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On a problem of team hiring

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Abstract

Given two positive integers n and k with $k \leq n$, let \mathcal{X} denote the family of all the k -subsets of $[n] := \{1, 2, \dots, n\}$. In this paper, assuming that the $\binom{n}{k}$ k -subsets hold some order of ranks, we consider the expected number of left-to-right maxima in some sequential ordering of \mathcal{X} , under a random permutation of $[n]$. In the case $k = 1$, it is well-known that the answer is the n th harmonic number $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} = \ln n + O(1)$. For general k , the contribution of this paper is about $E_{\text{LEX}}^*(n, k)$ and $E_{\text{COL}}^*(n, k)$, the expected numbers of left-to-right maxima respectively in the lexicographical and colexicographical orderings of \mathcal{X} , achieved when the k -subsets in \mathcal{X} hold their respective worst orders of ranks (to make the two values as large as possible). We show that $E_{\text{LEX}}^*(n, k) = E_{\text{COL}}^*(n, k)$, and give an exact formula (not in closed form) for them. For bounding them, we further show that when $k' := \min\{k, n - k\} = O(1)$, they are in $\Theta((\ln n)^{k'})$.

The problem we consider here can be viewed as an extension to the assistant-hiring problem, presented in the textbook *Introduction to Algorithms* by Cormen, Leiserson, Rivest, and Stein.

Mathematics Subject Classifications: 68W40, 05A15

1. Introduction

In their legendary textbook *Introduction to Algorithms* [1, Section 5.1, both the second and third editions], Cormen et al take an assistant-hiring problem as an example to illustrate probabilistic analysis of algorithms. In this paper, we extend that hiring problem to a team-hiring scenario, which can be described as follows. Suppose that an organization want to find the best-qualified k -person team from n applicants and hire them permanently. The qualification of a team can only be determined by a sophisticated test (for collaborative capability, etc.), and the result of the test is unpredictable. The organization are required to adopt an exhaustive testing-and-hiring process to find the best-qualified team. At the beginning of the process, the n applicants are assumed to have been randomly numbered and a least-qualified dummy team has been hired. Then, every k -person team is tested one by one in accordance with some ordering of all the $\binom{n}{k}$ teams. If the team just tested is better than the current hired team, they discard the current hired team and hire the new

team for a period of time (except that the lastly hired team will be hired permanently). Now, under the assumption that the qualifications of all the teams are distinct, and the cost for team hiring is greatly larger than the cost for team testing, we have two concerns in this team-hiring problem: the time and space requirements for generating the ordering of the teams, and the expected number of hired teams in the whole testing-and-hiring process. Clearly, the latter is affected by two factors: the order in which we test the teams, and the order of qualifications (ranks) that the $\binom{n}{k}$ teams hold.

Although this problem is fictitious, the answer to it corresponds to the probabilistic analysis of a combinatorial algorithm for finding an optimal k -subset of a n -set with two oracles, which may really exist in computer science (though not found by the author so far). In the algorithm, we exhaustively call a faster decision oracle to decide whether the current k -subset is more optimal than the previous best, and if yes, we call a slower computation oracle to actually compute its optimal value.

In the case $k = 1$, as analyzed in the textbook [1] and noted in many other references (e.g., [3]), the expected number of hired teams is $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} = \ln n + O(1)$, no matter in what order we test the n applicants, and no matter what is the order of ranks that the n applicants hold. In the case $k > 1$, if we assume that the $\binom{n}{k}$ teams hold a random order of ranks, the expected number of hired teams is ideally $H_{\binom{n}{k}} = O(k' \ln n)$ with $k' := \min\{k, n - k\}$, no matter in what order we test the teams (see Proposition 1). However, for a sequential order to test the teams, this result is not of probabilistic analysis to the team-hiring problem, because the assumption that “the teams hold a random order of ranks”, unlike the assumption that “the applicants are randomly numbered”, is not a proper assumption on the input (of which the “teams” are not a part).

If we wish to achieve the goal that the expected number of hired teams is always $H_{\binom{n}{k}}$ for $k > 1$ without any assumption on the order of ranks that the teams hold, to test the teams in a random order may be of first choice, such that we get a randomized process. To fulfil this, we need the Knuth shuffle (or called Fisher-Yates Shuffle) [8, Algorithm P] to generate a random ordering of all the teams, either by shuffling all the k -subsets of $\{1, 2, \dots, n\}$ directly, or by shuffling the set $\{1, 2, \dots, \binom{n}{k}\}$ firstly and then reconstructing the k -subsets with an UNRANK procedure which follows some sequential ordering [10, Algorithms 2.8, 2.10, or 2.12]. Although the Knuth shuffle depends on a pseudorandom number generator for implementation on computers, it is widely accepted as a uniform random permutation generator [1]. The drawback of applying it here is that the shuffling process needs $\Omega(\binom{n}{k})$ working space.

If the above random ordering cannot be accepted for the sake of space requirements, some pseudorandom or quasi-random orderings may be of consideration. To generate such orderings, we need in a stateless way to generate a pseudorandom or quasi-random permutation of $\{1, 2, \dots, \binom{n}{k}\}$, where we can use the *linear congruential sequence* with period length $\binom{n}{k}$ [8, Theorem A], or use the truncated *bit-reversal permutation* [11, Section V]. With such methods, the space requirements are cut down, but we still need an UNRANK procedure to reconstruct each k -subset, in which a quantity of binomial coefficients need to be computed and big numbers must be involved if $\binom{n}{k}$ is big (see [10] for detail). What is worse is that

we (at least the author) are not sure that these pseudorandom or quasi-random orderings can achieve the same or a similar result as the real random ordering.

In this paper, we focus on two well-known sequential orderings: the lexicographical (or lexical) and colexicographical (or colex) orderings. Compared with the above random, pseudorandom or quasi-random orderings, the sequences of k -subsets obeying these two orderings can be generated very efficiently in both time and space. As shown in [9], [10], [13], and [16], the current k -subset to be generated is only dependent on its predecessor, so all the k -subsets can be generated with a SUCCESSOR procedure, in $O(k)$ time per subset (indeed amortized constant time if the time for output is excluded), and in $O(k)$ working space, with no big numbers involved if n is not big. Although they are quite nice in this regard, we have the intuition that they should be bad in term of the expected number of hired teams they induce in the worst case. But how bad are they? This paper tries to give a quantitative answer to this question.

1.1. Problem formalization and our results

Throughout this paper, an *ordering* of a family of subsets, is referred to the sequence of these subsets generated in some *ordering* way. Here, we follow [10] (rather than [9]), for the normal forms of the lexical and colex orderings of all the k -subsets of the set $\{1, 2, \dots, n\}$, which can be seen from the examples hereafter. Given a sequence of elements from a well-order set, a (strong) *left-to-right maximum* of the sequence is the element which is greater than any element preceding it.

Given two positive integer n and k with $k \leq n$, the team-hiring problem in this paper is only determined by n and k , so we call it (n, k) -THP for short. We formalize it with the following notations:

$[n] := \{1, 2, \dots, n\}$;

\mathcal{X} : the family of all the $\binom{n}{k}$ k -subsets of $[n]$;

ord: the variable tag for an ordering way to generate an ordering of \mathcal{X} ;

\mathcal{X}_{ord} : the ordering of \mathcal{X} generated in the way of *ord*;

$\sigma : [n] \rightarrow [n]$, a permutation of $[n]$;

S_n : the symmetric group of degree n ;

$\{t_1, t_2, \dots, t_k\}^\sigma := \{\sigma(t_1), \sigma(t_2), \dots, \sigma(t_k)\}$, supposing that $\{t_1, t_2, \dots, t_k\} \in \mathcal{X}$;

$\mathcal{X}_{ord}^\sigma := \langle X_1^\sigma, X_2^\sigma, \dots, X_{\binom{n}{k}}^\sigma \rangle$, supposing that $\mathcal{X}_{ord} = \langle X_1, X_2, \dots, X_{\binom{n}{k}} \rangle$;

$f : \mathcal{X} \rightarrow [\binom{n}{k}]$, a bijective ranking function on \mathcal{X} ;

$\mathcal{M}_{ord}^{\sigma, f} := \{X_j^\sigma \in \mathcal{X}_{ord}^\sigma : 1 \leq j \leq \binom{n}{k}, f(X_i^\sigma) < f(X_j^\sigma) \text{ for all } i < j\}$, the collection of left-to-right maximum k -subsets in \mathcal{X}_{ord}^σ , under a fixed σ and a fixed f ;

$E_{ord}^f(n, k) := (1/n!) \sum_{\sigma \in S_n} |\mathcal{M}_{ord}^{\sigma, f}|$, the expected number of left-to-right maximum k -subsets in \mathcal{X}_{ord}^σ , under a random $\sigma \in S_n$ and a fixed f .

Additionally, we let “LEX” and “COL” be the tags of the lexical and colex orderings respectively, and (from now on) we use the convention that a permutation of $[n]$ is written as a n -word and a k -subset of $[n]$ is written as a k -word with elements in the ascending order. (When the ascending order cannot be guaranteed, we still use the form $\{\dots\}$ for a subset.)

For a ranking function f on \mathcal{X} , we generally write it as an ordering of \mathcal{X} to express it. That is, $f = \langle X_1, X_2, \dots, X_{\binom{n}{k}} \rangle$ means that $f(X_1) = 1, f(X_2) = 2, \dots, f(X_{\binom{n}{k}}) = \binom{n}{k}$. Besides, the inequality $f(X_1) < f(X_2)$ is generally written as $X_1 <_f X_2$.

