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MASTER EQUATION FOR THE FINITE STATE SPACE PLANNING PROBLEM

CHARLES BERTUCCI ¹, JEAN-MICHEL LASRY ², PIERRE-LOUIS LIONS^{2,3}

ABSTRACT. We present results of existence, regularity and uniqueness of solutions of the master equation associated with the mean field planning problem in the finite state space case, in the presence of a common noise. The results hold under monotonicity assumptions, which are used crucially in the different proofs of the paper. We also make a link with the trajectories induced by the solution of the master equation and start a discussion on the case of boundary conditions.

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INTRODUCTION

This paper is concerned with the study of the mean field games planning problem in finite state space, in the presence of a common noise. The cases of a continuous state

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space and of generalized optimal transport in the presence of a common noise shall be treated in a future work.

General introduction. The planning problem is a mathematical formulation of a model in which an infinite number of non-atomic and identical players face a time dependent game in which the final distribution of players is constrained. It can be seen as a sort of mean field game (MFG for short). We present briefly the MFG theory for a better understanding of the planning problem. MFG are games in which an infinite number of small agents face a differential game in which the costs paid by the players only depends on aggregate quantities. In [8], the last two authors gave a precise mathematical framework to study such games. The main advantage of the game with infinite players in comparison to a N players game, for N large, is that Nash equilibria of the game can often be computed by solving a system of partial differential equations (PDE for short) with low dimensionality. When the noises affecting the players are independent and when the state space is continuous, a typical form taken by this system is the following

$$(1) \quad \begin{cases} (i) - \partial_t u - \nu \Delta u + H(x, \nabla u, m) = 0 \text{ in } (0, T) \times \mathbb{R}^d; \\ (ii) \partial_t m - \nu \Delta m - \operatorname{div}(D_p H(x, \nabla u, m)m) = 0 \text{ in } (0, T) \times \mathbb{R}^d; \\ m(0) = m_0; u(T) = G(m(T)) \text{ in } \mathbb{R}^d. \end{cases}$$

In (1), u denotes the value function of a generic player over the state space $(0, T) \times \mathbb{R}^d$, m the density of the players, $H(x, p, z)$ is the Hamiltonian associated to the game and $\nu \geq 0$ is a parameter describing the intensity of the noise. The density m_0 is the initial repartition of players and G is a function describing the final cost paid by the players. The interpretation of such a system, is that, under an anticipation for the density of players m , the problem of the players can be resumed by the Hamilton-Jacobi-Bellman equation (i). On the other hand, if all the players have the value function u , and play accordingly, the evolution of their density m is given by the Fokker-Planck equation (ii). A solution (u, m) of the coupled system (1) is thus interpreted as a Nash equilibria of the MFG.

In several practical applications, the notion of a final cost for the players is not relevant and the players behave as if the final density of players is constrained. The latter constraint is of course non-classical and raises different modeling problems. The most serious one being that the final constraint on the distribution of players cannot be imposed on those players, which, because they are non-atomic, cannot affect the distribution. Formally, the planning problem addresses the previous issue as follows : there are strong incentives in the cost functions of the players to induce a behavior which is consistent with the constraint. That is the players have strong incentives to "do their best" to help the final density to respect this (sort of) constraint.

The planning problem originates from the work of the last two authors and was first presented in the lectures of the third author at Collège de France [9]. When the noises affecting the players are independent and when the state space is continuous, a system of PDE has been established in [9] to characterize the equilibria of the planning problem.

The typical form of this system is

$$(2) \quad \begin{cases} -\partial_t u - \nu \Delta u + H(x, \nabla u, m) = 0 & \text{in } (0, T) \times \mathbb{R}^d; \\ \partial_t m - \nu \Delta m - \operatorname{div}(D_p H(x, \nabla u, m)m) = 0 & \text{in } (0, T) \times \mathbb{R}^d; \\ m(0) = m_1; m(T) = m_2 & \text{in } \mathbb{R}^d. \end{cases}$$

This system is almost the same as (1) except for the fact that the final constraint on u has been replaced by a final constraint on m . A subclass of the planning problem of the utmost importance is of course the optimal transport problem. The analogy is almost transparent when comparing (2) with the so-called Benamou-Brenier formulation of the optimal transport problem [3]. Let us comment on the fact that from a modeling point of view, the main difference between the mean field planning problem and optimal transport is that optimal transport is associated to a centralized optimization problem, whereas the mean field planning problem deals with a game in which non-atomic players, or infinitesimal masses, are supposed to have a behavior which results from individual decisions.

The system (2) has now been quite extensively studied, especially by analogy with his mean field game (MFG for short) counterpart. The point of view being to impose the terminal condition

$$(3) \quad u(T) = \frac{1}{\epsilon}(m(T) - m_2) \text{ in } \mathbb{R}^d$$

instead of the condition $m(T) = m_2$, and to let ϵ goes to 0. The first results on (2) dates back from [9] and concern either the deterministic case $\nu = 0$ or the quadratic one $H(x, p, z) = |p|^2 - f(z)$. Later on, a more general case has been studied in [11] with the notion of weak solutions. Numerical methods for this type of systems have been presented in [1]. More recently, the works [6, 10] studied (2) in the potential case. In the so-called potential approach, (2) is interpreted as the optimality conditions of an infinite dimensional optimal control problem. As the aim of this paper is to study the case of a common noise, we cannot generalize the approach which consists in looking at a forward-backward system similar to (2), instead we have to study the master equation associated to the planning problem.

