Periodic Homogenization for Weakly Elliptic Hamilton-Jacobi-Bellman Equations with Critical Fractional Diffusion

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PERIODIC HOMOGENIZATION FOR WEAKLY ELLIPTIC HAMILTON-JACOBI-BELLMAN EQUATIONS WITH CRITICAL FRACTIONAL DIFFUSION

ADINA CIOMAGA*, DARIA GHILLI§, AND ERWIN TOPP#

ABSTRACT. In this paper we establish periodic homogenization for Hamilton-Jacobi-Bellman (HJB) equations, associated to nonlocal operators of integro-differential type. We consider the case when the fractional diffusion has the same order as the drift term, and is weakly elliptic. The outcome of the paper is two-fold. One one hand, we provide Lipschitz regularity results for weakly elliptic nonlocal HJB, extending the results previously obtained in [8]. On the other hand, we establish a convergence result, based on half relaxed limits and a comparison principle for the effective problem. The latter strongly relies on the regularity and the ellipticity properties of the effective Hamiltonian, for which a fine Lipschitz estimate of the corrector plays a crucial role.

Keywords: regularity of generalized solutions, viscosity solutions, nonlinear elliptic equations, partial integro-differential equations, homogenization

AMS Classification: 35D10, 35D40, 35J60, 35R09

1. INTRODUCTION

In this paper we are interested in periodic homogenization of parabolic nonlocal Hamilton-Jacobi equations of the form

$$\begin{cases} u_t^\varepsilon(x,t) + H(x, \dfrac{x}{\varepsilon}, Du^\varepsilon, u^\varepsilon(\cdot,t)) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{1}$$

where $T > 0$, the initial condition $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a bounded uniformly continuous function and H is a continuous Hamiltonian, periodic with respect to its fast variable $\xi = x/\epsilon$. The unknown functions $u^\varepsilon : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ depend on a homogenization scale $\varepsilon > 0$. The function $H = H(x,\xi,p,\phi)$ is a Hamilton-Jacobi-Bellman operator $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathscr{C}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \to \mathbb{R}$, depending nonlocally on a function $\phi \in C^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, through an integro-differential operator associated to Lévy processes. More precisely, given a compact metric space $\mathscr{A}$, the Hamiltonian takes the form

$$H(x,\xi,p,\phi) = \sup_{a \in \mathscr{A}} \Big\{ -\mathscr{L}^a(x,\xi,\phi) - b^a(x,\xi)\cdot p - f^a(x,\xi) \Big\}. \tag{2}$$

The integro-differential operator is given by

$$\mathscr{L}^a(x,\xi,\phi) = \int_{\mathbb{R}^d} \big(\phi(x+z) - \phi(x) - \mathbf{1}_B(z) D\phi(x)\cdot z\big) K^a(\xi,z)\,dz, \tag{3}$$

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where $\mathbf{1}_B$ denotes the indicator function of the unit ball B in $\mathbb{R}^d$, and $K^a(\cdot) = K(a,\cdot)$ is a family of kernels generated by a continuous function $K : \mathscr{A} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$. The kernels are possibly singular at the origin, satisfying the uniform Lévy condition

$$\sup_{a\in\mathscr{A}} \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \min\left(1, |z|^2\right) K^a(\xi, z)\, dz < +\infty.$$

Similarly to $(K^a)_{a\in\mathscr{A}}$, the families of functions $(f^a)_{a\in\mathscr{A}}$ and $(b^a)_{a\in\mathscr{A}}$ are given respectively by $f : \mathscr{A} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $b : \mathscr{A} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, bounded and continuous functions.

Nonlocal equations find applications in mathematical finance and occur in the theory of Lévy jump-diffusion processes. The theory of viscosity solutions has been extended for a rather long time to integro-differential equations. Some of the first papers are due to Soner [35, 36] in the context of stochastic control jump diffusion processes. The connection of such nonlocal equations with deterministic and stochastic singular perturbations of optimal control problems appears in [1], [15], [5]. Existence and comparison results for second order degenerate Hamilton-Jacobi-Bellman equations were provided by Benth, Karlsen and Reikvam in [16]. The viscosity theory for general partial integro-differential operators has been recently revisited and extended to solutions with arbitrary growth at infinity by Barles and Imbert [11].

In this paper, we deal with Hamilton-Jacobi-Bellman equations where the diffusion is given by a general Lévy nonlocal operator, with a kernel depending on the space variable x and we would like to place ourselves in a "critical" regime, where both the nonlocal diffusion and the Hamiltonian are of order 1. A key issue is the establishment of the concept of the "order" of the diffusion. It is known [17] that the behaviour of the kernel near the origin determines such an order. The typical example of an integro-differential operator of order 1 is given by *the square root of the Laplacian*, whose kernel $K(\xi, z) = 1/|z|^{d+1}$ is symmetric, and independent of ξ :

$$\begin{aligned}(-\Delta)^{1/2} u(x) &= \int_{\mathbb{R}^d} (u(x+z) - u(x) - \mathbf{1}_B Du(x)\cdot z)|z|^{-(d+1)}\, dz \\ &= \text{P.V.} \int_{\mathbb{R}^d} (u(x+z) - u(x))|z|^{-(d+1)}\, dz,\end{aligned}$$

where P.V. stands for the Cauchy Principal Value, see [23]. More generally, *uniformly elliptic* kernels could be considered, i.e. kernels for which there exist a constant $C_K > 0$ such that

$$\frac{1}{C_K|z|^{d+1}} \le K^a(\xi, z) \le \frac{C_K}{|z|^{d+1}} \quad \text{for all } z \in B \setminus \{0\}. \tag{4}$$

In the "critical" regime of uniformly elliptic kernels satisfying equation (4), the nonlocal and gradient terms in (2) have the same scaling properties, and therefore the diffusive role of $\mathscr{L}^a$ enters into competition with the transport effect of the drift term. The critical regime was already studied by Silvestre in [33] and [34], where regularity of solutions is shown and the result is used to establish the existence of classical solutions. The above ellipticity assumption is the equivalent of its local version, which roughly speaking requires all the eigenvalues associated to the diffusion matrix to stay bounded away from zero. We aim at dealing with more general kernels, where the pointwise ellipticity assumption (4) is replaced by an integral condition. We require kernels to be *weakly elliptic* only, i.e. there exists a constant $C_K > 0$ such that for any given direction $p \in \mathbb{R}^d$, there exist an

ellipticity cone $\mathscr{C}_{\eta,\rho}(p) := \{z \in B_\rho; (1-\eta)|z||p| \le |p\cdot z|\}$ of aperture $\eta \in (0,1)$ where

$$\int_{\mathscr{C}_{\eta,\rho}(p)} |z|^2 K^a(\xi,z)\,dz \ge C_K \eta^{\frac{d-1}{2}}\ \rho, \text{ for any } \xi \in \mathbb{R}^d.$$

Here, the quantity $\eta^{\frac{d-1}{2}}$ measures the volume of the cone in the unit ball relative to the volume of the unit ball, while ρ is related to the order/scaling of the nonlocal operator (see Example 1 in [8] for more details). In particular, any uniformly elliptic operator is weakly elliptic. Solutions associated with this type of weakly elliptic kernels are shown to be Lipschitz [8] in the case when the nonlocal diffusion has order larger than 1; nonetheless, the critical case remained open.

The setup we consider is in striking contrast with previous available results in homogenization of integro-differential problems. In [2, 3], Arisawa analyzed periodic homogenization for equations with purely Lévy operators, and rather light interaction between the slow and fast variable. Homogenization results for nonlocal equations with variational structure have been recently studied in [26, 30]. This paper is closely related to [32], where periodic homogenization for uniformly elliptic Bellman-Isaacs equations was obtained by Schwab. Later on these results were extended to stochastic homogenization in [31]. The arguments in both papers are completely different than ours, and are based on the obstacle problem method, previously introduced in [18, 19] in order to establish stochastic homogenization and rates of convergence for fully nonlinear, uniformly elliptic partial differential equations. Periodic homogenization for nonlocal Hamilton-Jacobi equations with coercive gradient terms has been addressed in [6], where techniques similar to ours appear, except that here we cannot rely on the gradient coercivity.

We show that the family of solutions $\left(u^\varepsilon\right)_\varepsilon$ of the Cauchy problem (1) converges locally uniformly on $\mathbb{R}^d \times [0,T]$, as $\varepsilon \to 0$, to the solution u of an *effective problem*

$$\begin{cases} u_t(x,t) + \overline{H}(x, Du, u(\cdot,t)) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{5}$$

where the limiting Hamiltonian $\overline{H}$ is to be implicitly defined. This main result is presented in Theorem 5.3. The program is classical, and falls into the lines of the celebrated preprint of Lions Papanicolau and Varadhan [29] and the seminal papers of Evans [24, 25]. We write the oscillatory solution as $u^\varepsilon(x,t) = \bar{u}(x,t) + \varepsilon\psi(x/\varepsilon) + \cdots$, and find the effective Hamiltonian $\overline{H}$ by solving a *cell problem* whose solution is the (periodic) corrector ψ, then establish properties of $\overline{H}$ that ensure well-posedness of the limiting problem (5) and finally conclude the convergence.

Though the result itself is standard in periodic homogenization, a series of difficulties arise, due to the general form and weak ellipticity of the nonlocal operator (3): (i) the implicit definition of $\overline{H}$ which does not say much about its nonlocal dependence on the whole function u, (ii) the absence of comparison principles for equations with integro-differential operators having general x- dependent kernels, and in particular the lack of comparison results for the limiting problem and (iii) the lack of Lipschitz regularity of the oscillatory solutions and of the corrector. We discuss each of these points in turn and the interplay in-between.

Homogenization occurs in two steps. The first step is the study of the cell problem and accordingly the construction the *effective Hamiltonian* $\overline{H}$, which here reads: given $x, p \in \mathbb{R}^d$ and a

function $u \in \mathscr{C}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ show that there exists a unique constant $\lambda = \overline{H}(x,p,u)$ so that the following problem has a Lipschitz continuous, periodic, viscosity solution

$$\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi) - \tilde{b}^a(\xi;x)\cdot D\psi(\xi) - \tilde{f}^a(\xi;x,p,u)\} = \lambda \quad \text{in } \mathbb{R}^d,$$

where $\tilde{b}^a$ and $\tilde{f}^a$ are to be computed. We note that, in the critical case, general non-symmetric kernels give rise to an extra drift term in the cell problem and $\tilde{b}^a = \tilde{b}^a + b_K$, for some b_K carefully determined from the properties of K, and this is due to the presence of the compensator term $\mathbf{1}_B(z)Du(x)\cdot z$ in (3). Contrarily, in the case of symmetric kernels, the lack of the compensator term keeps the drift term unchanged. In both scenarios, we give a Lipschitz regularity result for the corrector, with a fine estimate of the Lipschitz seminorm. This will play a crucial role in establishing properties of the effective Hamiltonian, which themselves have an important echo in the proof of convergence.

Several properties of the original Hamiltonian H given by (2) are translated into the effective one. If on one hand it is natural that $\overline{H}$ inherits the nonlocal nature in its third variable, on the other hand no explicit formula can be obtained in general. Some examples of explicit nonlocal effective equations can be found in [6] and [28], but we stress that these methods cannot be applied in the setting and/or the generality presented here. In particular, we establish a non-trivial ellipticity-growth condition for $\overline{H}$ that further allows to manipulate the effective problem in spite of not knowing its explicit form.

The second step is solving the effective problem (5) and showing the convergence of the sequence $\left(u^\varepsilon\right)_\varepsilon$. Well posedness for the limit problem (5) is not obvious, in view of the absence of explicit formulas for $\overline{H}$ and the lack of general comparison results for nonlocal problems with x-dependent kernels. This is overcome by a linearization of the effective Hamiltonian $\overline{H}$ via the extremal Pucci operators, and is intimately related to the Lipschitz regularity of the corrector and the ellipticity growth property of the effective Hamiltonian. Once comparison for the effective problem is proven, the homogenization result is standard and it follows from the perturbed test function method applied to half relaxed limits.

As pointed out above, both solving the cell problem and showing the convergence requires Lipschitz regularity of solutions.To the best of our knowledge, no Lipschitz regularity result had been proven before for this kind of equations in their full generality. In [8], Lipschitz regularity is proven for equations involving fractional diffusions with order in the whole range $(1,2]$, except when the order is one. We complete these results and establish Lipschitz regularity of solutions by Ishii-Lions method, making use of a non standard test function which behaves radially like $r + r\log^{-1}(r)$. We give a rather general Lipschitz regularity result for weakly elliptic integro-differential operators, which has an interest in its own, extending to the critical case Lipschitz estimates obtained in [8].

We stress that the methods presented in this article can be extended to other nonlocal homogenization problems and they are not exclusively circumscribed to the critical case described here. We emphasize on the “linearization" of the effective Hamiltonian, which reveals important information about the limiting problem. Related to this, it would be interesting to describe the effective problem in terms of an associated optimal control problem. This has been addressed in the deterministic case via the so-called limit occupational measures, see [7, 37] and references therein.

Finally, note that the results presented do not rely on the convexity of H, and therefore they can be readily adapted to Hamiltonians H of Bellman-Isaacs type, related to differential games (see [4]).

The paper is organized as follows: in Section 2 we introduce some notation and define the notion of solution to our problems. In Section 3 we establish a Lipschitz regularity result for integro-differential equations dealing with nonlocal Lévy operators of order one. In Section 4 we solve the cell problem and provide useful regularity and ellipticity properties of the effective Hamiltonian. In Section 5 we establish the homogenization result associated to equation (1).

2. Preliminaries and assumptions.

2.1. **Notations.** We denote the $d-$dimensional Euclidean space by $\mathbb{R}^d$, and by $\Pi^d = \mathbb{R}^d/\mathbb{Z}^d$ the thorus on $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $\rho > 0$ we denote $B_\rho(x)$ the ball centered at x with radius ρ, and we simply write B if $x = 0$ and $\rho = 1$. We use the notation $\mathbf{1}_B$ for the indicator function of the unit ball B in $\mathbb{R}^d$. By abuse of notation, we denote the cylinder $B_\rho(x,t) := B_\rho(x) \times (t-\rho, t+\rho)$. For a metric space X we denote respectively $USC(X)$ and $LSC(X)$ the sets of real-valued upper and lower semicontinuous functions on X, $BUC(X)$ the set of bounded uniformly continuous real-valued functions on X. The set of $\tau-$ Hölder functions on X is written $\mathscr{C}^{0,\tau}(X)$, the set of continuous functions is written $\mathscr{C}(X)$ and we denote $\mathscr{C}^r(X)$ the set of functions, with continuous differentials of order $r > 0$. The space of essentially bounded measurable functions on X is denoted $L^\infty(X)$ and its norm $||\cdot||_\infty$.

2.2. **Viscosity solutions.** To cope with the difficulties imposed by behaviour of the measure at infinity, as well as its singularity at the origin, we often split the nonlocal term into

$$\mathscr{L}(x,\xi,\phi) = \mathscr{L}[B_\rho](x,\xi,\phi) + \mathscr{L}[{B_\rho}^c](x,\xi,\phi),$$

with $0 < \rho < 1$, where for any $D \subset \mathbb{R}^d$ measurable, we write

$$\mathscr{L}[D](x,\xi,\phi) = \int_D \big(\phi(x+z) - \phi(x) - \mathbf{1}_B(z) D\phi(x)\cdot z\big) K^a(\xi,z)\,dz.$$

We work in the setting of viscosity solutions, as described in [11]. In this setup, the nonlocal term is evaluated in terms of a smooth test function on B_ρ and on the function itself on B_ρ^c. We give below the definition for a slightly modified equation

$$\begin{cases} u_t(x,t) + \mathscr{H}(x,Du,u) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{6}$$

where $\mathscr{H}$ is to be properly defined in each context (for the original oscillating problem (1), for the cell problem (17), and for the limiting problem (5)).

Definition 1 (*Viscosity solutions)*.

(1) *We say an upper semi-continuous (usc) function $u : \mathbb{R}^d \times (0,T] \to \mathbb{R}$ is a* viscosity subsolution *of* (6) *iff for any $\phi \in \mathscr{C}^2(\mathbb{R}^d \times [0,T])$, if (x,t) is a maximum of $u - \phi$ in $B_\rho(x,t)$ then*

$$\phi_t(x,t) + \mathscr{H}(x, D\phi(x,t), \mathbf{1}_{B_\rho(x)}\phi(\cdot,t) + \mathbf{1}_{B_\rho^c(x)} u(\cdot,t)) \le 0.$$

(2) *We say a lower semi-continuous (lsc) function* $u : \mathbb{R}^d \times (0,T] \to \mathbb{R}$ *is a* viscosity supersolution *of* (6) *iff for any* $\phi \in \mathscr{C}^2(\mathbb{R}^d \times [0,T])$*, if* (x,t) *is a minimum of* $u-\phi$ *in* $B_\rho(x,t)$ *then*

$$\phi_t(x,t) + \mathscr{H}(x, D\phi(x,t), \mathbf{1}_{B_\rho(x)}\phi(\cdot,t) + \mathbf{1}_{B_\rho^c(x)}u(\cdot,t)) \geq 0.$$

(3) *We say* u *is a* viscosity solution *if it is both a viscosity subsolution and supersolution.*

This definition has been formulated so it literally applies to the effective Hamiltonian $\overline{H}$, provided we show before hand that $\overline{H}$ is well defined. A similarly definition can be given for the stationary case and henceforth, for the cell-problem.

2.3. **Formal expansion.** In order to introduce the set of assumptions, and make precise our results we begin with the usual formal asymptotic expansion

$$u^\varepsilon(x,t) = \bar{u}(x,t) + \varepsilon\psi\left(\frac{x}{\varepsilon}\right) + ...$$

where $\bar{u}(x,t)$ is the average profile and $\psi(\xi)$ is the periodic corrector. Though this computation already appears in [6], for the readers' convenience we develop it here, in order to emphasize on (i) the interference between the order of the nonlocal operator and the homogenization scale ε and (ii) the need to distinguish within the set of assumptions between the symmetric and non-symmetric case and the fact that in the case of non-symmetric kernels the expansion gives rise to an extra drift term in the corrector equation.

