



HAL
open science

BV EXPONENTIAL STABILITY FOR NETWORKS OF SCALAR CONSERVATION LAWS USING LINEAR OR SATURATED CONTROLS

Mathias Dus

► **To cite this version:**

Mathias Dus. BV EXPONENTIAL STABILITY FOR NETWORKS OF SCALAR CONSERVATION LAWS USING LINEAR OR SATURATED CONTROLS. 2020. hal-02481725v2

HAL Id: hal-02481725

<https://hal.science/hal-02481725v2>

Preprint submitted on 6 Mar 2020 (v2), last revised 2 Feb 2021 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

BV EXPONENTIAL STABILITY FOR NETWORKS OF SCALAR CONSERVATION LAWS USING LINEAR OR SATURATED CONTROLS

MATHIAS DUS*

Abstract. In this paper, we investigate the BV exponential stability of general networks of scalar conservation laws with positive velocities and under dissipative boundary conditions. The paper is divided in two parts, the first one focusing on linear controls while the last one deals with saturated laws. For the linear case, the global exponential BV stability is proven. For the saturated case, we argue that we cannot expect to have a basin of attraction larger than the region of linearity in a BV context. We rather prove an L^∞ local stability result. An explicit estimate of the basin of attraction is given. The Lyapunov functional is inspired from Glimm's seminal work [13] reconsidered in [7].

Key words. Bounded variations, stabilization, feedback, saturation, wavefront tracking method.

AMS subject classifications. 93D05, 93D15, 93D20

1. Introduction. In this paper, we are interested in the exponential stabilization of some 1D hyperbolic systems using linear or saturated feedback control laws. More precisely, we are interested in networks of scalar conservation laws with strictly positive characteristic velocities. In Figure 1, we give an example of such networks.

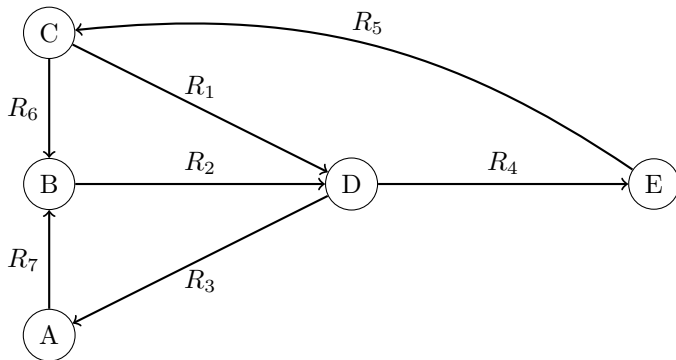


Figure 1: An example of network

Quantities (R_1, \dots, R_7) are scalar functions of space $x \in [0, 1]$ and time $t \geq 0$ and are all subject to a scalar nonlinear transport equation (to be defined) with positive velocities. To hope for a well-posed problem, we need to define $(R_1(t, 0), \dots, R_7(t, 0))$. It is given by transfer operators associated to nodes A, B, C, D, E . For example, at node B , one could impose:

$$R_2(t, 0) = R_6(t, 1) + R_7(t, 1) := g_B(R_6(t, 1), R_7(t, 1)).$$

More generally, the system we consider is of the form:

*Univ. Paul Sabatier, Institut de Mathématiques de Toulouse, 118 route de Narbonne, 31062 Toulouse Cedex 9 (mathias.dus@math.univ-toulouse.fr).

$$(1.1) \quad \begin{cases} R_t + [f(R)]_x & = 0 \\ R(t, 0) & = g(R(t, 1)) \\ R(0, x) & = R_0(x) \end{cases}$$

where $R : \mathbb{R}^+ \times [0, 1] \mapsto \mathbb{R}^d$ ($d \in \mathbb{N}^*$) and $f, g : \mathbb{R}^d \mapsto \mathbb{R}^d$.

For coherence, we impose that all characteristic velocities are positive and consequently, the boundary condition in (1.1) is adapted.

In [3, Chapter 1], typical examples of systems modeled by hyperbolic PDEs with feedback boundary conditions are cited; the telegrapher equations for electrical lines, the shallow water (Saint-Venant) equations for open channels, the isothermal Euler equations for gas flow in pipelines or even the Aw-Rascle equations for road traffic.

More specifically, we are interested in the stabilization of (1.1) using feedback control laws at the boundary. The problem is equivalent to find sufficient conditions on g such that for any R_0 initial data, the solution to (1.1) converges exponentially fast toward zero in the sense that

$$\forall t \geq 0, \|R(t, \cdot)\|_X \leq C e^{-\gamma t} \|R_0\|_X$$

where $C, \gamma > 0$ are constants independent on t and $\|\cdot\|_X$ is a norm on a Banach space X .

Two types of feedback laws are analyzed in this article: the linear law and the saturated one.

1.1. Linear feedback. For the case where $g = H \in M_d(\mathbb{R})$ is a linear operator, the literature is quite rich.

When the flux is linear; $f(R) = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_d)$ with $\lambda_1, \dots, \lambda_d > 0$, the problem of stabilization can be treated for the following classical functional spaces X :

- a. Sobolev spaces $W^{m,p}([0, 1])$ for $m \in \mathbb{N}$ and $p \in [1, +\infty]$.
- b. Spaces $C^m([0, 1])$ with ($m \in \mathbb{N}$).
- c. $BV([0, 1])$.

Indeed, in [14, Theorem 3.5 p. 275], the authors prove that 0 is globally exponentially stable in spaces X defined above if and only if there exists $\delta > 0$ such that

$$(1.2) \quad \left\{ z \in \mathbb{C} \mid \det(I_d - \text{diag}(e^{-z/\lambda_1}, \dots, e^{-z/\lambda_d})H) = 0 \right\} \subset \{z \in \mathbb{C} \mid \text{Re}(z) < -\delta\}.$$

However, the criteria (1.2) is not stable with respect to Λ . Indeed, when we take a H, Λ verifying (1.2), we cannot guaranty that the same holds for $\tilde{\Lambda}$ with $\tilde{\Lambda}$ diagonal and arbitrarily close to Λ [14, p. 285].

In the same book, Silkowski [14, Theorem 6.1 p. 286] proves that for all Banach spaces X listed above, 0 is globally exponentially stable and that this stability is robust with respect to Λ if and only if

$$(1.3) \quad \rho_0(H) := \max \{ \rho(\text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})H) \mid \theta_i \in \mathbb{R} \} < 1$$

where ρ designates the usual spectral radius.

Condition (1.3) is stronger than (1.2). For some years, many results came out generalizing to nonlinear fluxes.

When the flux is not linear, only sufficient conditions of stability are given and most of the time this stability is only proven to be local:

- a. For $X = C^m([0, 1])$ with $m \in \mathbb{N}^*$, a sufficient condition [10, 19, 21] is:

$$(1.4) \quad \rho_\infty(H) := \inf_{\Delta \in D_d^+(\mathbb{R})} |\Delta H \Delta^{-1}|_\infty < 1$$

where $|\cdot|_\infty$ is the canonical infinity norm of matrices and $D_d^+(\mathbb{R})$ is the set of diagonal positive matrices.

It should be mentioned that in [10, 19, 21], the stability was proven for $m = 1$ but the argument can be adapted for any integer $m > 0$.

- b. For Sobolev space $W^{m,p}([0, 1])$ a sufficient condition for stability writes:

$$\rho_p(H) := \inf_{\Delta \in D_d^+(\mathbb{R})} |\Delta H \Delta^{-1}|_p < 1$$

where $|\cdot|_p$ is the canonical p norm of matrices.

The case $p = 2$ was treated in [6] and the general case $p \geq 1$ was treated in [8]. Also, it should be mentioned that in [6, 8], the stability was proven for $m = 2$ but the argument can be adapted for any integer $m > 0$.

- c. For $BV([0, 1])$, few results are known. To the authors' knowledge only [7] deals with this case. They take a 2×2 system of conservation laws and give a sufficient condition on H to ensure the local BV stability.

In this article, we also place ourselves in a BV context and find a sufficient condition on H to ensure a global BV stability. Contrary to [7], we will not consider vectorial fluxes. In this case, solutions are only proven to exist for small initial data. This is why, we will rather consider scalar decentralized fluxes (see section 1.3) for which solutions exist for whatever initial data in BV . This hypothesis on the flux is all the more important that when we will study saturated feedback laws, we will also estimate the basin of attraction. This would not be possible with solutions defined only for small initial data.

1.2. Saturated control law. We take a matrix $H \in M_d(\mathbb{R})$ potentially unstable in the sense that $\rho_\infty(H) > 1$ (see (1.4)). Then we assume that there exist matrices $B, K \in M_d(\mathbb{R})$ such that $\rho_\infty(H + BK) < 1$. Finally, we consider the following system:

$$(1.5) \quad \begin{cases} R_t + [f(R)]_x & = 0 \\ R(t, 0) & = HR(t, 1) + B\sigma(KR(t, 1)) \\ R(0, x) & = R_0(x) \end{cases}$$

with σ defined as a saturation by component *ie* there exists a $\sigma_s > 0$ such that:

$$\forall i \in \llbracket 1, d \rrbracket, x \in \mathbb{R}, \begin{cases} \sigma_i(x) = x & \text{if } |x| \leq \sigma_s \\ \sigma_i(x) = \text{sign}(x)\sigma_s & \text{otherwise.} \end{cases}$$

From criterion (1.4), the system without saturation is locally stable in $C^m([0, 1])$ with $m \in \mathbb{N}^*$. It is natural to ask ourselves if this property of stability is conserved through the saturation. Apart from this theoretical interest, this problem has gained attention in the last few years because of the increasing need of precision for modeling real actuators. Physical controllers cannot provide infinite energy and sometimes, they saturate rendering classical unsaturated models restrictive. To avoid such situations, engineers choose controllers powerful enough to avoid saturation when the system operates in standard conditions. However, over-dimensioning actuators is not optimal in term of mass and cost of operation for many sophisticated systems as satellites for example. Moreover, in some exceptional configurations, actuators could saturate and lead to very dangerous situations; unpredictable via linear theory.

Very few papers consider the effect of saturation on hyperbolic systems. To our knowledge, only [18] deals with this question in an H^1 context and for the wave equation. Fortunately, the theory is much more developed for finite dimensional systems where polytopic and deadzone techniques were designed [22].

In this paper, we will argue that in a BV context, it is not possible to get a basin of attraction bigger than the region of linearity. We rather prove an L^∞ local stability result with an estimation of the basin of attraction. Then, we deduce the exponential decay of the BV norm for solutions whose initial data belongs to the L^∞ basin of attraction.

1.3. Scalar conservation laws. The feedback laws being presented, we can now focus on the partial differential equation in itself. In fact, we consider a particular form of systems of conservation laws (1.1). More precisely, we take a flux f verifying Hypothesis 1.1.

Hypothesis 1.1. The flux f is in $C^1(\mathbb{R}^d)$ and there exist scalar fluxes $f_i \in C^1(\mathbb{R})$ such that:

$$\forall R \in \mathbb{R}^d, \forall i \in \llbracket 1, d \rrbracket, [f(R)]_i = f_i(R_i).$$

Moreover, f_i is such that there exist $\alpha_i, \beta_i > 0$;

$$\forall i \in \llbracket 1, d \rrbracket, \alpha_i \leq f'_i \leq \beta_i.$$

Such hypothesis allows to define the maximal and the minimal velocity:

$$(1.6) \quad \begin{cases} c_{\max} & := \max_{i \in \llbracket 1, d \rrbracket} \beta_i \\ c_{\min} & := \min_{i \in \llbracket 1, d \rrbracket} \alpha_i. \end{cases}$$

The aim of this section is to give a very short introduction to scalar conservation laws without giving any proof (see [4] for more details).

1.3.1. The set of functions with bounded variations. It is well-known that the space BV is well-adapted for conservation laws (see [4] for instance). This is why, we give the definition and main properties of such a space here:

DEFINITION 1.2. Let $R : [0, 1] \mapsto \mathbb{R}^d$ be a vector valued function. We say that R has bounded variations if

$$\forall n \in \mathbb{N}, \forall x_1 < \dots < x_n \in [0, 1], \sum_{i=1}^{n-1} |R(x_{i+1}) - R(x_i)| < \infty.$$

We denote $TV_{[0,1]}(R) = \sup_{n, (x_1, \dots, x_n)} \left\{ \sum_{i=1}^{n-1} |R(x_{i+1}) - R(x_i)| \right\}$ the total variation of R . $BV([0, 1])$ is the space of vector valued functions with bounded variations and it is a Banach space when $BV([0, 1])$ is embedded with the norm $\|\cdot\|_{BV([0,1])}$ defined as

$$(1.7) \quad \forall R \in BV([0, 1]), \quad \|R\|_{BV([0,1])} = TV_{[0,1]}(R) + \|R\|_{L^1([0,1])}.$$

The reason why we consider this space is because any function with bounded variations has a left and a right limit at each point x of $[0, 1]$. Hence, it is easy to define the trace operator and impose a boundary condition. Moreover, $BV([0, 1])$ has a very interesting property of compactness which will be very useful when we will pass to the limit in the Lyapunov analysis of approximating solutions. These properties are summed up in a lemma and a theorem:

LEMMA 1.3. Let $R : [0, 1] \mapsto \mathbb{R}^d$ with bounded variations. Then for all $x \in (0, 1)$, the left and right limit

$$R(x^-) = \lim_{y \rightarrow x^-} R(y), \quad R(x^+) = \lim_{y \rightarrow x^+} R(y)$$

exist.