To illustrate the above notations, let's consider (4, 2)-THP. If we choose $f = \langle 13, 23, 34, 24, 12, 14 \rangle$, then for a fixed $\sigma = 3214$, we have

$$\begin{aligned}\mathcal{X}_{\text{LEX}} &= \langle 12, 13, 14, 23, 24, 34 \rangle, & \mathcal{X}_{\text{COL}} &= \langle 12, 13, 23, 14, 24, 34 \rangle, \\ \mathcal{X}_{\text{LEX}}^\sigma &= \langle 23, 13, 34, 12, 24, 14 \rangle, & \mathcal{X}_{\text{COL}}^\sigma &= \langle 23, 13, 12, 34, 24, 14 \rangle, \\ \mathcal{M}_{\text{LEX}}^{\sigma, f} &= \{23, 34, 12, 14\}, & \mathcal{M}_{\text{COL}}^{\sigma, f} &= \{23, 12, 14\}, \\ & & |\mathcal{M}_{\text{LEX}}^{\sigma, f}| &= 4, \quad |\mathcal{M}_{\text{COL}}^{\sigma, f}| = 3.\end{aligned}$$

Summarizing over all $\sigma \in S_4$, we can further get that

$$E_{\text{LEX}}^f(4, 2) = 61/24, E_{\text{COL}}^f(4, 2) = 59/24.$$

Let \mathcal{F} be the collection of all the ranking functions on \mathcal{X} . Although \mathcal{F} can be partitioned into at most $\binom{n}{k}!/n!$ equivalence classes (see Section 2.5), where the functions f in the same class induce the same $E_{\text{ord}}^f(n, k)$ for any fixed ord , we simply think of \mathcal{F} as $|\mathcal{F}| = \binom{n}{k}!$. Here, we concentrate on the average and the worst behavior of $f \in \mathcal{F}$, and only is the latter essential to the probabilistic analysis of the team-hiring problem. Precisely, we have two definitions that

$$E_{\text{ord}}^-(n, k) := (1/\binom{n}{k}!) \sum_{f \in \mathcal{F}} E_{\text{ord}}^f(n, k), \text{ and } E_{\text{ord}}^*(n, k) := \max_{f \in \mathcal{F}} E_{\text{ord}}^f(n, k).$$

Our first result of this paper is about how much $E_{\text{ord}}^-(n, k)$ is, which is both obvious and intriguing. It belongs to the rare cases in the world that the ‘‘average’’ is easier to solve than the ‘‘worst’’.

Proposition 1. *For any ordering ord of \mathcal{X} , $E_{\text{ord}}^-(n, k) = H_{\binom{n}{k}}$.*

Proof. For any fixed $\sigma \in S_n$, $(1/\binom{n}{k}!) \sum_{f \in \mathcal{F}} |\mathcal{M}_{\text{ord}}^{\sigma, f}| = H_{\binom{n}{k}}$. □

Our main result of this paper is an exact formula (not in closed form) for $E_{\text{LEX}}^*(n, k)$ and $E_{\text{COL}}^*(n, k)$.

Theorem 1. *We have*

$$E_{\text{LEX}}^*(n, k) = E_{\text{COL}}^*(n, k) = \sum_{1 \leq t_1 < t_2 < \dots < t_k \leq n} P(t_1..t_k),$$

where

$$P(t_1..t_k) = \begin{cases} 1/t_1 & \text{if } k = 1 \\ (1/t_k) \sum_{1 \leq j \leq k} P(t_1..t_{j-1}(t_{j+1}-1)..(t_k-1)) & \text{otherwise.} \end{cases} \quad (1)$$

In recurrence (1), the expression $t_1..t_{j-1}(t_{j+1}-1)..(t_k-1)$ is understood to be a $(k-1)$ -subset of $[n-1]$ obtained from the k -subset $t_1..t_k$ of $[n]$, by removing t_j and decrementing the elements larger than t_j (if any), and $P(\cdot)$ can be temporally viewed as a function which can take any nonempty subset of $[n]$ as its argument. In Section 2, we will see that $P(t_1..t_k)$ really corresponds to the probability that the subset $t_1..t_k$ achieves left-to-right maxima in \mathcal{X}_{LEX} or \mathcal{X}_{COL} . To illustrate recurrence (1), letting 247 be a 3-subset, we have $P(247) = \frac{1}{7}[P(24) + P(26) + P(36)] = \frac{1}{7}[\frac{1}{4}(\frac{1}{2} + \frac{1}{3}) + \frac{1}{6}(\frac{1}{2} + \frac{1}{5}) + \frac{1}{6}(\frac{1}{3} + \frac{1}{5})]$.

We can immediately check that Theorem 1 well fits the cases $k = 1$ and $n = k$. Applying Theorem 1 to (4, 2)-THP, we have

$$\begin{aligned} E_{\text{LEX}}^*(4, 2) = E_{\text{COL}}^*(4, 2) &= P(12) + P(13) + P(23) + P(14) + P(24) + P(34) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{5}{24} + \frac{1}{6} \\ &= \frac{61}{24}, \end{aligned}$$

which can be verified by a simple computer program. We will give the proof of Theorem 1 for the general case in Section 2. From Theorem 1 we know that $E_{\text{LEX}}^*(n, k) = E_{\text{COL}}^*(n, k)$, thus we use a unified $E_{\text{L(C)}}^*(n, k)$ to denote both $E_{\text{LEX}}^*(n, k)$ and $E_{\text{COL}}^*(n, k)$. Now, it is not difficult to derive a formula in closed form for $E_{\text{L(C)}}^*(n, 2)$, a formula in summation form for $E_{\text{L(C)}}^*(n, 3)$, and the subsequent asymptotics for them as well.

Corollary 1. *We have*

$$E_{\text{L(C)}}^*(n, 2) = \frac{1}{2}(H_n^2 - H_n^{(2)}) + H_n - 1 = \frac{1}{2}\ln^2 n + O(\ln n), \text{ and}$$

$$E_{\text{L(C)}}^*(n, 3) = \sum_{3 \leq i \leq n} [\frac{1}{2i}(H_{i-2}^2 - H_{i-2}^{(2)}) + (\frac{1}{i} + \frac{1}{i-1})H_{i-2} + \frac{1}{i(i-1)}] = \frac{1}{6}\ln^3 n + O(\ln^2 n),$$

where $H_n^{(2)} = \sum_{1 \leq i \leq n} 1/i^2 < \frac{\pi^2}{6}$, which is the second-order n th harmonic number.

The sequence of $n!E_{\text{L(C)}}^*(n, 2)$ for $n \geq 2$ is $\langle 2, 11, 61, 379, 2668, \dots \rangle$, and the sequence of $n!E_{\text{L(C)}}^*(n, 3)$ for $n \geq 3$ is $\langle 6, 50, 379, 3023, 26193, \dots \rangle$. These two sequences have been submitted to the On-line Encyclopedia of the Integer Sequences(OEIS) [17], and published at the entries A308729 and A308860.

For general k and n , analogous to the symmetry of binomial coefficients, $E_{\text{L(C)}}^*(n, k)$ is symmetric in k and $n - k$, as might be expected. And it holds the following simple bounds in terms of unsigned Stirling numbers of the first kind, which leave a lot of room for improving.

Corollary 2. $E_{\text{L(C)}}^*(n, k) = E_{\text{L(C)}}^*(n, n - k)$, and $\frac{1}{n!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \leq E_{\text{L(C)}}^*(n, k) \leq \frac{k!}{n!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$.

We can immediately check that the inequality in Corollary 2 tightly holds for $E_{\text{L(C)}}^*(n, 1) = H_n = \frac{1}{n!} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}$ and for $E_{\text{L(C)}}^*(k, k) = 1 = \frac{k!}{k!} \begin{bmatrix} k+1 \\ k+1 \end{bmatrix}$. A complete proof of Corollary 2 will be given Section 3. By applying the well-known asymptotic formula $\frac{1}{n!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \sim \frac{1}{k!} (\ln n)^k$ for $k = O(1)$ (see [15, Table 4.6]), we can immediately give a conditional asymptotic bound in terms of big-theta for $E_{\text{L(C)}}^*(n, k)$:

Corollary 3. *When $k' := \min\{k, n - k\} = O(1)$, $E_{\text{L(C)}}^*(n, k) = \Theta((\ln n)^{k'})$.*

1.2. Related work

Historically, the left-to-right maxima of some sequences have been studied in various names such as *éléments saients* (French), left-to-right maxima, records, and outstanding elements. Wilf [19] attributed to A. Rényi the result that the number of permutations of n letters that have exactly k left-to-right maxima is $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$, an unsigned Stirling number of the first kind. It seems to be of the folklore that the expected number of left-to-right maxima in a random permutation of n letters is H_n , and the variance is $H_n - H_n^{(2)}$. Wilf [19] presented plenty of other properties of left-to-right maxima in permutations of n letters. In recent years, the (strong or weak) left-to-right maxima in other types of sequences have also been analyzed by many authors. For example, see [6], [7], [12], and [14].

For a special problem (identifying stars from a set U of n spikes with queries on k -subsets of U), Mueller et al. [11] compared several orderings of k -subsets with experiments, and showed that the quasi-random ordering based on bit reversals is superior to the lexical ordering, in terms of the time for discovering all possible sets of stars.

There are another class of “hiring problems” (e.g., [5] and references therein), where an applicant is either irrevocably rejected or permanently hired at once after an evaluation of her/his capabilities, and the objective is to hire one or several partly good employees. These problems originated from the famous “secretary problem” [2], and from the practical point of view, what they model is much nearer to realistic hiring situations.