Plugging the previous expression into the nonlocal term, it follows that

$$\begin{aligned}
\mathscr{L}^a\left(x, \frac{x}{\varepsilon}, u^\varepsilon(\cdot,t)\right) &= \int_{\mathbb{R}^d} \left(u^\varepsilon(x+z,t) - u^\varepsilon(x,t) - \mathbf{1}_B(z) Du^\varepsilon(x,t)\cdot z\right) K^a\left(\frac{x}{\varepsilon}, z\right) dz \\
&= \int_{\mathbb{R}^d} (\bar{u}(x+z,t) - \bar{u}(x,t) - \mathbf{1}_B(z) D\bar{u}(x,t)\cdot z)\, K^a\left(\frac{x}{\varepsilon}, z\right) dz + \\
&\quad \varepsilon \int_{\mathbb{R}^d} \left(\psi\left(\frac{x+z}{\varepsilon}\right) - \psi\left(\frac{x}{\varepsilon}\right) - \mathbf{1}_B(z) D\psi\left(\frac{x}{\varepsilon}\right)\cdot\frac{z}{\varepsilon}\right) K^a\left(\frac{x}{\varepsilon}, z\right) dz.
\end{aligned}$$

Therefore, denoting the fast variable $x/\varepsilon = \xi$, we can write the nonlocal term as

$$\mathscr{L}^a\left(x, \xi, u^\varepsilon(\cdot,t)\right) = \mathscr{L}^a(x,\xi,\bar{u}(\cdot,t)) + \mathscr{I}_\varepsilon^a(\xi,\psi),$$

where

$$\mathscr{I}_\varepsilon^a(\xi,\psi) = \varepsilon^{d+1} \int_{\mathbb{R}^d} \left(\psi(\xi+z) - \psi(\xi) - \mathbf{1}_{B_{1/\varepsilon}} D\psi(\xi)\right) K^a(\xi, \varepsilon z)\, dz.$$

To keep the ideas clear in this formal expansion assume the kernel is of the following form, regardless its symmetry

$$K^a(\xi,z) = \frac{k(\xi,z)}{|z|^{d+1}}.$$

Note further that

(i) if $k^a(\xi,\varepsilon z) = k^a(\xi)$,the compensator term in the nonlocal expression $\mathscr{I}_\varepsilon^a(\xi,\psi)$ vanishes and

$$\mathscr{I}_\varepsilon^a(\xi,\psi) = \mathscr{L}^a(\xi,\xi,\psi) = k^a(\xi)(-\Delta)^{1/2}\psi(\xi).$$

(ii) if $k^a(\xi,\varepsilon z)$ is not independent of z, we employ a modulus of continuity of k

$$\omega_k(r) = \sup_{a\in\mathscr{A}} \sup_{\xi\in\Pi^d} \sup_{|z|\leq r} \left|k^a(\xi,z) - k^a(\xi,0)\right|.$$

to separate the nonlocal term into

$$\mathscr{I}_\varepsilon^a(\xi,\psi) = k^a(\xi,0)(-\Delta)^{1/2}\psi(\xi) + \mathscr{J}_\varepsilon^a(\xi,\psi),$$

where

$$\mathscr{J}_\varepsilon^a(\xi,\psi) = \int_{\mathbb{R}^d} \left(\psi(\xi+z) - \psi(\xi) - \mathbf{1}_{B_{1/\varepsilon}}(z) D\psi(\xi)\cdot z\right) \frac{k^a(\xi,\varepsilon z) - k^a(\xi,0)}{|z|^{d+1}}\, dz.$$

The term $\mathscr{J}_\varepsilon^a(\xi,\psi)$ can be split into

$$\mathscr{J}_\varepsilon^a(\xi,\psi) = \mathscr{J}_\varepsilon^a[B](\xi,\psi) + \mathscr{J}_\varepsilon^a[B_{1/\varepsilon}\setminus B](\xi,\psi) + \mathscr{J}_\varepsilon^a[B_{1/\varepsilon}^c](\xi,\psi),$$

where we use the notation $\mathscr{J}[D]$ to indicate the domain on which the integral is computed. Assuming that $\psi \in \mathscr{C}^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ with bounded $||D\psi||_\infty$ and $||D^2\psi||_\infty$ the following estimates hold

$$\begin{aligned}
|\mathscr{J}_\varepsilon^a[B](\xi,\psi)| &\leq \frac{1}{2}||D^2\psi||_\infty \int_B |z|^2 \frac{|k^a(\xi,\varepsilon z) - k^a(\xi,0)|}{|z|^{d+1}}\, dz \\
&\leq \frac{1}{2}||D^2\psi||_\infty \omega_k(\varepsilon) \int_B |z|^2 \frac{dz}{|z|^{d+1}} = o_\varepsilon(1), \\
|\mathscr{J}_\varepsilon^a[B_{1/\varepsilon}^c](\xi,\psi)| &\leq 4||\psi||_\infty ||k||_\infty \int_{B_{1/\varepsilon}^c} \frac{dz}{|z|^{d+1}} = o_\varepsilon(1),
\end{aligned}$$

whereas

$$\begin{aligned}
\mathscr{J}_\varepsilon^a[B_{1/\varepsilon}\setminus B](\xi,\psi) &= \int_{B_{1/\varepsilon}\setminus B} \left(\psi(\xi+z)-\psi(\xi)\right) \frac{k^a(\xi,\varepsilon z) - k^a(\xi,0)}{|z|^{d+1}}\, dz + \\
&\quad \int_{B_{1/\varepsilon}\setminus B} D\psi(\xi)\cdot z \frac{k(\xi,\varepsilon z) - k^a(\xi,0)}{|z|^{d+1}}\, dz \\
&= o_\varepsilon(1) + \int_{B\setminus B_\varepsilon} D\psi(\xi)\cdot z \frac{k^a(\xi,z) - k^a(\xi,0)}{|z|^{d+1}}\, dz \\
&= o_\varepsilon(1) + D\psi(\xi)\cdot b_K^a(\xi),
\end{aligned}$$

where

$$b_K^a(\xi) = \int_B (k^a(\xi,z) - k^a(\xi,0)) \frac{z}{|z|^{d+1}}\, dz$$

is well-defined provided that $\displaystyle\int_0^1 \frac{\omega_k(r)}{r} dr < \infty$. To conclude, we have that

$$\mathscr{I}_\varepsilon^a(\xi,\psi) = k^a(\xi,0)(-\Delta)^{\frac{1}{2}}\psi(\xi) + D\psi(\xi)\cdot b_K^a(\xi) + o_\varepsilon(1).$$

Plugging everything in (1), we arrive to the following equation which must be satisfied both with respect to the slow variable x and the fast variable ξ simultaneously

$$\begin{aligned}
u_t(x,t) + \sup_{a\in\mathscr{A}}\Big\{ &- \mathscr{L}^a(x,\xi,\bar{u}(\cdot,t)) - k^a(\xi,0)(-\Delta)^{1/2}\psi(\xi) \\
&- b^a(x,\xi)\cdot D\bar{u}(x,t) - \left(b^a(x,\xi) + b_K(\xi)\right)\cdot D\psi(\xi) - f^a(x,\xi)\Big\} = 0.
\end{aligned}$$

We are lead, in this context, to solving first the following cell problem: given $x, p \in \mathbb{R}^d$ and a function $u \in \mathscr{C}^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ show that there exists a unique constant $\lambda \in \mathbb{R}$ so that the following

problem has a Lipschitz continuous, periodic, viscosity solution

$$\sup_{a\in\mathscr{A}}\{-k^a(\xi,0)(-\Delta)^{1/2}\psi(\xi)-\tilde{b}^a(\xi;x)\cdot D\psi(\xi)-\tilde{f}^a(\xi;x,p,u)\}=\lambda,$$

where the source term is given by $\tilde{f}^a(\xi;x,p,u)=f^a(x,\xi)+b^a(x,\xi)\cdot p+\mathscr{L}^a(x,\xi,u)$, and the drift adds an extra term $\tilde{b}^a(\xi;x)=b^a(x,\xi)+b^a_K(\xi)$. The constant λ is known in the literature as the effective Hamiltonian and denoted by $\lambda=\overline{H}(x,p,u)$. This implicitly defines the effective equation (or the limit equation) (5), which is shown to be satisfied by the average profile $\bar{u}$. Once well posedness is established for the effective equation, the convergence of the whole sequence $\left(u^\varepsilon\right)_{\varepsilon>0}$ towards the average profile $\bar{u}$ is shown.

Going back to the points raised in (i) and (ii), we have seen above that nonlocal terms having kernels with a general dependence on the fast and slow variables give rise to an extra drift term. This is due on one hand to the fact that the homogenization scale ε has the same order as the nonlocal diffusion (in occurence 1) and on the other hand to the fact that the kernel has a non-symmetric behaviour in the slow variable. This is not the case if the kernel is symmetric, when the compensator is not needed.

2.4. **Assumptions.** Homogenization results are established both for symmetric and non-symmetric kernels, though the formal expansion has been given only for the non-symmetric case. To this end, we make two set of assumptions, corresponding to each setup.

(Ks) For each $a\in\mathscr{A}$, K^a is *symmetric* with respect to z, i.e. for all $\xi\in\mathbb{R}^d$ and $z\in R^d\setminus\{0\}$,

$$K^a(\xi,z)=K^a(\xi,-z)$$

and *homogeneous* with respect to z, i.e. for all $\xi\in\mathbb{R}^d$, $z\in R^d\setminus\{0\}$ and any $\epsilon>0$,

$$K^a(\xi,\epsilon z)=\frac{1}{\epsilon^{(d+1)}}K^a(\xi,z).$$

(Kns) For each $a\in\mathscr{A}$, there exists $k^a\in\mathscr{C}(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d})$ such that, for all $\xi\in\mathbb{R}^d$ and $z\in B\setminus\{0\}$,

$$K^a(\xi,z)=\frac{k^a(\xi,z)}{|z|^{d+1}},$$

and there exists a constant $C_K>1$ such that

$$\sup_{a\in\mathscr{A}}\sup_{\xi\in\Pi^d}\int_0^1\sup_{|z|\le r}\left|k^a(\xi,z)-k^a(\xi,0)\right|\frac{dr}{r}\le C_K.$$

To the scaling and symmetry assumptions above, we add a series of assumptions for the family of Lévy kernels, in order to ensure periodicity, existence of solutions, comparison results and regularity. These have now become classical, see [8, 10, 11].

($K0$) For any $a\in\mathscr{A}$, the mapping $\xi\mapsto K^a(\xi,z)$ is $\mathbb{Z}^d$ periodic, for all $z\in\mathbb{R}^d$.

($K1$) There exists a constant $C_K>0$ such that,

$$\sup_{a\in\mathscr{A}}\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{R}^d}\min(1,|z|^2)K^a(\xi,z)dz\le C_K.$$

($K2$) There exist a constant $C_K > 0$ such that for any $p \in \mathbb{R}^d$, there exist a $0 < \eta < 1$ such that the following holds for all $a \in \mathscr{A}$, for any $\xi \in \mathbb{R}^d$ and for all $\rho > 0$,

$$\int_{\mathscr{C}_{\eta,\rho}(p)} |z|^2 K^a(\xi, z)\, dz \geq C_K \eta^{\frac{d-1}{2}}\, \rho,$$

with $\mathscr{C}_{\eta,\rho}(p) := \{z \in B_\rho; (1-\eta)|z||p| \leq |p \cdot z|\}$.

($K3$) There exist a constant $C_K > 0$ and an exponent $\gamma \in (0,1]$ such that for all $a \in \mathscr{A}$, for any $\xi_1, \xi_2 \in \mathbb{R}^d$ and all $\rho > 0$,

$$\begin{aligned}
\int_{B_\rho} |z|^2 |K^a(\xi_1, z) - K^a(\xi_2, z)|\, dz &\leq C_K |\xi_1 - \xi_2|^\gamma \rho \\
\int_{B \setminus B_\rho} |z| |K^a(\xi_1, z) - K^a(\xi_2, z)|\, dz &\leq C_K |\xi_1 - \xi_2|^\gamma |\ln \rho| \\
\int_{\mathbb{R}^d \setminus B_\rho} |K^a(\xi_1, z) - K^a(\xi_2, z)|\, dz &\leq C_K |\xi_1 - \xi_2|^\gamma \rho^{-1}.
\end{aligned}$$

Finally, we assume the following for the drift term and the running cost.

($H0$) For each $a \in \mathscr{A}$, the mappings $\xi \mapsto f^a(x, \xi)$, $\xi \mapsto b^a(x, \xi)$ are $\mathbb{Z}^d$ periodic, for all $x \in \mathbb{R}^d$.

($H1$) Let $f^a : \mathbb{R}^{\tilde{d}} \to \mathbb{R}$ and $b^a : \mathbb{R}^{\tilde{d}} \to \mathbb{R}^{\tilde{d}}$ be two families of bounded functions. There exist two constants $C_f, C_b > 0$ and exponents $\alpha, \beta \in (0,1]$ such that, for all $a \in \mathscr{A}$ and $x_1, x_2 \in \mathbb{R}^{\tilde{d}}$,

$$|f^a(x_1) - f^a(x_2)| \leq C_f |x_1 - x_2|^\alpha, \;\; |b^a(x_1) - b^a(x_2)| \leq C_b |x_1 - x_2|^\beta.$$

This continuity assumption is a classical condition to conclude the existence of global solutions of Bellman equations related to finite/infinite horizon control problems. We write assumption ($H1$) in the previous general form, since we alternatively use it on variables x and ξ.

2.5. **Examples.** Here are some typical examples of kernels that correspond to our setup.

Example 1. *Let $(K^a)_{a \in \mathscr{A}}$ be a family of kernels of the form*

$$K^a(\xi, z) = \frac{1}{|M^a(\xi) z \cdot z|^{(d+1)/2}} \quad \xi \in \mathbb{R}^d, z \in \mathbb{R}^d \setminus \{0\},$$

where $M^a : \mathbb{R}^d \to \mathbf{S}^d$ is a family of periodic $\mathscr{C}^1$ matrices, and with eigenvalues uniformly bounded above and below: there exists $c_K > 1$ such that for each $a \in \mathscr{A}, \xi \in \mathbb{R}^d$, all the eigenvalues of $M^a(\xi)$ belong to the interval $[1/c_K, c_K]$.

Example 2. *Let $(K^a)_{a \in \mathscr{A}}$ be a family of kernels of the form*

$$K^a(\xi, z) = \frac{k^a(\xi, z/|z|)}{|z|^{d+1}} \quad \xi \in \mathbb{R}^d, z \in \mathbb{R}^d \setminus \{0\},$$

where $k^a : \mathbb{R}^d \times \mathbf{S}^{d-1} \to \mathbb{R}$ is a family of bounded continuous functions, periodic and Hölder continuous with respect to their first variable and symmetric with respect to their second variable.

Example 3. *Let $(K^a)_{a \in \mathscr{A}}$ be a family of kernels of the form*

$$K^a(\xi, z) = \frac{k^a(\xi) e^{-i\pi_i(z)}}{|z|^{d+1}} \quad \xi \in \mathbb{R}^d, z \in \mathbb{R}^d \setminus \{0\},$$

where $k^a : \mathbb{R}^d \to \mathbb{R}$ is a family of bounded Hölder continuous and periodic functions, and $\pi_i : \mathbb{R}^d \to \mathbb{R}$ is the projection function onto the $i-th$ component, $\pi_i(z_1, \cdots, z_d) = z_i$.

Finally, as announced in the introduction, we aim at dealing with degenerate kernels, such as kernels whose measure is supported only in half space, as in the example below.

Example 4. *Let $(K^a)_{a\in\mathscr{A}}$ be a family of kernels of the form*

$$K(\xi,z)=\mathbf{1}_{\{z_i>0\}}\frac{k^a(\xi)}{|z|^{d+1}}\quad z\in\mathbb{R}^d\setminus\{0\},$$

where, as before, $k^a:\mathbb{R}^d\to\mathbb{R}$ is a family of bounded Hölder continuous and periodic functions, and z_i is the $i-th$ component of z.

3. Regularity Estimates.

In this section we establish Lipschitz regularity of viscosity solutions of nonlocal Hamilton Jacobi equations, when the order of the integro-differential operator is one. To this end, we apply Ishii-Lions's method, as for previously obtained results in [8, 10]. If in the case of fractional diffusions of order larger than one (also known as subcritical) it was necessary to show first that the solution is $C^{0,\tau}$ for some small $\tau>0$, and employ this estimate to get Lipschitz, the technique failed for the critical case. We now complete this work and show below that, with a proper choice of control function, Lipschitz estimates can be directly obtained in the critical regime for drift fractional-diffusion equations, and their extension to Bellman equations. This will be further used when solving the cell problem, and establishing the homogenization results.

Consider for any $\delta\geq 0$, the following stationary problem

$$\delta u+\mathscr{H}(x,Du,u)=0\quad\text{in }\mathbb{R}^d,\tag{7}$$

where the Hamiltonian takes the Bellman form

$$\mathscr{H}(x,p,u)=\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(x,u)-b^a(x)\cdot p-f^a(x)\},\tag{8}$$

with the nonlocal operator given by

$$\mathscr{I}^a(x,u)=\int_{\mathbb{R}^d}\big(u(x+z)-u(x)-\mathbf{1}_B(z)Du(x)\cdot z\big)K^a(x,z)dz.\tag{9}$$

The main Lipschitz regularity result is given in the theorem below. Note that we do not assume periodicity. Assumptions (Ks) and (Kns) play no role in establishing the regularity of solutions, whereas the weak regularity assumption $(K2)$ is crucial.