Moreover, $R(0^+)$ and $R(1^-)$ are also well defined and R has at most countably many point of discontinuities.

Proof. This is an adaptation of [4, Lemma 2.1]. □

Defining the value of R at each jump by $R(x) = R(x^+)$, we can say that R is right continuous in the L^1 equivalence class. The following theorem is from Helly and states the compactness of $BV([0, 1])$ in $L^1_{loc}(\mathbb{R}^+, L^1([0, 1]))$.

THEOREM 1.4. [4, Theorem 2.4] Let $(R_\nu)_\nu$ be a sequence of functions from $\mathbb{R}^+ \times [0, 1]$ into \mathbb{R}^d such that there exist constants C , M and L satisfying

$$(1.8) \quad \forall \nu > 1, \forall x \in [0, 1], \forall t \geq 0, TV_{[0,1]}(R_\nu(t, \cdot)) \leq C, |R_\nu(t, x)| \leq M,$$

and

$$(1.9) \quad \forall 0 \leq t, s \leq T, \|R_\nu(t, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} \leq L|t - s|.$$

Then there exists a subsequence $(R_{\mu})_\mu$ converging strongly toward a certain R in $L^1_{loc}(\mathbb{R}^+, L^1([0, 1]))$ and this limit satisfies (1.8)-(1.9) with R_ν replaced by R .

1.3.2. Entropy. The concept of entropy is primordial in order to guaranty uniqueness of solutions to conservation laws. This is why we recall some basic definitions in this section.

If one considers the conservation law $R_t + [f(R)]_x = 0$ in the usual weak sense:

$$\forall \phi \in C_c^1((0, T) \times (0, 1); \mathbb{R}^d), \int_0^T \int_0^1 (\phi_t R + \phi_x f(R)) = 0,$$

it is commonly known that this PDE (associated with fixed boundary and initial conditions) can have several weak solutions (see Example 4.3 from [4]). In order to restrain the set of solutions, an entropy functional was introduced ([4], [9], [16]) and is defined as follows:

DEFINITION 1.5. *A continuously differentiable convex function $\eta : \mathbb{R}^d \mapsto \mathbb{R}$ is called an entropy for the conservation law $R_t + [f(R)]_x = 0$ with entropy flux $q : \mathbb{R}^d \mapsto \mathbb{R}$, if*

$$\forall R \in \mathbb{R}^d, D\eta(R) \cdot Df(R) = Dq(R).$$

For scalar conservation laws of the form $u_t + [f_1(u)]_x = 0$, the usual choice of entropy is $\eta(u) := |u - k|$ with flux $q(u) := (f_1(u) - f_1(k))\text{sign}(u - k)$ where k is an arbitrary real. Knowing this, we introduce the notion of entropy solution to (1.1).

DEFINITION 1.6. *Under Hypothesis 1.1, we say that $R \in L_{loc}^\infty(\mathbb{R}^+, BV([0, 1]))$ is an entropy solution on $[0, T]$ to the system*

$$(1.10) \quad \begin{cases} R_t + [f(R)]_x = 0 \\ R(\cdot, 0) = g(R(\cdot, 1)) \\ R(0, \cdot) = R_0 \in BV([0, 1]), \end{cases}$$

if:

•

$$(1.11)$$

$$\forall k \in \mathbb{R}^d, \sum_{i=1}^d \int_0^T \int_0^1 \{ |R_i - k_i| \phi_t + (f_i(R_i) - f_i(k_i)) \text{sign}(R_i - k_i) \phi_x \} dx dt \geq 0$$

for all $\phi \geq 0$ and $\phi \in C_c^1((0, T) \times (0, 1); \mathbb{R})$.

- $R(0, \cdot) = R_0$ in the almost everywhere sense.
- $R(\cdot, 0^+) = g(R(\cdot, 1^-))$ in the almost everywhere sense.

Remark 1.7. Here the entropy functional and its flux are defined for all k in \mathbb{R}^d by

$$(1.12) \quad \forall R \in \mathbb{R}^d, \eta_k(R) = \sum_{i=1}^d |R_i - k_i|, \quad q_k(R) = \sum_{i=1}^d (f_i(R_i) - f_i(k_i)) \text{sign}(R_i - k_i).$$

Moreover, equation (1.11) can be rewritten as

$$\eta_k(R)_t + q_k(R)_x \leq 0$$

in a weak sense. Hence entropy solutions are the solutions of (1.1) which make the entropy η decrease.

Remark 1.8. Equation (1.11) is stronger than the usual definition of weak solutions. Indeed, if one takes k such that $k_i < \text{ess inf}(R_i)$ and $k_j > \text{ess sup}(R_j)$ for $j \neq i$, then

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt - \sum_{j \neq i} \int_0^T \int_0^1 \{R_j \phi_t + f_j(R_j) \phi_x\} dx dt \geq 0.$$

Also, taking $k_j < \text{ess inf}(R_j)$ for $j \neq i$, one gets:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt + \sum_{j \neq i} \int_0^T \int_0^1 \{R_j \phi_t + f_j(R_j) \phi_x\} dx dt \geq 0.$$

Summing the last two inequalities, one gets:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt \geq 0$$

Similarly, one can obtain:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt \leq 0.$$

Hence, for all $\phi \in C_c^1((0, T) \times (0, 1))$ with $\phi \geq 0$:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt = 0.$$

Replacing ϕ with $-\phi$, it is also true for $\phi \in C_c^1((0, T) \times (0, 1))$ with $\phi \leq 0$. Now take a $\phi \in C_c^1((0, T) \times (0, 1))$, $\phi = \phi^+ + \phi^-$ where ϕ^+, ϕ^- are respectively the positive and negative parts of ϕ . As $\phi^+, \phi^- \in C_c^1((0, T) \times (0, 1))$, we get:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt = 0.$$

As a consequence, each entropy solution is also a weak one.

1.4. The contribution. Now that all the notions have been introduced, we can be more specific concerning the main contributions of this paper:

- State and prove a well-posedness result of (1.1) in a BV context.
To help us in the task, we will use front tracking techniques from DiPerna [15] and Bressan [4] to get an entropy solution in the domain considered. To deal with the boundary condition, the article [7] will be the reference work.
- State and prove a global exponential stability result for linear feedback laws.
This is the first key result. To our knowledge, no global stabilization result holds for feedback laws of the form $R(t, 0) = HR(t, 1)$ in a BV entropy context. The article [17] proposes also a feedback law of the form $R(t, 0) = g(\|R(t, \cdot)\|_{L^1})$. However, in physical systems the L^1 norm of the solution is not always accessible by observations. Additionally, the article [7] which considers a 2×2 system of conservation laws gives only a local stabilization result for an entropy solution.
- State and prove a local exponential stability result for saturated feedback laws. We will see that this is not possible in a BV context.
To our knowledge, only [12] has studied this kind of saturated feedback laws in an L^∞ context and for the case of constant characteristic velocities.

1.5. Outline. In Section 2, we will present and prove an approximation and a well-posedness result for the entropy BV solution to (1.1). The technique of front tracking will be mainly used. Then in Section 3, a sufficient condition for global BV stability will be given in the case of a linear feedback. Additionally, we give a sufficient condition for the local L^∞ stability in the case of a saturated feedback with an estimation of the basin of attraction. Finally, Section 4 is devoted to concluding remarks and perspectives.

Notation: For all $R \in \mathbb{R}^d$, $|R|$ designates the canonical euclidean norm of R . For matrices $M \in M_d(\mathbb{R})$, $|M| = \sup_{|R|=1, R \in \mathbb{R}^d} |MR|$. For all matrices $M \in M_d(\mathbb{R})$, $|M|_\infty := \max_{i=1..d} \sum_{j=1}^d |M_{i,j}|$. $D_d^+(\mathbb{R})$ is the set of diagonal strictly positive matrices. The value $\rho_\infty(M)$ for matrices $M \in M_d(\mathbb{R})$ is defined by $\rho_\infty(M) := \inf_{\Delta \in D_d^+(\mathbb{R})} |\Delta M \Delta^{-1}|_\infty$. L^p spaces on $[0, 1]$ ($1 \leq p \leq \infty$, $p \in \mathbb{N}$) are embedded with their canonical norms $\|\cdot\|_{L^p}$. For all matrices $P \in D_d^+(\mathbb{R})$ and $R \in L^\infty([0, 1])$, $\|R\|_{\infty, P} := \|PR\|_{L^\infty}$. The function $E : \mathbb{R} \mapsto \mathbb{N}$ is the integer part function and the function sign is the usual sign function with $\text{sign}(0) = 0$.

2. Well-posedness and approximation results. This section is devoted to the well-posedness of (1.10). Additionally, we prove the existence of an approximation by piecewise constant functions of the solution to (1.10).

2.1. Piecewise constant entropy solutions. Piecewise constant functions play an important role in the theory of BV solutions to conservation laws. Let us recall the definition of what a piecewise constant function is in our context.

DEFINITION 2.1. *An element R of $L_{loc}^\infty(\mathbb{R}^+, BV([0, 1]))$ is piecewise constant if for all $T > 0$, R viewed as a function defined on $[0, T] \times [0, 1]$ is constant on a finite number of polyhedra. The edges of such polyhedra are called the fronts of R . Additionally, the absolute value of the jump across the front is called the intensity of the front.*

In this paper, we use the concept of approximating sequence of piecewise constant functions (PCF).

DEFINITION 2.2. $(R_\nu)_\nu$ is an approximating sequence of PCFs of an entropy solution R to (1.10) if:

- For $\nu > 1$ fixed, R_ν is piecewise constant in the sense of Definition 2.1 and takes its values in $2^{-(n+1)\nu}\mathbb{Z}$ on strips

$$\{(x, t) \mid 0 \leq x \leq 1, \max\{(x+n-1)/c_{\max}, 0\} \leq t \leq (x+n)/c_{\max}\}$$

for $n \in \mathbb{N}$. The velocities of fronts are all bounded from below by c_{\min} and from above by c_{\max} (see (1.6) for the definition of c_{\min} and c_{\max}).

- For $\nu > 1$ fixed, no more than one front at a time can interact with the right boundary.
- For $\nu > 1$ fixed, if at a time $t \geq 0$ several fronts interact, the sum of intensities of outgoing fronts is inferior to the sum of intensities of ingoing fronts.
- The sequence $(R_\nu(0, \cdot))_\nu$ converges toward R_0 in $BV([0, 1])$.
- The approximated boundary condition is verified:

$$(2.1) \quad \forall n \in \mathbb{N}, \forall t \text{ s.t. } \frac{n}{c_{\max}} \leq t \leq \frac{n+1}{c_{\max}}, R_\nu(t, 0^+) = g_{(n+2)\nu}(R_\nu(t, 1^-))$$

where:

$$(2.2) \quad \forall R \in \mathbb{R}^d, \forall \nu > 1, g_\nu(R) = 2^{-\nu} (E(2^\nu g(R))).$$

- $\forall t \geq 0, \Delta t > 0,$

$$TV_{[0,1]}(R(t, \cdot)) \leq \limsup_{\nu \rightarrow +\infty} \sup_{s \in [t, t+\Delta t]} TV_{[0,1]}(R_\nu(s, \cdot))$$

and

$$(2.3) \quad \|R(t, \cdot)\|_{L^\infty([0,1])} \leq \limsup_{\nu \rightarrow +\infty} \sup_{s \in [t, t+\Delta t]} \|R_\nu(s, \cdot)\|_{L^\infty([0,1])}.$$

2.2. The result of well-posedness and approximation. Now we give the first result of this paper:

THEOREM 2.3. *Under Hypothesis 1.1 and for all $R_0 \in BV([0, 1])$, $g \in Lip(\mathbb{R}^d, \mathbb{R}^d)$, there exists a unique entropy solution $R \in L_{loc}^\infty(\mathbb{R}^+, BV([0, 1]))$ to (1.10). Moreover, there exists an approximating sequence of PCF $(R_\nu)_\nu$ of the entropy solution R .*

Proof. This theorem is proven in Appendix A for the existence and Appendix B for the uniqueness. \square

3. Lyapunov analysis. Before going into the stability analysis, we introduce the functional TV_H on the space BV . For all matrices H in $M_d(\mathbb{R})$, it is defined as follows:

$$(3.1) \quad \forall R \in BV([0, 1]), TV_H(R) = TV_{[0,1]}(R) + |HR(1^-) - R(0^+)|,$$

where $R(1^-)$ and $R(0^+)$ has to be understood as the left and right limits of the function R at $x = 1$ and $x = 0$.

Moreover, we introduce Hypothesis 3.1:

Hypothesis 3.1. The feedback matrix H verifies:

$$\rho_\infty(H) < 1.$$

The following lemma ensures the equivalence between TV_H and $\|\cdot\|_{BV([0,1])}$.