The master equation approach. In this paper we study a different setting than the one we just mentioned. We study the master equation associated to the planning problem, and we place ourselves in the finite state space case in which the study of the master equation is less difficult and less technical than in the continuous case. We shall address the continuous state space case in a future work.

The study of the master equation is necessary for MFG (and thus for the planning problem) when the noise affecting the players is not i.i.d. In such a situation, the approach of a forward-backward system such as (1) fails as the evolution of m cannot be captured by a deterministic PDE like (ii). The idea of the master equation is to consider a value function for the players as a function of both their personal state and the density

of the other players. The master equation is then the PDE satisfied by this value function. Let us note that for such a value function to be well defined, several assumptions on the game have to be made (we do not detail them here).

In the setting of a finite state space, a typical form taken by the master equation with d states in the MFG context is :

$$(4) \quad \begin{cases} \partial_t U + (F(x, U) \cdot \nabla_x)U + \lambda(U - (DT)^*U(Tx)) = G(x, U) & \text{in } (0, \infty) \times \mathbb{R}^d; \\ U(0, x) = U_0(x) & \text{in } \mathbb{R}^d; \end{cases}$$

where, to simplify notations time has been reversed, F and G are mappings from \mathbb{R}^{2d} onto \mathbb{R}^d which describes respectively the evolution of the distribution of players and the costs paid by the players, T is a mapping from \mathbb{R}^d into itself which describes the common noise and $\lambda \geq 0$ describes the intensity of the common noise. The solution $U = (U^i)_{1 \leq i \leq d} : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the value function of the players, meaning that $U^i(t, x)$ is the value for a generic player in state i when the distribution of players in the d states is given by $x \in \mathbb{R}^d$ and there remains a time interval of length t in the game.

In this paper we want to analyse the solution of (4) in the context of the planning problem. Thus the initial condition in (4) has to be replaced with an appropriate characterization. We shall replace the initial condition by a suitable penalization, by analogy with (3), and then pass to the limit in this penalization. We shall show that this approach leads to the construction of the value function of the players in the planning problem. We expect a very singular behavior for the solution of the planning problem's master equation at $t = 0$ and we show in this paper how to characterize precisely this singularity.

Let us also mention that in the classical MFG setting, i.e. when there is no blow-up at the time $t = 0$, the system (4) has been studied. Since its introduction by the last two authors, their results of well-posedness in the case $\lambda = 0$ have been presented in [9]. The case of common noise, i.e. $\lambda > 0$, has been treated in [5]. Let us also note that other types of noise have been studied in [2], which contrary to the noise we study here, do not propagate monotonicity but yield some kind of ellipticity for the master equation.

Structure of the paper. The rest of the paper is structured as follows. In section 1 we prove some key estimates which allow us to pass to the limit in a penalized master equation. In section 2 we show that we can characterize the solution of the master equation associated to the planning problem by using the Yosida's regularization when this master equation is posed on \mathbb{R}^d . In section 3 we show that the techniques of section 2 can be adapted to a case in which the master equation is posed on a subdomain of \mathbb{R}^d .

1. NOTATIONS AND PRELIMINARY RESULTS

1.1. **Notations.** We begin with some notations and definitions.

- We shall use the notations $|\cdot|$ for the norm of vectors and $\|\cdot\|$ for the norm of operators.
- The identity of applications is denoted Id and the identity of matrices I .

- An application A from an Hilbert space $(H, \langle \cdot, \cdot \rangle)$ into itself is α -monotone ($\alpha \geq 0$) if for all $x, y \in H$

$$\langle A(x) - A(y), x - y \rangle \geq \alpha \langle x - y, x - y \rangle.$$

- An application A is simply called monotone if the previous inequality holds with $\alpha = 0$.
- For any element x of an Hilbert space H , A_x denotes the multivalued operator whose domain is $\{x\}$ and such that $A(x) = H$.
- The Yosida approximation $S_\delta A$ of parameter δ of a monotone operator A is defined by

$$S_\delta A = A \circ (Id + \delta A)^{-1}.$$

1.2. Preliminary results. The first result we present in this section is a result of existence of solutions of (4). This result is fairly simple, once some a priori estimates has been established as was already mentioned in [9]. However, as the question of existence is central in this paper, we wish to state and prove a specific result for the sake of completeness.

Proposition 1. *Assume that*

- F, G and U_0 are Lipschitz applications.
- $(G, F) : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ and $U_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are monotone.
- either (G, F) is α monotone in its first variable and U_0 is α monotone or (G, F) is α monotone in its second variable for some $\alpha > 0$.
- either $\lambda = 0$ or T is an affine application.