Theorem 3.1. *Let $(f^a)_{a\in\mathscr{A}}$, $(b^a)_{a\in\mathscr{A}}$ two families of bounded functions on $\mathbb{R}^d$ satisfying $(H1)$ with Hölder exponents respectively $\alpha,\beta\in(0,1]$ and constants C_f, C_b, and $(K^a)_{a\in\mathscr{A}}$ be a family of kernels satisfying $(K1)-(K3)$ with Hölder exponent $\gamma\in(0,1]$ and constant C_K. Then any viscosity solution $u\in BUC(\mathbb{R}^d)$ of (7) is Lipschitz continuous, satisfying the following estimate: for every $\sigma\in(0,\alpha)$ there exists a constant $C_\sigma>0$ such that, for all $x,y\in\mathbb{R}^d$,*

$$|u(x)-u(y)|\leq C_\sigma C_f^{\frac{1}{1+\sigma}}|x-y|.\tag{10}$$

The constant C_σ depends on $\alpha,\|u\|_\infty$, and on the constants C_f, C_b, C_K, but is independent of δ,β,γ.

Proof of Theorem 3.1. The method, which has now become classical, consists in shifting the solution u and showing that the corresponding difference can be uniformly controlled by a concave function. This translates into a doubling of variables technique, leading to viscosity solutions equations estimates. The proof will be divided in several steps.

Step 1. Doubling of variables. Let

$$\Phi(x,y) = u(x) - u(y) - L\phi(x-y) - \psi_\zeta(x),$$

where ϕ is radial function $\phi(z) = \varphi(|z|)$ with a suitable choice of a smooth, increasing, concave function φ, and ψ_ζ is a smooth localisation term. The penalization function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is given here by

$$\varphi(r) = \begin{cases} 0 & r = 0 \\ r + r\log^{-1}(r) & r \in (0, r_0] \\ \varphi(r_0) & r \geq r_0, \end{cases}$$

where $r_0 \in (0, 0.04)$, so that the function φ is concave and increasing, and for all $r \in (0, r_0]$,

$$\begin{aligned} r/2 < \quad \varphi(r) \quad &< r, \\ 1/2 \leq \quad \varphi'(r) \quad &< 1 \\ -(r\log^2(r))^{-1} \leq \quad \varphi''(r) \quad &\leq -(r\log^2(r))^{-1}/2. \end{aligned}$$

The localisation term is given by $\psi_\zeta(x) = \psi(\zeta x)$, where $\psi \in \mathscr{C}^2(\mathbb{R}^d;\mathbb{R}_+)$ with bounded ψ, $D\psi$ and $D^2\psi$ on $\mathbb{R}^d$, such that

$$\psi(x) = \begin{cases} 0 & |x| \leq 1 \\ 3\,\mathrm{osc}_{\mathbb{R}^d}(u) & |x| \geq 2. \end{cases}$$

Our aim is to show that there exists an $L > 0$ such that

$$|u(x) - u(y)| \leq L\phi(x-y) \text{ if } |x-y| \leq r_0.$$

We argue by contradiction and assume that, for any choice of $L > 2\|u\|_\infty$ large enough, and $\zeta \in (0,1)$ small enough, Φ has a positive maximum, that we denote

$$M_L = \sup_{x,y\in\mathbb{R}^d} \Phi(x,y) = \Phi(\bar{x},\bar{y}) > 0.$$

To simplify the notation we drop the dependence on L and ζ for the point $(\bar{x},\bar{y})$ where the maximum is attained. It is immediate to see that

$$\begin{aligned} L|\bar{x}-\bar{y}|/2 \leq L\varphi\left(|\bar{x}-\bar{y}|\right) \leq 2||u||_\infty, \\ L|\bar{x}-\bar{y}|/2 \leq L\varphi\left(|\bar{x}-\bar{y}|\right) \leq \omega_u\left(|\bar{x}-\bar{y}|\right), \end{aligned} \tag{11}$$

where $\omega_u(\cdot)$ is the modulus of continuity of u (the solution being uniformly continuous). This implies in particular that $|\bar{x}-\bar{y}|$ is uniformly bounded above and away from zero as $\zeta \to 0$, and $|\bar{x}-\bar{y}| \to 0$ as $L \to \infty$, but also that $L|\bar{x}-\bar{y}| \to 0$ as $L \to \infty$. In addition

$$M_L \leq u(\bar{x}) - u(\bar{y}) \leq \omega_u(|\bar{x}-\bar{y}|). \tag{12}$$

Step 2. The viscosity inequalities. Let

$$\bar{p} = \bar{x} - \bar{y},\ \hat{p} = \bar{p}/|\bar{p}|,\ p = D\phi(\bar{p}) = \varphi'(|\bar{p}|)\hat{p},\ q = D\psi_\zeta(\bar{x}),$$

$$\phi_y(x) = L\phi(x-y) + \psi_\zeta(x) \quad \text{and} \quad \phi_x(y) = -L\phi(x-y).$$

Note that $u - \phi_{\bar{y}}$ has a global maximum at $\bar{x}$, respectively $u - \phi_{\bar{x}}$ has a global minimum at $\bar{y}$ and $D\phi_{\bar{y}}(\bar{x}) = D\phi_{\bar{x}}(\bar{y}) = Lp$. It follows from the viscosity inequalities that, for any $\nu > 0$, there exists $a \in \mathscr{A}$ such that, for all $0 < \rho' < 1$, we have

$$\begin{aligned}
&\delta u(\bar{x}) - \mathscr{I}^a[B_{\rho'}](\bar{x}, \phi_{\bar{y}}) - \mathscr{I}^a[B^c_{\rho'}](\bar{x}, u) - Lb^a(\bar{x})\cdot p - f^a(\bar{x}) \le 0\\
&\delta u(\bar{y}) - \mathscr{I}^a[B_{\rho'}](\bar{y}, \phi_{\bar{x}}) - \mathscr{I}^a[B^c_{\rho'}](\bar{y}, u) - Lb^a(\bar{y})\cdot p - f^a(\bar{y}) > -\nu,
\end{aligned}$$

where we have used the notation $\mathscr{I}^a[D](x,u)$ to denote the nonlocal operator (9) computed on the set D. Denote

$$\begin{aligned}
\mathscr{T}^a[B_{\rho'}](\bar{x},\bar{y},\phi) &:= \mathscr{I}^a[B_{\rho'}](\bar{x},\phi_{\bar{y}}) - \mathscr{I}^a[B_{\rho'}](\bar{y},\phi_{\bar{x}})\\
\mathscr{T}^a[B^c_{\rho'}](\bar{x},\bar{y},u) &:= \mathscr{I}^a[B^c_{\rho'}](\bar{x},u) - \mathscr{I}^a[B^c_{\rho'}](\bar{y},u).
\end{aligned}$$

Subtract the two inequalities and use the regularity assumption $(H1)$ and (12), to get that

$$\begin{aligned}
\delta M_L - \Big(\mathscr{T}^a[B_{\rho'}](\bar{x},\bar{y},\phi) + \mathscr{T}^a[B^c_{\rho'}](\bar{x},\bar{y},u)\Big) &< \nu + L\big(b^a(\bar{x}) - b^a(\bar{y})\big)\cdot p + f^a(\bar{x}) - f^a(\bar{y})\\
&< \nu + LC_b|\bar{x}-\bar{y}|^\beta|p| + C_f|\bar{x}-\bar{y}|^\alpha \qquad (13)\\
&< \nu + LC_b|\bar{p}|^\beta + C_f|\bar{p}|^\alpha.
\end{aligned}$$

Step 3. The nonlocal estimate. We first let $\rho' \to 0$ and see that the term $\mathscr{T}^a[B_{\rho'}](\bar{x},\bar{y},\phi)$ is $o_{\rho'}(1)$. We then let $\zeta \to 0$ and we note that the nonlocal terms corresponding to ψ_ζ are of order $o_\zeta(1)$. In what follows, we drop the dependence and *all terms in ρ' and ζ*. To simplify notations, we write $\mathscr{T}^a(\bar{x},\bar{y},u)$ instead of $\mathscr{T}^a[\mathbb{R}^d](\bar{x},\bar{y},u)$. It is useful to already see that the maximum of Φ gives the following bounds for the expressions in u, appearing as the integrant of the nonlocal terms composing $\mathscr{T}^a(\bar{x},\bar{y},u)$. Namely, for all $z \in \mathbb{R}^d$,

$$\begin{aligned}
u(\bar{x}+z) - u(\bar{x}) - p\cdot z &\le L\big(\phi(\bar{p}+z) - \phi(\bar{p}) - p\cdot z\big)\\
u(\bar{y}) - u(\bar{y}+z) + p\cdot z &\le L\big(\phi(\bar{p}-z) - \phi(\bar{p}) + p\cdot z\big). \qquad (14)
\end{aligned}$$

Here again, we dropped the terms in ψ_ζ to simplify the presentation.

It is within the nonlocal difference $\mathscr{T}^a(\bar{x},\bar{y},u)$ that we will see the role of the critical fractional diffusion in obtaining the right Lipschitz estimates. The key bound comes from the weak ellipticity in the gradient direction, given by assumption $(K2)$. To make this clear, we proceed as usual (see [8, 10]) and split the nonlocal difference into

$$\begin{aligned}
\mathscr{T}^a(\bar{x},\bar{y},u) &= \mathscr{T}^a[\mathscr{C}_{\eta,\rho}(\bar{p})](\bar{x},\bar{y}) + \mathscr{T}^a[B_\rho \setminus \mathscr{C}_{\eta,\rho}(\bar{p})](\bar{x},\bar{y}) + \qquad (15)\\
&\quad \mathscr{T}^a[B\setminus B_\rho](\bar{x},\bar{y}) + \mathscr{T}^a[B^c](\bar{x},\bar{y}),
\end{aligned}$$

where $\mathscr{C}_{\eta,\rho}(\bar{p})$ is the ellipticity cone in the direction of the gradient, given by $(K2)$ with $\bar{p} = \bar{x} - \bar{y}$, and $\eta \in (0,1)$ and $\rho > 0$ yet to be determined.

Lemma 3.2 (Nonlocal estimate on the ellipticity cone)**.** *Assume $(K2)$ holds with the ellipticity cone $\mathscr{C}_{\eta,\rho}(\bar{p})$ and let $\rho = c_1|\bar{p}|\log^{-2}(|\bar{p}|)$, $\eta = c_2\log^{-2}(|\bar{p}|)$, with $c_1, c_2 > 0$ sufficiently small. Then, there exist a constant $C > 0$ such that, for all $a \in \mathscr{A}$,*

$$\mathscr{T}^a[\mathscr{C}_{\eta,\rho}(\bar{p})](\bar{x},\bar{y}) \le -CL\big|\log(|\bar{p}|)\big|^{-(d+3)}.$$

Proof. Fix $a \in \mathscr{A}$. Note that, in view of (14),

$$\begin{aligned}\mathscr{T}^a[\mathscr{C}_{\eta,\rho}(\bar p)](\bar x,\bar y) &\le L\int_{\mathscr{C}_{\eta,\rho}(\bar p)} \big(\phi(\bar p+z)-\phi(\bar p)-D\phi(\bar p)\cdot z\big)\,K^a(\bar x,z)\,dz+\\ &\quad L\int_{\mathscr{C}_{\eta,\rho}(\bar p)} \big(\phi(\bar p-z)-\phi(\bar p)+D\phi(\bar p)\cdot z\big)\,K^a(\bar y,z)\,dz.\end{aligned}$$

Using Taylor's integral formula, the term above can be further bounded by

$$\mathscr{T}^a[\mathscr{C}_{\eta,\rho}(\bar p)](\bar x,\bar y)\le \sup_{a\in\mathscr{A}}\frac{L}{2}\int_{\mathscr{C}_{\eta,\rho}(\bar p)}\sup_{|s|\le 1}\big(D^2\phi(\bar p+sz)z\cdot z\big)\,(K^a(\bar x,z)+K^a(\bar y,z))\,dz.$$

Recall that $\phi(z)=\varphi(|z|)$ and use the notation $\hat z = z/|z|$. It follows that

$$\begin{aligned}D\phi(|z|) &= \varphi'(|z|)\hat z\\ D^2\phi(|z|) &= \varphi''(|z|)\hat z\otimes\hat z+\frac{\varphi'(|z|)}{|z|}(I-\hat z\otimes\hat z),\end{aligned}$$

and in particular

$$D^2\phi(\bar p+sz)z\cdot z=\varphi''(|\bar p+sz|)|\widehat{(\overline{p}+sz)}\cdot z|^2+\frac{\varphi'(|\bar p+sz|)}{|\bar p+sz|}\Big(|z|^2-|\widehat{(\overline{p}+sz)}\cdot z|^2\Big).$$

Taking into account that $\varphi''<0$ and $\varphi'>0$, we establish below a lower bound for the first term in the sum above, and an upper bound for the latter term. Take $\rho=|\bar p|\rho_0$ with $\rho_0\in(0,1)$, yet to be determined. Then, for all $z\in B_\rho$ and for all $s\in(-1,1)$, we have

$$|\bar p|(1-\rho_0)\le|\bar p+sz|\le|\bar p|(1+\rho_0),$$

whereas, for all $z\in\mathscr{C}_{\eta,\rho}(\bar p)=\{z\in B_\rho;(1-\eta)|z||p|\le|p\cdot z|\}$ and for all $s\in(-1,1)$,

$$\big|\big(\bar p+sz\big)\cdot z\big|\ge(1-\eta-\rho_0)|\bar p||z|.$$

These upper and lower bounds lead to the following estimate

$$D^2\phi(\bar p+sz)z\cdot z \le c(\eta,\rho_0)^2\varphi''(|\bar p+sz|)|z|^2+\big(1-c(\eta,\rho_0)^2\big)\frac{\varphi'(|\bar p+sz|)}{|\bar p+sz|}|z|^2,$$

with $c(\eta,\rho_0)=(1-\eta-\rho_0)/(1+\rho_0)$. Note that $c(\eta,\rho_0)^2\ge 1-2(\eta+2\rho_0)/(1+\rho_0)\ge 1/2$ for $\eta>0$ and $\rho_0>0$ sufficiently small. This implies that

$$\begin{aligned}D^2\phi(\bar p+sz)z\cdot z &\le \frac12\varphi''(|\bar p+sz|)|z|^2+2(\eta+2\rho_0)\frac{\varphi'(|\bar p+sz|)}{|\bar p+sz|}|z|^2,\\ &\le -\frac14\frac{|z|^2}{|\bar p+sz|\log^2|\bar p+sz|}+2(\eta+2\rho_0)\frac{|z|^2}{|\bar p+sz|}\\ &\le -\frac14\frac{|z|^2}{|\bar p|(1+\rho_0)\log^2\big(|\bar p|(1+\rho_0)\big)}+\frac{2(\eta+2\rho_0)|z|^2}{|\bar p|(1-\rho_0)}.\end{aligned}$$

For the choice of constants $\rho_0=c_1\log^{-2}(|\bar p|)$ and $\eta=c_2\log^{-2}(|\bar p|)$, with $c_1,c_2\in(0,0.001)$ sufficiently small, there exists a constant $c>0$, such that, the following estimate holds uniformly for $s\in(-1,1)$,

$$D^2\phi(\bar p+sz)z\cdot z \le -\frac{1}{64}\frac{|z|^2}{|\bar p|\log^2|\bar p|}+\frac{(8c_1+4c_2)|z|^2}{|\bar p|\log^2(|\bar p|)}\le -c\frac{|z|^2}{|\bar p|\log^2|\bar p|}.$$

Finally, in view of assumption $(K2)$, there exists $C>0$ such that

$$\begin{aligned}\mathscr{T}^a[\mathscr{C}_{\eta,\rho}(\bar{p})](\bar{x},\bar{y}) &\leq \sup_{a\in\mathscr{A}}\frac{L}{2}\int_{\mathscr{C}_{\eta,\rho}(\bar{p})}\left(-\frac{c}{|\bar{p}|\log^2|\bar{p}|}\right)|z|^2\left|K^a(\bar{x},z)-K^a(\bar{y},z)\right|dz\\ &\leq -Lc\left(|\bar{p}|\log^2|\bar{p}|\right)^{-1}C_K\left(c_2\log^{-2}(|\bar{p}|)\right)^{\frac{d-1}{2}}c_1|\bar{p}|\log^{-2}(|\bar{p}|)\\ &\leq -CL\left(\log^{-2}(|\bar{p}|)\right)^{\frac{d+3}{2}}.\end{aligned}$$

□

The nonlocal kernel is not bounded in B, but it only has a bounded second momentum. Outside the ellipticity cone, it is necessary to keep the estimate small. In order to obtain an optimal bound for the rest of the terms, we will use a measure decomposition as in [8, 10], that we briefly discuss next for completeness. Let

$$\Delta K^a(z) := \Delta K^a(\bar{x},\bar{y},z) = K^a(\bar{x},z)-K^a(\bar{y},z),$$

which is now a changing sign singular kernel. Define K^a_+, K^a_- as the nonnegative, mutually singular kernel measures satisfying $\Delta K^a = K^a_+ - K^a_-$ and let $\Theta^a = \mathrm{supp}(K^a_+)$. Let $K^a_{\min}$ be the minimum of the two kernels, with support $\mathbb{R}^d$. It follows that

$$K^a(\bar{x},z) = K^a_{\min}(z)+K^a_+(z) \text{ and } K^a(\bar{y},z) = K^a_{\min}(z)+K^a_-(z),$$

where we have dropped the $(\bar{x},\bar{y})$ dependence on the kernels, to keep the notation short. Note that for each pair of appropriate measurable functions $l_1, l_2 : \mathbb{R}^d \to \mathbb{R}$ and $D\subset\mathbb{R}^d$ measurable we can write

$$\begin{aligned}&\int_D l_1(z)K^a(\bar{x},z)dz - \int_D l_2(z)K^a(\bar{y},z)dz\\ &= \int_D (l_1(z)-l_2(z))\,K^a_{\min}(z)dz + \int_D l_1(z)K^a_+(z)dz - \int_D l_2(z)K^a_-(z)dz.\end{aligned} \tag{16}$$