LEMMA 3.2. *Assume Hypothesis 3.1. The functional TV_H defined in (3.1) is a norm on $BV([0, 1])$ equivalent to the norm $\|\cdot\|_{BV([0,1])}$ defined in (1.7). Moreover, there exists a constant $C > 0$ such that*

$$(3.2) \quad \forall R \in BV([0, 1]), \|R\|_{L^\infty([0,1])} \leq C TV_H(R).$$

Proof. We first prove the following claim:

$$(3.3) \quad \forall R \in \mathbb{R}^d, \quad |R| \leq C|R - HR|.$$

Let $P \in D_d^+(\mathbb{R})$ such that

$$|PHP^{-1}|_\infty < 1.$$

The map $\|\cdot\|_\infty: \begin{cases} M_d(\mathbb{R}) & \rightarrow \mathbb{R}^+ \\ M & \mapsto |PMP^{-1}|_\infty \end{cases}$ defines an algebra norm on $M_d(\mathbb{R})$ and $\|H\|_\infty < 1$. Hence, $I_d - H$ is invertible, which gives (3.3) with $C := |(I - H)^{-1}|$.

$$\begin{aligned} TV_H(R) &= TV_{[0,1]}(R) + |HR(1^-) - R(0^+)| \\ &\leq TV_{[0,1]}(R) + |HR(1^-) - HR(0^+)| + |HR(0^+) - R(0^+)| \\ &\leq TV_{[0,1]}(R) + |H||R(1^-) - R(0^+)| + |H - I_d||R(0^+)| \\ &\leq (1 + |H|)TV_{[0,1]}(R) + |H - I_d||R(0^+)|. \end{aligned}$$

Take $x \in [0, 1]$, by the triangle inequality,

$$\begin{aligned} TV_H(R) &\leq (1 + |H|)TV_{[0,1]}(R) + |H - I_d||R(0^+) - R(x)| + |H - I_d||R(x)| \\ &\leq (1 + |H| + |H - I_d|)TV_{[0,1]}(R) + |H - I_d||R(x)|. \end{aligned}$$

Integrating with respect to x on $[0, 1]$, one obtains:

$$\begin{aligned} TV_H(R) &\leq (1 + |H| + |H - I_d|)TV(R) + |H - I_d|\|R\|_{L^1([0,1])} \\ &= C\|R\|_{BV([0,1])}. \end{aligned}$$

where $C = 1 + |H| + |H - I_d|$.

To get the converse inequality, we remark that by (3.3),

$$|R(1^-)| \leq C|HR(1^-) - R(1^-)|.$$

As a consequence,

$$\begin{aligned} \|R\|_{BV([0,1])} &= TV_{[0,1]}(R) + \|R\|_{L^1([0,1])} \\ &\leq TV_{[0,1]}(R) + |R(1^-)| + \|R - R(1^-)\|_{L^1([0,1])} \\ &\leq 2TV_{[0,1]}(R) + C|HR(1^-) - R(1^-)| \\ &\leq 2TV_{[0,1]}(R) + C|HR(1^-) - R(0^+)| + C|R(0^+) - R(1^-)| \\ &\leq (2 + C)TV_{[0,1]}(R) + C|HR(1^-) - R(0^+)| \end{aligned}$$

and both norms are equivalent. Concerning the L^∞ estimate (3.2), take a couple $(x, y) \in [0, 1]^2$ and using again the triangle inequality

$$|R(x)| \leq |R(x) - R(y)| + |R(y)| \leq TV_{[0,1]}(R) + |R(y)|.$$

Integrating with respect to y on $[0, 1]$, one gets

$$|R(x)| \leq TV_{[0,1]}(R) + \|R\|_{L^1([0,1])} = \|R\|_{BV([0,1])}.$$

And as this is true for all x in $[0, 1]$,

$$\|R\|_{L^\infty([0,1])} \leq \|R\|_{BV([0,1])}.$$

The equivalence between the norms $\|\cdot\|_{BV([0,1])}$ and TV_H proven earlier allows to get (3.2). \square

3.1. Lyapunov analysis for the unsaturated system. In this section, we consider the following system

$$(3.4) \quad \begin{cases} R_t + [f(R)]_x & = 0 \\ R(\cdot, 0) & = HR(\cdot, 1) \\ R(0, \cdot) & = R_0 \in BV([0, 1]) \end{cases}$$

where the feedback operator g presented in the introduction is replaced by a matrix $H \in M_d(\mathbb{R})$.

The main theorem of this section is presented here:

THEOREM 3.3. *Under Hypothesis 3.1 and if $0 < \gamma < -\log(\rho_\infty(H))$, then the unique entropy solution of (3.4) satisfies*

$$\forall t \geq 0, \|R\|_{BV([0,1])} \leq Ce^{-\gamma c_{\min} t} \|R_0\|_{BV([0,1])}$$

where $C > 0$ is a constant which does not depend on R_0 and t .

A candidate Lyapunov functional first introduced by Glimm [13] and then by Coron et al [7] applies well to piecewise constant functions and is defined by:

DEFINITION 3.4. *Let R be a piecewise constant function on $[0, 1]$ and taking its values in \mathbb{R}^d . Take $i \in \llbracket 1, d \rrbracket$:*

- *We denote $x_{i,1} < x_{i,2} < \dots < x_{i,n_i}$ the discontinuities of R_i (n_i being the number of discontinuities).*
- *For all $j \in \llbracket 1, n_i \rrbracket$, $r_{i,j}^l, r_{i,j}^r$ designate the respective left and right state of R_i around $x_{i,j}$.*

The Lyapunov functional \mathcal{L} evaluated at R writes

$$(3.5) \quad \mathcal{L}(R) = \sum_{i=1}^d P_i \sum_{j=1}^{n_i} |r_{i,j}^r - r_{i,j}^l| e^{-\gamma x_{i,j}} + \sum_{i=1}^d P_i |[HR]_i(1^-) - R_i(0^+)|$$

where $\gamma > 0$ and $P = \text{diag}\{P_i, i \in \llbracket 1, d \rrbracket\} \in D_d^+(\mathbb{R})$ will be selected later.

Remark 3.5. Obviously, there exists a constant $C(H, P, \gamma) > 1$ such that for all R piecewise constant, we have:

$$(3.6) \quad \frac{\mathcal{L}(R)}{C(H, P, \gamma)} \leq TV_H(R) \leq C(H, P, \gamma) \mathcal{L}(R).$$

Theorem 3.3 will be proven using a piecewise approximation of the solution for which the exponential decay of the Lyapunov functional \mathcal{L} will be proven. As a last step, we will pass to the limit.

Proof. We consider $(R_\nu)_\nu$ an approximating sequence of PCFs of the entropy solution R in the sense of Definition 2.2. Such a sequence exists by Theorem 2.3. The following lemma asserts the exponential stability of the approximation:

LEMMA 3.6. *If $0 < \gamma < -\log(\rho_\infty(H))$. Then, for all $P \in D_d^+(\mathbb{R}^d)$ such that $|PHP^{-1}|_\infty < e^{-\gamma}$, there exists $\tilde{\nu}(P, H, \gamma)$ such that*

$$(3.7) \quad \forall \nu > \tilde{\nu}, \forall t \geq 0, \mathcal{L}(R_\nu) \leq e^{-\gamma c_{\min} t} \mathcal{L}(R_{0,\nu}) + \frac{1}{2^\nu} \sum_{i=1}^d P_i.$$

Proof. Fix $\nu > 1$, $P \in D_d^+(\mathbb{R}^d)$ such that $|PHP^{-1}|_\infty < e^{-\gamma}$ and time $0 \leq t \leq 1/c_{\max}$.

Three cases are to be considered:

- (Case 1) If at time t there is no interaction between two fronts nor between a front and the boundary, then $\mathcal{L}(R_\nu)$ is differentiable and

$$\begin{aligned} \frac{d\mathcal{L}(R_\nu(t, \cdot))}{dt} &= -\gamma \sum_{i=1}^d P_i \sum_{j=1}^{n_i} \frac{dx_{i,j}}{dt} |r_{i,j}^r - r_{i,j}^l| e^{-\gamma x_{i,j}} \\ &\leq -\gamma c_{\min} \sum_{i=1}^d P_i \sum_{j=1}^{n_i} |r_{i,j}^r - r_{i,j}^l| e^{-\gamma x_{i,j}}. \end{aligned}$$

Here, we used the fact that for all integers $i \in \llbracket 1, d \rrbracket$, characteristic velocities $\frac{dx_{i,j}}{dt}$ are bounded from below by $c_{\min} > 0$. Finally, by the definition of $\mathcal{L}(R_\nu(t, \cdot))$

$$(3.8) \quad \begin{aligned} \frac{d\mathcal{L}(R_\nu(t, \cdot))}{dt} &\leq -\gamma c_{\min} \mathcal{L}(R_\nu(t, \cdot)) \\ &\quad + \gamma c_{\min} \sum_{i=1}^d P_i |[HR_\nu]_i(t, 1^-) - R_{\nu,i}(t, 0^+)| \\ &\leq -\gamma c_{\min} \mathcal{L}(R_\nu(t, \cdot)) + \frac{\gamma c_{\min}}{2^\nu} \sum_{i=1}^d P_i \end{aligned}$$

where we used (2.1) with g replaced by H to get last equation.

- (Case 2) When a front interaction happens, the total variation is non increasing by construction and as a consequence

$$\mathcal{L}(R_\nu(t^+, \cdot)) - \mathcal{L}(R_\nu(t^-, \cdot)) \leq 0.$$

Here we used the third point of Definition 2.2.

- (Case 3) When an interaction of a front with the boundary happens, computations are a bit more difficult. Suppose that such a front is of type $i \in \llbracket 1, d \rrbracket$ and has $(R_{i,l}, R_{i,r})$ as respective left and right state (see Figure 2). We note its intensity by $I_i := |R_{i,l} - R_{i,r}|$. Note that as R_ν takes its values in $2^{-\nu} \mathbb{Z}$ on the triangle $\{(x, t) \mid 0 < t < x/c_{\max}\}$, we have:

$$(3.9) \quad I_i \geq 2^{-\nu}.$$

Moreover, recall that simultaneous interactions of fronts with the boundary are forbidden by construction. Using the approximate boundary condition (2.1) with g replaced by the linear operator H , we get

$$(3.10) \quad \mathcal{L}(R_\nu(t^+, \cdot)) - \mathcal{L}(R_\nu(t^-, \cdot)) \leq \sum_{j=1}^d P_j |H_{j,i}(R_{i,r} - R_{i,l})| - e^{-\gamma} I_i P_i + 2^{-2\nu+2} \sum_{j=1}^d P_j.$$

The second term on the right-hand side of (3.10) corresponds to the leaving front (which is of type i). The first term results from the entering fronts at the left boundary. Note that an entering front of type $j \in \llbracket 1, d \rrbracket$ may rather be a fan of fronts (see Figure 2). This is not problematic because the sum of the intensities of the fronts composing the fan is equal to the difference of extremal states of the fan by construction (see Appendix A.2 for details). The last term in (3.10) corresponds to the approximation of the boundary condition (2.1).

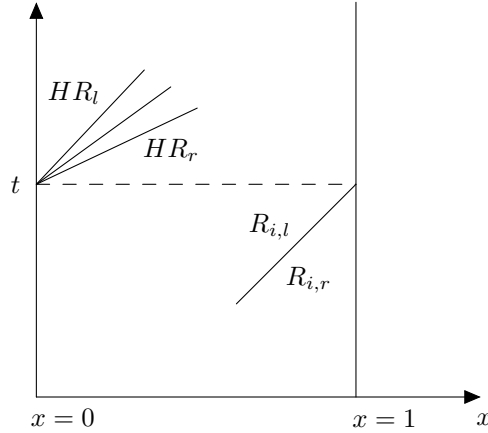


Figure 2: Case 3

Then, using the definition of $|\cdot|_\infty$ and (3.9), one gets:

$$\begin{aligned} \mathcal{L}(R_\nu(t^+, \cdot)) - \mathcal{L}(R_\nu(t^-, \cdot)) &\leq \left(\sum_{j=1}^d \frac{P_j}{P_i} |H_{j,i}| - e^{-\gamma} \right) P_i I_i \\ &\quad + 2^{-\nu+2} \sum_{j=1}^d P_j I_i \\ &\leq \left(|PH^T P^{-1}|_\infty + 2^{-\nu+2} \sum_{j=1}^d P_j / P_i \right. \\ &\quad \left. - e^{-\gamma} \right) P_i I_i. \end{aligned}$$

Remark 3.7. Here we see why the approximated boundary condition (2.1) is essential. Thanks to it, the error term $2^{-2\nu+2} \sum_{j=1}^d P_j$ coming from the approximation of g by g_ν can be bounded by the intensity $I_i \geq 2^{-\nu}$ of the front hitting the right boundary.