Then, there exists a function U defined on $\mathbb{R}_+ \times \mathbb{R}^d$, solution of (4) almost everywhere which is monotone and Lipschitz in the x variable, and locally Lipschitz in (t, x) .

Remark 1. *In fact, the following argument yields existence of a solution of (4) in a more precise sense : the one of monotone solutions introduced in [4]. Under the assumptions of the Proposition, uniqueness of a monotone solution holds. We refer to the aforementioned paper for more details on monotone solutions of MFG master equations.*

Proof. We present a proof by regularization to obtain existence of solutions. We add the elliptic term $-\epsilon \Delta$ in all the lines of the system (4) and we then pass to the limit $\epsilon \rightarrow 0$. The passage to the limit $\epsilon \rightarrow 0$ is justified with an a priori estimate.

Under the standing assumptions, for any $t_f > 0$, there exists $C > 0$ depending only on G, F, λ, T and U_0 , such that for any $\epsilon > 0$, for any smooth solution U of

$$(5) \quad \begin{cases} \partial_t U - \epsilon \Delta U + (F(x, U) \cdot \nabla_x)U + \lambda(U - (DT)^*U(Tx)) = G(x, U) \text{ in } (0, t_f) \times \mathbb{R}^d; \\ U(0, x) = U_0(x) \text{ in } \mathbb{R}^d; \end{cases}$$

the following holds

$$(6) \quad \forall t \leq t_f, \forall x \in \mathbb{R}^d, \|D_x U(t, x)\| \leq C.$$

This results is a direct generalization of the estimates established in [9, 5] which is obtained by remarking that adding the term $-\epsilon \Delta$ leaves the proof of the result unchanged.

For a fixed $\epsilon > 0$, existence of a solution of (5) is classical and we do not present it in full details but sketch a strategy of proof. Let us insist that, because of the presence of the Laplacian, existence of such solutions is somehow classical (especially because the non linearities have a linear growth). From chapter VII, paragraph 7 in [7], we know that there exists solutions of (5) when the space domain is any bounded smooth open set instead of \mathbb{R}^d (with appropriate boundary conditions). It is fairly easy to check that, using the proof of the estimate (6) in [5], we can show that it is also satisfied by the solutions of (5) in a bounded domain. Then, by considering a sequence of increasing domains (with respect to the inclusion), which tend to cover \mathbb{R}^d , we obtain the existence of a solution of (5) in the whole space.

We now want to show some compactness on the sequence $(U_\epsilon)_{\epsilon>0}$ of solutions of (5). The gradient estimate (6) of course holds and we only need to prove some estimate on the time derivative of the solutions. We first show that for all $\epsilon > 0$ there exists $C_1 > 0, \eta > 0$ such that for t sufficiently small, for all $x \in \mathbb{R}^d$

$$|U_\epsilon(t, x) - U_0(x)| \leq C_1 t(1 + |x|) + \eta.$$

We begin by showing the previous inequality component by component. Let us assume that such an inequality does not hold. Then for any $C_1 > 0, \eta > 0$, there exists $t_0 > 0, x_0 \in \mathbb{R}^d, i \in \{1; \dots; d\}$ with t_0 sufficiently small such that

$$\begin{cases} U_\epsilon^i(t_0, x_0) = U_0^i(x_0) + C_1 t_0(1 + |x_0|) + \eta; \\ \partial_t U_\epsilon^i(t_0, x_0) \geq C_1(1 + |x_0|); \\ \nabla_x U_\epsilon^i(t_0, x_0) = \nabla_x U_0^i(x_0) + C_1 t_0; \\ -\Delta U_\epsilon^i(t_0, x_0) \geq 0. \end{cases}$$

Thus evaluating (5) at (t_0, x_0) we deduce

$$C_1(1 + |x_0|) \leq \delta(1 + |x_0| + |U_\epsilon(t_0, x_0)|)$$

for some $\delta > 0$ which depends only on F, G, λ, T and the constant C from (6), in particular it does not depend on η or C_1 . From which we easily deduce that there exists $C_1 > 0$, such that for all $\eta > 0$,

$$|U_\epsilon(t, x) - U_0(x)| \leq C_1 t(1 + |x|) + \eta.$$

From which we deduce that it is also true for $\eta = 0$. Therefore we obtain

$$|\partial_t U_\epsilon(0, x)| \leq C_1(1 + |x|).$$

From this we deduce that the sequence $(U_\epsilon)_{\epsilon>0}$ is locally Lipschitz in $[0, \infty) \times \mathbb{R}^d$, uniformly in ϵ . From this uniform continuity and Ascoli-Arzelà Theorem, we know that along a subsequence, $(U_\epsilon)_{\epsilon>0}$ converges uniformly toward a locally Lipschitz function U , solution of (4), which satisfies (6). \square

We now state a regularizing result associated to the structure of the system we are studying. To our knowledge, this is the first result of this type on such a master equation.

Theorem 1. *Let U be a solution of (4). Assume that*

- U_0 and (G, F) are monotone.
- F and G are globally Lipschitz.
- (G, F) is α monotone in its second argument for some $\alpha > 0$.