2. Proof of Theorem 1

2.1. Preliminaries and proof outline

First, let’s consider general *ord* and general f . Let $\mathcal{X}_{ord} = \langle X_1, X_2, \dots, X_{\binom{n}{k}} \rangle$, and let $f = \langle Y_1, Y_2, \dots, Y_{\binom{n}{k}} \rangle$. Then, we say that $T_{ord}^f(X_i, Y_l)$ is the *contribution* of Y_l to X_i , with $T_{ord}^f(X_i, Y_l)$ being defined as follows:

$$T_{ord}^f(X_i, Y_l) := |\{\sigma \in S_n : X_i^\sigma = Y_l, X_i^\sigma \in \mathcal{M}_{ord}^{\sigma, f}\}|.$$

Observation 1. (i) $T_{ord}^f(X_i, Y_l) = 0$ for $l < i$. (ii) Suppose that f is variable. Then $T_{ord}^f(X_i, Y_l), T_{ord}^f(X_i, Y_{l+1}), \dots, T_{ord}^f(X_i, Y_{\binom{n}{k}})$ all remain unchanged if the consecutive subsequence $\langle Y_l, Y_{l+1}, \dots, Y_{\binom{n}{k}} \rangle$ remains unchanged.

In terms of contributions, we have another formula for $E_{ord}^f(n, k)$:

$$E_{ord}^f(n, k) = (1/n!) \sum_{1 \leq i \leq l \leq \binom{n}{k}} T_{ord}^f(X_i, Y_l).$$

Next, let’s consider two special ranking functions which are respectively consistent with \mathcal{X}_{LEX} and \mathcal{X}_{COL} in their normal forms. We respectively name them *lex* and *col*. That is, in our simplified language, we can say that $lex = \mathcal{X}_{LEX}$ and $col = \mathcal{X}_{COL}$, or say that $<_{lex}$ and $<_{col}$ are two strict well-order relations on \mathcal{X} . Without any confusion, we always stipulate

that $<_{\text{lex}}$ and $<_{\text{col}}$ can be used to compare any two distinct subsets of $[n]$ with the same cardinality.

Now let $s_1..s_k$ and $t_1..t_k$ be two distinct k -subsets of $[n]$. The definitions of $<_{\text{lex}}$ and $<_{\text{col}}$ can be given recursively as follows:

- $s_1..s_k <_{\text{lex}} t_1..t_k$ if $s_1 < t_1$, or $s_1 = t_1$ and $s_2..s_k <_{\text{lex}} t_2..t_k$.
- $s_1..s_k <_{\text{col}} t_1..t_k$ if $s_k < t_k$, or $s_k = t_k$ and $s_1..s_{k-1} <_{\text{col}} t_1..t_{k-1}$.

Observation 2. *Let $c \in s_1..s_k \cap t_1..t_k$. Then both $<_{\text{lex}}$ and $<_{\text{col}}$ hold that*

$$s_1..s_k <_{\text{lex(col)}} t_1..t_k \iff s_1..s_k \setminus \{c\} <_{\text{lex(col)}} t_1..t_k \setminus \{c\}.$$

Let $d \in [n]$. Then the mapping $\delta_d : [n] \setminus \{d\} \rightarrow [n-1]$ is called d -pivoted decrement (of order n) if it satisfies

$$\delta_d(i) = \begin{cases} i & \text{if } i < d \\ i - 1 & \text{otherwise.} \end{cases}$$

Clearly, δ_d is a bijection. Letting $d \notin t_1..t_k$, like the labeling of a permutation on a subset, we use $(t_1..t_k)^{\delta_d}$ to denote the subset $\delta_d(t_1)\delta_d(t_2).. \delta_d(t_k)$. (It is guaranteed that $\delta_d(t_1) < \delta_d(t_2) < \dots < \delta_d(t_k)$ if $d \notin t_1..t_k$.) Then it is not difficult to verify the following observation.

Observation 3. *Let $d \in [n]$ and $d \notin s_1..s_k \cup t_1..t_k$. Then both $<_{\text{lex}}$ and $<_{\text{col}}$ hold that*

$$s_1..s_k <_{\text{lex(col)}} t_1..t_k \iff (s_1..s_k)^{\delta_d} <_{\text{lex(col)}} (t_1..t_k)^{\delta_d}.$$

Now let $c \in t_1..t_k$. Then $(t_1..t_k \setminus \{c\})^{\delta_c}$ is a valid $(k-1)$ -subset of $[n-1]$. For convenience, we use $t_1..t_k \ominus c$ to denote it. That is, when $c \in t_1..t_k$, we have the definition that

$$t_1..t_k \ominus c := (t_1..t_k \setminus \{c\})^{\delta_c} := t_1..t_{i-1}(t_{i+1}-1)..(t_k-1),$$

where i satisfies $t_i = c$. Combining Observations 3 and 2 together, we immediately have the following Observation 4.

Observation 4. *Let $c \in [n]$ and $c \in s_1..s_k \cap t_1..t_k$. Then both $<_{\text{lex}}$ and $<_{\text{col}}$ hold that*

$$s_1..s_k <_{\text{lex(col)}} t_1..t_k \iff s_1..s_k \ominus c <_{\text{lex(col)}} t_1..t_k \ominus c.$$

After presenting the above preliminaries, the following part of this section focuses on showing that lex and col are just the worst ranking functions for \mathcal{X}_{LEX} and \mathcal{X}_{COL} respectively, and they induce the quantity given in Theorem 1. We present the proof by sequentially arguing the following assertions:

- (I) $E_{\text{COL}}^{\text{col}}(n, k) = \sum_{1 \leq t_1 < t_2 < \dots < t_k \leq n} P(t_1..t_k)$, with $P(t_1..t_k)$ holding recurrence (1);
- (II) for any $f \in \mathcal{F}$, $E_{\text{COL}}^f(n, k) \leq E_{\text{COL}}^{\text{col}}(n, k)$;
- (III) $E_{\text{LEX}}^{\text{lex}}(n, k) = E_{\text{COL}}^{\text{col}}(n, k)$;
- (IV) for any $f \in \mathcal{F}$, $E_{\text{LEX}}^f(n, k) \leq E_{\text{LEX}}^{\text{lex}}(n, k)$.

2.2. Proof of (I)

We now formally define $P(t_1..t_k)$ to be the probability that the subset $t_1..t_k$ in \mathcal{X}_{COL} achieves left-to-right maxima under a random σ and the fixed ranking function col . Precisely, we have the definition that

$$P(t_1..t_k) := (1/n!) |\{\sigma \in S_n : (t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, \text{col}}\}|.$$

Since the summation $\sum_{1 \leq t_1 < t_2 < \dots < t_k \leq n} P(t_1..t_k)$ traverses all the k -subsets of $[n]$, what we need to argue in the rest of this subsection, is only that in term of its probability meaning, recurrence (1) really holds. We now prove this by induction on n and k .

From recurrence (1), we have $P(t_1) = 1/t_1$ when $k = 1$, and $P(t_1..t_k) = P(1..k) = 1$ when $n = k$, so recurrence (1) holds for the basic $(n, 1)$ -THP and (k, k) -THP. In the inductive step, our assumption is that recurrence (1) holds for $(n-1, k-1)$ -THP and $(n-1, k)$ -THP with $1 < k < n$, and our goal is to argue that it holds for (n, k) -THP. Now we carry out the induction by separately considering two cases.

Case 1: $t_k = n$. In this case, the induction is from $(n-1, k-1)$ -THP. To make the exposition non-confusing, for $(n-1, k-1)$ -THP, we use the notations $\hat{\mathcal{X}}$, $\hat{\sigma}$, and $\hat{\mathcal{M}}_{\text{COL}}^{\hat{\sigma}, \text{col}}$, corresponding to \mathcal{X} , σ , and $\mathcal{M}_{\text{COL}}^{\sigma, \text{col}}$ respectively.

First, let's consider the k -subset $t_1..t_k$ with $t_k = n$ in the left-hand side of recurrence (1). It appears in both \mathcal{X}_{COL} and col , which are the same sequence indeed. By the property of the colex ordering, we know that all the last $\binom{n-1}{k-1}$ k -subsets in \mathcal{X}_{COL} include n , one of which is $t_1..t_k$; by Observation 1, we know that only the k -subsets in the consecutive subsequence of col starting from $t_1..t_k$ can have contribution to $t_1..t_k$ in \mathcal{X}_{COL} . So if $(t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, \text{col}}$, then there must be $1 \leq j \leq k$ such that $\sigma(t_j) = n$. Therefore in case 1, it holds that

$$P(t_1..t_k) = (1/n!) \sum_{1 \leq j \leq k} |\{\sigma \in S_n : \sigma(t_j) = n, (t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, \text{col}}\}|. \quad (2)$$

Next, for $1 \leq j \leq k$, let's consider the $(k-1)$ -subset corresponding to the current j in the right-hand side of recurrence (1). We observe that $t_1..t_{j-1}(t_{j+1}-1)..(t_k-1)$ is just

$$t_1..t_k \ominus t_j.$$

Then according to the assumption that recurrence (1) holds for $(n-1, k-1)$ -THP, we have

$$P(t_1..t_k \ominus t_j) = (1/(n-1)!) |\{\hat{\sigma} \in S_{n-1} : (t_1..t_k \ominus t_j)^{\hat{\sigma}} \in \hat{\mathcal{M}}_{\text{COL}}^{\hat{\sigma}, \text{col}}\}|. \quad (3)$$

Now substituting their formulas (respectively in (2) and (3)) for $P(t_1..t_k)$ and $P(t_1..t_k \ominus t_j)$ in recurrence (1), we find that the holding of recurrence (1) for (n, k) -THP is implied by the following equation for $1 \leq j \leq k$:

$$|\{\sigma \in S_n : \sigma(t_j) = n, (t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, \text{col}}\}| = |\{\hat{\sigma} \in S_{n-1} : (t_1..t_k \ominus t_j)^{\hat{\sigma}} \in \hat{\mathcal{M}}_{\text{COL}}^{\hat{\sigma}, \text{col}}\}|. \quad (4)$$