As $|PH^T P^{-1}|_\infty - e^{-\gamma} < 0$ by assumption, we can take ν sufficiently large say $\nu \geq \tilde{\nu}(P, H, \gamma)$ such that

$$\mathcal{L}(R_\nu(t^+, \cdot)) - \mathcal{L}(R_\nu(t^-, \cdot)) \leq 0$$

(Case 2) and (Case 3) can occur only a finite number of times on finite time intervals because R_ν is piecewise constant in the sense of Definition 2.1. Consequently, one can integrate (3.8) with respect to time to get:

$$\forall 0 \leq t \leq 1/c_{\max}, \quad \mathcal{L}(R_\nu(t, \cdot)) \leq e^{-\gamma c_{\min} t} \mathcal{L}(R_{0,\nu}) + \frac{1}{2^\nu} \sum_{i=1}^d P_i.$$

The proof for time $n/c_{\max} \leq t \leq (n+1)/c_{\max}$ where n is an integer can be dealt in a similar way. This ends the proof of Lemma 3.6. \square

Now, we conclude on the proof of Theorem 3.3 taking $t \geq 0$ fixed. By (3.7) and (3.6), there exists a constant $C > 0$ such that

$$\forall \nu > 0, \quad TV_H(R_\nu(t, \cdot)) \leq C \left(e^{-\gamma c_{\min} t} TV_H(R_{0,\nu}) + \frac{1}{2^\nu} \sum_{i=1}^d P_i \right).$$

Using the equivalence between the norm TV_H and the norm $\|\cdot\|_{BV([0,1])}$,

$$(3.11) \quad \forall \nu > 0, \quad \|R_\nu(t, \cdot)\|_{BV([0,1])} \leq C \left(e^{-\gamma c_{\min} t} \|R_{0,\nu}\|_{BV([0,1])} + \frac{1}{2^\nu} \sum_{i=1}^d P_i \right)$$

where we may have changed the constant $C > 0$.

As $(R_\nu)_\nu$ is an approximating sequence of PCFs of R , one has:

$$\left\{ \begin{array}{l} \lim_{\nu \rightarrow \infty} R_\nu(0, \cdot) = R_0 \in BV([0, 1]) \\ \forall \tau \geq 0, \quad d\tau > 0, \quad TV_{[0,1]}(R(\tau, \cdot)) \leq \limsup_{\nu \rightarrow \infty} \sup_{s \in [\tau, \tau + d\tau]} TV_{[0,1]}(R_\nu(s, \cdot)). \end{array} \right.$$

Moreover, by Remark A.4,

$$\forall \tau \geq 0, \quad \lim_{\nu \rightarrow \infty} \|R_\nu(\tau, \cdot) - R(\tau, \cdot)\|_{L^1([0,1])} = 0.$$

We have for all $dt > 0$,

$$\begin{aligned} \|R(t, \cdot)\|_{BV([0,1])} &\leq \limsup_{\nu \rightarrow \infty} \left(\sup_{s \in [t, t+dt]} TV_{[0,1]}(R_\nu(s, \cdot)) + \|R_\nu(t, \cdot)\|_{L^1([0,1])} \right) \\ &\leq \limsup_{\nu \rightarrow \infty} \sup_{s \in [t, t+dt]} \left(TV_{[0,1]}(R_\nu(s, \cdot)) + \|R_\nu(s, \cdot)\|_{L^1([0,1])} \right) \\ &= \limsup_{\nu \rightarrow \infty} \sup_{s \in [t, t+dt]} \|R_\nu(s, \cdot)\|_{BV([0,1])} \\ &\leq C \limsup_{\nu \rightarrow \infty} \left(e^{-\gamma c_{\min} t} \|R_{0,\nu}\|_{BV([0,1])} + \frac{1}{2^\nu} \sum_{i=1}^d P_i \right) \\ &= C e^{-\gamma c_{\min} t} \|R_0\|_{BV([0,1])} \end{aligned}$$

where we have used (3.11) to get the fourth equation.

This finishes the proof of Theorem 3.3. \square

3.2. Stability analysis for the saturated system. In this section, we consider the following system

$$(3.12) \quad \begin{cases} R_t + [f(R)]_x &= 0 \\ R(., 0) &= [H \cdot + B\sigma(K \cdot)]R(., 1) \\ R(0, \cdot) &= R_0 \in BV([0, 1]). \end{cases}$$

We introduce the deadzone function defined by

$$(3.13) \quad \forall R \in \mathbb{R}^d, \phi(R) = \sigma(R) - R$$

and Hypothesis 3.8:

Hypothesis 3.8. The matrices H, B, K are chosen such that:

$$\rho_\infty(H + BK) < 1.$$

Here the main result is different since we prove local exponential stability (Proposition 3.10). We cannot directly study the problem of BV stability because of the lack of contractivity of the saturation σ .

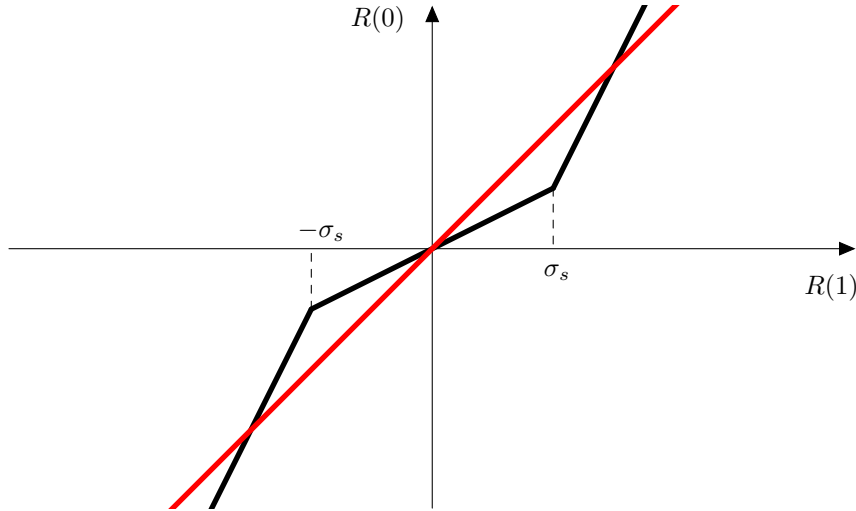


Figure 3: The feedback operator (black line) compared with the graph of the function $R(0) = R(1)$ (red line)

Motivating example 3.9. In Figure 3, we represent the boundary operator $H \cdot + B\sigma(K \cdot)$ for $d = 1$, $H = 2$, $B = 1$, $K = -1.5$ and $\sigma_s = 2$. Except for the zone of linearity, the boundary operator is only 2-Lipschitz. As a consequence, it is possible to construct a front whose left/right states are arbitrary close to the zone of linearity

and whose intensity increases after a passage through the feedback operator. This is why it is not possible to get a basin of attraction in BV norm larger than the zone of linearity. We will rather prove the L^∞ local stability with a basin of attraction in L^∞ .

This section is devoted to the proof of the following proposition and theorem (the definition of $\|\cdot\|_{\infty,P}$ is given in the section notation):

PROPOSITION 3.10. *Under Hypothesis 3.8, if $0 < \gamma < -\log(\rho_\infty(H+BK))$. Then, for all $P \in D_d^+(\mathbb{R}^d)$ such that $|P(H+BK)P^{-1}|_\infty \leq e^{-\gamma}$, there exists a constant C depending on (H, B, K, P, γ) such that if*

$$(3.14) \quad \|R_0\|_{\infty,P} < \frac{|PBP^{-1}|_\infty P_{\min}\sigma_s}{\left| |P(H+BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma} \right|}.$$

Then, the unique entropy solution $R \in L_{loc}^\infty(\mathbb{R}^+, BV([0,1]))$ of (3.12) satisfies,

$$(3.15) \quad \forall t \geq 0, \|R(t, \cdot)\|_{L^\infty([0,1])} \leq Ce^{-\gamma c_{\min} t} \|R_0\|_{L^\infty([0,1])}$$

where C depends on the parameters of the problem but not R_0 .

For cases where $\rho_\infty(H) > 1$, the denominator in (3.14) is not zero:

Remark 3.11. If $\rho_\infty(H) > e^{-\gamma}$, then we claim that for all $P \in D_d^+(\mathbb{R})$:

$$|P(H+BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma} > 0.$$

Proof of the claim of Remark 3.11

Let P be in $D_d^+(\mathbb{R})$. As $\rho_\infty(H) > e^{-\gamma}$,

$$|PHP^{-1}|_\infty > e^{-\gamma}.$$

This gives by the triangle inequality:

$$|P(H+BK)P^{-1}|_\infty + |PBKP^{-1}|_\infty > e^{-\gamma}.$$

Finally, by the fact that

$$\forall A, B \in M_d(\mathbb{R}), |PABP^{-1}|_\infty \leq |PAP^{-1}|_\infty |PBP^{-1}|_\infty,$$

we have:

$$|P(H+BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PBP^{-1}|_\infty > e^{-\gamma}$$

and the claim is proven.

The following theorem is a consequence of Proposition 3.10.

THEOREM 3.12. *Under the conditions of Proposition 3.10,*

$$\forall t \geq 0, \|R(t, \cdot)\|_{BV([0,1])} \leq Ce^{-\gamma c_{\min} t} \|R_0\|_{BV([0,1])}$$

where C depends on the parameters of the problem but not R_0 .

Let us assume for the time being Proposition 3.10 and prove Theorem 3.12:

Proof of Theorem 3.12. Equation (3.15) implies that at a certain time denoted t^* depending on $\|R_0\|_{L^\infty([0,1])}$, the solution enters in the zone of linearity and stays in it. Then, Theorem 3.3 implies:

$$(3.16) \quad \forall t \geq t^*, \|R(t, \cdot)\|_{BV([0,1])} \leq C e^{-\gamma c_{\min}(t-t^*)} \|R(t^*, \cdot)\|_{BV([0,1])}$$

where C depends on $H, B, K, P, \gamma, \sigma_s$.

Then, for $t \leq t^*$, one can prove using the same techniques from Section 3.1 that:

$$(3.17) \quad \forall 0 \leq t \leq t^*, \|R(t, \cdot)\|_{BV([0,1])} \leq e^{\nu t} \|R_0\|_{BV([0,1])}$$

where $\nu > 0$ is a constant depending on c_{\max}, γ and a Lipschitz constant of the feedback operator $H + B\sigma(K)$.

From (3.17) and (3.16), one gets:

$$\forall t \geq 0, \|R(t, \cdot)\|_{BV([0,1])} \leq C e^{-\gamma c_{\min} t} \|R_0\|_{BV([0,1])}$$

where C depends on the parameters of the problem and on $\|R_0\|_{L^\infty([0,1])}$.

As we have the bound (3.14), we can conclude that C does not depend on $\|R_0\|_{L^\infty([0,1])}$ and the corollary is proven. \square

The following lemma will be useful for the proof of Proposition 3.10.

LEMMA 3.13. *Let $R \in \mathbb{R}^d$ be such that:*

$$(3.18) \quad |PR|_\infty \leq \frac{|PBP^{-1}|_\infty P_{\min} \sigma_s}{|P(H+BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma}}.$$

Then,

$$|P(HR + B\sigma(KR))|_\infty \leq e^{-\gamma} |PR|_\infty.$$

Proof. Let i be in $\llbracket 1, d \rrbracket$. If $\text{sat}_i(R) := \{j \in \llbracket 1, d \rrbracket, |[KR]_j| > \sigma_s \text{ and } B_{i,j} \neq 0\}$ is empty, then:

$$\begin{aligned} P_i |HR + B\sigma(KR)|_i &= P_i |(H+BK)R|_i \\ &\leq |P(H+BK)P^{-1}|_\infty |PR|_\infty \\ &\leq e^{-\gamma} |PR|_\infty. \end{aligned}$$

If the set $\text{sat}_i(R)$ is not empty, then:

$$\begin{aligned} P_i |HR + B\sigma(KR)|_i &= P_i |(H+BK)R + B\phi(KR)|_i \\ &\leq \sum_{j=1}^d P_i |(H+BK)_{i,j} R_j| \\ &\quad + \sum_{j \in \text{sat}_i(R)} P_i |B_{i,j}| (|[KR]_j| - \sigma_s) \\ &\leq \sum_{j=1}^d P_i |(H+BK)_{i,j} \frac{1}{P_j} P_j R_j| \\ &\quad + \sum_{j \in \text{sat}_i(R)} P_i |B_{i,j}| \frac{P_j}{P_j} (|[KR]_j| - \sigma_s) \\ &\leq |P(H+BK)P^{-1}|_\infty |PR|_\infty \\ &\quad + |PBP^{-1}|_\infty (|PKP^{-1}|_\infty |PR|_\infty - P_{\min} \sigma_s) \\ &\leq e^{-\gamma} |PR|_\infty \end{aligned}$$

where we have used the hypothesis (3.18) to get the last inequality. \square

Now we can focus on the proof of Proposition 3.10.

Proof of Proposition 3.10. We take $P \in D_d^+(\mathbb{R})$ such that $|P(H + BK)P^{-1}|_\infty < e^{-\gamma}$ and $R_0 \in BV([0, 1])$ satisfying (3.14). We consider $(R_\nu)_\nu$ an approximating sequence of PCFs of the entropy solution R in the sense of Definition 2.2. Such a sequence exists because of Theorem 2.3. Then, we analyze the exponential damping of R_ν for a fixed $\nu > 1$. As $(R_{0,\nu})_\nu$ converges towards R_0 in $BV([0, 1])$, we have for ν sufficiently large:

$$(3.19) \quad \|R_{0,\nu}\|_{\infty,P} \leq \frac{|PBP^{-1}|_\infty P_{\min} \sigma_s}{|P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma}}$$

We first recall the definition of c_{\min}, c_{\max} the respective minimum and maximum velocity, in (1.6). Let $t \leq 1/c_{\min}$ and $x > c_{\max}t$ be in $[0, 1]$. Constructing the light cone enclosed by line with slopes $1/c_{\min}$ and $1/c_{\max}$ and passing through (t, x) , we get that:

$$(3.20) \quad |PR_\nu(t, x)|_\infty \leq \|R_{0,\nu}\|_{\infty,P}$$

The argument of the light cone can be justified by the fact that the L^∞ norm does not increase by fronts interaction (see Appendix A.2.3) and because fronts velocities belongs to $[c_{\min}, c_{\max}]$.

