 0$.

Remark 5.2.6. *The profiles, up to linear isomorphism, are analytic invariants of the isolated singularity. In dimension* 2*, Zariski defined analytic invariant of an irreducible curve of equation $f(x, y) = 0$ using relevant coefficients of the Puiseux expansion of the curve (see [Za]). The dimension of Zariski's module is larger that the dimension provided by the profiles. It would be interesting to understand which information the profiles give about Zariski's invariants.*

6 What does Newton lenses tell about the level $f = 0$?

6.1 The singular level $\{f = 0\} \subset \mathbb{C}^2$

Remark 6.1.1. *We know that f is reduced, as a multiple factor will introduce a curve of singular points. The induction should be stopped when the induction concerning df shows a profile. We may as well suppose that f is irreducible, and study the possible component one by one.*

The tangent cone of the singular level $\{f = 0\} \subset \mathbb{C}^2$ is where the homogeneous polynomial sum of the terms of lowest degree of f is zero; it is a finite union of lines. The interesting lines for us are the lines of multiplicity at least 2.

The change of variables Φ_1 is defined by

$$x = x_0 + \rho_1 X_1$$
$$y = \rho_1 Y_1$$

Let us chose one side of slope $-1 < 1/r_1 < 0$ of the Newton-lens polygon of f (case 1) or a vertex bounded by two sides of slope $1/r0,1$ and $1/r0,2$ $-1 < 1/r_{0,1} < 1/r_{0,2} < 0$ (case 2).

Then, taking $\rho_1 = x_0^r$, $r = r_1$ in the first case, $1/r_{0,1} < 1/r < 1/r_{0,2}$ in the second case, consider

$$\widetilde{\Phi_1}^*(f)(X_1, Y_1) = f(x_0 + x_0^r X_1, x_0^r Y_1)$$

Proposition 6.1.2. *Choosing $\rho = x_0^r$ the sum of the terms corresponding to points of this side of the Newton-lens polygon provides, after factorizing the highest possible power of x_0, an equation $f_1(Y_1)$ (which does not depend of the variable X_1). Choosing a slope corresponding to a support line of a vertex of the first Newton-lens polygon, we get $f_1(Y_1) = Y_1^p$.*

Proof. Writing $f(x, y) = \sum a_{p,q} x^p y^q$, a non zero monomial $a_{p,q} x^p y^q$ gives a polynomial $a_{i,j}(x_0 + \rho_1 X_1)^p \rho_1^q Y_1^q$ in $f(x_0 + \rho_1 X_1, \rho_1 Y_1)$.

One point comes from the monomial $x_0^p \rho_1^q Y_1^q$.

The other points introduced by the terms of $a_{p,q}(x_0 + \rho_1 X_1)^i \rho_1^j Y_1^j$ containing X_1 form a *tail* of slope -1 on the left of (p, q, ℓ) (in black in Figure 8, here we apply the same reasoning to f). □

Two possibilities can occur:

- 1) the equation $f_1(Y_1) = 0$ has only simple roots.

- 2) some roots are multiple, more precisely, some factors of $f_1(Y_1)$ have an exponent strictly greater than 1; $f_1 = f_{1,1}^{a_1} \cdot f_{1,2}^{a_2} \cdot \cdots$.

Example with only simple roots $f(x, y) = x^3 - y^2$.

$\Phi_1^*(f) = (x_0 + \rho_1 X_1)^3 - (\rho_1 Y_1)^2$; Choosing $\rho_1 = x_0^{3/2}$, we get $f_1 = 1 - Y_1^2$.

Remembering that $y = x_0^{3/2} Y_1$ and following a circle of radius $|x_0|$ centered at the origin in the x-plane, we recover a Lê carrousel (see [Le]), and a trefoil knot in $S^1_{|x_0|} \times (y\text{-axis})$ (the y-axis is a complex line, that is of real dimension 2).