Lemma 3.3 (Nonlocal estimate outside the ellipticity cone in B_ρ)**.** *Assume $(K3)$ holds with $\gamma\in(0,1]$ and let $\mathscr{C}_{\eta,\rho}(\bar{p})$ as in $(K2)$, and $\rho\in(0,1)$ be as in Lemma 3.2. Then there exists a constant $C>0$ such that, for all $a\in\mathscr{A}$,*

$$\mathscr{T}^a[B_\rho\setminus\mathscr{C}_{\eta,\rho}(\bar{p})](\bar{x},\bar{y}) \leq CL|\bar{p}|^\gamma\log^{-2}(|\bar{p}|).$$

Proof. Note that, in view of (14), and remark (16) above, the nonlocal term outside the ellipticity cone in B_ρ is bounded by

$$\begin{aligned}\mathscr{T}^a[B_\rho\setminus\mathscr{C}_{\eta,\rho}(\bar{p})](\bar{x},\bar{y}) &\leq L\int_{B_\rho\setminus\mathscr{C}_{\eta,\rho}(\bar{p})}\left(\phi(\bar{p}+z)-\phi(\bar{p})-D\phi(\bar{p})\cdot z\right)K^a_+(z)dz +\\ &\quad L\int_{B_\rho\setminus\mathscr{C}_{\eta,\rho}(\bar{p})}\left(\phi(\bar{p}-z)-\phi(\bar{p})+D\phi(\bar{p})\cdot z\right)K^a_-(z)dz.\end{aligned}$$

Using a second-order Taylor expansion of ϕ and taking into account that φ is smooth, $\varphi'\geq 0$ and $\varphi''\leq 0$, the following bound holds

$$\begin{aligned}\mathscr{T}^a[B_\rho\setminus\mathscr{C}_{\eta,\rho}(\bar{p})](\bar{x},\bar{y}) &\leq L\int_{B_\rho\setminus\mathscr{C}_{\eta,\rho}(\bar{p})}\sup_{|s|\leq 1}\left(D^2\phi(\bar{p}+sz)z\cdot z\right)\left(K^a_+(z)+K^a_-(z)\right)dz\\ &\leq L\int_{B_\rho\setminus\mathscr{C}_{\eta,\rho}(\bar{p})}\sup_{|s|\leq 1}\frac{\varphi'(|\bar{p}+sz|)}{|\bar{p}+sz|}|z|^2\left|K^a(\bar{x},z)-K^a(\bar{y},z)\right|dz.\end{aligned}$$

In view of assumption $(K3)$, it follows that there exists $C>0$ such that

$$\begin{aligned}\mathscr{T}^a[B_\rho\setminus\mathscr{C}_{\eta,\rho}(\bar p)](\bar x,\bar y) &\le \frac{L}{|\bar p|-\rho}\int_{B_\rho\setminus\mathscr{C}_{\eta,\rho}(\bar p)}|z|^2\left|K^a(\bar x,z)-K^a(\bar y,z)\right|dz\\ &\le \frac{L}{|\bar p|-\rho}C_K|\bar p|^\gamma\rho = C_KL|\bar p|^\gamma\frac{c_1|\bar p|\log^{-2}(|\bar p|)}{|\bar p|\left(1-c_1\log^{-2}(|\bar p|)\right)}\\ &\le CL|\bar p|^\gamma\log^{-2}(|\bar p|).\end{aligned}$$

□

Lemma 3.4 (Nonlocal estimate on the circular crown $B\setminus B_\rho$)**.** *Assume* $(K3)$ *holds with* $\gamma\in(0,1]$ *and let* $\rho\in(0,1)$ *be as in Lemma 3.2. Then there exists a constant* $C>0$ *such that, for all* $a\in\mathscr{A}$*,*

$$\mathscr{T}^a[B\setminus B_\rho](\bar x,\bar y)\le CL|\bar p|^\gamma\left|\log(|\bar p|)\right|.$$

Proof. As before, in view of (14), and remark (16) above, the nonlocal term on the circular crown is bounded by

$$\begin{aligned}\mathscr{T}^a[B\setminus B_\rho](\bar x,\bar y) &\le L\int_{B\setminus B_\rho}\left(\phi(\bar p+z)-\phi(\bar p)-D\phi(\bar p)\cdot z\right)K^a_+(z)dz+\\ &\quad L\int_{B\setminus B_\rho}\left(\phi(\bar p-z)-\phi(\bar p)+D\phi(\bar p)\cdot z\right)K^a_-(z)dz.\end{aligned}$$

Using the monotonicity, the concavity and the Lipschitz continuity of φ, the following holds

$$\begin{aligned}\mathscr{T}^a[B\setminus B_\rho](\bar x,\bar y) &\le L\int_{B\setminus B_\rho}\left(\varphi(|\bar p|+|z|)-\varphi(|\bar p|)+\varphi'(|\bar p|)|\hat p||z|\right)\left(K^a_+(z)+K^a_-(z)\right)dz\\ &\le L\int_{B\setminus B_\rho}2\varphi'(|\bar p|)|z|\left|K^a(\bar x,z)-K^a(\bar y,z)\right|dz.\end{aligned}$$

Employing now the regularity assumption $(K3)$, this further leads to the existence of a constant $C>0$ so that

$$\begin{aligned}\mathscr{T}^a[B\setminus B_\rho](\bar x,\bar y) &\le 2L\int_{B\setminus B_\rho}|z|\left|K^a(\bar x,z)-K^a(\bar y,z)\right|dz\\ &\le 2L\,C_K|\bar p|^\gamma\left|\ln\left(c_1|\bar p|\log^{-2}(|\bar p|)\right)\right|\\ &\le CL|\bar p|^\gamma\left|\log(|\bar p|)\right|.\end{aligned}$$

□

It is immediate to see that, in view of the integrability assumption, we have a uniform bound outside the unit ball.

Lemma 3.5 (Nonlocal estimate outside the unit ball)**.** *Assume* $(K3)$ *holds with* $\gamma\in(0,1]$*. Then there exists a constant* $C>0$ *such that, for all* $a\in\mathscr{A}$*,*

$$\mathscr{T}^a[B^c](\bar x,\bar y)\le CL|\bar p|^\gamma.$$

Proof. The same measure decomposition as before, gives

$$\begin{aligned}\mathscr{T}^a[B^c](\bar{x},\bar{y}) &\leq L\int_{B^c}\big(\phi(\bar{p}-z)-\phi(\bar{p})\big)K_+^a(z)dz + L\int_{B^c}\big(\phi(\bar{p}+z)-\phi(\bar{p})\big)K_-^a(z)dz\\ &\leq 4L||\phi||_\infty\left(\int_{B^c}\left|K^a(\bar{x},z)-K^a(\bar{y},z)\right|dz\right)\leq 4LC_K||\phi||_\infty|\bar{p}|^\gamma.\end{aligned}$$

□

Step 4. The conclusion. Plugging the estimates obtained in the previous lemmas into (15), we conclude that there exists a universal constant $C>0$, depending only on the constants given by assumptions $(K1)-(K3)$, such that, for $|\bar{p}|$ sufficiently small,

$$\begin{aligned}\mathscr{T}^a(\bar{x},\bar{y},u) &\leq -CL\left|\log(|\bar{p}|)\right|^{-(d+3)} + CL|\bar{p}|^\gamma\log^{-2}(|\bar{p}|) + CL|\bar{p}|^\gamma\left|\log(|\bar{p}|)\right| + CL|\bar{p}|^\gamma\\ &\leq -CL\left(\log^{-2}(|\bar{p}|)\right)^{\frac{d+3}{2}} + CL|\bar{p}|^\gamma|\log(|\bar{p}|)| + CL|\bar{p}|^\gamma.\end{aligned}$$

Plugging the above inequality into (13), it follows that

$$\delta M_L + CL\left|\log(|\bar{p}|)\right|^{-(d+3)} - CL|\bar{p}|^\gamma|\log(|\bar{p}|)| - CL|\bar{p}|^\gamma \quad < \quad \nu + C_bL|\bar{p}|^\beta + C_f|\bar{p}|^\alpha.$$

Recalling that in view of (12), $|\bar{p}|\to 0$ when $L\to\infty$, and taking into account that for any $\bar{\beta}>0$ we have that $\lim_{|\bar{p}|\to 0}\left(|\bar{p}|^{\bar{\beta}}|\log(|\bar{p}|)|\right)=0$, it follows that, up to a modification of the universal constant $C>0$, for sufficiently large L,

$$\delta M_L + CL\left|\log(|\bar{p}|)\right|^{-(d+3)} \quad < \quad \nu + C_f|\bar{p}|^\alpha.$$

Recalling that in view of (12), $M_L\to 0$ when $L\to\infty$, and ν can be chosen arbitrarily small, the previous inequality leads to

$$CL\left|\log(|\bar{p}|)\right|^{-(d+3)} \quad \leq \quad C_f|\bar{p}|^\alpha.$$

In particular, for any $0<\sigma<\alpha$, it follows that $CL|\bar{p}|^\sigma\leq C_f|\bar{p}|^\alpha$. Employing further inequality (11) we have $|\bar{p}|\leq CL^{-1}$, from where the following constraint holds for L, (up to a modification of the universal constant C)

$$L\leq\frac{C_f}{C}|\bar{p}|^{\alpha-\sigma}\leq\frac{C_f}{C}L^{-\alpha+\sigma}.$$

Let $\theta=1/(1+\alpha-\sigma)\in(1/(1+\alpha),1)$. Choosing then $L>(C_f/C)^\theta+1$, we arrive to a contradiction. This concludes the proof. □

Remark 1. *It is easy to see, from the proof above, that the Hölder continuity of the data can be weakened to a logarithmical modulus of continuity.*

Remark 2. *Notice that, if we assume $\alpha=1$, then σ in the statement of the theorem can be chosen arbitrarily close to 1, and the exponent $1/(1+\sigma)$ in the Lipschitz bounds is arbitrarily close to $1/2$. This is a crucial estimate to be used in the next section.*

The proof previously developed applies literally to parabolic integro-differential equations. The following holds.

Theorem 3.6. *Let $(f^a)_{a\in\mathscr{A}}$, $(b^a)_{a\in\mathscr{A}}$ two families of bounded functions on $\mathbb{R}^d$ satisfying $(H1)$ with Hölder exponents respectively $\alpha,\beta\in(0,1]$ and constants C_f, C_b, and $(K^a)_{a\in\mathscr{A}}$ be a family of kernels satisfying $(K1)-(K3)$ with Hölder exponent $\gamma\in(0,1]$ and constant C_K. Let $u\in BUC(\mathbb{R}^d\times[0,T])$ be a viscosity solution of*

$$\begin{cases} u_t+\mathscr{H}(x,Du,u)=0 & in\ \mathbb{R}^d\times(0,T]\\ u(x,0)=u_0(x) & in\ \mathbb{R}^d, \end{cases}$$

with $\mathscr{H}$ is as in (7). If $u_0\in Lip(\mathbb{R}^d)$, then u is Lipschitz continuous with respect to x uniformly on $[0,T]$, satisfying estimate (10) with a Lipschitz constant depending only on α, $\|u\|_\infty$, and on the constants C_f, C_b, C_K, but is independent of β,γ.

Proof of Theorem 3.6. We proceed similarly to the proof of Theorem 3.1, with the following function which doubles the variables

$$\Phi(x,y,t,s)=u(x,t)-u(y,s)-L\phi(x-y)-C|t-s|-\psi_\zeta(x),$$

where $C>0$ is a constant and ϕ is defined as in the proof of Theorem 3.1. The previous proof literally adapts to the parabolic case, since the non linearity $\mathscr{H}$ is independent of time. □

4. The cell problem and the effective Hamiltonian

In this section we establish the well-posedness of the cell problem and give a *fine* Lipschitz regularity estimate for the corrector, that will later play a crucial role in the proof of convergence. Further, we set forth a series of properties for the effective Hamiltonian, which shall have an implicit nonlocal dependence on the the averaged profile.

4.1. **The cell problem.** As made precise in Section 2, the cell problem both in the symmetric and the non-symmetric case can be formulated as follows. Given $x,p\in\mathbb{R}^d$ and a function $u\in\mathscr{C}^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ show that there exists a unique constant $\lambda\in\mathbb{R}$ so that the following problem has a periodic, continuous viscosity solution

$$\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi)-\tilde{b}^a(\xi;x)\cdot D\psi(\xi)-\tilde{f}^a(\xi;x,p,u)\}=\lambda \quad \text{in } \mathbb{R}^d, \tag{17}$$

where the source term is given by

$$\tilde{f}^a(\xi;x,p,u)=f^a(x,\xi)+b^a(x,\xi)\cdot p+\mathscr{L}^a(x,\xi,u),$$

with $\mathscr{L}^a$ defined by (3). However, the nonlocal operator $\mathscr{I}^a(\xi,\psi)$ and the drift term $\tilde{b}^a$ are defined differently according to the symmetry of the nonlocal kernel.

(1) In the case of symmetric kernels - assumption (Ks), the nonlocal operator is given by

$$\mathscr{I}^a(\xi,\psi)=\int_{\mathbb{R}^d}\big(\psi(\xi+z)-\psi(\xi)-\mathbf{1}_B(z)D\psi(\xi)\cdot z\big)K^a(\xi,z)dz,$$

and the drift is $\tilde{b}^a(\xi;x)=b^a(x,\xi)$.

(2) In the non-symmetric case - assumption (Kns), the nonlocal operator is just

$$\mathscr{I}^a(\xi,\psi)=-k^a(\xi,0)(-\Delta)^{1/2}\psi(\xi)$$

whereas the drift adds an extra term $\tilde{b}^a(\xi;x)=b^a(x,\xi)+b^a_K(\xi)$, with $b^a_K:\mathbb{R}^d\to\mathbb{R}^d$ given by

$$b^a_K(\xi)=\int_B\big(k^a(\xi,z)-k^a(\xi,0)\big)\frac{z}{|z|^{d+1}}dz.$$

In what follows, proofs are nowhere different in the symmetric or the non-symmetric case. This explains why we want to keep everything under a unified notation.

The well-posedness of problem (17) is standard [6, 8, 29], except for few arguments due to the lack of comparison. We show that the corrector is Lipschitz continuous and give in addition a *fine estimate* for the Lipschitz constant. This estimate plays a central role in establishing a comparison principle for the effective equation, which in turn will be helpful in establishing homogenization.

Theorem 4.1. *Let $(f^a)_{a\in\mathscr{A}}$ and $(b^a)_{a\in\mathscr{A}}$ be two families of bounded functions on $\mathbb{R}^{2d}$, satisfying $(H0)$, $(H1)$ with respect to the fast variable ξ and with Hölder exponents respectively $\alpha,\beta\in(0,1]$. Let $(K^a)_{a\in\mathscr{A}}$ be a family of kernels satisfying $(K0)-(K3)$ with Hölder exponent $\gamma\in(1/2,1]$. Then, for any $x,p\in\mathbb{R}^d$ and $u\in\mathscr{C}^2(B_\rho(x))\cap L^\infty(\mathbb{R}^d)$ for some $\rho\in(0,1]$, there exists a unique constant $\lambda\in\mathbb{R}$ so that problem (17) has a Lipschitz continuous, periodic viscosity solution ψ. Moreover, ψ satisfies the following Lipschitz bound: there exists $\sigma\in(0,\min(\alpha,\beta,\gamma))$ such that, for all $\xi_1,\xi_2\in\mathbb{R}^d$,*

$$|\psi(\xi_1)-\psi(\xi_2)|\le C_\sigma(1+|p|+C_\rho^{x,u})^{\frac{1}{1+\sigma}}|\xi_1-\xi_2|, \tag{18}$$

where $C_\sigma>0$ is a constant depending on α, $||\psi||_\infty$, and $C_\rho^{x,u}$ is given by

$$C_\rho^{x,u}:=||D^2u||_{L^\infty(B_\rho(x))}\rho+|Du(x)||\ln(\rho)|+||u||_\infty\rho^{-1}. \tag{19}$$

Remark 3. *In the case of symmetric kernels, the compensator is not needed and the constant writes*

$$C_\rho^{x,u}:=||D^2u||_{L^\infty(B_\rho(x))}\rho+||u||_\infty\rho^{-1}.$$

Proof of Theorem 4.1. In view of the available regularity estimates, we rely on a new comparison principle for general Lévy measures, shown in Proposition 6.1 of the Appendix. Then, the proof follows the same arguments as for instance in [9, 12], where measures were of Lévy-Itô type and comparison was for free (see [11]). We provide here the main ideas of the proof.