When $x \leq c_{\max}t$, constructing the light cone enclosed by lines with slopes $1/c_{\min}$ and $1/c_{\max}$ and passing through (t, x) , one gets:

$$|PR_\nu(t, x)|_\infty \leq \max\{\|R_{0,\nu}\|_{\infty,P}, \sup_{t \in [0, 1/c_{\min}]} |PR_\nu(t, 0)|\}.$$

The boundary condition gives:

$$|PR_\nu(t, x)|_\infty \leq \max\{\|R_{0,\nu}\|_{\infty,P}, \sup_{t \in [0, 1/c_{\min}]} |P[H \cdot + B\sigma(K \cdot)]R_\nu(t, 1)|\}.$$

By (3.20) applied to $x = 1$ and (3.19), hypothesis of Lemma 3.13 are verified and we have:

$$|PR_\nu(t, x)|_\infty \leq \max\{\|R_{0,\nu}\|_{\infty,P}, e^{-\gamma}\|R_{0,\nu}\|_{\infty,P}\} \leq \|R_{0,\nu}\|_{\infty,P}.$$

Next we proceed by induction on intervals of the form $t \in [n/c_{\min}, (n+1)/c_{\min}]$ with $n \in \mathbb{N}$. Suppose that:

$$\forall t \in [n/c_{\min}, (n+1)/c_{\min}], \|R_\nu(t, \cdot)\|_{\infty,P} \leq e^{-\gamma n} \|R_{0,\nu}\|_{\infty,P}.$$

Let $(n+1)/c_{\min} \leq t \leq (n+2)/c_{\min}$ and x be in $[0, 1]$. Constructing the light cone enclosed by lines with slopes $1/c_{\min}$ and $1/c_{\max}$ and passing through (t, x) , one gets the existence of a $t^* \in [n/c_{\min}, (n+2)/c_{\min}]$ such that:

$$(3.21) \quad |PR_\nu(t, x)|_\infty \leq |PR_\nu(t^*, 0)|_\infty \leq |P[H \cdot + B\sigma(K \cdot)]R_\nu(t^*, 1)|.$$

Using same reasoning as in the case $n = 0$, one proves that:

$$\|R_\nu(t^*, \cdot)\|_{\infty, P} \leq \|R_\nu(n/c_{\min}, \cdot)\|_{\infty, P}.$$

Hence, by the hypothesis of induction:

$$(3.22) \quad |PR_\nu(t^*, 1)|_\infty \leq \|R_\nu(t^*, \cdot)\|_{\infty, P} \leq e^{-\gamma n} \|R_{0, \nu}\|_{\infty, P} \leq \|R_{0, \nu}\|_{\infty, P}.$$

As a consequence, by (3.19):

$$|PR_\nu(t^*, 1)|_\infty \leq \frac{|PBP^{-1}|_\infty P_{\min} \sigma_s}{\left| |P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma} \right|}$$

Thus, we can use Lemma 3.13 in (3.21) to get:

$$\begin{aligned} |PR_\nu(t, x)|_\infty &\leq e^{-\gamma} |PR_\nu(t^*, 1)|_\infty \\ &\leq e^{-\gamma} \|R_\nu(t^*, \cdot)\|_{\infty, P}. \end{aligned}$$

Hence,

$$\|R_\nu(t, \cdot)\|_{\infty, P} \leq e^{-\gamma} \|R_\nu(t^*, \cdot)\|_{\infty, P} \leq e^{-\gamma(n+1)} \|R_{0, \nu}\|_{\infty, P}$$

where we have used (3.22).

To conclude, we have:

$$\forall t \geq 0, \|R_\nu(t, \cdot)\|_{\infty, P} \leq \max\{1, e^{-\gamma(c_{\min} t - 1)}\} \|R_{0, \nu}\|_{\infty, P}.$$

It remains to prove the exponential decay for the solution R . It suffices to use property (2.3) and to take a sequence of initial data piecewise constant such that:

$$\forall \nu > 1, \|R_{0, \nu}\|_{\infty, P} \leq \|R_0\|_{\infty, P}.$$

Owing this, we can pass to the limit as ν goes to infinity to get:

$$\forall t \geq 0, \|R(t, \cdot)\|_{\infty, P} \leq \max\{1, e^{-\gamma(c_{\min} t - 1)}\} \|R_0\|_{\infty, P}.$$

This ends the proof of Proposition 3.10. \square

3.3. Numerical results. Here, we study a numerical example with saturation and show the relevance of the estimation of the region of attraction (3.14).

3.3.1. Relevance of the estimation of the basin of attraction. In this section, we analyze an example of network of scalar conservation laws for $d = 2$ with saturated feedback control law. Matrices are defined as follows.

$$H = \begin{pmatrix} 0 & 1.1 \\ 1 & 0 \end{pmatrix}, \quad B = I_2, \quad K = \begin{pmatrix} 0 & -0.1050 \\ -0.1045 & 0 \end{pmatrix}.$$

We take a linear flux $f(R) = \Lambda R + 0.2(\arctan(R_1), \arctan(R_2))$ with

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

The open-loop system can be represented by the graph given in Figure 4:

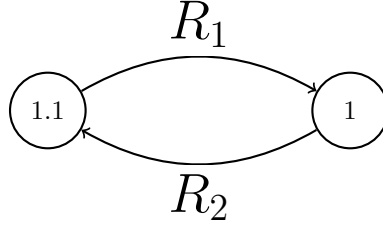


Figure 4: The open-loop system

We recall the estimation of the basin of attraction for $\gamma > 0$ and $P \in D_d^+(\mathbb{R})$:

$$(3.23) \quad \|R_0\|_{\infty, P} \leq \frac{|PBP^{-1}|_{\infty} \sigma_s}{\left| |P(H+BK)P^{-1}|_{\infty} + |PBP^{-1}|_{\infty} |PKP^{-1}|_{\infty} - e^{-\gamma} \right|}.$$

We calculate $P \in D_d^+(\mathbb{R})$ such that $|P(H+BK)P^{-1}|_{\infty}$ is minimal. We obtain:

$$P = \begin{pmatrix} 0.974 & 0 \\ 0 & 1.026 \end{pmatrix}.$$

To estimate the largest region of attraction, we take $\gamma = 0$ in (3.23) which gives the following criteria of stability:

$$(3.24) \quad \|R_0\|_{\infty, P} \leq \frac{|PBP^{-1}|_{\infty} \sigma_s}{\left| |P(H+BK)P^{-1}|_{\infty} + |PBP^{-1}|_{\infty} |PKP^{-1}|_{\infty} - 1 \right|}.$$

3.3.2. Numerical simulations. Still keeping the matrices from previous section, we take a certain range of initial data R_0 constant on $[0, 1]$ belonging to the estimated region of attraction and simulate the behavior of the solution. For example, one can take R_0 constant with value in $(-40, 40)^2$ and look if the solution does not blow up at infinite time in BV norm. For information, we have used the classical superbee limiter scheme for our simulations. In the same graph, we plot the estimated region of attraction given by (3.24) and the region where the solution saturates at $t = 0$.

In Figure 5, contours correspond to the rate of exponential decay of the numerical solution. If it is negative, the solution decays exponentially in norm. If it is positive, we have exponential divergence. The largest square is the estimated region of attraction while the smallest one is the zone where saturation does not occur at $t = 0$. We also pick a initial data R_0 in the estimated region of attraction and observe the dynamic of the solution. For example, one can take $R_0(x) = (15, -15)$ on $[0, 1]$. The black dot in Figure 5 corresponds to this initial data. The value of the control is plotted in Figure 7 while the BV norm of the solution is given in Figure 6. We see that there is indeed saturation from $t = 0$ until time $t \approx 13$, then we enter in the zone of linearity.

The initial jump of the BV norm is due to boundary condition which immediately creates a jump at $x = 0$ when $t = 0^+$. Here the BV norm behaves well in the zone of saturation *i.e* it does not exponentially blow and even decay quite fast. Then, in the zone of linearity we recover the exponential decay pattern.

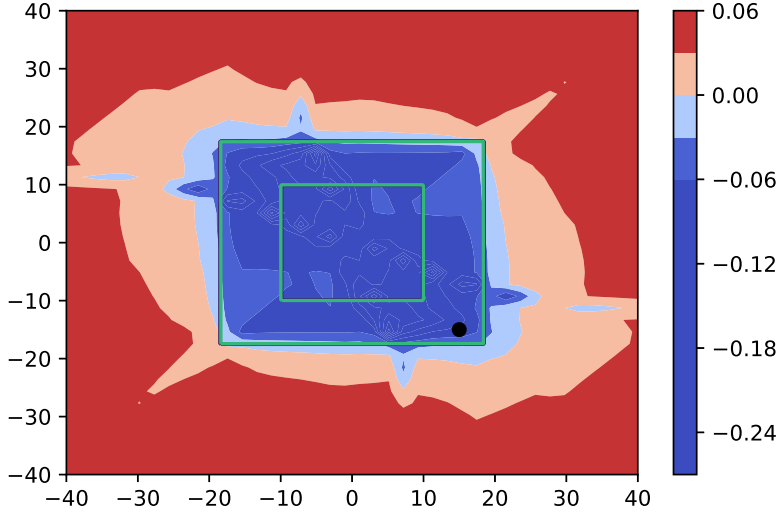


Figure 5: The basin of attraction

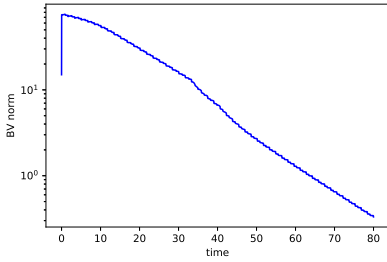
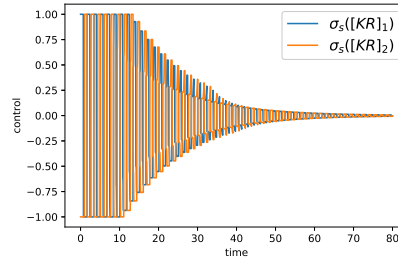
Figure 6: The BV norm of the solution

Figure 7: The control

4. Conclusion. The well-posedness for a wide class of systems of scalar conservation laws with boundary unsaturated and saturated feedback laws was established. The ρ_∞ criteria was established in the BV context for linear feedback laws. Then, for saturated feedback laws, we proved with an example that estimating a basin of attraction in BV was not relevant. We rather gave an estimation of the basin of attraction in L^∞ and deduce the exponential decay of the BV norm of solutions whose initial data belongs to this basin of attraction. Some questions remain open. The estimation (3.14) may not be optimal. Moreover, a method of maximizing the basin of attraction where the matrix K is the variable of optimization is not given in this article. This is not an easy task since criterion (3.14) is not convex with respect to K . Finally, the other big gap to bridge is the stabilization of general systems of conservation laws the main difficulty coming from the well-posedness. The initial-boundary value problem for hyperbolic systems of conservation laws is indeed a very delicate

matter, even when no characteristic speed vanishes. We refer to [20], [1], [5], [2], [11], [7] and the references therein.

Appendix A. Existence of a solution. All this section is dedicated to the proof of the existence result of Theorem 2.3.

A.1. The approximated problem. Let $\nu \geq 1$, i in $\llbracket 1, d \rrbracket$ and define $f_{\nu,i}$ the piecewise affine approximation of f_i coinciding with f_i at all $2^{-\nu}j$ nodes ($j \in \mathbb{Z}$) by:

$$f_{\nu,i}(s) = \frac{s - 2^{-\nu}j}{2^{-\nu}} f_i(2^{-\nu}(j+1)) + \frac{2^{-\nu}(j+1) - s}{2^{-\nu}} f_i(2^{-\nu}j) \text{ for } s \in [2^{-\nu}j, 2^{-\nu}(j+1)].$$

The sequence $(f_\nu)_\nu$ is introduced in order to construct a piecewise constant entropy solution. The following lemma gives its main properties:

LEMMA A.1. *For all $T > 0$, there exists a constant $C(g, T)$ such that for all $\nu > 1$ and $R_{0,\nu}$ piecewise constant taking its values in $2^{-\nu}\mathbb{Z}$, there exists R_ν piecewise constant in the sense of Definition 2.1 verifying the following assertions:*

- *The approximated boundary condition (2.1) is verified.*
- *Two fronts cannot interact simultaneously with the right boundary.*
- $\forall k \in \mathbb{R}^d, \phi \in C_c^1((0, T) \times (0, 1); \mathbb{R})$:

$$(A.1) \quad \int_0^T \int_0^1 \eta_k(R_\nu) \phi_t + q_k(R_\nu) \phi_x dx dt \geq -C(g, T) \frac{TV(R_{0,\nu})}{\nu} \|\phi\|_{L^\infty(\mathbb{R}^+ \times [0, 1])}.$$

where :

$$(A.2) \quad \begin{cases} \eta_k(R_\nu(t, x)) &= \sum_{i=1}^d |R_{\nu,i}(t, x) - k_i| \\ q_k(R_\nu(t, x)) &= \sum_{i=1}^d |f_i(R_{\nu,i}(t, x)) - f_i(k_i)|. \end{cases}$$

- *The following bounds hold:*

$$\forall t \leq T, \quad TV_{[0,1]}(R_\nu(t, \cdot)) \leq C(g, T) TV_{[0,1]}(R_{0,\nu}).$$

$$(A.3) \quad \forall t \leq T, \quad \|R_\nu(t, \cdot)\|_{L^\infty([0,1])} \leq C(g, T) \|R_{0,\nu}(t, \cdot)\|_{L^\infty([0,1])}.$$

Proof. See Appendix A.2 □

It is relatively easy to construct piecewise constant functions that make the entropy decrease. The main rules of construction are presented in the following lemma.