6.2 Multiple roots of $f_1(Y_1) = 0$

Example with multiple roots $f(x, y) = y^4 - 2x^3 y^2 - 4x^5 y + x^6 - x^7$ (example taken from [Bri])

From the construction of the Newton-lens polygon, we get again $\rho_1 = x_0^{3/2}$ and $f_1 = (Y_1^2 - 1)^2$; the two roots $Y_1 = \pm 1$ are double roots.

Next step provides $\rho_2 = x_0^{1/4}$ and $f_2(Y_2) = Y_2^2 - 1$, equation which have simple roots $Y_2 = \pm 1$.

When $f_1 = 0$) has multiple roots, we need to proceed and perform another change of variables Φ_2, with c_1 a roots of one of the multiples factors $f_{1,i}^{a_i}$.

$$X_1 = \rho_2 X_2$$
$$Y_1 = c_1 + \rho_2 Y_2$$

We get a polynomial in the variables x_0, ρ_2 from the expression $\Phi_2^* \widetilde{\Phi_1}^*(f)$. Choosing a side of the Newton-lens polygon indicates the enlarging rate $(1/\rho_2)$. If the polynomial obtained from the sum of the terms of $\Phi_2^* \widetilde{\Phi_1}^*(f)$ corresponding to point of the side of the second Newton-lens polygon, after factorizing the maximal power of x_0, has only simple roots the induction stops.

The form of the two successive changes of variables in the y-plane

$$y = \rho_1 Y_1$$
$$Y_1 = c_1 + \rho_2 Y_2$$

provides, following $x_0 = |x_0| e^{i\theta}$ in the x-plane, a two-step Lê carousel in the y-plane. It provides also an iterated torus knot, component of $\{f = 0\} \cap S^1_{|x_0|} \times y -$ plane. In general, the number of steps of the induction provides the number of iterations of the torus knot $\{(f = 0) \cap \mathbb{S}^3_\varepsilon\}$.

6.3 In $\mathbb{C}^3$, $\{f = 0\}$

The use of Newton lenses leads, when the tangent cone of $\{f = 0\}$ has only isolated singular rays, to a decomposition of the link $K = \{f = 0\} \cap \mathbb{S}^5_\varepsilon$ in components which are either $\mathbb{S}^1$-fiber bundles over an algebraic curves deprived of a finite number of discs or fiber bundles over $\mathbb{S}^1$, glued along tori.

The tangent cone of $\{f = 0\}$ defines a curve C_{tang} of $\mathbb{CP}^2$ with isolated singularities $\{\sigma_i\}$; deleting small balls $B^4_{\sigma_i,\varepsilon}$ of radius ε centered at the singular points $\{\sigma_i\}$, we get a regular complex curve $C_{tang,\varepsilon}$. Its inverse image by the Hopf map $\mathbb{S}^5 \to \mathbb{CP}^2$ is a $\mathbb{S}^1$ fiber bundle A bounded by tori product of a Hopf fiber and a component of one of the (usual) links $\kappa_i = \mathbb{S}^3_{\sigma_i,\varepsilon} \cap C_{tang}$.

Let us now suppose as usual that the x-axis is an isolated singular ray of the tangent cone of $\{f = 0\}$. Consider the change of variables

$$\begin{aligned} x &= x_0 + \rho_1 X_1 \\ y &= \rho_1 Y_1 \\ z &= \rho_1 Z_1 \end{aligned} \tag{7}$$

Then

$\Phi_1^*(f)(X_1, Y_1, Z_1) = f(x_0 + \rho_1 X_1, \rho_1 Y_1, \rho_1 Z_1)$.

Lemma 6.3.1. *Factorizing the maximal power of x_0 from the sum of the terms corresponding to a side of the Newton-lens polygon of slope $-1/r_1$ and choosing $\rho_1 = x_0^{r_1}$, we get a polynomial $f_1(Y_1, Z_1)$ which does not depend on the variable X_1.*

The proof is the same as the proof of Proposition 4.3.1 (see Figure 8).