- T is affine.

Then there exists a constant $C > 0$ (independent of U_0) such that for all $t \leq 1$

$$(7) \quad \|D_x U(t)\| \leq \frac{C}{t}.$$

Remark 2. Let us insist on the fact that no assumption is made on the regularity of $U_0 = U(0)$.

Proof. This proof starts by following the same argument as in [9, 5]. We note $T = S + e$ with S a linear map. We take two functions $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ to be defined later on and we introduce the auxiliary function $Z_{\beta, \gamma}$ defined by

$$Z_{\beta}(t, x, \xi) = \langle \xi, \nabla_x W(t, x, \xi) \rangle - \beta(t) |\nabla_x W(t, x, \xi)|^2 + \gamma(t) |\xi|^2;$$

where W is defined by

$$W(t, x, \xi) = \langle U(t, x), \xi \rangle.$$

The chain rule yields that $Z_{\beta, \gamma}$ satisfies the PDE :

$$(8) \quad \begin{aligned} & \partial_t Z_{\beta, \gamma} + \langle F(x, \nabla_\xi W), \nabla_x Z_{\beta, \gamma} \rangle + \langle D_p F(x, \nabla_\xi W) \nabla_\xi Z_{\beta, \gamma}, \nabla_x W \rangle - \langle D_p G(x, \nabla_\xi W) \nabla_\xi Z_{\beta, \gamma}, \xi \rangle \\ & + \lambda(Z_{\beta, \gamma} - Z_{\beta, \gamma}(t, Tx, T\xi - e)) \\ & = \langle D_x G(x, \nabla_\xi W) \xi, \xi \rangle - \langle D_p G(x, \nabla_\xi W) \nabla_x W, \xi \rangle - \langle D_x F(x, \nabla_\xi W) \nabla_x W, \xi \rangle \\ & + \langle D_p F(x, \nabla_\xi W) \nabla_x W, \nabla_x W \rangle \\ & - 2\beta \langle D_x G(x, \nabla_\xi W) \xi, \nabla_x W \rangle + 2\beta \langle D_x F(x, \nabla_\xi W) \nabla_x W, \nabla_x W \rangle \\ & + \beta \lambda \left(|\nabla_x W|^2 - 2 \langle \nabla_x W(t, Tx, T\xi - e), S \nabla_x W \rangle + |\nabla_x W(t, Tx, T\xi - e)|^2 \right) \\ & - \frac{d}{dt} \beta |\nabla_x W|^2 + 2\gamma (\langle D_p F(x, \nabla_\xi W) \xi, \nabla_x W \rangle - \langle D_p G(x, \nabla_\xi W) \xi, \xi \rangle) \\ & + \frac{d}{dt} \gamma |\xi|^2 + \lambda \gamma (|\xi|^2 - |T\xi - e|^2). \end{aligned}$$

Using the monotonicity assumption on (G, F) and the Lipschitz assumptions, we deduce

$$(9) \quad \begin{aligned} & \partial_t Z_{\beta, \gamma} + \langle F(x, \nabla_\xi W), \nabla_x Z_{\beta, \gamma} \rangle + \langle D_p F(x, \nabla_\xi W) \nabla_\xi Z_{\beta, \gamma}, \nabla_x W \rangle - \langle D_p G(x, \nabla_\xi W) \nabla_\xi Z_{\beta, \gamma}, \xi \rangle \\ & + \lambda(Z_{\beta, \gamma} - Z_{\beta, \gamma}(t, Tx, S\xi)) \\ & \geq \alpha |\nabla_x W|^2 - \beta \|D_x G\| \cdot |\xi|^2 - \beta \|D_x G\| \cdot |\nabla_x W|^2 - 2\beta \|D_x F(\nabla_\xi W)\| \cdot |\nabla_x W|^2 \\ & + \beta \lambda (|\nabla_x W|^2 - |S \nabla_x W|^2) - \frac{d}{dt} \beta |\nabla_x W|^2 - \gamma (\|D_p F\| - 2\|D_p G\|) |\xi|^2 \\ & - \gamma \|D_p F\| \cdot |\nabla_x W|^2 + \frac{d}{dt} \gamma |\xi|^2 + \lambda \gamma (|\xi|^2 - |S\xi|^2). \end{aligned}$$

The right hand side of the previous equation is positive if γ and β satisfy

$$(10) \quad \begin{cases} \alpha - \beta [\|D_x G\| + 2\|D_x F\| - \lambda(1 - \|S\|^2)] - \frac{d}{dt} \beta - \gamma \|D_p F\| \geq 0; \\ \frac{d}{dt} \gamma + \gamma [\lambda(1 - \|S\|^2) - \|D_p F\| - 2\|D_p G\|] - \beta \|D_x G\| \geq 0. \end{cases}$$

Let us now define β and γ with

$$(11) \quad \begin{cases} \beta(t) = \frac{\alpha}{2}t; \\ \gamma(t) = \|D_x G\| \alpha t^2. \end{cases}$$