To prove equation (4), we note that for $1 \leq j \leq k$, there is a bijection $\Delta_j : \{\sigma \in S_n : \sigma(t_j) = n\} \rightarrow S_{n-1}$ defined by

$$\sigma \mapsto \sigma(1)\sigma(2)\dots\sigma(t_j-1)\sigma(t_j+1)\sigma(t_j+2)\dots\sigma(n).$$

In other words, $\Delta_j(\sigma)$ is obtained by removing $\sigma(t_j) = n$, and shifting the sub-word $\sigma(t_j + 1)\sigma(t_j + 2)\dots\sigma(n)$ left (if any). Let $(\sigma, \hat{\sigma}) \in \Delta_j$, and $r_1..r_k$ be a k -subset of $[n]$ with some $r_i = t_j$. Then we can see bijection Δ_j holds that

$$\begin{aligned}
(r_1..r_k)^\sigma &= (r_1..r_{i-1}t_jr_{i+1}..r_k)^\sigma \\
&= \{\sigma(r_1), \dots, \sigma(r_{i-1}), \sigma(r_{i+1}), \dots, \sigma(r_k)\} \cup \{\sigma(t_j)\} \\
&= \{\hat{\sigma}(r_1), \dots, \hat{\sigma}(r_{i-1}), \hat{\sigma}(r_{i+1}-1), \dots, \hat{\sigma}(r_k-1)\} \cup \{n\} \\
&= (r_1..r_{i-1}(r_{i+1}-1)..(r_k-1))^{\hat{\sigma}} \cup \{n\} \\
&= (r_1..r_k \ominus t_j)^{\hat{\sigma}} \cup \{n\}
\end{aligned} \tag{5}$$

With bijection Δ_j , to prove equation (4), what we need to argue is only the following assertion for $1 \leq j \leq k$ and $(\sigma, \hat{\sigma}) \in \Delta_j$:

$$(t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, \text{col}} \iff (t_1..t_k \ominus t_j)^{\hat{\sigma}} \in \hat{\mathcal{M}}_{\text{COL}}^{\hat{\sigma}, \text{col}}. \tag{6}$$

To prove assertion (6), we note that for $1 \leq j \leq k$, there is a bijection $\Lambda_j : \{r_1..r_k \in \mathcal{X} : t_j \in r_1..r_k\} \rightarrow \hat{\mathcal{X}}$ defined by

$$r_1..r_k \mapsto r_1..r_k \ominus t_j.$$

The bijectiveness of Λ_j is based on Observation 3: In terms of $<_{\text{lex}}$ or $<_{\text{col}}$, Λ_j is order-preserving. So we immediately see that the mapping rule $r_1..r_k \mapsto r_1..r_k \ominus t_j$ also establish the bijection

$$\Lambda'_j : \{s_1..s_k \in \mathcal{X} : t_j \in s_1..s_k, s_1..s_k <_{\text{col}} t_1..t_k\} \rightarrow \{\hat{\mathcal{X}} : r_1..r_{k-1} \in \hat{\mathcal{X}} : r_1..r_{k-1} <_{\text{col}} t_1..t_k \ominus t_j\}.$$

To illustrate bijection Λ'_j , let's consider (6, 3)-THP, and suppose that $t_1t_2t_3 = 236$ and the current $j = 2$ (i.e., $t_j = 3$). Then, bijection Λ'_j is as follows:

$$\begin{array}{cccccccccccccc}
123 & (124) & 134 & 234 & (125) & 135 & 235 & (145) & (245) & 345 & (126) & 136 & (236) \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & & \downarrow & & \downarrow & \\
12 & & 13 & 23 & & 14 & 24 & & & 34 & & 15 & (25)
\end{array}$$

With bijection Δ_j and Λ'_j , we are ready for proving assertion (6). Combining equation (5) and Observation 2, we have the following assertion for $1 \leq j \leq k$, $(\sigma, \hat{\sigma}) \in \Delta_j$, and $(s_1..s_k, s_1..s_k \ominus t_j) \in \Lambda'_j$:

$$\begin{aligned}
(s_1..s_k)^\sigma <_{\text{col}} (t_1..t_k)^\sigma &\iff (s_1..s_k \ominus t_j)^{\hat{\sigma}} \cup \{n\} <_{\text{col}} (t_1..t_k \ominus t_j)^{\hat{\sigma}} \cup \{n\} \quad (\text{by eq. (5)}) \\
&\iff (s_1..s_k \ominus t_j)^{\hat{\sigma}} <_{\text{col}} (t_1..t_k \ominus t_j)^{\hat{\sigma}} \quad (\text{by Obs. 2}). \tag{7}
\end{aligned}$$

Since bijection Λ'_j establishes an injection from $\{r_1..r_{k-1} \in \hat{\mathcal{X}} : r_1..r_{k-1} <_{\text{col}} t_1..t_k \ominus t_j\}$ to $\{s_1..s_k \in \mathcal{X} : s_1..s_k <_{\text{col}} t_1..t_k\}$, we find that assertion (7) directly implies the \Rightarrow direction of assertion (6). How about its \Leftarrow direction? Consider such $s_1..s_k$ with $s_1..s_k <_{\text{col}} t_1..t_k$ and

$t_j \notin s_1..s_k$. We find that $n \notin (s_1..s_k)^\sigma$, and thus it inherently holds that $(s_1..s_k)^\sigma <_{\text{col}} (t_1..t_k)^\sigma$. Therefore, the \Leftarrow direction of assertion (6) also holds.

With assertion (6) being proved, we accomplish the argument for case 1.

Case 2: $t_k < n$. In this case, we know that recurrence (1) has already held for (t_k, k) -THP, by the conclusion of case 1. What we need to argue is only that $P(t_1..t_k)$ remains unchanged if we extend the problem from (t_k, k) -THP to (n, k) -THP. We argue this also by induction. That is, under the assumption that $P(t_1..t_k)$ remains unchanged from (t_k, k) -THP to $(n-1, k)$ -THP, our goal is to argue that it remains unchanged from $(n-1, k)$ -THP to (n, k) -THP.

By the property of the colex ordering, we know that every $s_1..s_k$ with $s_1..s_k <_{\text{col}} t_1..t_k$ remains unchanged if we extend the problem from $(n-1, k)$ -THP to (n, k) -THP, and it doesn't include n . So what we need to argue for case 2 is implied by the following equation for $1 \leq d \leq n$:

$$|\{\sigma \in S_n : \sigma(n) = d, (t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, \text{col}}\}| = |\{\tilde{\sigma} \in S_{n-1} : (t_1..t_k)^{\tilde{\sigma}} \in \tilde{\mathcal{M}}_{\text{COL}}^{\tilde{\sigma}, \text{col}}\}|, \quad (8)$$

where $\tilde{\mathcal{M}}_{\text{COL}}^{\tilde{\sigma}, \text{col}}$ is understood to be the notation of $(n-1, k)$ -THP, corresponding to $\mathcal{M}_{\text{COL}}^{\sigma, \text{col}}$.

To prove equation (8), we note that for $1 \leq d \leq n$, there is a bijection $\mathcal{Y}_d : \{\sigma \in S_n : \sigma(n) = d\} \rightarrow S_{n-1}$ defined by

$$\sigma \mapsto \delta_d(\sigma(1))\delta_d(\sigma(2))\dots\delta_d(\sigma(n-1)).$$

In other words, $\mathcal{Y}_d(\sigma)$ is obtained by removing $\sigma(n) = d$ and decrementing all the elements larger than d . Now, letting $(\sigma, \tilde{\sigma}) \in \mathcal{Y}_d$, and $r_1..r_k$ be a k -subset of $[n]$ with $r_k < n$ (i.e., $d \notin (r_1..r_k)^\sigma$), we can see bijection \mathcal{Y}_d holds that

$$\begin{aligned} ((r_1..r_k)^\sigma)^{\delta_d} &= \{\delta_d(\sigma(r_1)), \delta_d(\sigma(r_2)), \dots, \delta_d(\sigma(r_k))\} \\ &= \{\tilde{\sigma}(r_1), \tilde{\sigma}(r_2), \dots, \tilde{\sigma}(r_k)\} \\ &= (r_1..r_k)^{\tilde{\sigma}}. \end{aligned} \quad (9)$$

Then, combining equation (9) and Observation 3, we have the following assertion for $1 \leq d \leq n$, $(\sigma, \tilde{\sigma}) \in \mathcal{Y}_d$, and any $s_1..s_k <_{\text{col}} t_1..t_k$:

$$\begin{aligned} (s_1..s_k)^\sigma <_{\text{col}} (t_1..t_k)^\sigma &\iff ((s_1..s_k)^\sigma)^{\delta_d} <_{\text{col}} ((t_1..t_k)^\sigma)^{\delta_d} && \text{(by Observation 3)} \\ &\iff (s_1..s_k)^{\tilde{\sigma}} <_{\text{col}} (t_1..t_k)^{\tilde{\sigma}} && \text{(by equation (9))}. \end{aligned}$$

That directly implies equation (8).

With the argument for case 2 being completed, we accomplish the proof of (I).