Remark 6.3.2. *The vertices of the Newton-lens polygon also give rise to an homogeneous polynomial f_1 of the two variables Y_1 and Z_1 only. The equation $f_1 = 0$ represents in this case a finite number of lines.*

Corollary 6.3.3. *The knot K is, when f is irreducible, when the singular rays of the tangent cone of f are isolated , and when one step is sufficient to provide a smooth non-multiple transverse profile, obtained from a fiber space A over $C_{tang,\varepsilon}$ gluing along the boundary tori of A a fiber space over $\mathbb{S}^1$ (a circle in a complex line parallel to the x-axis) with fiber an algebraic curve deprived of a finite number of discs (a bounded part of an algebraic curve of $\mathbb{C}^2$).*

7 Further steps, polynomial profiles

7.1 The second step for the level $f = 0$

Recall that the x-axis is singular for ω.

Chose first a regular point (c_1, c_2) of the algebraic curve of equation $f_1(Y_1, Z_1) = 0$.

Let us prove the

Proposition 7.1.1. *When the point (c_1, c_2) is regular, the function $f_2(X_2, Y_2, Z_2)$ is cylindrical, that is, depends only on the variables Y_2 and Z_2.*

Proposition 5.2.4 shows that the assertion is true for df. In fact, the weaker statement: the plane field $ker\,\omega_2$ is invariant by translation parallel to the X_2-axis is sufficient to prove Proposition 7.1.1. We know also that the level $f_2 = 0$ need to be of the form $A_2 \times x$-axis, as it is the limit of approximations of the level $f = 0$ of f.

Proof: As we look at analytic curves tangent to the x-axis at the origin, after the second enlargement, a small neighborhood of a point will look more and more like a piece of the x-axis. Theorem 5.2.4 shows that the coefficients of the forms ω_2 do not depend on the variable X_2. The places where $ker\,\omega$ differs significantly from a direction containing Ox is a finite union of zones union of branches of polar curves of bouquets. A bouquet is visible through a Newton lens constructed at the last step of the Newton-lens induction, when the slope of the bottom right segment of the Newton-lens polygon of f is (negative and) smaller than the bottom right segment of the Newton-lens df (see Figure 12). Then the last enlargement obtained by an induction adapted to f does not catch the nearest bouquet and the previous ones do not see them as their limit is a line parallel to the Ox-axis above a root of the equations $(B_1 = C_1 = 0)$ (see the end of Subsection 5.2). Notice that we already know that $f_2 = 0$ can contain only lines parallel to the x-axis and that $(df)_2$ does not depend on X_2. Therefore f_2 cannot depend on the variable X_2. □

When the slope $-1 < s < 0$ of the support line is such that the support point is a vertex bounded by sides of slope $-1/r_{2,1}$ and $-1/r_{2,2}$ of a second Newton-lens polygon, the function f_2 is homogeneous in the two variables Y_2 and Z_2. In the plane $x = x_0$, the equation $f_2 = 0$ has then as solutions a finite number of lines through the origin.

Lemma 7.1.2. - *When the enlarging ratio $\rho_2 = x_0^{r_{2,1}}$, the regular points of $f_2(Y_1, Z_1)$ provide a transverse Lê carousels and therefore fibered pieces of a link K, defined by the action of S^1 obtained following a circle $S^1_{x_0} = \{x_o e^{i\theta}\}$.*

- a vertex bounded by sides of slope $-1 < s < 0$ provides 3-dimensional regions of the form (annulus) $\times\, \mathbb{S}^1$.

The arguments necessary to continue the induction, if required, are similar.

The hypothesis “isolated singularity” rules out the possibility of multiple components, therefore the induction stops when $f_p(Y_p, Z_p)$ has no multiple points. Simultaneously, following the same path of inductive steps, p is the step where a profile of $\omega = df$ appears (see Diagram 7.2).