Let us now observe that there exists $t_f > 0$ such that if β and γ are defined by (11), then (10) holds for all time $t \leq t_f$. Let us now remark that at $t = 0$, $Z_{\beta,\gamma}$ satisfies

$$Z_{\beta,\gamma}(t, x, \xi) = \langle \xi, D_x U_0 \xi \rangle \geq 0;$$

from the monotonicity assumption we made on U_0 . Thus we deduce from Lemma 3 in appendix of [5], that $Z_{\beta,\gamma}$ stays positive for all time $t \leq t_f$. From this we deduce that for all $t \leq t_f, \xi \neq 0$:

$$\beta(t) \frac{|\nabla_x W|^2}{|\xi|^2} \leq \gamma(t) + \frac{|\nabla_x W|}{|\xi|}.$$

Hence we obtain

$$\frac{|\nabla_x W|}{|\xi|} \leq \frac{\sqrt{1 + 4\beta(t)\gamma(t)}}{\beta(t)};$$

from which we derive

$$\|D_x U(t)\| \leq \frac{\sqrt{1 + 4\beta(t)\gamma(t)}}{\beta(t)}.$$

The result then easily follows. \square

2. PLANNING PROBLEM MASTER EQUATION IN \mathbb{R}^d

2.1. Statement of the problem. In this section we show that, under some assumptions, the sequence $(U_\epsilon)_{\epsilon>0}$ of solutions of

$$(12) \quad \begin{cases} \partial_t U + (F(x, U) \cdot \nabla_x)U + \lambda(U - (DT)^*U(Tx)) = G(x, U) \text{ in } (0, t_f) \times \mathbb{R}^d; \\ U(0, x) = \frac{1}{\epsilon}(x - x_0) \text{ in } \mathbb{R}^d; \end{cases}$$

converges toward a function U , which can be interpreted as the value function of a generic player for the mean field planning problem described by F, G, λ and T and constrained in x_0 at the final time. Before presenting the proof of this result, let us recall briefly the interpretation in terms of modeling of the master equation. In this model, as already mentioned, U represents the value function of the players. In the case $\lambda = 0$, given this value function, the discrete density of players x is assumed to evolve through the term F . This means that a particular density starting from x_1 at time t_1 evolves according to (time has been reversed) :

$$(13) \quad \begin{cases} \frac{d}{dt}x(t) = F(x(t), U(t, x(t))) \text{ for } 0 \leq t \leq t_1; \\ x(t_1) = x_1. \end{cases}$$

The evolution of the value function of the players is given by the function G and the terminal cost is $U(0, \cdot)$, this means that it satisfies

$$(14) \quad \begin{cases} \frac{d}{dt}U(t, x(t)) = G(x(t), U(t, x(t))) \text{ for } 0 \leq t \leq t_1; \\ U(0, x(0)) = \frac{1}{\epsilon}(x(0) - x_0). \end{cases}$$

The main idea of the penalization term is that it induces sufficient incentives so that the final density $x(0)$ shall be close to x_0 .

In the case $\lambda > 0$, we model a situation in which at random times given by a poisson process of intensity λ , all the population is affected by the transformation T , we refer to [5] for more details on this type of noise.

2.2. Properties of the Yosida approximation. In this section, we present an argument which explains how we can pass to the limit in (12). Namely we show some compactness on the sequence of Yosida approximations of the solution of the penalized problem. Let us mention that the use of the Yosida approximation may seem arbitrary but the definition of solution that we give in the next section does not involve the choice of this approximation. Also let us mention that we could have avoided the use of Yosida approximations and worked only with the regularizing result established in the first section but we believe the present approach is more instructive.

From Proposition 1, we know that $(U_\epsilon)_{\epsilon>0}$ is a well defined sequence of locally Lipschitz functions which are monotone in space for all time. For any $\epsilon > 0, \delta > 0$, we define $V_{\epsilon,\delta} = S_\delta U_\epsilon$, the Yosida regularization of U_ϵ of parameter $\delta > 0$. Equivalently, we could have defined $V_{\delta,\epsilon}$ by $V_{\delta,\epsilon}(t, x) = W_{\epsilon,t}(s, x)|_{s=\delta}$ where W is the solution of

$$(15) \quad \begin{cases} \partial_s W_{\epsilon,t} + W_{\epsilon,t} \cdot \nabla_x W_{\epsilon,t} = 0; \\ W_{\epsilon,t}(0, x) = U_\epsilon(t, x). \end{cases}$$

We present the PDE satisfied by $V_{\delta,\epsilon}$ in the following result.

Proposition 2. *For any $\delta > 0, \epsilon > 0$, $V_{\delta,\epsilon}$ is a solution of*

$$(16) \quad \begin{cases} \partial_t V + F((Id - \delta V), V) \cdot \nabla_x V = \\ \quad (G((Id - \delta V), V) - \lambda[V - T^* \circ V \circ (Id - \delta V)^{-1} \circ T \circ (Id - \delta V)])(I - \delta D_x V); \\ V(0, x) = \frac{1}{\delta + \epsilon}(x - x_0). \end{cases}$$

Proof. This claim easily follows from the chain rule together with the fact that $Id + \delta U_\epsilon = (Id - \delta V_{\delta,\epsilon})^{-1}$.