LEMMA A.2 (Characterization of entropy piecewise constant functions).

A piecewise constant R in the sense of Definition 2.1 verifies the condition of entropy decay (1.11) if and only if for all integers i in $\llbracket 1, d \rrbracket$ and all fronts $\gamma(t)$ of R_i ,

- *The Rankine-Hugoniot condition holds for R_i^l the left state and R_i^r the right state:*

$$(A.4) \quad \dot{\gamma}(t)[R_i^r(t) - R_i^l(t)] = f_i(R_i^r(t)) - f_i(R_i^l(t)).$$

- If $R_i^l(t) < R_i^r(t)$ then:

$$(A.5) \quad \forall \alpha \in [0, 1], f_i(\alpha R_i^r(t) + (1 - \alpha)R_i^l(t)) \geq \alpha f_i(R_i^r(t)) + (1 - \alpha)f_i(R_i^l(t)).$$

- If $R_i^l(t) > R_i^r(t)$ then:

$$(A.6) \quad \forall \alpha \in [0, 1], f_i(\alpha R_i^r(t) + (1 - \alpha)R_i^l(t)) \leq \alpha f_i(R_i^r(t)) + (1 - \alpha)f_i(R_i^l(t)).$$

Proof. This corresponds to [4, Theorem 4.4]. \square

We call conditions (A.5)-(A.6), the entropy decay conditions of fronts; it selects values before and after the front such that the entropy of the solution decreases with time. If a front verifies such conditions, we say that the front is **entropic**.

A.2. Proof of Lemma A.1. Now we prove Lemma A.1 constructing step by step a piecewise constant solution. We begin by solving a Riemann problem to get a solution near $t = 0$.

A.2.1. The Riemann problem. Let i be an integer of $\llbracket 1, d \rrbracket$, $\nu > 1$ and $R_i^l, R_i^r \in \mathbb{R}$ be two states. We recall techniques from [4, pp .108-113] to solve the Riemann problem associated to (R_i^l, R_i^r) when taking f_ν as flux. There are two cases to consider:

- If $R_i^l < R_i^r$. Then, we consider f_i^* the largest convex function inferior to $f_{\nu,i}$ on $[R_i^r, R_i^l]$. Denote also $w_0 := R_i^l < w_1 < w_2 < \dots < w_n := R_i^r$ the states where $f_i^{*,l}$ jumps. We give an example for $n = 2$ on Figure 8:

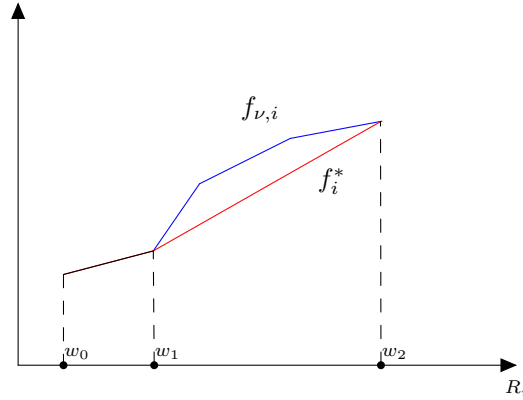


Figure 8: The case $R_i^l < R_i^r$

Introducing the speeds

$$(A.7) \quad \lambda_l = \frac{f_{\nu,i}(w_l) - f_{\nu,i}(w_{l-1})}{w_l - w_{l-1}}, \quad l \in \llbracket 1, n \rrbracket,$$

we define the solution to the Riemann problem as:

$$(A.8) \quad R_{\nu,i}(t, x) = \begin{cases} w_0 & \text{if } x < t\lambda_1 \\ w_l & \text{if } t\lambda_l < x < t\lambda_{l+1}, \quad l \in \llbracket 1, n-1 \rrbracket \\ w_n & \text{if } x > t\lambda_n. \end{cases}$$

This solution is entropic because it is piecewise constant, all fronts are entropic (A.5) and satisfy the Rankine-Hugoniot condition (A.4).

- If $R_i^l > R_i^r$. Then, we consider f_i^* the smallest concave function larger than $f_{\nu,i}$. Denote also $w_0 := R_i^l > w_1 > w_2 > \dots > w_n := R_i^r$ the states where $f_i^{*,l}$ jumps. We give an example for $n = 2$ on the Figure 9.

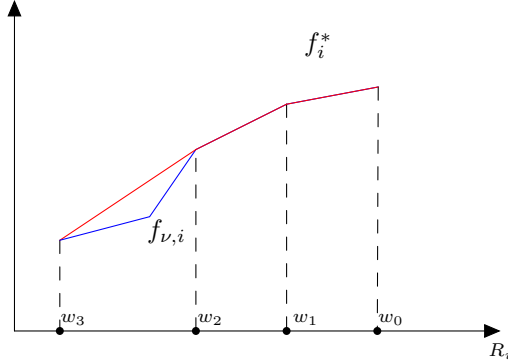


Figure 9: The case $R_i^l > R_i^r$

Defining velocities $(\lambda_l)_{l \in \llbracket 1, q \rrbracket}$ as in (A.7), we define the local solution also as:

$$(A.9) \quad R_{\nu,i}(t, x) = \begin{cases} w_0 & \text{if } x < t\lambda_1 \\ w_l & \text{if } t\lambda_l < x < t\lambda_{l+1}, l \in \llbracket 1, n-1 \rrbracket \\ w_n & \text{if } x > t\lambda_n. \end{cases}$$

A.2.2. Local in time solution. Take a fixed $\nu > 1$. Let us define what we will call the limit line $t \mapsto c_{\max}t$ with maximal speed. Thanks to the Riemann solver defined in the previous section, we can find an entropy solution until a front interaction happens. The corresponding picture is given in Figure 10.

A.2.3. Dealing with shock interactions. We recall the method described in [4, pp. 111-112]. Two cases have to be considered:

- (Case 1) All the incoming jumps have the same sign. Suppose they are all positive and let us denote $w_0 < w_1 < \dots < w_n$ ($n \in \mathbb{N}$) the consecutive “incoming” states. As all incoming fronts are entropic, we have:

$$(A.10) \quad \forall \alpha \in [0, 1], f(\alpha w_i + (1 - \alpha)w_{i+1}) \geq \alpha f(w_i) + (1 - \alpha)f(w_{i+1}).$$

The fact that we have converging fronts gives that the function h built from lines passing through points $(w_i, f(w_i))_{i \in \llbracket 0, n \rrbracket}$ is concave. Moreover, by (A.10):

$$(A.11) \quad \forall w \in [w_0, w_n], h(w) \leq f(w).$$

Hence, by the concavity of h and (A.11):

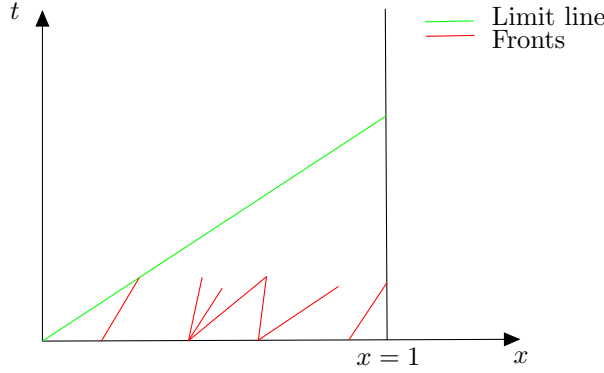


Figure 10: The local in time solution

$$\forall \alpha \in [0, 1], f(\alpha w_0 + (1-\alpha)w_n) \geq h(\alpha w_0 + (1-\alpha)w_n) \geq \alpha f(w_0) + (1-\alpha)f(w_n).$$

Thus, it is possible to link the extremal (w_0 and w_n) states by a unique entropic front whose jump intensity is strictly equal to the sum of the intensities of incoming jumps. Hence, in this case the total variation is conserved and so is the L^∞ norm.

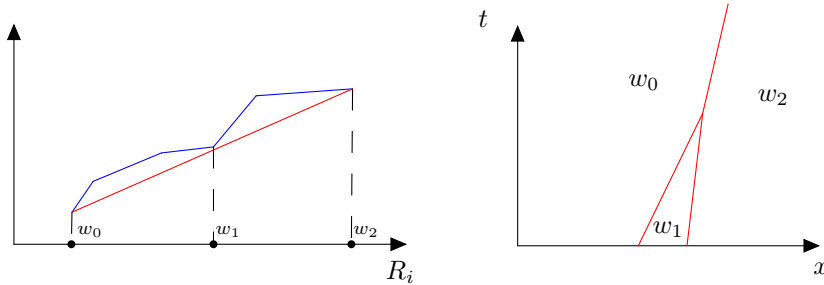


Figure 11: All jumps have the same sign

- (Case 2) Not all jumps have the same sign. Let us denote w_0, w_1, \dots, w_n ($n \in \mathbb{N}$) the consecutive “incoming” states. It is possible to link the extremal (w_0 and w_n) states using fronts whose jumps have the same sign. To do so, it suffices to solve a Riemann problem between extremal states w_0 and w_n as in Section A.2.1. Moreover, by the triangle inequality, the total variation decreases at least by $2 \times 2^{-\nu}$. Concerning the L^∞ norm, it is conserved. This is because the $f_{\nu,i}$ s are non decreasing.

(Case 1) creates a unique front and let the total variation unchanged whereas (Case 2) can create several fronts but the total variation decreases by at least by $2 \times 2^{-\nu}$. Consequently, (Case 2) can happen only a finite number of times and the

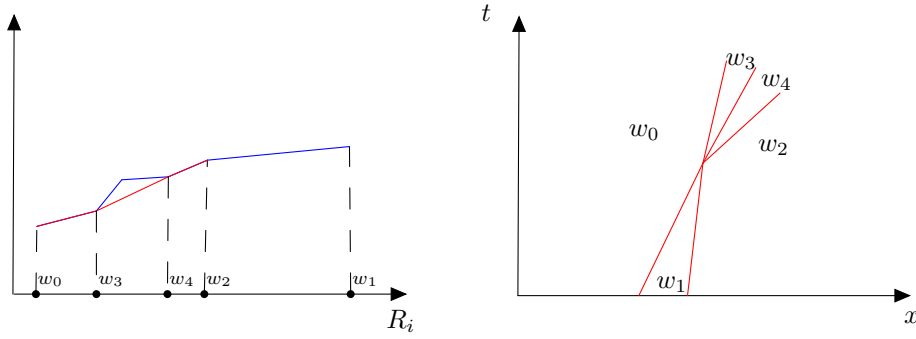


Figure 12: Not all jumps have the same sign

number of fronts remains bounded as time evolves. As a consequence, it is possible to construct an entropy piecewise constant approximate solution under the limit line verifying

$$(A.12) \quad \begin{cases} \forall 0 \leq t \leq 1/c_{\max}, TV_{[c_{\max}t, 1]}(R_{\nu, i}(t, \cdot)) \leq TV_{[0, 1]}(R_{0, \nu, i}) \\ \forall x \in [0, 1], TV_{[0, x/c_{\max}]}(R_{\nu, i}(\cdot, x)) \leq TV_{[0, 1]}(R_{0, \nu, i}) \end{cases}$$

The corresponding picture is given in Figure 13:

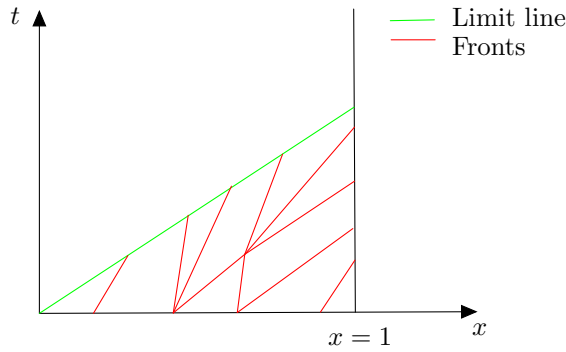


Figure 13: The solution under the limit line

Remark A.3. If $n \geq 2$ ($n \in \mathbb{N}$) fronts interacts exactly at $x = 1$ for some time $t > 0$. Then, we modify a bit the velocity of $n - 1$ fronts to prevent this situation. Taking one of such fronts, we denote λ and $\tilde{\lambda}$ the respective former and new velocities. We can choose them such that $|\lambda - \tilde{\lambda}| \leq \frac{1}{\nu}$.