Lemma 7.1.2 implies Theorem 7.1.3

Theorem 7.1.3. *Let f be an algebraic hypersurface with an isolated singular point at the origin. Suppose also that its tangent cone has only isolated singular rays. Let S^5_ε be the sphere centered at the origin of radius ε. When ε is small enough, the link $K = \mathbb{S}^5_\varepsilon \cap \{f = 0\}$ is obtained from the $\mathbb{S}^1$-fiber bundle A defined at the beginning of the subsection by gluing a sequence of fiber bundles over $\mathbb{S}^1$ of fiber the intersection of a complex curve deprived of neighborhoods of its singular points with a large 4-ball of $\mathbb{C}^2$, and at the end of each sequence, a fiber bundle over S^1 obtained following a circle $S^1_{x_0} = \{x_o e^{i\theta}\}$, and of fiber the intersection of an algebraic curve of $\mathbb{C}^2$ with no singularity and a large 4-ball .*

7.2 Polynomial profile when $\omega = df$

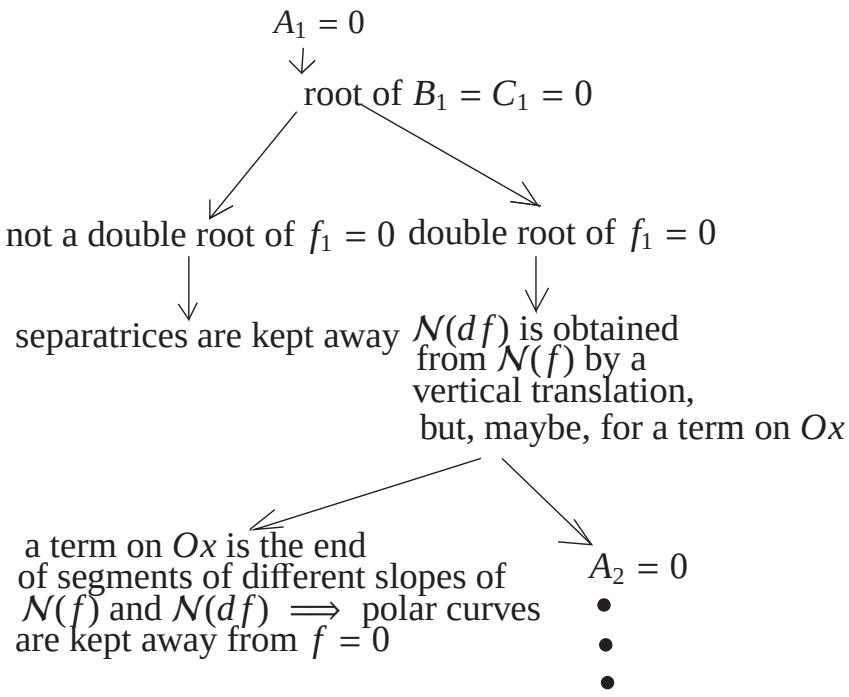
Theorem 7.2.1. *When $\omega = df$, f a polynomial with an isolated singularity at the origin and such that Ox is an isolated singular ray of the tangent cone, then profiles are graphs of polynomials.*

Theorem 7.2.1 is already proved in $\mathbb{C}^2$ (see [La-Si]).

Remark 7.2.2. *Theorem 7.2.1 is closely related to the position of the bouquets of generic polar curves with respect to the singular level $f = 0$, generalizing the 2-dimensional result of Merle (see [Me].*

Given a form ω, the profile may be given by differential equations with coefficient rational functions. The saddle-node of equation $\omega = ydx - x^2dy$ is already an example of this phenomenon in $\mathbb{C}^2$ (see figure **??**).

The diagram 7.2 is a scheme of the proof of Theorem7.2.1. The notation for Newton polygons is $\mathcal{N}$.