2.3. Proof of (II)

In this subsection, we continue to use the definition of $P(t_1..t_k)$ as in Section 2.2, i.e., it is the probability that the subset $t_1..t_k$ in \mathcal{X}_{COL} achieves left-to-right maxima under a random σ and the fixed col, and it holds recurrence (1) by the conclusion of Section 2.2. When

the ranking function f is general, we use $R_{\text{COL}}^f(t_1..t_k)$ to denote the probability that the subset $t_1..t_k$ in \mathcal{X}_{COL} achieves left-to-right maxima under a random σ . Precisely, we have the definition that

$$R_{\text{COL}}^f(t_1..t_k) := (1/n!) |\{\sigma \in S_n : (t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, f}\}|.$$

(That is, under this new definition, $P(t_1..t_k)$ is just $R_{\text{COL}}^{\text{col}}(t_1..t_k)$.) Then we can see that the inequality $E_{\text{COL}}^f(n, k) \leq E_{\text{COL}}^{\text{col}}(n, k)$ we want to prove is definitely implied by the following stronger inequality for any $t_1..t_k \in \mathcal{X}$:

$$R_{\text{COL}}^f(t_1..t_k) \leq P(t_1..t_k). \quad (10)$$

So the remainder of this subsection rests on proving inequality (10), by using an induction similar to the one in the proof of (I).

We first note that inequality 10 holds for the basic $(n, 1)$ -THP and (k, k) -THP with equalities. In the inductive step, our goal is to argue that it holds for (n, k) -THP under the assumption that it holds for $(n-1, k-1)$ -THP and $(n-1, k)$ -THP with $1 < k < n$. We achieve this goal also by separately considering two cases.

Case 1: $t_k = n$. In this case, the induction is also from $(n-1, k-1)$ -THP. To make the notations distinguished, we still add $\hat{\cdot}$ to the relevant symbols of $(n-1, k-1)$ -THP.

Letting $f = \langle Y_1, Y_2, \dots, Y_{\binom{n}{k}} \rangle$, we now consider two subcases for case 1.

Case 1.1: There is $1 \leq c \leq n$ such that $c \in \bigcap_{\binom{n-1}{k} < l \leq \binom{n}{k}} Y_l$. In other words, in this subcase, c appears only in the last $\binom{n-1}{k-1}$ k -subsets of f . Let's construct $\hat{f} = \langle \hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_{\binom{n-1}{k-1}} \rangle$ from f with the rule

$$\hat{Y}_l := Y_{\binom{n-1}{k} + l} \ominus c$$

for $1 \leq l \leq \binom{n-1}{k-1}$. For example, if $f = \langle \dots, 234, 125, 245, 124, 123, 235 \rangle$ (i.e., $c = 2$) in $(5, 3)$ -THP, then $\hat{f} = \langle 23, 14, 34, 13, 12, 24 \rangle$. This mapping rule establishes a surjection

$$\Phi_c : \{f \in \mathcal{F} : c \in \bigcap_{\binom{n-1}{k} < l \leq \binom{n}{k}} Y_l\} \rightarrow \hat{\mathcal{F}},$$

since similar to bijection Λ_j , the bijection from $\{Y_{\binom{n-1}{k}+1}, Y_{\binom{n-1}{k}+2}, \dots, Y_{\binom{n}{k}}\}$ to $\{\hat{Y}_1, \hat{Y}_2, \dots, \hat{Y}_{\binom{n-1}{k-1}}\}$ is order-preserving in terms of $<_f$ and $<_{\hat{f}}$. So we have the following assertion for $c \in r_1..r_k \cap s_1..s_k$ and $(f, \hat{f}) \in \Phi_c$:

$$r_1..r_k <_f s_1..s_k \iff r_1..r_k \ominus c <_{\hat{f}} s_1..s_k \ominus c. \quad (11)$$

In $(n-1, k-1)$ -THP, for general $\hat{f} \in \hat{\mathcal{F}}$ and $1 \leq j \leq k$, it is naturally understood that

$$\hat{R}_{\text{COL}}^{\hat{f}}(t_1..t_k \ominus t_j) = (1/(n-1)! |\{\hat{\sigma} \in S_{n-1} : (t_1..t_k \ominus t_j)^{\hat{\sigma}} \in \hat{\mathcal{M}}_{\text{COL}}^{\hat{\sigma}, \hat{f}}\}|, \quad (12)$$

and according to the assumption on $(n-1, k-1)$ -THP, it holds that

$$\hat{R}_{\text{COL}}^{\hat{f}}(t_1..t_k \ominus t_j) \leq P(t_1..t_k \ominus t_j). \quad (13)$$

Now we claim that inequality (10) is implied by the following equation for $(f, \hat{f}) \in \Phi_c$:

$$R_{\text{COL}}^f(t_1..t_k) = (1/n) \sum_{1 \leq j \leq k} \hat{R}_{\text{COL}}^{\hat{f}}(t_1..t_k \ominus t_j). \quad (14)$$

That is because if equation (14) is true, then we immediately have

$$\begin{aligned} R_{\text{COL}}^f(t_1..t_k) &\leq (1/n) \sum_{1 \leq j \leq k} P(t_1..t_k \ominus t_j) && \text{(by inequality (13))} \\ &= P(t_1..t_k) && \text{(by recurrence (1)).} \end{aligned}$$

All that remains for case 1.1 is to prove equation (14). In fact, it can be proved along the way to argue case 1 of Section 2.2. Since $t_k = n$, $t_1..t_k$ is one of the last $\binom{n-1}{k-1}$ k -subsets in \mathcal{X}_{COL} ; so if $(t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, f}$, then there must be $1 \leq j \leq k$ such that $\sigma(t_j) = c$. Therefore in case 1.1, it holds that

$$R_{\text{COL}}^f(t_1..t_k) = (1/n!) \sum_{1 \leq j \leq k} |\{\sigma \in S_n : \sigma(t_j) = c, (t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, f}\}|. \quad (15)$$

Substituting their formulas (respectively in (15) and (12)) for $R_{\text{COL}}^f(t_1..t_k)$ and $\hat{R}_{\text{COL}}^{\hat{f}}(t_1..t_k \ominus t_j)$ in equation (14), we find that the holding of equation (14) is implied by the following equation for $1 \leq j \leq k$ and $(f, \hat{f}) \in \Phi_c$:

$$|\{\sigma \in S_n : \sigma(t_j) = c, (t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, f}\}| = |\{\hat{\sigma} \in S_{n-1} : (t_1..t_k \ominus t_j)^{\hat{\sigma}} \in \hat{\mathcal{M}}_{\text{COL}}^{\hat{\sigma}, \hat{f}}\}|. \quad (16)$$

Now we can check the similarity between (16) and (4); nevertheless, a self-contained argument for equation (16) is necessary, though it looks somewhat redundant.

To prove equation (16), we first need a bijection $\Psi_{j,c} : \{\sigma \in S_n : \sigma(t_j) = c\} \rightarrow S_{n-1}$ defined by

$$\sigma \mapsto \delta_c(\sigma(1))\delta_c(\sigma(2))\dots\delta_c(\sigma(t_j-1))\delta_c(\sigma(t_j+1))\delta_c(\sigma(t_j+2))\dots\delta_c(n),$$

which can be viewed as a generalization of bijection Δ_j from the special case $c = n$ to the general case. Letting $(\sigma, \hat{\sigma}) \in \Psi_{j,c}$, and $r_1..r_k$ be a k -subset with some $r_i = t_j$, we can see bijection $\Psi_{j,c}$ holds that

$$\begin{aligned} (r_1..r_k)^\sigma \ominus c &= ((r_1..r_{i-1}t_jr_{i+1}..r_k)^\sigma \setminus \{\sigma(t_j)\})^{\delta_c} \\ &= \{\delta_c(\sigma(r_1)), \dots, \delta_c(\sigma(r_{i-1})), \delta_c(\sigma(r_{i+1})), \dots, \delta_c(\sigma(r_k))\} \\ &= \{\hat{\sigma}(r_1), \dots, \hat{\sigma}(r_{i-1}), \hat{\sigma}(r_{i+1}-1), \dots, \hat{\sigma}(r_k-1)\} \\ &= (r_1..r_{i-1}(r_{i+1}-1)..(r_k-1))^{\hat{\sigma}} \\ &= (r_1..r_k \ominus t_j)^{\hat{\sigma}}. \end{aligned} \quad (17)$$

Another bijection we need is the bijection

$$\Lambda'_j : \{s_1..s_k \in \mathcal{X} : t_j \in s_1..s_k, s_1..s_k <_{\text{col}} t_1..t_k\} \rightarrow \{r_1..r_{k-1} \in \hat{\mathcal{X}} : r_1..r_{k-1} <_{\text{col}} t_1..t_k \ominus t_j\},$$

which is the same as the one in Section 2.2.

With bijections $\Psi_{j,c}$ and Λ'_j , as well as surjection Φ_c , we are ready for proving equation (16). Combining equation (17) and assertion (11), we have the following assertion for $1 \leq j \leq k$, $(f, \hat{f}) \in \Phi_c$, $(\sigma, \hat{\sigma}) \in \Psi_{j,c}$, and $(s_1..s_k, s_1..s_k \ominus t_j) \in \Lambda'_j$:

$$\begin{aligned} (s_1..s_k)^\sigma <_f (t_1..t_k)^\sigma &\iff (s_1..s_k)^\sigma \ominus c <_{\hat{f}} (t_1..t_k)^\sigma \ominus c && \text{(by assertion (11))} \\ &\iff (s_1..s_k \ominus t_j)^{\hat{\sigma}} <_{\hat{f}} (t_1..t_k \ominus t_j)^{\hat{\sigma}} && \text{(by equation (17)).} \end{aligned} \quad (18)$$

Assertion (18) directly implies

$$(t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma,f} \Rightarrow (t_1..t_k \ominus t_j)^{\hat{\sigma}} \in \hat{\mathcal{M}}_{\text{COL}}^{\hat{\sigma},\hat{f}},$$

since $|\{s_1..s_k \in \mathcal{X} : s_1..s_k <_{\text{col}} t_1..t_k\}| > |\{r_1..r_{k-1} \in \hat{\mathcal{X}} : r_1..r_{k-1} <_{\text{col}} t_1..t_k \ominus t_j\}|$. How about such $s_1..s_k$ with $s_1..s_k <_{\text{col}} t_1..t_k$ and $t_j \notin s_1..s_k$? We can find it inherently holds that $(s_1..s_k)^\sigma <_f (t_1..t_k)^\sigma$, because $c \notin (s_1..s_k)^\sigma$ and $c \in (t_1..t_k)^\sigma$. Therefore, it also holds that

$$(t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma,f} \Leftarrow (t_1..t_k \ominus t_j)^{\hat{\sigma}} \in \hat{\mathcal{M}}_{\text{COL}}^{\hat{\sigma},\hat{f}}.$$

This completes the proof of equation (16), and completes the argument for case 1.1 as well.