Fix $x,p\in\mathbb{R}^d$ and $u\in\mathscr{C}^2(B_\rho(x))\cap L^\infty(\mathbb{R}^d)$ with $\rho\in(0,1]$. Let $\delta>0$ and consider the approximated problem

$$\delta\psi^\delta+\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi^\delta)-\tilde{b}^a(\xi;x)\cdot D\psi^\delta(\xi)-\tilde{f}^a(\xi;x,p,u)\}=0. \tag{20}$$

Lemma 4.2. *There exists a Lipschitz continuous viscosity solution ψ^δ of problem (21).*

Proof of Lemma 4.2. We use a vanishing-coercivity argument in ordr to establish th existence of a uniformly continuous solution. More precisely, for any $\eta>0$, consider the coercive problem

$$\delta\psi^{\delta,\eta}+\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi^{\delta,\eta})-\tilde{b}^a(\xi;x)\cdot D\psi^{\delta,\eta}(\xi)-\tilde{f}^a(\xi;x,p,u)\}+\eta|D\psi^{\delta,\eta}|^2=0, \tag{21}$$

which in view of the results of [12] admits a Hölder continuous viscosity solution. In view of Theorem 3.1 the solutions are Lipschitz continuous, with a Lipschitz norm independent of η. Indeed, in order to cope with the quadratic (but autonomous) gradient term, one should look at the approximated equation with $|D\psi^{\delta,\eta}|$ replaced by $\max(|D\psi^{\delta,\eta}|,R)$, for $R>0$, and remark that its solutions are Lipschitz continuous, with the Lipschitz norm independent of R. Moreover, if we denote $M=\sup_{a\in\mathscr{A}}||\tilde{f}^a||_\infty$, we note that $||\psi^{\delta,\eta}||_\infty\le M/\delta$. Thus, passing to the limit, it follows that there exists a Lipschitz continuous solution of (21) which satisfies $||\psi^\delta||_\infty\le M/\delta$. □

Consider the sequence of functions

$$\tilde{\psi}^{\delta}(\xi) := \psi^{\delta}(\xi) - \psi^{\delta}(0),$$

which satisfy the equation

$$\delta\tilde{\psi}^{\delta} + \sup_{a\in\mathscr{A}}\{-\mathscr{I}^{a}(\xi,\tilde{\psi}^{\delta}) - \tilde{b}^{a}(\xi;x)\cdot D\tilde{\psi}^{\delta}(\xi) - \tilde{f}^{a}(\xi;x,p,u)\} = -\delta\psi^{\delta}(0).$$

In view of the strong maximum principle (see [21]), it can be shown as in [9] that the above family of functions is precompact. Indeed, the following holds.

Lemma 4.3. *The sequence $\{\tilde{\psi}^{\delta}(\cdot)\}_{\delta}$ is uniformly bounded and uniformly Lipschitz continuous.*

Proof of Lemma 4.3. We argue by contradiction and assume there exists a subsequence for which the associated sequence of norms blows up, i.e. $||\tilde{\psi}^{\delta}||_{\infty}\to\infty$, as $\delta\to 0$. Consider the renormalized functions

$$\hat{\psi}^{\delta}(\xi) = \frac{\tilde{\psi}^{\delta}(\xi)}{||\tilde{\psi}^{\delta}||_{\infty}},$$

which satisfy the equation

$$\delta\hat{\psi}^{\delta} + \sup_{a\in\mathscr{A}}\left\{-\mathscr{I}^{a}(\xi,\hat{\psi}^{\delta}) - \tilde{b}^{a}(\xi;x)\cdot D\hat{\psi}^{\delta}(\xi) - \frac{\tilde{f}^{a}(\xi;x,p,u)}{||\tilde{\psi}^{\delta}||_{\infty}}\right\} = -\frac{\delta\psi^{\delta}(0)}{||\tilde{\psi}^{\delta}||_{\infty}}.$$

Since the renormalized functions all have norm $||\hat{\psi}^{\delta}||_{\infty}=1$, it follows from Theorem 3.1 that the family is equi-Lipschitz continuous. Thus, by the Ascoli-Arzela theorem, there exists a subsequence of periodic functions $\{\hat{\psi}^{\delta_n}(\cdot)\}_{\delta_n}$ which converges locally uniformly - and globally in view of the periodicity -, to a function $\hat{\psi}$ satisfying the equation

$$\sup_{a\in\mathscr{A}}\left\{-\mathscr{I}^{a}(\xi,\hat{\psi}) - \tilde{b}^{a}(\xi;x)\cdot D\hat{\psi}(\xi)\right\} = 0.$$

The latter equation satisfies the strong maximum principle (see [21]), while its solution has $||\hat{\psi}||_{\infty} = 1$ and $\hat{\psi}(0)=0$, which leads to a contradiction. Thus, the sequence of functions $\{\tilde{\psi}^{\delta_n}(\cdot)\}_{\delta_n}$ is uniformly bounded. In view of Theorem 3.1, the family is also uniformly Lipschitz continuous. □

In view of Ascoli-Arzela theorem, there exists a subsequence $\left(\tilde{\psi}^{\delta_n}\right)_{\delta_n}$ which converges locally uniformly (and globally due to periodicity) to a periodic, Lipschitz continuous function

$$\psi = \lim_{\delta_n\to 0}\psi^{\delta_n}.$$

Moreover $\left(\delta_n\psi^{\delta_n}(0)\right)_{\delta_n}$ is bounded and, up to a subsequence, there exists a constant $\lambda\in\mathbb{R}$, so that

$$\lambda = -\lim_{\delta_n\to 0}\delta\psi^{\delta_n}(0).$$

The uniqueness of the constant λ follows from the comparison principle stated in Proposition 6.1.

Furthermore, in view of Theorem 3.1, we obtain the following Lipschitz estimate for the corrector. In view of $(K3)$, there exists a constant $C>0$ such that, for any $a\in\mathscr{A}$, and for all $\xi_1,\xi_2\in\mathbb{R}^d$,

$$\begin{aligned}
|\mathscr{L}^a(x,\xi_1,u)-\mathscr{L}^a(x,\xi_2,u)| &\leq \|D^2u\|_{L^\infty(B_\rho(x))}\int_{B_\rho}|z|^2|K^a(\xi_1,z)-K(\xi_2,z)|dz+\\
&\quad |Du(x)|\int_{B\setminus B_\rho}|z||K^a(\xi_1,z)-K(\xi_2,z)|dz+\\
&\quad 2\|u\|_{L^\infty(B^c_\rho(x))}\int_{\mathbb{R}^d\setminus B_\rho}|K^a(\xi_1,z)-K(\xi_2,z)|dz\\
&\leq C_K\Big(||D^2u||_{L^\infty(B_\rho(x))}\rho+|Du(x)||\ln(\rho)|+2||u||_\infty\rho^{-1}\Big)|\xi_1-\xi_2|^\gamma\\
&\leq 2C_K\,C_\rho^{x,u}\,|\xi_1-\xi_2|^\gamma,
\end{aligned}$$

where $C_\rho^{x,u}$ is given by (19). In view of assumption $(H1)$ it follows that, for any $a\in\mathscr{A}$, and for all $\xi_1,\xi_2\in\mathbb{R}^d$,

$$\begin{aligned}
\left|\tilde f^a(\xi_1;x,p,u)-\tilde f^a(\xi_1;x,p,u)\right| &\leq \left(C_f|\xi_1-\xi_2|^\alpha+C_b|p||\xi_1-\xi_2|^\beta+2C_K\,C_\rho^{u,x}|\xi_1-\xi_2|^\gamma\right)\\
&\leq \max(C_b,C_f,2C_K)\left(1+|p|+C_\rho^{x,u}\right)|\xi_1-\xi_2|^{\min(\alpha,\beta,\gamma)}.
\end{aligned}$$

Thus, $\tilde f^a$ is Hölder continuous in ξ, with Hölder coefficient $\tilde\alpha=\min(\alpha,\beta,\gamma)$. In view of Theorem 3.1, we conclude that for each $\sigma\in(0,\min(\alpha,\beta,\gamma))$, there exists $C_\sigma>0$ depending on $||\psi||_\infty$ such that, for all $\xi_1,\xi_2\in\mathbb{R}^d$, it holds

$$|\psi(\xi_1)-\psi(\xi_2)|\leq C_\sigma\left(1+|p|+C_\rho^{x,u}\right)^{\frac{1}{1+\sigma}}|\xi_1-\xi_2|.$$

□

Remark 4. *The Lipschitz estimate* (4.1) *holds for the approximate corrector ψ^δ as well.*

4.2. **The effective Hamiltonian.** The ergodic constant in Theorem 4.1 has a local dependence on $x,p\in\mathbb{R}^d$, and a nonlocal dependence with respect to $u\in\mathscr{C}^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. To display explicitly this dependence, we hereafter write

$$\lambda=\overline{H}(x,p,u),$$

and call $\overline{H}$ the *effective Hamiltonian*, which is well defined as a global function

$$\overline{H}:\mathbb{R}^d\times\mathbb{R}^d\times\mathscr{C}^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\to\mathbb{R}.$$

Remark 5. *In fact, in view of Theorem 4.1, for fixed $(x,p)\in\mathbb{R}^{2d}$, the effective Hamiltonian is well defined for functions which are only in $\mathscr{C}^2(B_\rho(x))\cap L^\infty(\mathbb{R}^d)=:\mathscr{E}^x_\rho$, for some $\rho\in(0,1]$. Denote $\mathscr{E}^x=\bigcup_{\rho>0}\mathscr{E}^x_\rho$ and introduce the space*

$$\mathscr{E}:=\left\{(x,u)\in\mathbb{R}^d\times L^\infty(\mathbb{R}^d):\ \textit{there exists } \rho>0 \textit{ s.t. } u\in\mathscr{C}^2(B_\rho(x))\right\}$$

One could consider $\overline{H}$ as a function

$$\begin{aligned}
&\overline{H}:\mathbb{R}^d\times\mathscr{E}\to\mathbb{R}\\
&\overline{H}(p,(x,u))=\lambda.
\end{aligned}$$

This turns out to be useful when viscosity solutions associated to the effective Hamiltonian are employed. Similar to viscosity solutions associated to the original problem (1)*, or its stationary variant,*

when dealing with the nonlocal term it is often convenient to replace $\mathscr{C}^2(\mathbb{R}^d)$ test functions ϕ by their local truncation around x in a small neighbourhood, namely by $\mathbf{1}_{B_\rho(x)}\phi + \mathbf{1}_{B_\rho(x)}u$. However, since the nonlocal dependence of the effective Hamiltonian is not explicit, we will not to be able to give (later on) equivalent definitions of viscosity solutions in terms of smooth or less regular test functions. In this sense, it is crucial for $\overline{H}$ to make sense for locally $\mathscr{C}^2(B_\rho(x))$ functions.

Remark 6. *Note in addition that, for fixed $p \in \mathbb{R}^d$, one can write $\overline{H}$ as a function*

$$\overline{H}_p : \mathscr{C}^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \to \mathscr{F}(\mathbb{R}^d)$$
$$\overline{H}_p[u](x) = \overline{H}(x,p,u),$$

where $\mathscr{F}(\mathbb{R}^d)$ is the space of all functions $h : \mathbb{R}^d \to \mathbb{R}$.

We will see below that in fact $\overline{H}$ maps $\mathscr{C}^2$ functions into continuous functions, is convex in u and in p, and it satisfies a global comparison principle. We will use the space $\mathscr{C}^{2,\sigma}(\mathbb{R}^d)$ to be the collection of functions u, with continuous second derivatives on $\mathbb{R}^d$ with $||u||_{\mathscr{C}^{0,\sigma}(\mathbb{R}^d)}$, $||Du||_{\mathscr{C}^{0,\sigma}(\mathbb{R}^d)}$, $||D^2u||_{\mathscr{C}^{0,\sigma}(\mathbb{R}^d)}$ all finite. More precisely, the following structural properties hold for $\overline{H}$.

Proposition 4.4. *Let $(f^a)_{a\in\mathscr{A}}$ and $(b^a)_{a\in\mathscr{A}}$ be two families of bounded functions on $\mathbb{R}^{2d}$, satisfying $(H0)$ and $(H1)$ with respect to both variables with $\tilde{d} = 2d$ and with Hölder exponents respectively $\alpha, \beta \in (0,1]$. Let $(K^a)_{a\in\mathscr{A}}$ be a family of kernels satisfying $(K0)-(K3)$ with Hölder exponent $\gamma \in (1/2, 1]$. Then, the effective Hamiltonian satisfies the following properties.*

(1) *Fix $(x,p) \in \mathbb{R}^d$ and let $u_1, u_2 \in \mathscr{E}^x$. Then*

$$\overline{H}(x,p,u_1) - \overline{H}(x,p,u_2) \geq -\sup_{a\in\mathscr{A}}\sup_{\xi\in\mathbb{R}^d}\left(\mathscr{L}^a(x,\xi,u_1) - \mathscr{L}^a(x,\xi,u_2)\right).$$

In particular, $\overline{H}$ satisfies the global comparison principle *: if $u_1, u_2 \in \mathscr{E}^x$ such that $u_1 \leq u_2$ in $\mathbb{R}^d$ and $u_1(x) = u_2(x)$, then $\overline{H}(x,p,u_1) \geq \overline{H}(x,p,u_2)$.*

(2) *For any $(x,p) \in \mathbb{R}^{2d}$, $\overline{H}(x,p,\cdot)$ is convex, i.e. for any $u_1, u_2 \in \mathscr{E}^x$ and $s \in (0,1)$,*

$$\overline{H}(x,p,su_1 + (1-s)u_2) \leq s\overline{H}(x,p,u_1) + (1-s)\overline{H}(x,p,u_2).$$

(3) *There exists a constant $B > 0$ such that for all $x \in \mathbb{R}^d$, $u \in \mathscr{E}^x$ and $p_1, p_2 \in \mathbb{R}^d$,*

$$\left|\overline{H}(x,p_1,u) - \overline{H}(x,p_2,u))\right| \leq B\left|p_1 - p_2\right|.$$

(4) *Fix $p \in \mathbb{R}^d$. Then $\overline{H}_p : C^{2,\sigma}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \to C^{0,\sigma}(\mathbb{R}^d)$, for any $\sigma \in (0, \min(\alpha,\beta,\gamma))$, i.e. for any $u \in C^{2,\sigma}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a constant $C = C(p,u) > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^d$,*

$$\left|\overline{H}(x_1,p,u) - \overline{H}(x_2,p,u)\right| \leq C|x_1 - x_2|^\sigma.$$

Remark 7. *In most cases, little can said about the nonlocal structure of the nonlocal operator. It is known for instance, that if a nonlocal operator satisfies the global maximum principle, is* linear *and maps $C^2(\mathbb{R}^d)$ into $C(\mathbb{R}^d)$, then it takes the Courrège form (see Theorem 1.5 in [22]). In our setup, the mapping of $\overline{H}$ from $C^{2,\sigma}(\mathbb{R}^d)$ to $C^{0,\sigma}(\mathbb{R}^d)$ is convex, so it is natural to expect that $\overline{H}$ takes the Bellman form over the Courrége operators. However, no rigorous result is proven in this respect.*

Proof. We show each of these points separately, though a global argument could be applied.

(1) Fix $(x,p)\in\mathbb{R}^d\times\mathbb{R}^d$ and let $u_1,u_2\in\mathscr{E}^x$. Consider the triplets (x,p,u_1) and (x,p,u_2) and denote their corresponding approximate correctors ψ_1^δ and ψ_2^δ, which solve the equations

$$\begin{aligned}\delta\psi_1^\delta(\xi)+\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_1^\delta)-\tilde{b}^a(\xi;x)\cdot D\psi_1^\delta(\xi)-\tilde{f}^a(\xi;x,p,u_1)\} &= 0,\\ \delta\psi_2^\delta(\xi)+\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_2^\delta)-\tilde{b}^a(\xi;x)\cdot D\psi_2^\delta(\xi)-\tilde{f}^a(\xi;x,p,u_2)\} &= 0.\end{aligned}$$

It is easy to see that ψ_1^δ is a viscosity subsolution for

$$\begin{aligned}\delta\psi_1^\delta(\xi) \;+\; &\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_1^\delta)-\tilde{b}^a(\xi;x)\cdot D\psi_1^\delta(\xi)-\tilde{f}^a(\xi;x,p,u_2)\}\\ \le\; &\sup_{a\in\mathscr{A}}\left(\mathscr{L}^a(x,\xi,u_1)-\mathscr{L}^a(x,\xi,u_2)\right).\end{aligned}$$

Taking into account that x is a local maximum of u_1-u_2 and that $u_1,u_2\in\mathscr{C}^2(B_\rho(x))$, it follows that $Du_1(x)=Du_2(x)$ and thus, for all $a\in\mathscr{A}$,

$$\mathscr{L}^a(x,\xi,u_1)-\mathscr{L}^a(x,\xi,u_2)\le 0.$$

Therefore, ψ_1^δ is a viscosity subsolution of

$$\delta\psi_1^\delta(\xi)+\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_1^\delta)-\tilde{b}^a(\xi;x)\cdot D\psi_1^\delta(\xi)-\tilde{f}^a(\xi;x,p,u_2)\}\le 0.$$

Since the approximate correctors are Lipschitz, it follows from the comparison principle for Lipschitz functions given in Proposition 6.1, that $\psi_1^\delta\le\psi_2^\delta$ in $\mathbb{R}^d$, which further leads to

$$\overline{H}(x,p,u_1)=-\lim_{\delta\to0}\delta\psi_1^\delta(0)\ge-\lim_{\delta\to0}\delta\psi_2^\delta(0)=\overline{H}(x,p,u_2).$$

(2) In order to prove convexity, under the same notations as above, consider as well for any $s\in(0,1)$ the triplet $(x,p,(1-s)u_1+su_2)$ and its approximate corrector ψ_s^δ, which solves the equation

$$\delta\psi_s^\delta(\xi)+\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_s^\delta)-\tilde{b}^a(\xi;x)\cdot D\psi_s^\delta(\xi)-\tilde{f}^a(\xi;x,p,(1-s)u_1+su_2)\}=0.$$

It is standard to check that $(1-s)\delta\psi_1^\delta+s\delta\psi_2^\delta$ is a viscosity subsolution of the above equation. In view of the comparison principle given in Proposition 6.1, it follows that

$$(1-s)\delta\psi_1^\delta+s\delta\psi_2^\delta\le\delta\psi_s^\delta,$$

which implies, as $\delta\to0$, the convexity of $\overline{H}$ with respect to u.