7.3 A proof of Theorem 7.2.1

Lemma 7.3.1. *The change of variables of the Newton-lens induction and differentiation commute when the points chosen in the (Y_p, Z_p)-plane used to continue the induction are the same for f_p and A_p. In other terms*

$$d_{X_p,Y_p,Z_p}\Phi_p \circ \varphi_{p-1} \circ \Phi_1 f = \Phi_p \circ \Phi_{p-1} \circ \Phi_1 d_{x,y,z} f$$

.

This fact is sufficient to compare efficiently the Newton-lens polygons of f and df when the induction needs more than one step. We need now at each extra step to compare the multiple roots of the equation $f_p(Y_p, Z_p) = 0$ and the roots of the system of equations $B_p(Y_p, Z_p) = 0$ and $C_p(Y_p, Z_p) = 0$.

The two-dimensional case will guide us. We had noticed in Section 6, observing Newton-lens polygon at different stages that the double roots of $f_p(Y_p) = 0$ are roots of $B_p(Y) = 0$.

Lemma 7.3.2. *Again, let us suppose that $\omega = df$.*

- When $A_p = 0$, the Newton-lens cloud of $\Phi_p \circ \Phi_{p-1} \circ \Phi_1 df$ can be obtained from the Newton-lens cloud of $\Phi_p \circ \Phi_{p-1} \circ \Phi_1 f$ using a vertical translation of vector $(0, -1)$ as the Newton-lens cloud of df can be obtained from the Newton-cloud of f when $A_1 = 0$.

- When $A_p \neq 0$ its only contribution to the Newton-lens polygon of $\widetilde{df_p}$ correspond to a point on the x-axis. This point is then the extremity of a segment of (negative) slope larger than the slope of the bottom segment of the Newton-lens polygon of $\widetilde{f_p}$.

When $\omega = df$, if the polynomial $A_1(X_2, Y_2, Z_2)$ is non-zero, it is reduced to a constant term. The proof is the same as in first step we considered parallel side of the first Newton-lens polygons of f and df. Moreover the double roots of f_1 are the roots of $B_1 = C_1 = 0$. Therefore $(\Phi_2 \circ \Phi_1)^* df = d(\Phi_2 \circ \Phi_1)^* f$. The only term of A_2 should then come from the term of $(\Phi_2 \circ \Phi_1)^* \frac{\partial f}{\partial x} f$ on the x_0-axis. We get a profile from the differential equations

$$\begin{array}{l} \frac{\partial X_2}{\partial Y_2} = B_2(Y_2, Z_2)/A_2 \\ \frac{\partial X_2}{\partial Z_2} = C_2(Y_2, Z_2)/A_2 \end{array}$$

when the polynomial A_2 is a non-zero constant. If $A_2 = 0$, we should continue the induction.

7.4 Merle-type results (2)

Theorem 7.4.1. *When $\omega = df$, the bouquets of polar curves tangent to the x-axis are separated from the singular level $f = 0$ by a security funnel.*

Proof: The computation above shows that the small ball of radius $|x_0|^{r_1+\ldots+r_p^{df}}$ containing points of the polar curves of the bouquet is much smaller that the ball of radius $|x_0|^{r_1+\ldots+r_p^f}$ containing points of $f = 0$ nearby. Then the bouquet of polar branches contained in the zone $Z_{Ox,r_p^{df}}$ is separated by a zone $Z_{Ox,r_p^f,r_p^{df}}$ from the component of $f = 0$ corresponding to the path leading to the profile corresponding to the bouquet of polar branches we a considering. □

The zone $Z_{Ox,r_p^f,r_p^{df}}$ (see Remark 4.4.2) guarantees also the separation of the bouquet of polar branches from components of $f = 0$ obtained by other paths, in particular if the component of $f = 0$ we want to avoid is not tangent to the x-axis at the origin.