Case 1.2: $\bigcap_{\binom{n-1}{k} < l \leq \binom{n}{k}} Y_l = \emptyset$. Let v be the smallest number such that $\bigcap_{v \leq l \leq \binom{n}{k}} Y_l \neq \emptyset$, and for convenience, let $\mathcal{X}_{\text{COL}} = \langle C_1, C_2, \dots, C_{\binom{n}{k}} \rangle$ and $t_1..t_k = C_u$ with $\binom{n-1}{k} < u \leq \binom{n}{k}$. We divide the representation of $R_{\text{COL}}^f(C_u)$ in terms of contributions into two parts:

$$R_{\text{COL}}^f(C_u) = (1/n!) \sum_{u \leq l < v} T_{\text{COL}}^f(C_u, Y_l) + (1/n!) \sum_{v \leq l \leq \binom{n}{k}} T_{\text{COL}}^f(C_u, Y_l).$$

First consider $(1/n!) \sum_{u \leq l < v} T_{\text{COL}}^f(C_u, Y_l)$. It is null if $v \leq u$; otherwise we claim that it is equal to 0. Why? From $\bigcap_{u \leq i \leq \binom{n}{k}} C_i \neq \emptyset$ (all including n), we know that $\bigcap_{u \leq i \leq \binom{n}{k}} C_i^\sigma \neq \emptyset$ for any $\sigma \in S_n$. However, for any l with $u \leq l < v$, $\bigcap_{l \leq i \leq \binom{n}{k}} Y_i = \emptyset$, so anyway we have $\{Y_l, Y_{l+1}, \dots, Y_{\binom{n}{k}}\} \not\subseteq \{C_u^\sigma, C_{u+1}^\sigma, \dots, C_{\binom{n}{k}}^\sigma\}$. That is, when $C_u^\sigma = Y_l$ with $u \leq l < v$, C_u^σ cannot be a left-to-right maximum in $\mathcal{X}_{\text{COL}}^\sigma$, under f .

Next, consider $(1/n!) \sum_{v \leq l \leq \binom{n}{k}} T_{\text{COL}}^f(C_u, Y_l)$. We claim that it cannot exceed $P(C_u)$. Indeed, let's construct a valid ranking function $g = \langle Z_1, Z_2, \dots, Z_{\binom{n}{k}} \rangle$ from f such that it satisfies

$$\langle Z_v, Z_{v+1}, \dots, Z_{\binom{n}{k}} \rangle = \langle Y_v, Y_{v+1}, \dots, Y_{\binom{n}{k}} \rangle \text{ and } \bigcap_{\binom{n-1}{k} < l \leq \binom{n}{k}} Z_l \neq \emptyset.$$

That is, g satisfies the condition of case 1.1, and keeps the last $\binom{n}{k} - v + 1$ k -subsets unchanged. Such construction is possible because $\bigcap_{v \leq l \leq \binom{n}{k}} Y_l \neq \emptyset$. Then we have

$$\begin{aligned} (1/n!) \sum_{v \leq l \leq \binom{n}{k}} T_{\text{COL}}^f(C_u, Y_l) &= (1/n!) \sum_{v \leq l \leq \binom{n}{k}} T_{\text{COL}}^g(C_u, Z_l) && \text{(by Obs. 1(ii))} \\ &\leq (1/n!) \sum_{u \leq l \leq \binom{n}{k}} T_{\text{COL}}^g(C_u, Z_l) && \text{(being } = \text{ if } v \leq u) \\ &= R_{\text{COL}}^g(C_u) \\ &\leq P(C_u) && \text{(from case 1.1).} \end{aligned}$$

Summarizing these two parts results in the inequality $R_{\text{COL}}^f(C_u) \leq P(C_u)$ (i.e., inequality (10)).

Case 2: $t_k < n$. In this case, we know that inequality (10) has already held for (t_k, k) -THP, by the conclusion of case 1. We still use induction to prove that it holds for (n, k) -THP. That is, under the assumption that inequality (10) holds for $(n-1, k)$ -THP, we argue its holding for (n, k) -THP. To distinguish the notations of $(n-1, k)$ -THP from (n, k) -THP, we add $\tilde{\cdot}$ to the relevant symbols of $(n-1, k)$ -THP.

Still let $f = \langle Y_1, Y_2, \dots, Y_{\binom{n}{k}} \rangle$. For $1 \leq d \leq n$, we note that there is a surjection $\Omega_d : \mathcal{F} \rightarrow \tilde{\mathcal{F}}$, in which $\tilde{f} = \langle \tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_{\binom{n-1}{k}} \rangle$ is obtained from f as follows: remove all the k -subsets including d and conduct δ_d to the remaining $\binom{n-1}{k}$ k -subsets. Precisely, letting $\langle Y_{i_1}, Y_{i_2}, \dots, Y_{i_{\binom{n-1}{k}}} \rangle$ with $i_1 < i_2 < \dots < i_{\binom{n-1}{k}}$ be the subsequence of f in which no k -subset includes d , we construct \tilde{f} by the rule

$$\tilde{Y}_j := (Y_{i_j})^{\delta_d}$$

for $1 \leq j \leq \binom{n-1}{k}$. For instance, in Ω_3 of $(5, 2)$ -THP,

$$\langle 15, \underline{34}, \underline{35}, \underline{23}, 25, 24, \underline{13}, 12, 45, 14 \rangle \mapsto \langle 14, 24, 23, 12, 34, 13 \rangle,$$

where the underlined subsets are neglected. By the mapping rule of Ω_d , we have the following assertion for $d \notin r_1..r_k \cup s_1..s_k$ and $(f, \tilde{f}) \in \Omega_d$:

$$r_1..r_k <_f s_1..s_k \iff (r_1..r_k)^{\delta_d} <_{\tilde{f}} (s_1..s_k)^{\delta_d}. \quad (19)$$

Now we claim that the holding of inequality (10) for case 2 is implied by the following equation for $1 \leq d \leq n$ and $(f, \tilde{f}) \in \Omega_d$:

$$|\{\sigma \in S_n : \sigma(n) = d, (t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, f}\}| = |\{\tilde{\sigma} \in S_{n-1} : (t_1..t_k)^{\tilde{\sigma}} \in \tilde{\mathcal{M}}_{\text{COL}}^{\tilde{\sigma}, \tilde{f}}\}|, \quad (20)$$

which is quite similar to equation (8). Indeed, if equation (20) is true, we immediately have

$$\begin{aligned} R_{\text{COL}}^f(t_1..t_k) &= (1/n!) \sum_{1 \leq d \leq n} \text{(the left-hand side of (20))} \\ &= (1/(n-1)!) |\{\tilde{\sigma} \in S_{n-1} : (t_1..t_k)^{\tilde{\sigma}} \in \tilde{\mathcal{M}}_{\text{COL}}^{\tilde{\sigma}, \tilde{f}}\}| \quad \text{(by equation (20))} \\ &= \tilde{R}_{\text{COL}}^{\tilde{f}}(t_1..t_k) \\ &\leq \tilde{P}(t_1..t_k) && \text{(by the assumption)} \\ &= P(t_1..t_k) && \text{(from case 2 of Section 2.2).} \end{aligned}$$

To prove equation (20), we can use the same bijection $\mathcal{Y}_d : \{\sigma \in S_n : \sigma(n) = d\} \rightarrow S_{n-1}$ for $1 \leq d \leq n$ as the one in Section 2.2. Recall that for $(\sigma, \tilde{\sigma}) \in \mathcal{Y}_d$ and a k -subset $r_1..r_k$ with $r_k < n$, bijection \mathcal{Y}_d holds equation (9), i.e., $((r_1..r_k)^\sigma)^{\delta_d} = (r_1..r_k)^{\tilde{\sigma}}$, as being restated. Then, combining this equation and assertion (19), we have the following assertion for $1 \leq d \leq n$, $(\sigma, \tilde{\sigma}) \in \mathcal{Y}_d$, $(f, \tilde{f}) \in \Omega_d$, and any $s_1..s_k <_{\text{col}} t_1..t_k$:

$$\begin{aligned} (s_1..s_k)^\sigma <_f (t_1..t_k)^\sigma &\iff ((s_1..s_k)^\sigma)^{\delta_d} <_{\tilde{f}} ((t_1..t_k)^\sigma)^{\delta_d} && \text{(by assertion (19))} \\ &\iff (s_1..s_k)^{\tilde{\sigma}} <_{\tilde{f}} (t_1..t_k)^{\tilde{\sigma}} && \text{(by equation (9)).} \end{aligned} \quad (21)$$

To explain it further, we should see that when $\sigma(n) = d$, every $(s_1..s_k)^\sigma$ with $s_1..s_k <_{\text{col}} t_1..t_k$ doesn't include d , so the k -subsets in f including d are as ineffective as nothing.