(3) Fix $x\in\mathbb{R}^d$, let $u\in\mathscr{E}^x$ and $p_1,p_2\in\mathbb{R}^d$. Denote by ψ_1^δ and ψ_2^δ the approximate correctors corresponding to p_1 and p_2, viscosity solutions of

$$\begin{aligned}\delta\psi_1^\delta(\xi)+\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_1^\delta)-\tilde{b}^a(\xi;x)\cdot D\psi_1^\delta(\xi)-\tilde{f}^a(\xi);x,p_1,u)\} &= 0,\\ \delta\psi_2^\delta(\xi)+\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_2^\delta)-\tilde{b}^a(\xi;x)\cdot D\psi_2^\delta(\xi)-\tilde{f}^a(\xi;x,p_2,u)\} &= 0.\end{aligned}$$

Then ψ_1 solves in the viscosity sense

$$\begin{aligned}\delta\psi_1^\delta(\xi) \;+\; &\sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_1^\delta)-\tilde{b}^a(\xi;x)\cdot D\psi_1^\delta(\xi)-\tilde{f}^a(\xi;x,p_2,u)\}\\ \le\; &\sup_{a\in\mathscr{A}}\left|\tilde{f}^a(\xi;x,p_1,u)-\tilde{f}^a(\xi;x,p_2,u)\right|\le\sup_{a\in\mathscr{A}}||b^a||_\infty|p_1-p_2|.\end{aligned}$$

In view of the comparison principle given in Proposition 6.1, it follows that, for $B = \sup_{a\in\mathscr{A}} ||b^a||_\infty$,

$$\delta\psi_1^\delta \le \delta\psi_2^\delta + B|p_1 - p_2|.$$

Reverting p_1 and p_2 we get the bound from below. Letting $\delta \to 0$, the conclusion follows.

(4) Fix $p \in \mathbb{R}^d$, let $u \in \mathscr{C}^{2,\sigma}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and take $x_1, x_2 \in \mathbb{R}^d$. Let ψ_1^δ and ψ_2^δ be the approximate correctors corresponding to x_1 and x_2, thus viscosity solutions of

$$\begin{aligned}
\delta\psi_1^\delta(\xi) + \sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_1^\delta) - \tilde{b}^a(\xi;x_1)\cdot D\psi_1^\delta(\xi) - \tilde{f}^a(\xi;x_1,p,u)\} &= 0,\\
\delta\psi_2^\delta(\xi) + \sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_2^\delta) - \tilde{b}^a(\xi;x_2)\cdot D\psi_2^\delta(\xi) - \tilde{f}^a(\xi;x_2,p,u)\} &= 0.
\end{aligned}$$

Then ψ_1^δ is a viscosity subsolution of

$$\begin{aligned}
\delta\psi_1^\delta(\xi) \;+\; & \sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_1^\delta) - b^a(\xi;x_2)\cdot D\psi_1^\delta(y) - \tilde{f}^a(\xi;x_2,u,p)\}\\
\le\; & \sup_{a\in\mathscr{A}}\left|b^a(\xi;x_1) - b^a(\xi;x_2)\right| ||D\psi_1^\delta||_\infty + |\tilde{f}^a(\xi;x_1,p,u) - \tilde{f}^a(\xi;x_2,p,u)|.
\end{aligned}$$

In view of assumption $(K1)$ the following holds, uniformly in $\xi, p \in \mathbb{R}^d$ and for all $a \in \mathscr{A}$,

$$\begin{aligned}
|\mathscr{L}^a(x_1,\xi,u) \;-\; & \mathscr{L}^a(x_2,\xi,u)|\\
\le\; & \int_0^1 (1-t)dt \int_{B_\rho} |D^2u(x_1+tz) - D^2u(x_2+tz)||z|^2\,|K^a(\xi,z)|dz +\\
& \int_0^1 tdt \int_{B\setminus B_\rho} |Du(x_1+tz) - Du(x_2+tz)||z|\,|K^a(\xi,z)|dz +\\
& \int_{B\setminus B_\rho} |Du(x_1) - Du(x_2)||z|\,|K^a(\xi,z)|dz +\\
& \int_{B^c} |(u(x_1+z) - u(x_2+z)) - (u(x_1) - u(x_2))|\,|K^a(\xi,z)|dz\\
\le\; & ||D^2u||_{C^{0,\sigma}(B(x_1))}|x_1-x_2|^\sigma \int_{B_\rho} |z|^2\,|K^a(\xi,z)|dz +\\
& 2||Du||_{C^{0,\sigma}(B(x_1))}|x_1-x_2|^\sigma \int_{B\setminus B_\rho} |z|\,|K^a(\xi,z)|dz +\\
& 2||u||_{C^{0,\sigma}(\mathbb{R}^d)}|x_1-x_2|^\sigma \int_{B^c} |K^a(\xi,z)|dz\\
\le\; & C||u||_{C^{2,\sigma}(\mathbb{R}^d)}|x_1-x_2|^\sigma,
\end{aligned}$$

where C is a universal constant. In view of the regularity assumption $(H1)$, the previous inequality leads to

$$\begin{aligned}
|\tilde{f}^a(\xi;x_1,p,u) - \tilde{f}^a(\xi;x_2,p,u)| &\le C_f|x_1-x_2|^\alpha + C_b|x_1-x_2|^\beta|p| + C||u||_{C^{2,\sigma}}|x_1-x_2|^\sigma\\
&\le C\Big(1+|p|+||u||_{C^{2,\sigma}(\mathbb{R}^d)}\Big)|x_1-x_2|^\sigma.
\end{aligned}$$

The Lipschitz regularity of the approximate corrector, implies that, for any $\rho \in (0,1)$,

$$||D\psi_1^\delta||_\infty \le C\left(1+|p|+C_\rho^{x_1,u}\right)^{\frac{1}{1+\sigma}}.$$

In particular for $\rho = 1$, we have $C_1^{x_1,u} \le ||u||_{C^2(\mathbb{R}^d)}$ and hence there exits $C > 0$ so that

$$||D\psi_1^\delta||_\infty \le C_\sigma \left(1 + |p| + ||u||_{C^2(\mathbb{R}^d)}\right)^{\frac{1}{1+\sigma}}.$$

Therefore, we conclude that ψ_1^δ is a viscosity subsolution, in $\mathbb{R}^d$, of

$$\begin{aligned} \delta\psi_1^\delta(\xi) \quad + \quad & \sup_{a\in\mathscr{A}}\{-\mathscr{I}^a(\xi,\psi_1^\delta) - b^a(\xi;x_2)\cdot D\psi_1^\delta(\xi) - \tilde{f}^a(\xi;x_2,u,p)\} \\ \le \quad & C\Big(\left(1 + |p| + ||u||_{C^2(\mathbb{R}^d)}\right)^{\frac{1}{1+\sigma}} + \left(1 + |p| + ||u||_{C^{2,\sigma}(\mathbb{R}^d)}\right)\Big)|x_1 - x_2|^\sigma, \end{aligned}$$

up to a modification of the universal constant C. In view of the comparison principle for Lipschitz functions, given in Proposition 6.1, it follows that there exists a constant

$$C(p,u) := C\Big(\left(1 + |p| + ||u||_{C^2(\mathbb{R}^d)}\right)^{\frac{1}{1+\sigma}} + \left(1 + |p| + ||u||_{C^{2,\sigma}(\mathbb{R}^d)}\right)\Big)$$

such that, uniformly in δ and ξ,

$$\delta\psi_1^\delta(\xi) - \delta\psi_2^\delta(\xi) \le C(p,u)|x_1 - x_2|^\sigma.$$

Reverting x_1 and x_2 we get the lower bound. Thus, letting $\delta \to 0$, the conclusion follows. □

We give in the following corollary the global behaviour of $\overline{H}$ with respect to all of its variables and give an ellipticity growth condition. This turns out to be fundamental in order to perform later on a linearization for the effective problem (see the following section). The result strongly relies on the Lipschitz estimate of the solution to the cell problem (17) given by Theorem 4.1.

Corollary 4.5. *Let the same assumptions as in Proposition 4.4 hold. For any $x_1, x_2, p_1, p_2 \in \mathbb{R}^d$, $u_1 \in \mathscr{E}_\rho^{x_1}$ and $u_2 \in \mathscr{E}_\rho^{x_2}$ with $\rho > 0$, the following holds, for any $\sigma \in (0, \min(\alpha,\beta,\gamma))$,*

$$\begin{aligned} \overline{H}(x_2,p_2,u_2) - \overline{H}(x_1,p_1,u_1) \quad \le \quad & C\Big(\left(1 + |p_1| + C_\rho^{x_1,u_1}\right)^{\frac{1}{1+\sigma}} + \left(1 + |p_1|\right)\Big)|x_1 - x_2|^{\min(\alpha,\beta)} \\ & + \sup_{a\in\mathscr{A}} ||b^a||_\infty |p_1 - p_2| + \sup_{\substack{a\in\mathscr{A} \\ \xi\in\Pi^d}} \left(-\mathscr{L}^a(x_2,\xi,u_2) + \mathscr{L}^a(x_1,\xi,u_1)\right), \end{aligned}$$

where $C_\rho^{x_1,u_1} = ||D^2u_1||_{L^\infty(B_\rho(x_1))}\rho + |Du_1(x_1)||\ln(\rho)| + ||u_1||_\infty\rho^{-1}$.

Proof. It is easy to see from the previous proof that, the following improved estimate holds for the global variables. This is due to the fact that we drop the estimate of the nonlocal terms $\mathscr{L}^a$ which appear in the definition of $\tilde{f}^a$. Indeed, the $C^{2,\sigma}$ norm of u appearing in the computation at the end of the proof of Proposition 4.4 and stemming from the estimate of the nonlocal terms does not appear in the statement of the corollary. However, we need to keep the original estimate of the Lipschitz constant for the corrector $D\psi_1^\delta$, namely $C_\rho^{x_1,u_1}$. □

5. The homogenization

We establish in this section the homogenization result for problem (1). More precisely, we show that the viscosity solutions $\left(u^\varepsilon\right)_{\varepsilon>0}$ of (1) converge locally uniformly to the solution of the averaged equation (5). The proof uses the perturbed test function method, which is standard and we do not detail here. Nonetheless, the uniqueness of the limit for convergent subsequences is not straightforward, since linearization does not go hand in hand with the viscosity solution theory approach

and difficulties imposed by the x dependence and the behaviour of the measure near the singularity might appear. This is due to the fact that the effective Hamiltonian is implicitly defined and its linearization is based on the variable-dependence given in Corollary 4.5. The Lipschitz regularity result and in particular the fine estimate of the Lipschitz constant play a central role in the linearization procedure.

We start by noting that, in view of Corollary 4.5, the regularity results for weakly elliptic nonlocal operators obtained in [8, 10] apply and solutions for the effective problem are Hölder continuous.

Proposition 5.1. *Let $(f^a)_{a\in\mathscr{A}}$ and $(b^a)_{a\in\mathscr{A}}$ be two families of bounded functions on $\mathbb{R}^{2d}$, satisfying $(H1)$ with respect to both variables with $\alpha,\beta\in(0,1]$. Let $(K^a)_{a\in\mathscr{A}}$ be a family of kernels satisfying $(K1)-(K3)$ with $\gamma\in(0,1]$. Then any bounded continuous viscosity solution $u:\mathbb{R}^d\times[0,T]\to\mathbb{R}$ of (5) is Hölder continuous in space, i.e. there exists $\tau\in(0,1)$ such that for all $t\in[0,T]$, $u(\cdot,t)\in\mathscr{C}^{0,\tau}(\mathbb{R}^d)$.*

Proof. Note that we cannot literally apply Theorem 1 of [10] as we do not have an explicit formulation for $\overline{H}$ and hence the ellipticity-growth condition (H) of [10] cannot be checked. Nonetheless, it is enough to remark that the right hand side of the ellipticity-growth condition (H) plays the central role in getting the regularity. Making use of Corollary 4.5 we get a similar expression for the effective Hamiltonian $\overline{H}$. Namely, in our case the functions $\Lambda_1\equiv 0$ and $\Lambda_2\equiv 1$ and the nonlocal difference l_1-l_2 in (H) is just the explicit expression $\sup_{\substack{a\in\mathscr{A}\\ \xi\in\Pi^d}}\left(\mathscr{L}^a(x_1,\xi,u_1)-\mathscr{L}^a(x_2,\xi,u_2)\right)$ (which could have also been directly written in [10]). The only term we need to exploit in our case is the (first) one having a nonlinear dependence between the space variable x, the gradient variable p and the function u - given in terms of the constant $C_\rho^{x,u}$.

Recall that in order to prove Hölder regularity a radial penalty function of the form $\varphi(|x-y|)=L|x-y|^\tau$ is considered and estimates are made within the viscosity inequalities. In our case, it is enough to consider the following parameters in Corollary 4.5 above, $\rho=\rho_0|x_1-x_2|$, and u_1 a fonction satisfying $p_1=Du_1(x_1)$, and $||D^2u_1||_{L^\infty(B_\rho(x_1))}\le C|p_1|\rho^{-1}$. The constant $C_\rho^{x_1,u_1}$ then becomes $C_\rho^{x_1,u_1}=C|p_1|+|p_1|\,|\ln\left(\rho_0|x_1-y_1|\right)|+C\rho_0^{-1}|x_1-y_1|$ and the first term in the bound of $\overline{H}(x_2,p_2,u_2)-\overline{H}(x_1,p_1,u_1)$ is given, up to a modification fo the universal constant C, by

$$C\Big(\left(1+|p_1|+|p_1|\,|\ln\left(\rho_0|x_1-y_1|\right)|+C\rho_0^{-1}|x_1-y_1|\right)^{\frac{1}{1+\sigma}}+\left(1+|p_1|\right)\Big)|x_1-x_2|^{\min(\alpha,\beta)}.$$

This is enough to reach the same conclusion. □

A priori regularity of the solution further permits to establish a linearization result for the effective problem, which is formulated in terms of the extremal Pucci operators

$$\mathscr{M}^+(x,\phi)=\sup_{a\in\mathscr{A}}\sup_{\xi\in\mathbb{R}^d}\mathscr{L}^a(x,\xi,\phi),\quad \mathscr{M}^-(x,\phi)=\sup_{a\in\mathscr{A}}\inf_{\xi\in\mathbb{R}^d}\mathscr{L}^a(x,\xi,\phi).$$

Proposition 5.2. *Let $(f^a)_{a\in\mathscr{A}}$ and $(b^a)_{a\in\mathscr{A}}$ be two families of bounded functions on $\mathbb{R}^{2d}$, satisfying $(H0)$ and $(H1)$ with respect to both variables with $\tilde{d}=2d$ and $\alpha=\beta=1$. Let $(K^a)_{a\in\mathscr{A}}$ be a family of kernels satisfying $(K0)-(K3)$ with $\gamma=1$. Assume in addition that (Ks) or (Kns) hold. Let $u\in USC(\mathbb{R}^d\times[0,T])$ and $v\in LSC(\mathbb{R}^d\times[0,T])$ be respectively a viscosity subsolution and viscosity supersolution of equation (5).*

(1) *If $v(\cdot,t)\in\mathscr{C}^{0,\tau}(\mathbb{R}^d)$ for all $t\in[0,T]$ with $\tau\in(0,1)$, then $w=u-v$ is a viscosity subsolution of*

$$w_t-\mathscr{M}^+(x,w(\cdot,t))-B|Dw|=0\quad in\quad \mathbb{R}^d\times[0,T],$$

(2) *If $u(\cdot,t)\in\mathscr{C}^{0,\tau}(\mathbb{R}^d)$ for all $t\in[0,T]$ with $\tau\in(0,1)$, then $w=v-u$ is a viscosity supersolution of*

$$w_t+\mathscr{M}^-(x,w(\cdot,t))+B|Dw|=0 \quad \text{in} \quad \mathbb{R}^d\times[0,T],$$

where $B=\sup_{a\in\mathscr{A}}||b^a||_\infty$.

Proof. Fix $(x_0,t_0)\in\mathbb{R}^d\times(0,T)$ and $\rho'>0$ and let $\varphi\in\mathscr{C}^2(\mathbb{R}^d\times[0,T])$ such that $w-\varphi$ has a strict maximum at (x_0,t_0) in $B_{\rho'}(x_0,t_0)$. We want to show that

$$\varphi_t(x_0,t_0)-\mathscr{M}^+\big(x_0,\mathbf{1}_{B_{\rho'}(x_0)}\varphi(\cdot,t_0)+\mathbf{1}_{B^c_{\rho'}(x_0)}w(\cdot,t_0)\big)-B|D\varphi(x_0,t_0)|\le 0. \tag{22}$$

Consider, for $\epsilon>0$, the function

$$\phi(x,y,t,s)=\varphi(x,t)+\frac{|x-y|^2}{\epsilon^2}+\frac{(t-s)^2}{\epsilon^2}+\psi_\zeta(x),$$

where $\psi_\zeta(x):=\psi(\zeta x)$ is a localisation term, with a choice of a smooth function $\psi\ge 0$, satisfying $\psi=0$ in B and $\psi\ge 1+\|u\|_\infty+\|v\|_\infty+\|\varphi\|_\infty$ outside B_2.

Since (x_0,t_0) is a strict global maximum for $u(x,t)-v(x,t)-\varphi(x,t)$, for ϵ sufficiently small, there exists a sequence of points $(x_\epsilon,y_\epsilon,t_\epsilon,s_\epsilon)$ which are local maxima respectively for

$$\Phi(x,y,t,s):=u(x,t)-v(y,s)-\phi(x,y,t,s).$$

It follows, from the inequality $\Phi(x_\epsilon,x_\epsilon,t_\epsilon,s_\epsilon)\le\Phi(x_\epsilon,y_\epsilon,t_\epsilon,s_\epsilon)$ and the regularity of v, that

$$\frac{|t_\epsilon-s_\epsilon|^2}{\epsilon^2}\le v(x_\epsilon,s_\epsilon)-v(y_\epsilon,s_\epsilon)\le 2||v||_\infty,$$

and

$$\frac{|x_\epsilon-y_\epsilon|^2}{\epsilon^2}\le v(x_\epsilon,s_\epsilon)-v(y_\epsilon,s_\epsilon)\le C|x_\epsilon-y_\epsilon|^\tau.$$

Therefore, the following holds

$$|t_\epsilon-s_\epsilon|\le C\epsilon^2, \qquad |x_\epsilon-y_\epsilon|\le C\epsilon^{2/(2-\tau)}. \tag{23}$$

In particular, $(x_\epsilon,y_\epsilon,t_\epsilon,s_\epsilon)\to(x_0,x_0,t_0,t_0)$ as $\epsilon\to 0$ for any fixed $\zeta>0$. To simplify notation we dropped their dependence in ζ.