By the definition of $V_{\delta,\epsilon}$, the following holds

$$(17) \quad U_\epsilon(t, x) = V_{\delta,\epsilon}(t, x + \delta U_\epsilon(t, x)).$$

For general fields F and H , if a function U solves

$$(18) \quad \partial_t U + (F \cdot \nabla_x)U = H,$$

then its Yosida approximation V_δ solves

$$(19) \quad \partial_t V_\delta(t, x + \delta U(t, x)) + ((F + \delta H)(x, U(t, x)) \cdot \nabla_x) V_\delta(t, x + \delta U(t, x)) = H(x, U(t, x)),$$

Applying this relation to the equation (12) yields the required result. \square

Looking at the the system (16), it is easy to imagine how we can pass to the limit $\epsilon \rightarrow 0$ at the level of the Yosida approximation. We explain this passage to the limit in the next result.

Proposition 3. For any $\delta > 0$, the sequence $(V_{\delta,\epsilon})_{\epsilon>0}$ converges locally uniformly toward a function V_δ , solution of

$$(20) \quad \begin{cases} \partial_t V + F((Id - \delta V), V) \cdot \nabla_x V = \\ \quad (G((Id - \delta V), V) - \lambda[V - T^* \circ V \circ (Id - \delta V)^{-1} \circ T \circ (Id - \delta V)])(Id - \delta D_x V); \\ V(0, x) = \frac{1}{\delta}(x - x_0). \end{cases}$$

The function V_δ is such that $(Id - \delta V_\delta(t))^{-1}$ is well defined for $t > 0$.

Proof. First, let us note that from basic properties of the Yosida approximation, the following holds for any $\delta > 0, \epsilon > 0$ and $t > 0$:

$$\|D_x V_{\delta,\epsilon}\| \leq \frac{1}{\delta}.$$

In the case $\lambda = 0$, this estimate is enough to gain compactness on the sequence $(V_{\delta,\epsilon})_{\epsilon>0}$ using equation (16).

Let us now turn to the case $\lambda > 0$. From Theorem 1, we know that $(Id - \delta V_{\delta,\epsilon}(t))^{-1} = Id + \delta U_\epsilon(t)$ is Ct^{-1} Lipschitz for some constant $C > 0$ and time $t \leq t_f$ where t_f is independent of δ and ϵ . On the other hand, it is easy to check that $Id - \delta V_{\delta,\epsilon}(t)$ is $C\epsilon(\epsilon + \delta)^{-1}t$ Lipschitz for time $t \leq t_f$ where t_f does not depend on ϵ . Furthermore, the right hand side of (16) stays bounded uniformly near $(t, x) = (0, x_0)$. Thus from the fact that $V_{\delta,\epsilon}$ satisfies (16), we deduce compactness on the sequence $(V_{\delta,\epsilon})_{\epsilon>0}$. Extracting a subsequence if necessary, we note V_δ the limit of $(V_{\delta,\epsilon})_{\epsilon>0}$. The fact that $((Id - \delta V_{\delta,\epsilon}(t))^{-1})_{\epsilon>0}$ converges toward $(Id - \delta V_\delta(t))^{-1}$ for $t > 0$ is a simple exercise that we leave to the interested reader. □

2.3. The limit master equation. The previous result characterizes the behavior of the Yosida approximation of the solution of the master equation, in particular it gives information on the behavior of the solution near $\{t = 0\}$. This leads us to the following definition, that we comment below.

Definition 1. A solution of the master equation associated to the planning problem characterized by F, G, λ, T, x_0 is a function $U : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ solution of

$$(21) \quad \partial_t U + (F(x, U) \cdot \nabla_x)U + \lambda(U - (DT)^*U(Tx)) = G(x, U) \text{ in } (0, \infty) \times \mathbb{R}^d;$$

such that $(U(t))_{t \geq 0}$ converges toward A_{x_0} in the sense of graphs as t tends to 0.

Let us recall that a sequence of multivalued operators $(A_n)_{n \in \mathbb{N}}$ converges toward a multivalued operator A in the sense of graphs if for any sequence $(x_n, y_n)_{n \in \mathbb{N}}$ which converges toward $(x, y) \in H^2$ such that $y_n \in A_n(x_n)$ for all $n \geq 0$, the property $y \in A(x)$ holds. We do not comment on the fact that U is a solution of (21) in $(0, \infty) \times \mathbb{R}^d$. Let us note that the initial condition we impose is rather weak, but, as we shall see in Theorem 3, it is sufficient to establish uniqueness. In some sense, this condition is strong enough to capture the idea that trajectories of finite cost induced by the solution of the problem necessary start from x_0 . Moreover let us note that the convergence in the sense of graphs is natural to be expected as the sequence of initial conditions in the penalized problem converges in the sense of graphs toward A_{x_0} .