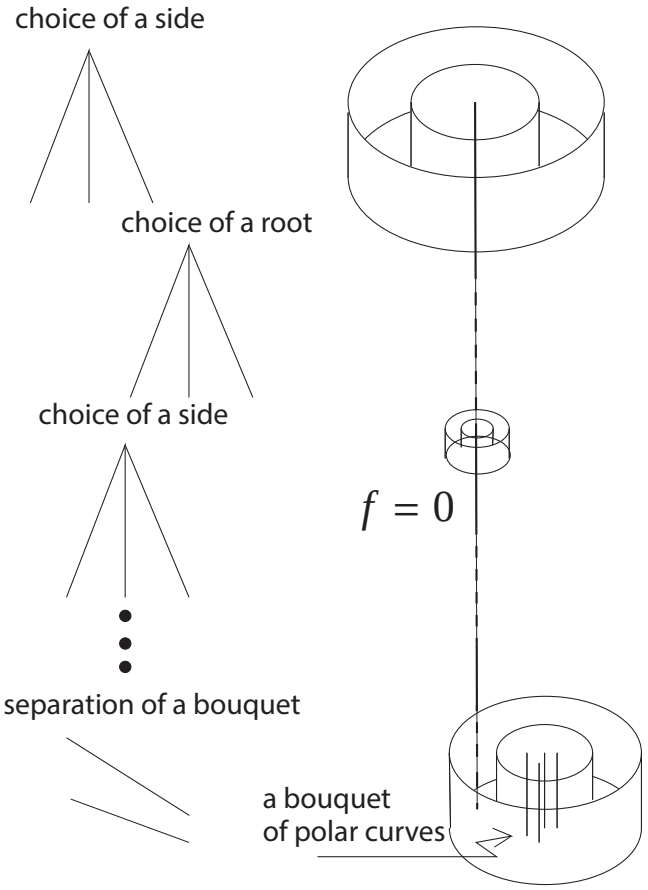


Figure 14: A bouquet of polar curves is separated from $f = 0$ by a security funnel

When, at the end of a branch of the Newton-lens induction tree, we get a profile, we simultaneously get a bouquet of branches of polar curves corresponding to directions in $\mathbb{CP}^2$ avoiding a neighborhood of the line of planes containing the x-axis. The diagram (see Figure 14) shows that bouquets of branches of polar curves corresponding to different branches of the Newton-lens induction tree are separated and separated from $f = 0$ by annular zones. This provide a "pedestrian" proof of Merle's result [Me].

Remark 7.4.2. *- The domains of these bouquets are good candidates to provide Lipschitz equivalence of the intrinsic and extrinsic distances on a level $f = \lambda$ (see [Neu-Pi2] and [Pi-Te]).*

- In $\mathbb{C}^2$ we can repeat Remark 4.7.2, giving an interpretation of Theorem 7.4.1 in terms of contact orders.

7.5 Separatrices

Definition 7.5.1. *A* separatrix *of ω is an analytic arc S ending at the origin such that*

$$\dot{S}(x(t), y(t), z(t)) \in ker\,\omega \quad \forall t.$$

In particular, when $\omega = df$, and when the origin is the only singular point and is contained in the level $f = 0$, the analytic curves contained in the level $f = 0$ ending at the origin are the separatrices.

In the saddle-node example, the x-axis is a separatrix. Notice that in this case the separatrix stays in the bouquet of polar curves Γ_h, h "far" from D_x.

In dimension 2, Rouillé (see [Rou3]) considered the role of separatrices in a particular case (quasi functions).

Definition 7.5.2. *The separatrices of a* 1*-form* ω *are* kept away from *the bouquets of polar curves* Γ_h*, h not too close from the curve* $D_x \subset \mathbb{CP}^2$ *of planes containing the x-axis, if, in a small enough neighborhood of the singular point, none of these bouquets contains a separatrix.*

Therefore, when ω is integrable, the "graph of polynomial" condition will be guaranteed if the separatrices tangent to the x-axis at the origin stay away from the very curved zones, or using Merle's ideas, are kept away from all the bouquets of polar curves Γ_h, h "far from" D_x.