Let

$$\phi^u(x,t)=v(y_\epsilon,s_\epsilon)+\phi(x,y_\epsilon,t,s_\epsilon),$$
$$\phi^v(y,s)=u(x_\epsilon,t_\epsilon)-\phi(x_\epsilon,y,t_\epsilon,s),$$

where for convenience of notations we have dropped the ϵ-dependence in ϕ^u and ϕ^v. Note that (x_ϵ,t_ϵ) is a maximum of $u-\phi^u$ in $B_\rho(x_\epsilon,t_\epsilon)$, whereas (y_ϵ,s_ϵ) is a minimum of $v-\phi^v$ in $B_\rho(y_\epsilon,s_\epsilon)$, for $\rho\in(0,\rho')$ sufficiently small. We will eventually choose $\rho=\epsilon^r$ with $r>0$ yet to be determined, and let $\epsilon\to 0$, then $\zeta\to 0$.

Let

$$\tilde u_\rho(\cdot,t)=\mathbf{1}_{B_\rho(x_\epsilon)}\phi^u(\cdot,t)+\mathbf{1}_{B^c_\rho(x_\epsilon)}u(\cdot,t),$$
$$\tilde v_\rho(\cdot,s)=\mathbf{1}_{B_\rho(y_\epsilon)}\phi^v(\cdot,s)+\mathbf{1}_{B^c_\rho(y_\epsilon)}v(\cdot,s).$$

The viscosity inequalities for the sub and supersolution then read

$$\phi^u_t(x_\epsilon,t_\epsilon)+\overline{H}\big(x_\epsilon,D\phi^u(x_\epsilon,t_\epsilon),\tilde u_\rho(\cdot,t_\epsilon)\big)\le 0,$$
$$\phi^v_t(y_\epsilon,s_\epsilon)+\overline{H}\big(y_\epsilon,D\phi^y(y_\epsilon,s_\epsilon),\tilde v_\rho(\cdot,s_\epsilon)\big)\ge 0.$$

Subtracting the two inequalities above, it follows, in view of Corollary 4.5, that

$$\begin{aligned}\varphi_t(x_\epsilon,t_\epsilon) &\leq \overline{H}\big(y_\epsilon, D\phi^v(y_\epsilon,s_\epsilon),\tilde{v}_\rho(\cdot,s_\epsilon)\big)-\overline{H}\big(x_\epsilon,D\phi^u(x_\epsilon,t_\epsilon),\tilde{u}_\rho(\cdot,t_\epsilon)\big) \\ &\leq \mathscr{Q}^u_\epsilon|x_\epsilon-y_\epsilon|+B\left|D\varphi(x_\epsilon,t_\epsilon)\right|+\sup_{\substack{a\in\mathscr{A}\\ \xi\in\mathbb{R}^d}}\Big\{\mathscr{L}^a(x_\epsilon,\xi,\tilde{u}_\rho(\cdot,t_\epsilon))-\mathscr{L}^a(y_\epsilon,\xi,\tilde{v}_\rho(\cdot,s_\epsilon))\Big\},\end{aligned} \tag{24}$$

where $B=\sup_{a\in\mathscr{A}}||b^a||_\infty$ and

$$\mathscr{Q}^u_\epsilon := C\big(1+|D\phi^u(x_\epsilon,t_\epsilon)|+C_{\rho,\epsilon}\big)^{\frac{1}{1+\sigma}}+1+|D\phi^u(x_\epsilon,t_\epsilon)|,$$

with $C>0$ a universal constant and $C_{\rho,\epsilon}$ a constant depending on $\tilde{u}_\rho$ given by (19).

Each of the terms above is further estimated as $\epsilon\to 0$. We start with the first term. Note that the constant $C_{\rho,\epsilon}$ herein translates into

$$\begin{aligned}C_{\rho,\epsilon} &= ||D^2\phi^u||_{L^\infty(B_\rho(x_\epsilon,t_\epsilon))}\,\rho+|D\phi^u(x_\epsilon,t_\epsilon)|\,|\ln(\rho)|+||u||_\infty\rho^{-1} \\ &\leq \tilde{C}\Big(\big(1+\epsilon^{-2}+o_\zeta(1)\big)\rho+\big(1+|p_\epsilon|+o_\zeta(1)\big)\,|\ln(\rho)|+\rho^{-1}\Big),\end{aligned}$$

where $p_\epsilon=(x_\epsilon-y_\epsilon)/\epsilon^2$ and $\tilde{C}>0$ is a constant depending on $||\varphi||_{C^2(B_{\rho'}(x_0,t_0))}$ and $||u||_\infty$. Using (23) and the fact that we will chose ρ of the form $\rho=\epsilon^r$ with $r>0$ such that all the terms will be bounded, it follows, up to a modification of the constant C, that

$$\begin{aligned}\mathscr{Q}^u_\epsilon|x_\epsilon-y_\epsilon| &\leq C\Big(1+|p_\epsilon|+\big(1+\epsilon^{-2}\big)\rho+\big(1+|p_\epsilon|+o_\zeta(1)\big)\,|\ln(\rho)|+\rho^{-1}+o_\zeta(1)\Big)^{\frac{1}{1+\sigma}}|x_\epsilon-y_\epsilon| \\ &\quad +C\big(1+|p_\epsilon|\big)|x_\epsilon-y_\epsilon|+o_\zeta(1) \\ &\leq C\Big(o_\epsilon(1)+|x_\epsilon-y_\epsilon|^{\sigma+1}\epsilon^{-2}\rho+|x_\epsilon-y_\epsilon|^{\sigma+2}\epsilon^{-2}|\ln(\rho)|+|x_\epsilon-y_\epsilon|^{\sigma+1}\rho^{-1}\Big)^{\frac{1}{1+\sigma}}+ \\ &\quad o_\epsilon(1)+o_\zeta(1) \\ &\leq C\Big(o_\epsilon(1)+\epsilon^{\frac{2(\sigma+1)}{2-\tau}-2+r}+\epsilon^{\frac{2(\sigma+2)}{2-\tau}-2}|\ln(\epsilon)|+\epsilon^{\frac{2(\sigma+1)}{2-\tau}-r}\Big)^{\frac{1}{1+\sigma}}+o_\epsilon(1)+o_\zeta(1).\end{aligned}$$

Let $r=\frac{2(\sigma+1)}{2-\tau}-\frac{\tau}{2-\tau}$ and choose $\sigma>1-\tau/2$. Note that we strongly rely on the estimate of the Lipschitz constant for the corrector to control the terms above : the exponent σ in $\mathscr{Q}^u_\epsilon$ can be chosen arbitrarily close to one. The above estimate then writes

$$\begin{aligned}\mathscr{Q}^u_\epsilon|x_\epsilon-y_\epsilon| &\leq C\Big(o_\epsilon(1)+\epsilon^{\frac{4\sigma+\tau}{2-\tau}}+\epsilon^{\frac{2(\sigma+\tau)}{2-\tau}}|\ln(\epsilon)|+\epsilon^{\frac{\tau}{2-\tau}}\Big)^{\frac{1}{1+\sigma}}+o_\epsilon(1)+o_\zeta(1) \\ &= o_\epsilon(1)+o_\zeta(1).\end{aligned} \tag{25}$$

We now estimate the nonlocal difference. To this end, we split the domain of integration into B_ρ, $B_{\rho'}\setminus B_\rho$ and $B^c_{\rho'}$ and evaluate $\mathscr{T}^a(x_\epsilon,y_\epsilon):=\mathscr{L}^a(x_\epsilon,\xi,\tilde{u}_\rho)-\mathscr{L}^a(y_\epsilon,\xi,\tilde{v}_\rho)$. As usual, we use the

notation $\mathcal{T}^a[D]$ to specify the domain of integration D on which the nonlocal difference is computed.

$$\begin{aligned}
\mathcal{T}^a[B_\rho](x_\epsilon, y_\epsilon) &= \int_{B_\rho} \big(\phi^u(x_\epsilon + z, t_\epsilon) - \phi^u(x_\epsilon, t_\epsilon) - D\phi^u(x_\epsilon, t_\epsilon)\cdot z\big)K(\xi, z)\,dz - \\
&\quad \int_{B_\rho} \big(\phi^v(y_\epsilon + z, s_\epsilon) - \phi^v(y_\epsilon, s_\epsilon) - D\phi^v(y_\epsilon, s_\epsilon)\cdot z\big)K(\xi, z)\,dz \\
&= \int_{B_\rho} \big(\varphi(x_\epsilon + z, t_\epsilon) - \varphi(x_\epsilon, t_\epsilon) - D\varphi(x_\epsilon, t_\epsilon)\cdot z\big)K(\xi, z)\,dz + \\
&\quad \frac{2}{\epsilon^2}\int_{B_\rho} |z|^2 K(\xi, z)\,dz + \int_{B_\rho} \big(\psi_\zeta(x_\epsilon + z) - \psi_\zeta(x_\epsilon) + D\psi_\zeta(x_\epsilon)\cdot z\big)K(\xi, z)\,dz \\
&\le \mathcal{L}^a[B_\rho](x_\epsilon, \xi, \varphi(\cdot, t_\epsilon)) + \frac{2}{\epsilon^2}\int_{B_\rho} |z|^2 K(\xi, z)\,dz + \mathcal{L}^a[B_\rho](x_\epsilon, \xi, \psi_\zeta).
\end{aligned}$$

To estimate the nonlocal difference on $B_{\rho'} \setminus B_\rho$ and on $B^c_{\rho'}$ we use again the maximum property and deduce from the inequality $\Phi(x_\epsilon + z, y_\epsilon + z, t_\epsilon, s_\epsilon) \le \Phi(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon)$ that

$$\begin{aligned}
\mathcal{T}^a[B_{\rho'} \setminus B_\rho](x_\epsilon, y_\epsilon) &= \int_{B_{\rho'}\setminus B_\rho} \big(u(x_\epsilon + z, t_\epsilon) - u(x_\epsilon, t_\epsilon) - \mathbf{1}_B(z) D\phi^u(x_\epsilon, t_\epsilon)\cdot z\big)K(\xi, z)\,dz - \\
&\quad \int_{B_{\rho'}\setminus B_\rho} \big(v(y_\epsilon + z, s_\epsilon) - v(y_\epsilon, s_\epsilon) - \mathbf{1}_B(z) D\phi^v(y_\epsilon, s_\epsilon)\cdot z\big)K(\xi, z)\,dz \\
&\le \int_{B_{\rho'}\setminus B_\rho} \Big(\big(\varphi(x_\epsilon + z, t_\epsilon) - \varphi(x_\epsilon, t_\epsilon)\big) + \big(\psi_\zeta(x_\epsilon + z) - \psi_\zeta(x_\epsilon)\big) \\
&\qquad\qquad -\mathbf{1}_B(z)\big(D\varphi(x_\epsilon, t_\epsilon) + D\psi_\zeta(x_\epsilon)\big)\cdot z\Big)K(\xi, z)\,dz \\
&= \mathcal{L}^a\big[B_{\rho'} \setminus B_\rho\big](x_\epsilon, \xi, \varphi(\cdot, t_\epsilon)) + \mathcal{L}^a\big[B_{\rho'} \setminus B_\rho\big](x_\epsilon, \xi, \psi_\zeta),
\end{aligned}$$

whereas

$$\begin{aligned}
\mathcal{T}^a[B^c_{\rho'}](x_\epsilon, y_\epsilon) = \int_{B^c_{\rho'}} \Big(&\big(u(x_\epsilon + z, t_\epsilon) - u(x_\epsilon, t_\epsilon)\big) - \big(v(y_\epsilon + z, s_\epsilon) - v(y_\epsilon, s_\epsilon)\big) - \\
&-\mathbf{1}_B(z)\big(D\varphi(x_\epsilon, t_\epsilon) + D\psi_\zeta(x_\epsilon)\big)\cdot z\Big)K(\xi, z)\,dz.
\end{aligned}$$

The overall estimate becomes

$$\begin{aligned}
\mathcal{T}^a(x_\epsilon, y_\epsilon) &\le \mathcal{L}^a[B_{\rho'}](x_\epsilon, \xi, \varphi(\cdot, t_\epsilon)) + \frac{2}{\epsilon^2}\int_{B_\rho} |z|^2 K(\xi, z)\,dz + o_\zeta(1) + \\
&\quad \int_{B^c_{\rho'}} \Big(\big(u(x_\epsilon + z, t_\epsilon) - u(x_\epsilon, t_\epsilon)\big) - \big(v(y_\epsilon + z, s_\epsilon) - v(y_\epsilon, s_\epsilon)\big) - \\
&\qquad\qquad -\mathbf{1}_B(z)\big(D\varphi(x_\epsilon, t_\epsilon)\big)\cdot z\Big)K(\xi, z)\,dz.
\end{aligned}$$

Let $\rho = \epsilon^{\frac{2(\sigma+1)-\tau}{2-\tau}}$ and $\sigma > 1 - \tau/2$ as above. In view of (Ks) or (Kns), it follows that

$$\frac{2}{\epsilon^2}\int_{B_\rho} |z|^2 K(\xi, z)\,dz \le C_K \frac{2}{\epsilon^2}\rho \le \tilde{C}_K \epsilon^{\frac{2\sigma}{(2-\tau)}-1} = o_\epsilon(1).$$

Employing the dominated convergence theorem and the semi-continuity of u and continuity of v, it follows that as $\epsilon \to 0$,

$$\begin{aligned}\limsup_{\epsilon\to 0} \mathcal{T}^a(x_\epsilon, y_\epsilon) &\leq \mathcal{L}^a[B_{\rho'}](x_0,\xi,\varphi(\cdot,t_0)) + \\ &\qquad \int_{B^c_{\rho'}} \Big(w(x_0+z,t_0) - w(x_0,t_0) - \mathbf{1}_B(z) D\varphi(x_0,t_0)\cdot z\Big) K(\xi,z)\,dz + o_\zeta(1) \\ &= \mathcal{L}^a\big[B_{\rho'}\big](x_0,\xi,\varphi(\cdot,t_0)) + \mathcal{L}^a\big[B^c_{\rho'}\big](x_0,\xi,w(\cdot,t_0)) + o_\zeta(1).\end{aligned}$$

Therefore, the following overall estimate holds for the nonlocal difference

$$\begin{aligned}&\limsup_{\epsilon\to 0} \sup_{\substack{a\in\mathscr{A},\\ \xi\in\mathbb{R}^d}} \Big\{\mathcal{L}^a(x_\epsilon,\xi,\tilde{u}_\rho(\cdot,t_\epsilon)) - \mathcal{L}^a(y_\epsilon,\xi,\bar{v}_\rho(\cdot,v_\epsilon))\Big\} \qquad (26)\\ &\qquad \leq \mathcal{M}^+\big(x_0,\varphi(\cdot,t_0)\mathbf{1}_{B_{\rho'}(x_0)} + w(\cdot,t_0)\mathbf{1}_{B^c_{\rho'}(x_0)}\big) + o_\zeta(1).\end{aligned}$$

We conclude, from equations (24)-(26), letting $\epsilon \to 0$ and then $\zeta \to 0$ that (22) holds. □

Remark 8. *Lipschitz regularity of the data is necessary to linearize. This appears already in [21] when a strong comparison between subsolutions and supersolutions is shown, for Lévy-Itô integro-differential equations. However, for more general Lévy measures, as above, the result is unknown. We are able here to prove it for the effective Hamiltonian since there is an explicit dependence on the Lipschitz bound of the corrector. Recalling that the Lipschitz estimate* (10) *depends only on the exponent α of the source term (and other constants), which in the case of the corrector is $\tilde{f}^a$ and hence it involves all the datum, it is crucial that $\sigma \in (0, \min(\alpha,\beta,\gamma))$ is as close as possible to* 1*, from where the requirement that α,β,γ ought to be* 1.

We are now in shape of proving the main homogenization result.

Theorem 5.3. *Let $(f^a)_{a\in\mathscr{A}}$ and $(b^a)_{a\in\mathscr{A}}$ be two families of bounded functions on $\mathbb{R}^{2d}$, satisfying $(H0)$ and $(H1)$ with respect to both variables with $\tilde{d} = 2d$ and with $\alpha = \beta = 1$. Let $(K^a)_{a\in\mathscr{A}}$ be a family of kernels satisfying $(K0)-(K3)$ with $\gamma = 1$. Assume in addition that (Ks) or (Kns) hold. Then, the viscosity solutions $\big(u^\varepsilon\big)_{\varepsilon>0}$ of* (1) *converge locally uniformly to the unique, bounded continuous viscosity solution u of* (5).

Proof. Note first that, by means of a vanishing coercivity argument (as in the proof of Lemma 4.2), for each $\varepsilon > 0$ problem (1) admits a bounded continuous viscosity solution u^ε, which in view of Theorem 3.6, is Lipschitz continuous for all times $t \in (0,T)$. Comparison principle given in Proposition 6.2 for the class of Lipschitz functions (in space) further asserts the uniqueness of u^ε.