A.2.4. Finishing the construction. To construct locally the solution above the limit line, we impose the boundary condition

$$\forall 0 \leq t \leq 1/c_{\max}, R_\nu(t, 0^+) = g_{2\nu}(R_\nu(t, 1^-))$$

where we recall that:

$$\forall R \in \mathbb{R}^d, \nu > 1, i \in \llbracket 1, d \rrbracket, g_{\nu,i}(R) = 2^{-\nu}(E(2^\nu g_i(R))).$$

Then, to construct a local solution, we solve the different Riemann problems as in section A.2.2 this time using the approximated flux $f_{2\nu}$. More precisely if at a time t , $R_i(t^-, 0^+) \neq R_i(t^+, 0^+)$, we solve the Riemann problem with $R_i(t^+, 0^+)$ as left state and $R_i(t^-, 0^+)$ as right state. Hence we are able to get a solution locally above the limit line taking its values in $2^{-2\nu}\mathbb{Z}$.

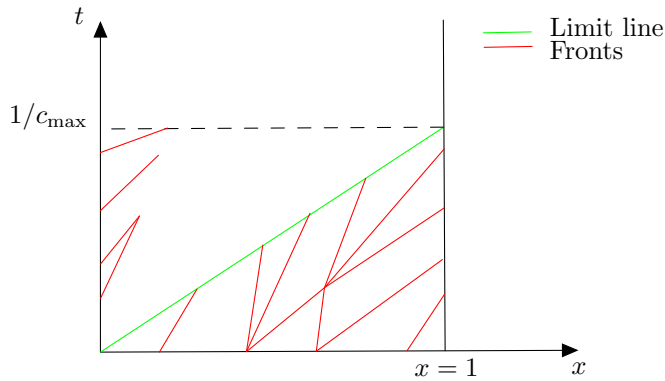


Figure 14: The local solution above the limit line

Finally, we extend fronts coming from the zone under the limit line and deal with front interactions as in section A.2.3 this time using the approximated flux $f_{2\nu}$. The final picture is given in Figure 15. This is very important to remark that the picture under the limit line cannot be modified by fronts coming from the left boundary. This is because the limit line has maximal velocity.

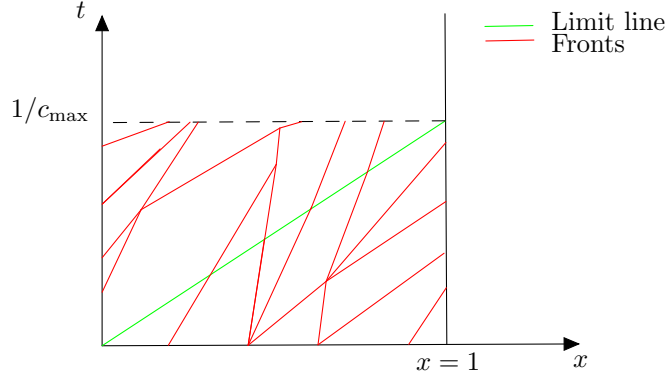


Figure 15: The solution

Concerning the total variation, the way we dealt with front interactions prevents the total variation from increasing when we compare the total strength of ongoing fronts with the one of outgoing fronts. As a consequence,

$$(A.13) \quad \begin{cases} \forall 0 \leq t \leq 1/c_{\max}, TV_{[0, c_{\max}t]}(R_{\nu, i}(t, \cdot)) \leq TV_{[0, 1]}(R_{0, \nu, i}) \\ \quad + TV_{[0, 1/c_{\max}]}(R_{\nu, i}(\cdot, 0^+)) \\ \forall x \in [0, 1], TV_{[x/c_{\max}, 1/c_{\max}]}(R_{\nu, i}(\cdot, x)) \leq TV_{[0, 1]}(R_{0, \nu, i}) \\ \quad + TV_{[0, 1/c_{\max}]}(R_{\nu, i}(\cdot, 0^+)) \end{cases}$$

A.2.5. Conclusion. All previous steps can be repeated on intervals $[k/c_{\max}, (k+1)/c_{\max}]$ and a solution defined for all time is built. Now let $T > 0$. There are several points to verify:

- (Boundary condition) The approximated boundary condition (2.1) is satisfied by construction.
- (Boundary interactions) Two fronts cannot interact simultaneously at the right boundary by construction.
- (Estimate on the total variation). Using (A.12), (A.13) and the fact that R_{ν} satisfies the approximated boundary condition (2.1), one can deduce that there exists a constant $C(g, T)$ (depending on the Lipschitz constant of g and T) such that

$$(A.14) \quad \forall 0 \leq t \leq T, TV_{[0, 1]}(R_{\nu}(t, \cdot)) \leq C(g, T)TV_{[0, 1]}(R_{0, \nu}(t, \cdot)).$$

- (Estimate on the entropy) Take a positive test function $\phi \in C_c^1((0, T) \times (0, 1))$, $T > 0$ and $k \in \mathbb{R}^d$. Then, by integration by parts, one obtains:

$$\int_0^T \int_0^1 [\eta_k(R_{\nu})\phi_t + q_k(R_{\nu})\phi_x] dx dt = \sum_{\alpha} [\gamma_{\alpha}(\eta_k(R_{\nu, \alpha}^r) - \eta_k(R_{\nu, \alpha}^l)) - (q_k(R_{\nu, \alpha}^r) - q_k(R_{\nu, \alpha}^l))] \phi(t, \gamma_{\alpha})$$

where α runs over the discontinuities $t \rightarrow (t, \gamma_{\alpha}(t))$ of R_{ν} .

We denote P the set of physical fronts *ie* the fronts for which the velocity has not been modified. NP designates the complement of P . As fronts of P are entropic by construction, we have:

$$\begin{aligned}
& \int_0^T \int_0^1 \eta_k(R_\nu) \phi_{i,t} + q_k(R_\nu) \phi_{i,x} dx dt \\
& \geq 0 + \sum_{\alpha \in NP} [\tilde{\gamma}_\alpha (\eta_k(R_{\nu,\alpha}^r) \eta_k(R_{\nu,\alpha}^l)) - (q_k(R_{\nu,\alpha}^r) - q_k(R_{\nu,\alpha}^l))] \phi(t, \gamma_\alpha) \\
& = \sum_{\alpha \in NP} [\tilde{\lambda}_\alpha (\eta_k(R_{\nu,\alpha}^r) - \eta_k(R_{\nu,\alpha}^l)) - (q_k(R_{\nu,\alpha}^r) - q_k(R_{\nu,\alpha}^l))] \phi(t, \gamma_\alpha)
\end{aligned}$$

where we replaced the notation $\tilde{\gamma}$ by $\tilde{\lambda}$ to emphasize the fact that it corresponds to a modified velocity (see Remark A.3); the unmodified "entropic" velocity being denoted λ . Hence,

$$\begin{aligned}
& \int_0^T \int_0^1 \eta_k(R_\nu) \phi_t + q_k(R_\nu) \phi_x dx dt \\
& \geq \sum_{\alpha \in NP} [\tilde{\lambda}_\alpha (\eta_k(R_{\nu,\alpha}^r) - \eta_k(R_{\nu,\alpha}^l)) - (q_k(R_{\nu,\alpha}^r) - q_k(R_{\nu,\alpha}^l))] \phi(t, \gamma_\alpha) \\
& = \sum_{\alpha \in NP} [\lambda_\alpha (\eta_k(R_{\nu,\alpha}^r) - \eta_k(R_{\nu,\alpha}^l)) - (q_k(R_{\nu,\alpha}^r) - q_k(R_{\nu,\alpha}^l)) \\
& \quad + (\tilde{\lambda}_\alpha - \lambda_\alpha) (\eta_k(R_{\nu,\alpha}^r) - \eta_k(R_{\nu,\alpha}^l))] \phi(t, \gamma_\alpha) \\
& \geq 0 - \frac{TV(R_\nu(t, \cdot))}{\nu} \|\phi\|_{L^\infty(\mathbb{R}^+ \times [0,1])} \\
& \geq -C(g, T) \frac{TV(R_{0,\nu})}{\nu} \|\phi\|_{L^\infty(\mathbb{R}^+ \times [0,1])}
\end{aligned}$$

where we have used chronologically:

- The fact that an unmodified velocity corresponds to an entropy front
- The equation $|\lambda - \tilde{\lambda}| \leq \frac{1}{\nu}$ from Remark A.3
- The fact that η_k is 1-Lipschitz
- The estimate (A.14) proven before.
- (L^∞ estimate). Remark that when we solved Riemann problem, the L^∞ norm did not increase. This is mainly because we are dealing with non decreasing fluxes. The only way for the L^∞ norm to increase is through the boundary condition. As a consequence, the estimate (A.3) holds.

This finishes the proof of Lemma A.1.

A.3. End of the proof of the existence result. To conclude on the existence, we will use Lemma A.1 and Helly's Theorem 1.4. There are several points to prove:

- (Entropy decay) Take $T > 0$, $R_0 \in BV([0,1])$ and a sequence $(R_{0,\nu})_\nu$ of piecewise constant functions converging to R_0 in BV (such a sequence exists by [4, Lemma 2.2]). For all $\nu > 1$, we denote $(R_\nu)_\nu$ the sequence of piecewise constant functions of Lemma A.1.

By Lemma A.1, there exists a $C(g, T) > 0$ such that

$$\forall 0 \leq t \leq T, TV_{[0,1]}(R_\nu(t, \cdot)) \leq C(g, T) TV_{[0,1]}(R_{0,\nu}).$$

As $\lim_{\nu \rightarrow \infty} R_{0,\nu} = R_0 \in BV([0,1])$,

$$(A.15) \quad \forall 0 \leq t \leq T, TV_{[0,1]}(R_\nu(t, \cdot)) \leq C(g, T, R_0).$$

Next by (A.3) and the fact that the L^∞ norm of the elements of $(R_{0,\nu})_\nu$ are bounded, we have

$$(A.16) \quad \forall 0 \leq t \leq T, \quad \|R_\nu(t, \cdot)\|_{L^\infty([0,1])} \leq C(g, T, R_0).$$

Finally for all $0 \leq s, t \leq T$ and by the finiteness of the speed of propagation:

$$(A.17) \quad \begin{aligned} \|R_\nu(t, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} &\leq c_{\max}(t-s) \max_{u \in [s,t]} TV_{[0,1]}(R_\nu(u, \cdot)) \\ &\leq c_{\max}(t-s)C(g, T, R_0) \end{aligned}$$

where we have used (A.15).

By Helly's Theorem (Theorem 1.4), there exists a subsequence of $(R_\nu)_\nu$ still denoted $(R_\nu)_\nu$ converging in $L^1_{loc}(\mathbb{R}^+, L^1([0,1]))$ to an element $R \in L^\infty_{loc}(\mathbb{R}^+, BV([0,1]))$. Moreover,

$$(A.18) \quad \forall 0 \leq s, t \leq T, \quad \|R(t, \cdot) - R(s, \cdot)\|_{L^1([0,1])} \leq c_{\max}(t-s)C(g, T, R_0).$$

As $(f_\nu)_\nu$ converges uniformly towards f on bounded intervals, we can pass to the limit in (A.1) to get (1.11).

- (Initial condition). Let $\varepsilon > 0$ and $s > 0$

$$\begin{aligned} \|R(0, \cdot) - R_\nu(0, \cdot)\|_{L^1([0,1])} &\leq \|R(0, \cdot) - R(s, \cdot)\|_{L^1([0,1])} \\ &\quad + \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} \\ &\quad + \|R_\nu(s, \cdot) - R_\nu(0, \cdot)\|_{L^1([0,1])} \\ &\leq 2C(g, R_0)s + \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])}. \end{aligned}$$

where we have used (A.18).

Integrating with respect to s on an interval $[0, t]$ for $0 \leq t \leq 1/c_{\max}$, one gets

$$\begin{aligned} \|R(0, \cdot) - R_\nu(0, \cdot)\|_{L^1([0,1])} &\leq C(g, R_0)t \\ &\quad + \frac{1}{t} \int_0^t \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} ds \\ &\leq C(g, R_0)t \\ &\quad + \frac{1}{t} \int_0^{1/c_{\max}} \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} ds \end{aligned}$$

Taking $t = \frac{\varepsilon}{2C(g, R_0)}$ and ν sufficiently large such that $\int_0^{1/c_{\max}} \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} ds \leq \frac{\varepsilon^2}{4C(g, R_0)}$, one finally obtains:

$$\|R(0, \cdot) - R_\nu(0, \cdot)\|_{L^1([0,1])} \leq \varepsilon.$$

By the fact that $(R_{0,\nu})_\nu$ converges towards R_0 in $L^1([0,1])$, we deduce that $R(0, \cdot) = R_0$ in a L^1 sense and $R(0, \cdot) = R_0$ almost everywhere.

Remark A.4. We can repeat the same procedure for any $t \geq 0$ and

$$\forall t \geq 0, \quad \lim_{\nu \rightarrow \infty} \|R_\nu(t, \cdot) - R(t, \cdot)\|_{L^1([0,1])} = 0.$$

- (Boundary condition). For the boundary condition, it suffices to consider the variable x as a time variable.