8 Non-isolated singular rays of the tangent cone, an example

• $f(x,y,z) = x^4 - y^3 + z^2$ (see Figure 15).
$\omega = df = 4x^3dx - 3y^2dy + 2zdz$

The equation of the tangent cone is $z^2 = 0$, all the rays of this plane are degenerate.

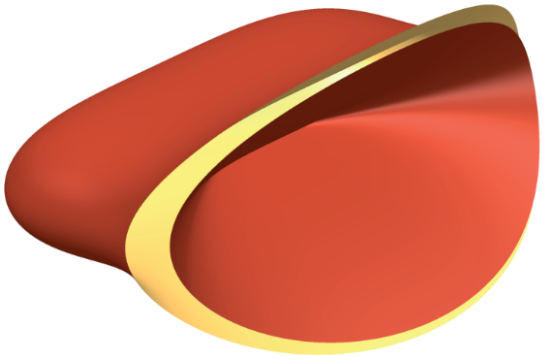


Figure 15: The surface of equation $x^4 - y^3 + z^2 = 0$ and a level $x^4 - y^3 + z^2 = \lambda$,

$f(x,y,z) = \lambda$. Let us compute x_0 and ρ_1 in terms of λ, using in fact [GarBar-Te] track.
$x_0^4 = \lambda$, $|x_0| = |\lambda|^{1/4}$, $|\rho_1| = |\lambda|^{3/4}$

Choosing the singular ray Oy , that is using the change of variables
$x = x_0 + \rho_1 X_1$, $y = y_0 + \rho_1 Y_1$, $z = \rho_1 Z_1$, we get
$\omega_1 = 4\rho_1^3 X_1^3 dx - 3(y_0 + \rho_1 Y_1)^2 dy + 2\rho_1 Z_1 dz$.
The Newton-lens polygon provides
$\omega_1 = -3dY_1 + 2Z_1 dZ_1$
Therefore

$$\frac{dY_1}{dZ_1} = (2/3)Z_1, \text{ so } Y_1 = \frac{Z_1^2}{3} + c$$

$f(x,y,z) = \lambda$. Let us compute y_0 and ρ_1 in terms of λ.
$-y_0^3 = \lambda$, $|y_0| = |\lambda|^{1/3}$, $|\rho_1| = |\lambda|^{2/3}$

In order to understand the "bag" shape of the levels $f = \lambda$, let us look at the critical points of the orthogonal projection of the levels on the tangent plane $z = 0$. These critical points have the

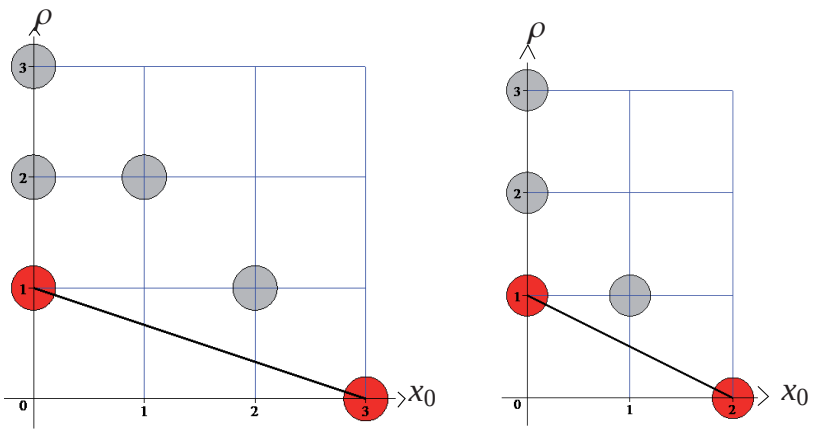


Figure 16: $f(x,y,z) = x^4 - y^3 + z^2$, Newton-lens polygon when $x_0 \in Ox$ and when $y_0 \in Oy$