Since assertion (21) directly implies equation (20), we accomplish the argument for case 2, and so accomplish the proof of (II) as well.

2.4. Proof of (III)

Obviously, the equation $E_{\text{LEX}}^{\text{lex}}(n, k) = E_{\text{COL}}^{\text{col}}(n, k)$ we want to prove is implied by the following stronger assertion for any $\sigma \in S_n$ and any $t_1..t_k \in \mathcal{X}$:

$$(t_1..t_k)^\sigma \in \mathcal{M}_{\text{LEX}}^{\sigma, \text{lex}} \iff (t_1..t_k)^\sigma \in \mathcal{M}_{\text{COL}}^{\sigma, \text{col}}. \quad (22)$$

We now prove this assertion by contradiction.

\Leftarrow : We have the assumption that $(s_1..s_k)^\sigma <_{\text{col}} (t_1..t_k)^\sigma$ for all $s_1..s_k <_{\text{col}} t_1..t_k$. We now suppose for the purpose of contradiction that there is a $r_1..r_k$ with $r_1..r_k <_{\text{lex}} t_1..t_k$ and $(t_1..t_k)^\sigma <_{\text{lex}} (r_1..r_k)^\sigma$. Our objective is to find a $s_1..s_k$ with $s_1..s_k <_{\text{col}} t_1..t_k$ and $(t_1..t_k)^\sigma <_{\text{col}} (s_1..s_k)^\sigma$, which contradicts the assumption.

To do this, we let $c_1..c_l = r_1..r_k \cap t_1..t_k$, and let $r'_1..r'_j = r_1..r_k \setminus c_1..c_l$, $t'_1..t'_j = t_1..t_k \setminus c_1..c_l$. Clearly, $j \geq 1$ and $j + l = k$. Then, by the property of lexical ordering, we have that $r'_1 < t'_1$, and $\min_{1 \leq i \leq j} \sigma(t'_i) < \min_{1 \leq i \leq j} \sigma(r'_i)$. Now, let $(t'_1..t'_j)^\sigma = \sigma(t''_1)\sigma(t''_2)\dots\sigma(t''_j)$ (with $\sigma(t''_1) < \sigma(t''_2) < \dots < \sigma(t''_j)$). Then, we claim that the k -subset $\{r'_1, t''_2, t''_3, \dots, t''_j, c_1, c_2, \dots, c_l\}$ is just what we want to find.

In fact, since $r'_1 < t'_1$ and $\{t''_2, t''_3, \dots, t''_j\} \subset t'_1..t'_j$, we have

$$\{r'_1, t''_2, t''_3, \dots, t''_j, c_1, c_2, \dots, c_l\} <_{\text{lex}(\text{col})} \{t'_1, t'_2, \dots, t'_j, c_1, c_2, \dots, c_l\} = t_1..t_k. \quad (23)$$

In inequality (23), as well as (24),(25), and (26) hereafter, the reason why both $<_{\text{lex}}$ and $<_{\text{col}}$ hold is that there is only one distinct element between those two k -subsets.

On the other hand, since $\sigma(t'_1) = \min_{1 \leq i \leq j} \sigma(t'_i) < \min_{1 \leq i \leq j} \sigma(r'_i) \leq \sigma(r'_1)$, we have

$$(t_1..t_k)^\sigma = \{t'_1, t'_2, \dots, t'_j, c_1, c_2, \dots, c_l\}^\sigma <_{\text{lex}(\text{col})} \{r'_1, t''_2, t''_3, \dots, t''_j, c_1, c_2, \dots, c_l\}^\sigma. \quad (24)$$

Inequalities (23) and (24) together form a contradiction to the assumption of the \Leftarrow direction.

\Rightarrow : This time we have the assumption that $(s_1..s_k)^\sigma <_{\text{lex}} (t_1..t_k)^\sigma$ for all $s_1..s_k <_{\text{lex}} t_1..t_k$. We now suppose for the purpose of contradiction that there is a $r_1..r_k$ with $r_1..r_k <_{\text{col}} t_1..t_k$

and $(t_1..t_k)^\sigma <_{\text{col}}(r_1..r_k)^\sigma$. We still let $c_1..c_l = r_1..r_k \cap t_1..t_k$, and let $r'_1..r'_j = r_1..r_k \setminus c_1..c_l$, $t'_1..t'_j = t_1..t_k \setminus c_1..c_l$. Then, by the property of colex ordering, we have that $r'_j < t'_j$, and $\max_{1 \leq i \leq j} \sigma(t'_i) < \max_{1 \leq i \leq j} \sigma(r'_i)$. Now, let $(r'_1..r'_j)^\sigma = \sigma(r''_1)\sigma(r''_2)...\sigma(r''_j)$ (with $\sigma(r''_1) < \sigma(r''_2) < \dots < \sigma(r''_j)$). Then in this time, we rely on the k -subset $\{t'_1, t'_2, \dots, t'_{j-1}, r''_j, c_1, c_2, \dots, c_l\}$ to derive contradiction.

Indeed, noting that $r''_j < t'_j$ (because $r''_j \in r'_1..r'_j$ and $r'_j < t'_j$), we have

$$\{t'_1, t'_2, \dots, t'_{j-1}, r''_j, c_1, c_2, \dots, c_l\} <_{\text{col(lex)}} \{t'_1, t'_2, \dots, t'_j, c_1, c_2, \dots, c_l\} = t_1..t_k. \quad (25)$$

On the other hand, since $\max_{1 \leq i \leq j} \sigma(t'_i) < \max_{1 \leq i \leq j} \sigma(r'_i) = \sigma(r''_j)$, we have

$$(t_1..t_k)^\sigma = \{t'_1, t'_2, \dots, t'_j, c_1, c_2, \dots, c_l\}^\sigma <_{\text{col(lex)}} \{t'_1, t'_2, \dots, t'_{j-1}, r''_j, c_1, c_2, \dots, c_l\}^\sigma. \quad (26)$$

Inequalities (25) and (26) together form a contradiction to the assumption of the \Rightarrow direction.

2.5. Proof of (IV)

We start by presenting two properties of $E_{ord}^f(n, k)$ for general ord and f . Then the assertion we want to prove here can be viewed as a corollary of assertion (II) plus these two properties.

Let $\mathcal{X}_{ord} = \langle X_1, X_2, \dots, X_{\binom{n}{k}} \rangle$ and $f = \langle Y_1, Y_2, \dots, Y_{\binom{n}{k}} \rangle$. We define a series of ‘‘complements’’ as follows:

$$\begin{aligned} \overline{X}_i &:= [n] \setminus X_i; \\ \overline{\mathcal{X}}_{ord} &:= \langle \overline{X}_1, \overline{X}_2, \dots, \overline{X}_{\binom{n}{k}} \rangle, \text{ the ordering of all the } (n-k)\text{-subsets of } [n] \text{ in the way of } \overline{ord}; \\ \overline{f} &:= \langle \overline{Y}_1, \overline{Y}_2, \dots, \overline{Y}_{\binom{n}{k}} \rangle, \text{ a ranking function of } (n, n-k)\text{-THP satisfying that } \overline{f}(\overline{X}_i) = f(X_i) \\ &\text{ for } 1 \leq i \leq \binom{n}{k}. \end{aligned}$$

We claim that $E_{ord}^f(n, k)$ holds the following complementary-equivalence property:

$$E_{ord}^f(n, k) = E_{\overline{ord}}^{\overline{f}}(n, n-k). \quad (27)$$

Indeed, for any $\sigma \in S_n$, we observe that $\overline{X}_i^\sigma = [n] \setminus X_i^\sigma = [n]^\sigma \setminus X_i^\sigma = ([n] \setminus X_i)^\sigma = \overline{X}_i^\sigma$, so we have $\langle f(X_i^\sigma) \rangle_{1 \leq i \leq \binom{n}{k}} = \langle \overline{f}(\overline{X}_i^\sigma) \rangle_{1 \leq i \leq \binom{n}{k}} = \langle \overline{f}(\overline{X}_i) \rangle_{1 \leq i \leq \binom{n}{k}}$, from which equation (27) follows.

Let $\pi, \omega \in S_n$. We define ord^π and f^ω as follows:

$$\begin{aligned} \mathcal{X}_{ord^\pi} &:= \langle X_1^\pi, X_2^\pi, \dots, X_{\binom{n}{k}}^\pi \rangle; \\ f^\omega &:= \langle Y_1^\omega, Y_2^\omega, \dots, Y_{\binom{n}{k}}^\omega \rangle, \text{ a ranking function of } (n, k)\text{-THP satisfying that } f^\omega(X_i^\omega) = f(X_i) \\ &\text{ for } 1 \leq i \leq \binom{n}{k}. \end{aligned}$$

We claim that $E_{ord}^f(n, k)$ holds the following permuting-equivalence property:

$$E_{ord}^f(n, k) = E_{ord^\pi}^{f^\omega}(n, k). \quad (28)$$

Indeed, for any $\sigma \in S_n$, we observe that $f(X_i^\sigma) = f((X_i^\pi)^{\pi^{-1} \cdot \sigma}) = f^\omega((X_i^\pi)^{\pi^{-1} \cdot \sigma \cdot \omega})$. That is, whenever there is a $\sigma \in S_n$ incurring the sequence $\langle f(X_i^\sigma) \rangle_{1 \leq i \leq \binom{n}{k}}$, there is only one $\sigma' = \pi^{-1} \cdot \sigma \cdot \omega$ such that $\langle f^\omega((X_i^\pi)^{\sigma'}) \rangle_{1 \leq i \leq \binom{n}{k}} = \langle f(X_i^\sigma) \rangle_{1 \leq i \leq \binom{n}{k}}$. Therefore, equation (28) follows.