It is easy to see that the sequence is uniformly bounded. If $M = \sup_{a\in\mathscr{A}} ||f^a||_\infty$, note that

$$\overline{u}(x,t) = ||u_0||_\infty + Mt, \quad \underline{u}(x,t) = -||u_0||_\infty - Mt$$

are respectively supersolutions and subsolutions of (1). Hence

$$\sup_{\varepsilon>0} ||u^\varepsilon||_\infty \leq ||u_0||_\infty + MT.$$

However, since the nonlocal operator is only weakly elliptic, in the sense of assumption $(K3)$, uniform Hölder or Lipschitz estimates are not available. In order to show that the sequence $\big(u^\varepsilon\big)_{\varepsilon>0}$

converges uniformly to a viscosity solution of the effective problem, we employ half-relaxed limits, introduced by Ishii [27] and Barles and Perthame [13, 14]. Let

$$u^*(x,t) = \limsup_{\substack{\varepsilon\to 0\\ (y,s)\to(x,t)}} u^\varepsilon(y,s) \quad u_*(x,t) = \liminf_{\substack{\varepsilon\to 0\\ (y,s)\to(x,t)}} u^\varepsilon(y,s).$$

Then u^* is bounded and upper semi-continuous, u_* is bounded and lower-continuous. By definition, for all $(x,t)\in\mathbb{R}^d\times(0,T)$,

$$u_*(x,t)\le u^*(x,t).$$

Moreover, in view of the comparison principle for equation (1) and the fact that u^ε are Lipschitz continuous in space, it follows that there exists a modulus of continuity independent of ε, given by $\omega(t)=Mt$ for $t\in[0,T]$, with M as above, such that

$$\sup_{x\in\mathbb{R}^d}|u^\varepsilon(x,t)-u_0(x)|\le\omega(t).$$

Hence, the following holds for the initial condition

$$u^*(x,0)\le u_0(x)\le u^*(x,0).$$

Employing the perturbed test function method, it is possible to check that u^* is viscosity subsolution and u_* is viscosity supersolution of the effective problem (5), in the sense of Definition 1. Nonetheless, the lack of comparison principle for the effective problem does not allow us to conclude directly that $u^*\ge u_*$.

To overcome this difficulty, we note that, in view of the structural properties of the effective Hamiltonian, the same vanishing coercivity argument as in the proof of Lemma 4.2 applies and we conclude, in view of Proposition 5.1, the existence of a viscosity solution u of the effective problem (5), which is $\tau-$Hölder continuous in space. The linearization result stated in Proposition 5.2, applied on one hand to u^* and u, and on the other hand to u and u_*, further gives

$$u^*(x,t)\le u(x,t)\le u_*(x,t),\ \text{for all } x\in\mathbb{R}^d,\ t\in[0,T].$$

Hence $u^*=u_*=u$ and the whole sequence converges locally uniformly in $\mathbb{R}^d\times[0,T]$. □

Remark 9. *In the uniformly elliptic case, i.e. when the kernel satisfies*

$$\frac{1}{C_K|z|^{d+1}}\le K^a(\xi,z)\le\frac{C_K}{|z|^{d+1}}\quad \text{for } \xi\in\mathbb{R}^d,\ z\in B\setminus\{0\},$$

the uniform convergence is immediate, in view of the space-time regularity results of Chang-Lara and Davila (see Corollary 7.1 in [20]). More precisely, in the proof above the equi-bounded family of solutions $\left(u^\varepsilon\right)_{\varepsilon>0}$ becomes uniformly τ-Hölder continuous in space and time, i.e. there exists $\tau\in(0,1)$ such that

$$\sup_{\varepsilon>0}||u^\varepsilon||_{\mathscr{C}^{0,\tau}(\mathbb{R}^d\times[0,T])}<\infty.$$

In view of Arzela-Ascoli theorem, there exists a subsequence $(\varepsilon_k)_{k>0}$, such that $\left(u_{\varepsilon_k}\right)_k$ converges locally uniformly in $\mathbb{R}^d\times[0,T]$ to a function $u\in\mathscr{C}^{0,\tau}(\mathbb{R}^d\times[0,T])\cap L^\infty(\mathbb{R}^d\times[0,T])$.

6. Appendix

General comparison results have been established by Barles and Imbert in [11] for Lévy-Itô integro-differential operators, but it continues to be an open problem for nonlocal Lévy operators with x dependent kernels. However, under *a priori* Lipschitz regularity assumption on the sub/super-solutions, comparison can be established by standard arguments. Though results apply for parabolic problems as well, we give a sketch of the proof in the stationary case, to simplify ideas.

Proposition 6.1. *Let $(f^a)_{a\in\mathscr{A}}$ and $(b^a)_{a\in\mathscr{A}}$ be two families of bounded functions on $\mathbb{R}^d$ satisfying $(H1)$ with $\alpha,\beta\in(0,1]$. Let $(K^a)_{a\in\mathscr{A}}$ be a family of kernels satisfying $(K1)$ and $(K3)$ with $\gamma\in(\frac{1}{2},1]$. If $u\in USC(\mathbb{R}^d)$ and $v\in LSC(\mathbb{R}^d)$ are respectively a bounded viscosity subsolution and a bounded viscosity supersolution of equation (7), such that $u\in C^{0,1}(\mathbb{R}^d)$ or $v\in\mathscr{C}^{0,1}(\mathbb{R}^d)$, then $u\le v$ on $\mathbb{R}^d$.*

Proof. We argue by contradiction and assume that $M:=\sup_{x\in\mathbb{R}^d}(u(x)-v(x))>0$. Doubling the variables we consider

$$M_{\epsilon,\zeta}=\sup\{u(x)-v(y)-\phi_\epsilon(x-y)-\psi_\zeta(x)\},$$

where ϵ,ζ are small parameters that will eventually go to 0. The penalization function $\phi_\epsilon:\mathbb{R}^d\to\mathbb{R}_+$ is given by

$$\phi_\epsilon(x-y):=\varphi\left(\frac{|x-y|^2}{\epsilon^2}\right),$$

where φ is a smooth nonnegative function on $\mathbb{R}_+$, with $||\varphi||_\infty$, $||\varphi'||_\infty$ and $||\varphi''||_\infty$ all finite and

$$\varphi(s)=\begin{cases} s & \text{if } s\le s_0\\ 2||u||_\infty+1 & \text{if } s\ge 2s_0.\end{cases}$$

The localization function ψ_ζ is given by $\psi_\zeta(x)=\psi(\zeta x)$, with $\psi\in\mathscr{C}^2(\mathbb{R}^d;\mathbb{R}_+)$ with $||\psi||_\infty$, $||D\psi||_\infty$ and $||D^2\psi||_\infty$ all finite, such that

$$\psi(x)=\begin{cases} 0 & |x|\le 1\\ ||u||_\infty+||v||_\infty+1 & |x|\ge 2.\end{cases}$$

Since the localization function only gives terms in $o_\zeta(1)$, we drop the dependence in ζ in what follows. In view of the properties of the localisation term ψ_ζ, the supremum $M_{\epsilon,\zeta}$ is actually a maximum, achieved at a point that we denote $(x_\epsilon,y_\epsilon)\in B_{s_0}\times B_{s_0}$. For ϵ small enough

$$\frac{M}{2}\le M_{\epsilon,\zeta}\le u(x_\epsilon)-v(y_\epsilon)\le||u||_\infty+||v||_\infty,$$

whereas the maximum property together with the assumption that $u\in C^{0,1}(\mathbb{R}^d)$ give

$$\frac{|x_\epsilon-y_\epsilon|^2}{\epsilon^2}\le|u(x_\epsilon)-u(y_\epsilon)|\le C|x_\epsilon-y_\epsilon|,$$

hence $|x_\epsilon-y_\epsilon|\le C\epsilon^2$. Let $a_\epsilon=x_\epsilon-y_\epsilon$, $p=\dfrac{x_\epsilon-y_\epsilon}{\epsilon^2}$, $q=D\psi_\zeta(y_\epsilon)$. Denote

$$\phi^u(x):=v(y_\epsilon)+\phi_\epsilon(x-y_\epsilon)+\psi_\zeta(x),$$

$$\phi^v(y):=u(x_\epsilon)-\phi_\epsilon(x_\epsilon-y)-\psi_\zeta(x_\epsilon),$$

and observe that

$$D\phi_\epsilon(a_\epsilon) = \varphi'\left(\frac{|a_\epsilon|^2}{\epsilon^2}\right)p, \quad D\phi^u(x_\epsilon) := D\phi_\epsilon(a_\epsilon) + q,\ D\phi^v(y_\epsilon) := D\phi_\epsilon(a_\epsilon).$$

In view of the maximum property, there exists $\rho \in (0, \min(1, s_0))$ sufficiently small such that x_ϵ is a local maximum for $u - \phi^u$ in $B_\rho(x_\epsilon)$, and y_ϵ is a local minimum for $v - \phi^v$ in $B_\rho(y_\epsilon)$. It follows from the viscosity inequalities that, for any $\nu > 0$, there exists $a \in \mathscr{A}$ such that, for all $0 < \rho' < \rho$,

$$\begin{aligned}
\delta u(x_\epsilon) - \mathscr{I}^a[B_{\rho'}](x_\epsilon, \phi^u(x_\epsilon)) - \mathscr{I}^a[B^c_{\rho'}](x_\epsilon, u) - b^a(x_\epsilon)\cdot D\phi^u(x_\epsilon) - f^a(x_\epsilon) &\leq 0,\\
\delta v(y_\epsilon) - \mathscr{I}^a[B_{\rho'}](y_\epsilon, \phi^v(y_\epsilon)) - \mathscr{I}^a[B^c_{\rho'}](y_\epsilon, v) - b^a(y_\epsilon)\cdot D\phi^v(y_\epsilon) - f^a(y_\epsilon) &\geq -\nu.
\end{aligned}$$

Denote

$$\begin{aligned}
\mathscr{T}^a[B_{\rho'}](x_\epsilon, y_\epsilon, \phi) &:= \mathscr{I}^a[B_{\rho'}](x_\epsilon, \phi^u(x_\epsilon)) - \mathscr{I}^a[B_{\rho'}](y_\epsilon, \phi^v(y_\epsilon)),\\
\mathscr{T}^a[B^c_{\rho'}](x_\epsilon, y_\epsilon, u, v) &:= \mathscr{I}^a[B^c_{\rho'}](x_\epsilon, u) - \mathscr{I}^a[B^c_{\rho'}](y_\epsilon, v).
\end{aligned}$$

Subtracting the two inequalities, it follows that

$$\begin{aligned}
\delta u(x_\epsilon) - \delta v(y_\epsilon) - \nu \leq\ & \mathscr{T}^a[B_{\rho'}](x_\epsilon, y_\epsilon, \phi) + \mathscr{T}^a[B^c_{\rho'}](x_\epsilon, y_\epsilon, u, v) + \\
& \varphi'\left(\frac{|a_\epsilon|^2}{\epsilon^2}\right)\big(b^a(x_\epsilon) - b^a(y_\epsilon)\big)\cdot p + b^a(x_\epsilon)\cdot q + \big(f^a(x_\epsilon) - f^a(y_\epsilon)\big),
\end{aligned} \tag{27}$$

whose last terms are further bounded by, in view of assumption $(H1)$ and previous notations,

$$C_b\varphi'\left(\frac{|a_\epsilon|^2}{\epsilon^2}\right)|a_\epsilon|^\beta|p| + ||b^a||_\infty|q| + C_f|a_\epsilon|^\alpha \leq C_b||\varphi'||_\infty\epsilon^{2\beta} + C_f\epsilon^{2\alpha} + o_\zeta(1) = o_\epsilon(1) + o_\zeta(1).$$

In order to estimate the nonlocal terms, we make use of the following inequalities coming from the maximum property

$$\begin{aligned}
u(x_\epsilon + z) - u(x_\epsilon) - \big(D\phi_\epsilon(a_\epsilon) + q\big)\cdot z \leq\ & \phi_\epsilon(a_\epsilon + z) - \phi_\epsilon(a_\epsilon) - D\phi_\epsilon(a_\epsilon)\cdot z + \\
& \psi_\zeta(y_\epsilon + z) + \psi_\zeta(y_\epsilon) - q\cdot z\\
-\big(v(y_\epsilon + z) - v(y_\epsilon) - D\phi_\epsilon(a_\epsilon)\cdot z\big) \leq\ & \phi(a_\epsilon - z) - \phi(a_\epsilon) + D\phi_\epsilon(a_\epsilon)\cdot z.
\end{aligned}$$

Letting first $\rho' \to 0$, it is immediate to see that the term $\mathscr{T}^a[B_{\rho'}](x_\epsilon, y_\epsilon, \phi)$ is $o_{\rho'}(1)$. To simplify notations hereafter, we write $\mathscr{T}^a(x_\epsilon, y_\epsilon, u, v)$ instead of $\mathscr{T}^a[\mathbb{R}^d](x_\epsilon, y_\epsilon, u, v)$ and split it into

$$\mathscr{T}^a(x_\epsilon, y_\epsilon, u, v) = \mathscr{T}^a[B_\rho](x_\epsilon, y_\epsilon, u, v) + \mathscr{T}^a[B^c_\rho](x_\epsilon, y_\epsilon, u, v).$$

Using the measure decomposition as in the proof of Theorem 3.1 with the total variation measure satisfying $|K(x_\epsilon, z) - K(y_\epsilon, z)| = K^a_+(z) + K^a_-(z)$, and in view of (16) and the above inequalities, the

following estimates hold.

$$\begin{aligned}\mathscr{T}^a[B_\rho](x_\epsilon,y_\epsilon,u,v) &\leq \int_{B_\rho}(\phi_\epsilon(a_\epsilon+z)-\phi_\epsilon(a_\epsilon)-D\phi_\epsilon(a_\epsilon)\cdot z)K_+^a(z)\,dz+\\ &\quad \int_{B_\rho}(\psi_\zeta(x_\epsilon+z)-\psi_\zeta(a_\epsilon)-D\psi_\zeta(a_\epsilon)\cdot z)K_+^a(z)\,dz+\\ &\quad \int_{B_\rho}(\phi_\epsilon(a_\epsilon-z)-\phi_\epsilon(a_\epsilon)+D\phi_\epsilon(a_\epsilon)\cdot z)K_-^a(z)\,dz\\ &= \frac{1}{\epsilon^2}\int_{B_\rho}|z|^2\,|K^a(x_\epsilon,z)-K^a(y_\epsilon,z)|\,dz+o_\zeta(1),\end{aligned}$$

which in view of assumption $(K3)$ and of the choice of φ, is further bounded above by

$$\mathscr{T}^a[B_\rho](x_\epsilon,y_\epsilon,u,v)\leq C_K\frac{1}{\epsilon^2}|a_\epsilon|^\gamma\rho+o_\zeta(1)\leq C_K\epsilon^{2\gamma-2}\rho+o_\zeta(1).$$

Similarly, we obtain

$$\begin{aligned}\mathscr{T}^a[B_\rho^c](x_\epsilon,y_\epsilon,u,v) &\leq \int_{B_\rho^c}(\phi_\epsilon(a_\epsilon+z)-\phi_\epsilon(a_\epsilon))K_+^a(z)\,dz-\int_{B\setminus B_\rho}D\phi_\epsilon(a_\epsilon)\cdot z\,K_+^a(z)\,dz\\ &\quad \int_{B_\rho^c}(\phi_\epsilon(a_\epsilon-z)-\phi_\epsilon(a_\epsilon))K_-^a(z)\,dz+\int_{B\setminus B_\rho}D\phi_\epsilon(a_\epsilon)\cdot z\,K_-^a(z)\,dz\\ &\leq 2||\phi_\epsilon||_\infty\int_{B_\rho^c}|K^a(x_\epsilon,z)-K^a(y_\epsilon,z)|\,dz+\\ &\quad 2|D\phi_\epsilon(a_\epsilon)|\int_{B\setminus B_\rho}|z|\,|K^a(x_\epsilon,z)-K^a(y_\epsilon,z)|\,dz+o_\zeta(1),\end{aligned}$$

which in view of assumption $(K3)$ and of the choice of φ, is further bounded above by

$$\begin{aligned}\mathscr{T}^a[B_\rho^c](x_\epsilon,y_\epsilon,u,v) &\leq 2C_K(||\varphi||_\infty|a_\epsilon|^\gamma\rho^{-1}+||\varphi'||_\infty|p||a_\epsilon|^\gamma|\ln(|\rho|)|)+o_\zeta(1)\\ &\leq C\Big(\epsilon^{2\gamma}\rho^{-1}+\epsilon^{2\gamma}|\ln(|\rho|)|\Big)+o_\zeta(1).\end{aligned}$$

Putting together all the previous estimates and taking $\rho=\rho_0\epsilon^{2r}$ with $r<\gamma$, it follows that

$$\mathscr{T}^a(x_\epsilon,y_\epsilon,u,v)\leq C\epsilon^{2\gamma}\Big(\epsilon^{2r-2}+|\ln(|\rho_0\epsilon^{2r}|)|+\epsilon^{-2r}\Big)+o_\zeta(1)=o_\epsilon(1)+o_\zeta(1).$$

Going back to (27),

$$0<\delta\frac{M}{2}\leq\delta u(x_\epsilon)-\delta v(y_\epsilon)-\nu\leq o_\epsilon(1)+o_\zeta(1),$$

and letting ϵ, ζ and ν go to zero we arrive to a contradiction. □

The proof previously shown applies literally to parabolic integro-differential equations and the following theorem holds.

Proposition 6.2. *Let $(K^a)_{a\in\mathscr{A}}$, $(b^a)_{a\in\mathscr{A}}$, $(f^a)_{a\in\mathscr{A}}$ satisfy the same assumptions as in Proposition 6.1. If $u\in USC(\mathbb{R}^d\times[0,T])$ and $v\in LSC(\mathbb{R}^d\times[0,T])$ are respectively a bounded viscosity subsolution and a bounded viscosity supersolution of equation* (6) *with $\mathscr{H}$ given by* (8) *such that for all times $t\in[0,T]$, $u(\cdot,t)\in C^{0,1}(\mathbb{R}^d)$ or $v(\cdot,t)\in\mathscr{C}^{0,1}(\mathbb{R}^d)$ and $u(x,0)\leq v(x,0)$, then $u(\cdot,t)\leq v(\cdot,t)$ for all $t\in[0,T]$.*

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