We now present a result of existence and one of uniqueness for such solutions.

Theorem 2. *Under the assumptions of Theorem 1, there exists a solution U of the planning problem master equation in the sense of definition 1.*

Proof. We consider the sequence $(U_\epsilon)_{\epsilon>0}$ of solutions of (12) and for some $\delta > 0$, $(V_{\delta,\epsilon})_{\epsilon>0}$ the corresponding Yosida approximations. Thanks to Proposition 3, extracting a subsequence if necessary, $(V_{\delta,\epsilon})_{\epsilon>0}$ converges toward a function V such as in Proposition 3. We define the function U by

$$(22) \quad U(t, x) = V(t, (Id - \delta V(t, \cdot))^{-1}(x)); t > 0; x \in \mathbb{R}^d.$$

The fact that U solves the PDE for $t > 0$ is a simple consequence of the chain rule. Let us now analyse the behavior of U near $t = 0$. Let us take a real sequence $(t_n)_{n \geq 0}$ converging toward 0 and two converging sequences of \mathbb{R}^d $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ such that for all $n \geq 0$ $y_n = U(t_n, x_n)$. Let us define for all $n \geq 0$, $z_n = (Id - \delta V(t_n, \cdot))^{-1}(x_n)$. By definition of U , we deduce that $(V(t_n, z_n))_{n \geq 0} = (y_n)_{n \geq 0}$ and thus that it is a converging sequence. Hence, $(z_n)_{n \geq 0} = (x_n + \delta V(t_n, z_n))_{n \geq 0}$ is also a converging sequence. From the behavior of V near $t = 0$, we deduce finally that $(x_n)_{n \geq 0}$ converges toward x_0 . Thus $(U(t))_{t \geq 0}$ converges toward A_{x_0} in the sense of graphs as t tends to 0. \square

Theorem 3. *Under the assumptions of Theorem 1, there is a unique U solution of the planning problem master equation in the sense of definition 1 with smooth Yosida approximations for small times. Moreover, U is the limit of the sequence $(U_\epsilon)_{\epsilon>0}$ of solutions of (12).*

Proof. Denote by U_1 and U_2 two solutions of the problem. The main argument of this proof consists in showing that the real function W defined for all $t > 0$, $x, y \in \mathbb{R}^d$ by

$$(23) \quad W(t, x, y) = \langle U_1(t, x) - U_2(t, y), x - y \rangle$$

is positive on $(0, \infty) \times \mathbb{R}^{2d}$. This function satisfies

$$(24) \quad \partial_t W + F(x, U_1) \cdot \nabla_x W + F(y, U_2) \cdot \nabla_y W + \lambda(W - W(t, Tx, T\xi)) \geq 0 \text{ on } (0, \infty) \times \mathbb{R}^{2d}.$$

From the convergence of U_1 and U_2 toward A_{x_0} as t tends to 0, and the fact that U_1 and U_2 solves the master equation for $t > 0$, we can prove that for all $x, y \in \mathbb{R}^d$,

$$(25) \quad \liminf_{t \rightarrow 0} W(t, x, y) \geq 0.$$

Indeed, let us consider $\delta > 0$ and let us write $V_1 = S_\delta U_1$ and $V_2 = S_\delta U_2$ and let us define

$$(26) \quad \tilde{W}(t, x, y) := \langle V_1(t, x) - V_2(t, y), x - y - \delta(V_1(t, x) - V_2(t, y)) \rangle,$$

which satisfies for all $t > 0$, $x, y \in \mathbb{R}^d$

$$(27) \quad W(t, x, y) = \tilde{W}(t, x + \delta U_1(t, x), y + \delta U_2(t, y)).$$

The function \tilde{W} is well defined up to $t = 0$ and satisfies $\tilde{W}|_{t=0} = 0$ and $\partial_t \tilde{W}|_{t=0} \geq 0$ on \mathbb{R}^{2d} . This fact is easily obtained from the fact that (20) simplifies at $t = 0$. From the convergence of U_1 and U_2 toward A_{x_0} in the sense of graphs, we obtain that for all $x, y \in \mathbb{R}^d$

$$(28) \quad \lim_{t \rightarrow 0} \tilde{W}(t, x, y) = 0.$$

From this we deduce that (25) holds.

Thus we obtain from a maximum principle result (see Lemma 3 in appendix of [5]) that $W \geq 0$ everywhere. This yields that $U = V$. The proof of the previous result guarantees that this solution is the limit of the solutions of the penalized problems. \square

Remark 3. *The assumption on the smoothness of the Yosida approximations does not seem to be necessary as the convergence in the sense of graphs for monotone operators yields the convergence everywhere of their Yosida approximations. However, we maintain this assumption to avoid technical difficulties, arising from the regularity of U . Moreover, let us mention that in a situation without such regularity, an approach similar to [4] seems to work, although it is outside the scope of this paper.*

2.4. Links with the induced trajectories. In this section, we indicate why the trajectories induced by the solution U of the master equation of the planning problem converge toward the constrained point x_0 . This convergence, explained in the next result, is a consequence of the behavior of U near $t = 0$ and of the monotonicity of F and G . We focus on the deterministic case (i.e. $\lambda = 0$) to avoid some technicalities which are due to the particular choice of noise we made, however the same type of approach can be developed in the stochastic case.