It is a common knowledge that the lexical and colex orderings are complementary to each other. If we only use their normal forms, letting $\pi = n(n-1)\dots 1$, we have that in symbols, $\text{COL} = \overline{\text{LEX}}^\pi$, $\text{col} = \overline{\text{lex}}^\pi$, and vice versa. Therefore for any f , we have

$$\begin{aligned} E_{\text{LEX}}^f(n, k) &= E_{\overline{\text{LEX}}^\pi}^{\overline{f}}(n, n-k) = E_{\text{COL}}^{\overline{f}}(n, n-k) \\ &\leq E_{\text{COL}}^{\text{col}}(n, n-k) = E_{\overline{\text{COL}}^\pi}^{\overline{\text{col}}}(n, k) = E_{\text{LEX}}^{\text{lex}}(n, k), \end{aligned}$$

where the inequality comes from assertion (II). This completes the proof of assertion (IV).

3. Proof of Corollary 2

Let us first prove the symmetry of $E_{\text{L}(C)}^*(n, k)$ in k and $n-k$. It should be an interesting and challenging exercise to give an algebraic proof of this property directly from the formulas in Theorem 1. But here, we base our proof simply upon the results of the previous section: By assertions (I)–(IV), we know that $E_{\text{L}(C)}^*(n, k) = E_{\text{LEX}}^{\text{lex}}(n, k) = E_{\text{COL}}^{\text{col}}(n, k)$, and from Section 2.5, we know that $E_{\text{LEX}}^{\text{lex}}(n, k) = E_{\text{COL}}^{\text{col}}(n, n-k)$. Consequently, combining these two results gives the equation $E_{\text{L}(C)}^*(n, k) = E_{\text{L}(C)}^*(n, n-k)$. Since we derive these results by a series of bijections, our proof here should be put into the category of combinatorial proof.

Next, let us prove the lower and upper bounds of $E_{\text{L}(C)}^*(n, k)$, directly from the formulas in Theorem 1. We start by claiming that for $s_1 \dots s_k, t_1 \dots t_k \in \mathcal{X}$ with $s_1 \leq t_1, s_2 \leq t_2, \dots, s_k \leq t_k$, it holds that

$$P(s_1 \dots s_k) \geq P(t_1 \dots t_k). \quad (29)$$

Indeed, inequality (29) can be easily proved by induction. Since $1/s_1 \geq 1/t_1$ in the case $k=1$, inequality (29) holds for the basic $(n, 1)$ -THP. We now assume that inequality (29) holds for $(n-1, k-1)$ -THP with $1 < k \leq n$. Under this assumption, we observe that

$$\begin{aligned} P(s_1 \dots s_k) &= (1/s_k) \sum_{1 \leq j \leq k} P(s_1 \dots s_{j-1} (s_{j+1} - 1) \dots (s_k - 1)) \\ &\geq (1/t_k) \sum_{1 \leq j \leq k} P(t_1 \dots t_{j-1} (t_{j+1} - 1) \dots (t_k - 1)) \\ &= P(t_1 \dots t_k). \end{aligned}$$

That is, inequality (29) holds for (n, k) -THP.

Then, looking at the right-hand side of recurrence (1), we find that

$$P(t_1 \dots t_{k-1}) = \max_{1 \leq j \leq k} P(t_1 \dots t_{j-1} (t_{j+1} - 1) \dots (t_k - 1)). \quad (30)$$

Therefore, by (30), we have

$$P(t_1 \dots t_k) \leq \frac{k}{t_k} P(t_1 \dots t_{k-1}) \leq \frac{k(k-1)}{t_k t_{k-1}} P(t_1 \dots t_{k-2}) \leq \dots \leq \frac{k!}{t_k t_{k-1} \dots t_1},$$

and at the same time, also have

$$P(t_1..t_k) \geq \frac{1}{t_k}P(t_1..t_{k-1}) \geq \frac{1}{t_k t_{k-1}}P(t_1..t_{k-2}) \geq \dots \geq \frac{1}{t_k t_{k-1} \dots t_1}.$$

So far, we have obtained the following bounds for $E_{L(C)}^*(n, k)$:

$$\sum_{1 \leq t_1 < t_2 < \dots < t_k \leq n} \frac{1}{t_1 t_2 \dots t_k} \leq E_{L(C)}^*(n, k) \leq k! \sum_{1 \leq t_1 < t_2 < \dots < t_k \leq n} \frac{1}{t_1 t_2 \dots t_k}.$$

At last, what we need to argue is the identity

$$\sum_{1 \leq t_1 < t_2 < \dots < t_k \leq n} \frac{1}{t_1 t_2 \dots t_k} = \frac{1}{n!} \binom{n+1}{k+1}, \quad (31)$$

which may have already been listed in some textbook, as a knowledge point or exercise. However, the author cannot find it in the famous textbooks such as [4], [15], and [18].

Indeed, recalling that $\binom{n+1}{k+1}$ is the coefficient of x^{k+1} in the polynomial $x(x+1)\dots(x+n)$, we have $\binom{n+1}{k+1} = \sum_{1 \leq s_1 < s_2 < \dots < s_{n-k} \leq n} s_1 s_2 \dots s_{n-k}$. Whenever there is a product $s_1 s_2 \dots s_{n-k}$ with $1 \leq s_1 < s_2 < \dots < s_{n-k} \leq n$, there is a fraction $n!/(t_1 t_2 \dots t_k)$ with $1 \leq t_1 < t_2 < \dots < t_k \leq n$, which is both one-to-one corresponding to and equal to it. Hence, identity (31) follows.

4. Concluding remarks

Letting $\mathcal{X}_{ord} = \langle X_i \rangle_{1 \leq i \leq \binom{n}{k}}$, we denote the sequence $\langle f(X_i^\sigma) \rangle_{1 \leq i \leq \binom{n}{k}}$ by $\mathcal{X}_{ord}^{\sigma, f}$. Then, under a uniformly random $\sigma \in S_n$ and for fixed ord and f , $\mathcal{X}_{ord}^{\sigma, f}$ forms a ‘‘partly random’’ permutation of $[\binom{n}{k}]$ when $k > 1$. In the previous sections, we only quantify the expected-value measure for one statistic of $\mathcal{X}_{ord}^{\sigma, f}$ —the number of left-to-right maxima—for two special pairs of ord and f : (LEX, lex) and (COL, col). Among all the other statistics discussed in the textbooks [4], [15], and [18] for permutations, what we can immediately answer are the expected numbers of *descents* and *inversions*: In terms of the expected-value measure of these two statistics, $\mathcal{X}_{ord}^{\sigma, f}$ is as ‘‘random’’ as a uniformly random permutation of $[\binom{n}{k}]$.

Proposition 2. *The expected number of descents in $\mathcal{X}_{ord}^{\sigma, f}$ is $(\binom{n}{k} - 1)/2$, and the expected number of inversions in $\mathcal{X}_{ord}^{\sigma, f}$ is $\binom{n}{k}(\binom{n}{k} - 1)/4$.*

Proof. Let $X_1, X_2 \in \mathcal{X}$ with $X_1 \neq X_2$, and let $Y_1, Y_2 \in \mathcal{X}$ with $Y_1 \neq Y_2$. We claim that

$$|\{\sigma \in S_n : X_1^\sigma = Y_1, X_2^\sigma = Y_2\}| = |\{\sigma \in S_n : X_1^\sigma = Y_2, X_2^\sigma = Y_1\}|.$$

Why? Let $|X_1 \cap X_2| = k - j$ with $1 \leq j \leq k$. Then, these two cardinalities are both equal to 0 if $|Y_1 \cap Y_2| \neq k - j$, and both equal to $(k - j)!(j!)^2(n - k - j)!$ otherwise. We have shown that for any pair of k -subsets X_1 and X_2 in \mathcal{X}_{ord} , the probability that $f(X_1^\sigma) > f(X_2^\sigma)$ is always $1/2$. \square

Notice that the equidistribution property of some pairs of statistics, which holds for a uniformly random permutation of $[n]$, can hardly hold for $\mathcal{X}_{ord}^{\sigma,f}$. That is, neither is $E_{ord}^f(n, k)$ possibly equal to the expected number of *cycles* in $\mathcal{X}_{ord}^{\sigma,f}$, nor does Proposition 2 possibly provide answers to the expected number of *excedances* and to the expected value of *major index*[18].

Other measures of these statistics on $\mathcal{X}_{ord}^{\sigma,f}$ seem more difficult to solve. For example, let us consider the variances of the number of left-to-right maxima respectively in $\mathcal{X}_{LEX}^{\sigma,lex}$ and $\mathcal{X}_{COL}^{\sigma,col}$, which most deserve to be explored. From assertion (22) in Section 2.4, we can know that these two variances are equivalent. However, it is not easy to quantify them, primarily because the set of the $\binom{n}{k}$ events that left-to-right maxima arise from some k -subset, are no longer mutually independent in the case $k > 1$.

Come back the team-hiring problem itself. Apart from the lexical and colex orderings for listing k -subsets of $[n]$, there is another well-known sequential ordering, called the *revolving-door* or *minimal-change* ordering (see [10, Section 2.3.3]). The sequence of k -subsets in accordance with this ordering can also be generated efficiently in both time and space, and it looks somewhat more “disorderly” than the lexical or colex ordering. Let “REV” be the tag of this revolving-door ordering. Is $E_{REV}^*(n, k)$ asymptotically smaller than $E_{L(c)}^*(n, k)$? That is the final question we are concerned with.

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