equation $df(0,0,1) = 0$, that is $z = 0$. These critical points are all in the horizontal plane. The form $\omega_{x,y} = 4x^3dx - 3y^2dy$ defines a foliation, in fact the levels of the function $f_{x,y} = x^4 - y^3$, which are the projection of the critical locus of the restriction of the orthogonal projection on the horizontal ($z = 0$) plane of the levels $f = \lambda$. We can study the 2-dimensional function

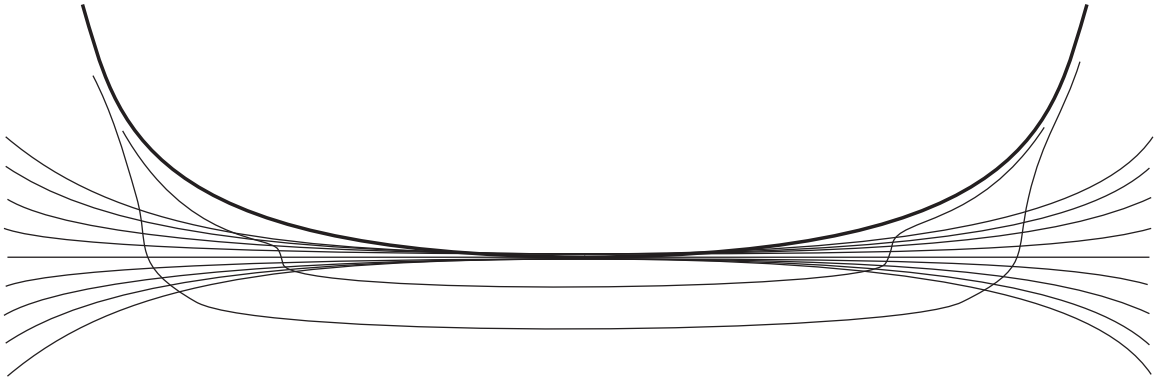


Figure 17: $f(x,y,z) = x^4 - y^3 + z^2$, real picture (sketch) of the critical locus of the projection on the (x,y)-plane of polar curves and of two real levels

$f_{x,y} = x^4 - y^3$. Its differential is $\omega_{x,y} = 4x^3dx - 3y^2dy$. The Newton-lens polygons of $f_{x,y}$ and $df_{x,y}$ indicate that the choice $\rho_1 = x_0^{3/2}$ will provide a profile.

We get the differential equation

$$\frac{dX_1}{dY_1} = \frac{3}{4}Y_1$$

providing the solutions

$$X_1 = \frac{1}{4}Y_1^3 + c$$

This explains the shape of the sides of the "bag". The flat bottom of the bag comes from the flat piece of the level through the point $(x_0, 0, 0)$, of equation $f_{x,y} = x_0^4$, which is at distance of the order $|x_0|^{4/3}$ of the origin.

Understanding the polar sets $V_\ell, \ell = H^\perp$, for planes H containing a non-isolated ray of the tangent cone of f, and the image by the orthogonal projection on H of the critical loci of the restriction to the levels $f = \lambda$ of the orthogonal projection on ℓ may help to understand better

the geometry of the neighborhood of an isolated singularity when the tangent cone have non-isolated singular rays. The symmetry of the example $f(x,y,z) = x^4 - y^3 + z^2$ with respect to the "horizontal plane" makes the study deceptively easy. The general case implies the study of the restriction of differential form ω to the surface V_ℓ, which makes sense even if ω is not of the form df.

This step will be generically unavoidable when we will study 1-forms in $\mathbb{C}^4$ or $\mathbb{C}^n$, $n \geq 4$.

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