Proposition 4. *Let U be a solution of the problem in the sense of definition 1 and assume that F and G are globally Lipschitz and that $\lambda = 0$, then for any $x_1 \in \mathbb{R}^d$ and $t_1 > 0$, the trajectory $(x(t))_{0 < t \leq t_1}$ defined (backwardly) by*

$$(29) \quad \begin{cases} \frac{d}{dt}x(t) = F(x(t), U(t, x(t))); \\ x(t_1) = x_1; \end{cases}$$

is such that $x(t)$ converges toward x_0 as t goes to 0.

Proof. Let us remark that because of the assumption we made, (29) defines a continuous path $(x(t))_{0 < t \leq t_1}$ in \mathbb{R}^d . Note that the following holds :

$$(30) \quad \begin{cases} \frac{d}{dt}U(t, x(t)) = G(x(t), U(t, x(t))); \\ U(t_1, x(t_1)) = U(t_1, x_1). \end{cases}$$

Thus $(x(t), U(t, x(t)))_{0 < t \leq t_1}$ is the unique solution of an ordinary differential equation of the form $\dot{y} = f(y)$ for some Lipschitz function f . Hence, we deduce that $(x(t), U(t, x(t)))$ is uniformly bounded for $0 < t \leq t_1$. From the convergence of $U(t)$ toward A_{x_0} as t tends to 0, in the sense of graphs, we deduce that

$$(31) \quad x(t) \rightarrow_{t \rightarrow 0} x_0.$$

\square

3. A COMMENT ON THE CASE OF A RESTRICTED DOMAIN

In this section, we briefly discuss how the idea developed in the previous section can be extended to situations involving bounded domains. We indicate an example in which the behavior of the solution of the planning problem is clear but we do not present a thorough study of all the different possible structures. Many different behaviors can be expected and more cases shall be treated in a future work.

3.1. Main differences with the previous case. The typical form of a master equation in a domain $\Omega \subset \mathbb{R}^d$ is

$$(32) \quad \partial_t U + (F(x, U) \cdot \nabla_x)U + \lambda(U - (DT)^*U(Tx)) = G(x, U) \text{ in } (0, \infty) \times \Omega;$$

without boundary conditions on the boundary of the domain $\partial\Omega$. We refer to [4] for precise results on such equations, in particular in the MFG setting the fact that this equation behaves nicely is a rather simple extension of similar results in the whole space.

Let us mention that, when $\partial\Omega$ is smooth, a natural (necessary) condition for (32) to be well-posed when equipped with an initial condition is

$$(33) \quad \langle F(x, p), n(x) \rangle \leq 0; x \in \partial\Omega; p \in \mathbb{R}^d;$$

where $n(x)$ is the unit normal vector to $\partial\Omega$ at x . The relation (33) is of course an obstacle to some uniform monotonicity assumption on F in its second variable.

3.2. The half space case with vanishing conditions. In this section, we investigate the case in which $\Omega = \{x_1 > 0\}$ and F_1 , the first component of F vanishes linearly near 0. More precisely, we assume that for $x_1 \leq 1$, the following holds

$$(34) \quad F_1(x, p) = x_1 \tilde{F}_1(x, p);$$

where \tilde{F}_1 is such that $\tilde{F} = (\tilde{F}_1, F_2, \dots, F_d)$ satisfies the assumption of the first part.

For the sake of simplicity, we work in the case $T(x) = (x_1, T'(x_2, \dots, x_d))$ with $T' \in \mathcal{L}(\mathbb{R}^{d-1}, \mathbb{R}^{d-1})$.

To sum up, the problem we are interested in is finding a function $U : (0, \infty) \times \Omega \rightarrow \mathbb{R}^d$ such that U is a solution of

$$(35) \quad \partial_t U + (F(x, U) \cdot \nabla_x)U + \lambda(U - (DT)^*U(Tx)) = G(x, U) \text{ in } (0, \infty) \times \{x_1 > 0\};$$

and $(U(t))_{t>0}$ converges toward $A_{x_0}|_\Omega$ in the sense of graphs, under the assumption (34).

Now, let us note that the function $V : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by

$$(36) \quad V(t, y_1, y_2, \dots, y_d) = \begin{cases} U(t, e^{y_1-1}, y_2, \dots, y_d) & \text{if } y_1 < 1; \\ U(t, y) & \text{else;} \end{cases}$$

satisfies a master equation of the type presented in section 2. Thus everything we did above applies immediately. Let us mention that in this case, we obtain from (36) and the behavior of V that the solution U of the master equation in the half space is unbounded near $\{x_1 = 0\}$, even for $t > 0$. More precisely, it behaves as $\ln(x_1)$ near $\{x_1 = 0\}$, which shows that Lipschitz estimates may not be true in the case of a boundary. Indeed they rely mainly on the monotone assumption on F .

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