Using (A.12), (A.13) and the approximated boundary condition (2.1), one can easily prove that

$$\forall x \in [0, 1], TV_{[0,1/c_{\max}]}(R_\nu(\cdot, x)) \leq C(g)TV_{[0,1]}(R_{0,\nu}(\cdot)).$$

As $(R_{0,\nu})_\nu$ is bounded in BV ,

$$(A.19) \quad \forall x \in [0, 1], TV_{[0,1/c_{\max}]}(R(\cdot, x)) \leq C(g, R_0).$$

Additionally, with (A.16) we get the L^∞ estimate

$$(A.20) \quad \forall \nu > 1, \forall x \in [0, 1], \|R_\nu(\cdot, x)\|_{L^\infty([0,1/c_{\max}])} \leq C(g, R_0).$$

Finally, using (A.19) and recalling the definition $c_{\min} := \min_i \alpha_i$ of front velocities, we have for $0 \leq x, y \leq 1$:

$$(A.21) \quad \forall \nu > 1, \|R_\nu(\cdot, x) - R_\nu(\cdot, y)\|_{L^1([0,1/c_{\max}])} \leq \frac{|x - y|}{c_{\min}} C(g, R_0).$$

By Helly's Theorem (Theorem 1.4), $(R_\nu)_\nu$ tends towards R in $L^\infty_{loc}([0, 1], L^1([0, 1/c_{\max}]))$. Using a similar argument as in the previous item of the proof, one shows that $R(\cdot, 0^+) = g(R(\cdot, 1^-))$ in the almost everywhere sense on $[0, 1/c_{\max}]$. We repeat the argument to get the same conclusion for all time.

Hence, R is a solution of (1.10) in the sense of Definition 1.6. It remains to prove that $(R_\nu)_\nu$ is an approximating sequence of PCFs of the entropy solution R in the sense of Definition 2.2. By construction, $(R_\nu)_\nu$ satisfies the first five points of Definition 2.2. It remains only to prove the bound

$$(A.22) \quad \forall t \geq 0, \delta t > 0, TV_{[0,1]}(R(t, \cdot)) \leq \limsup_{\nu \rightarrow \infty} \sup_{s \in [t, t + \delta t]} TV_{[0,1]}(R_\nu(s, \cdot)).$$

This is a consequence of Helly's Theorem. Indeed, take $t \geq 0$, $\Delta t > 0$ and $n \in \mathbb{N}^*$. Instead of applying Helly's Theorem on an interval of the form $[0, T]$ for the sequence $(R_\nu)_{\nu > 1}$, we apply it on the interval $[t, t + \Delta t]$ for the sequence $(R_\nu)_{\nu > n}$.

As

$$\forall s \in [t, t + \Delta t], \forall \nu > n, TV_{[0,1]}(R_\nu(s, \cdot)) \leq \sup_{\substack{u \in [t, t + \Delta t] \\ \nu > n}} TV_{[0,1]}(R_\nu(u, \cdot)),$$

we deduce by Helly's Theorem that

$$\forall s \in [t, t + \Delta t], TV_{[0,1]}(R(s, \cdot)) \leq \sup_{\substack{u \in [t, t + \Delta t] \\ \nu > n}} TV_{[0,1]}(R_\nu(u, \cdot)).$$

Passing to the limit as n goes to infinity gives (A.22). To get the estimate (2.3), the proof is similar. The existence part of Theorem 2.3 is proven.

Appendix B. Uniqueness. We will adapt the method of doubling variables of Kruzhkov to our boundary value problem. Let u, v be two entropy solutions of (1.10) with their respective initial data u_0, v_0 .

We will first show the uniqueness on the triangle T_1 :

$$T_1 := \{(t, x) \mid c_{\max}t \leq x \leq 1, 0 \leq t \leq 1/c_{\max}\}$$

To do so, let $0 < t \leq 1/c_{\max}$ and define the domain Ω_t by:

$$\Omega_t := \{(s, x); 0 \leq s \leq t, c_{\max}s \leq x \leq 1\}.$$

We give a graphical representation of Ω_t in Figure 16.

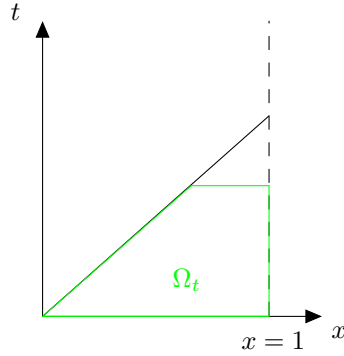


Figure 16: The domain Ω_t

Formally, as u in entropy on Ω_t , we have for all $k \in \mathbb{R}^d$

$$\begin{aligned} 0 &\geq \int \int_{\Omega_t} \eta_k(u)_t + q_k(u)_x dx dt \\ &= \int_{1-c_{\max}t}^1 \eta_k(u) dx - \int_0^1 \eta_k(u_0) dx \\ &\quad + \int_0^t c_{\max} \eta_k(u(\tau, c_{\max}\tau)) - q_k(u(\tau, c_{\max}\tau)) d\tau + \int_0^t q_k(u(1, s)) ds. \end{aligned}$$

The third term is positive because c_{\max} is superior to all the Lipschitz constants of the f_i s. The last term is positive since all the f_i s are non decreasing. Hence,

$$(B.1) \quad \int_{1-c_{\max}t}^1 \eta_k(u) dx \leq \int_0^1 \eta_k(u_0) dx.$$

It is equivalent to:

$$\forall k \in \mathbb{R}^d, \sum_{i=1}^d \int_{1-c_{\max}t}^1 |u_i(t, x) - k_i| dx \leq \sum_{i=1}^d \int_0^1 |u_{0,i}(x) - k_i| dx.$$

Kruzhkov's doubling variable method allows to replace the k_i by the v_i to give:

$$\sum_{i=1}^d \int_{1-c_{\max}t}^1 |u_i(t, x) - v_i(t, x)| dx \leq \sum_{i=1}^d \int_0^1 |u_{0,i}(x) - v_{0,i}(x)| dx.$$

Remark B.1. Rigorous justifications of previous computations can be found in the proof of [4, Theorem 6.2].

As a consequence, the solution is unique on the triangle T_1 .

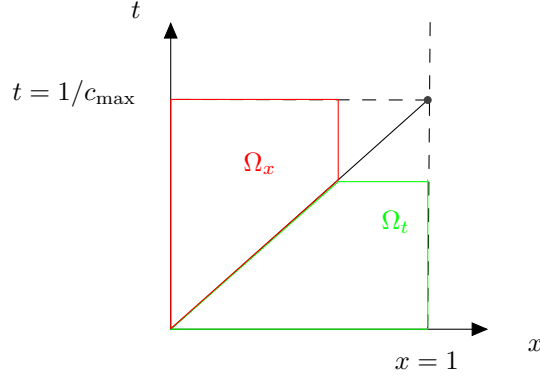


Figure 17: The domain Ω_x

Now let x be in $]0, 1[$, we apply the same strategy to the set

$$\Omega_x := \{(s, y); 0 \leq y \leq x, y/c_{\max} \leq s \leq 1/c_{\max}\}$$

represented in Figure 17.

Integrating $\eta_k(u)_t + q_k(u)_x \leq 0$ in Ω_x , one obtains: $\forall x \in [0, 1], k \in \mathbb{R}^d$,

$$\int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} q_k(u(t, x)) dt \leq \int_0^{1/c_{\max}} q_k(u(t, 0)) dt + \int_0^{x/c_{\max}} c_{\max} \eta_k(u(t, c_{\max}t)) - q_k(u(t, c_{\max}t)) dt.$$

This is equivalent to: $\forall x \in [0, 1], k \in \mathbb{R}^d$,

$$\begin{aligned} & \sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} (f_i(u_i) - f_i(k_i)) \text{sign}(u_i - k_i) dt \\ & \leq \sum_{i=1}^d \int_0^{1/c_{\max}} (f_i(u_i(t, 0)) - f_i(k_i)) \text{sign}(u_i(t, 0) - k_i) dt \\ & + \int_0^{x/c_{\max}} c_{\max} \eta_k(u(t, c_{\max}t)) - q_k(u(t, c_{\max}t)) dt. \end{aligned}$$

As all the f_i are non decreasing ($q_k \geq 0$) and all c_{\max} Lipschitz: $\forall x \in [0, 1]$, $k \in \mathbb{R}^d$,

$$\begin{aligned} \sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} |f_i(u_i) - f_i(k_i)| dt &\leq c_{\max} \sum_{i=1}^d \int_0^{1/c_{\max}} |u_i(t, 0) - k_i| dt \\ &\quad + c_{\max} \int_0^{x/c_{\max}} \eta_k(u(t, c_{\max}t)) dt. \end{aligned}$$

Following Kruzhkov's method, the k_i s can be replaced by the v_i s and for all x in $[0, 1]$:

$$\begin{aligned} \sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} |f_i(u_i) - f_i(v_i)| dt &\leq c_{\max} \sum_{i=1}^d \int_0^{1/c_{\max}} |u_i(t, 0) - v_i(t, 0)| dt \\ &\quad + c_{\max} \int_0^{x/c_{\max}} |u(t, c_{\max}t) - v(t, c_{\max}t)| dt. \end{aligned}$$

As u, v satisfy the boundary condition on $[0, 1/c_{\max}]$, there exists a constant $C(g, c_{\max})$ depending on the Lipschitz constant of g such that for all x in $[0, 1]$:

$$\begin{aligned} &\sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} |f_i(u_i) - f_i(v_i)| dt \\ &\leq C(g, c_{\max}) \sum_{i=1}^d \int_0^{1/c_{\max}} |u_i(t, 1) - v_i(t, 1)| dt \\ &\quad + c_{\max} \int_0^{x/c_{\max}} |u(t, c_{\max}t) - v(t, c_{\max}t)| dt. \end{aligned}$$

If $u_0 = v_0$, we have seen that u and v coincide on T_1 . This implies that if $u_0 = v_0$, u and v coincide on the segment $\{1\} \times [0, 1/c_{\max}]$ and on the line $(x = c_{\max}t, t)$ for $t \leq 1/c_{\max}$. As a consequence,

$$\forall x \in [0, 1], \sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} |f_i(u_i) - f_i(v_i)| dt = 0.$$

By the monotonicity of the f_i s, u and v coincide on the triangle T_2 defined by

$$T_2 := \{(t, x) \mid 0 \leq x \leq 1, x/c_{\max} \leq t \leq 1/c_{\max}\}.$$

To conclude, u and v coincide for $t \leq 1/c_{\max}$ and repeating this argument, we can prove the uniqueness for all time. This finishes the proof of the uniqueness.

Acknowledgments. I would like to sincerely thank F. Boyer, F. Ferrante and C. Prieur for our fruitful discussions.

REFERENCES

- [1] D. AMADORI AND R. COLOMBO, *Continuous dependence for 2×2 conservation laws with boundary*, Journal of Differential Equations, 138 (1997), pp. 229 – 266, <https://doi.org/https://doi.org/10.1006/jdeq.1997.3274>, <http://www.sciencedirect.com/science/article/pii/S0022039697932745>.
- [2] D. AMADORI AND R. COLOMBO, *Viscosity solutions and standard Riemann semigroup for conservation laws with boundary*, Rendiconti del Seminario Matematico della Università di Padova, 99 (1998), pp. 219–245, http://www.numdam.org/item/RSMUP_1998__99__219_0.

- [3] G. BASTIN AND J.-M. CORON, *Stability And Boundary Stabilization Of 1-D Hyperbolic Systems*, Progress in Nonlinear Differential Equations and Their Applications, Springer International Publishing, 2016.
- [4] A. BRESSAN, *Hyperbolic Systems of Conservation Laws: The One-dimensional Cauchy Problem*, Oxford Lecture Series in Mathematics, Oxford University Press, 2000.
- [5] R. COLOMBO AND G. GUERRA, *On general balance laws with boundary*, Journal of Differential Equations, 248 (2010), pp. 1017 – 1043, <https://doi.org/https://doi.org/10.1016/j.jde.2009.12.002>, <http://www.sciencedirect.com/science/article/pii/S0022039609004550>.
- [6] J.-M. CORON, B. D'ANDRÉA NOVEL, AND G. BASTIN, *A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws*, IEEE Transactions on Automatic Control, 52 (2004), pp. 2–11.
- [7] J.-M. CORON, S. ERVEDOZA, S. GHOSHAL, O. GLASS, AND V. PERROLLAZ, *Dissipative boundary conditions for 2×2 hyperbolic systems of conservation laws for entropy solutions in bv*, Journal of Differential Equations, 262 (2017), pp. 1 – 30, <https://doi.org/https://doi.org/10.1016/j.jde.2016.09.016>, <http://www.sciencedirect.com/science/article/pii/S0022039616302832>.
- [8] J.-M. CORON AND H. NGUYEN, *Dissipative boundary conditions for nonlinear 1-D hyperbolic systems: sharp conditions through an approach via time-delay systems*, SIAM Journal on Mathematical Analysis, 47 (2015), pp. 2220–2240, <https://doi.org/10.1137/140976625>.
- [9] C. DAFERMOS, *Hyperbolic conservation laws in continuum physics*, Grundlehren Math. Wissenschaften Series, Vol 325, Springer Verlag, Third Edition, (2010).
- [10] J. DE HALLEUX, C. PRIEUR, J.-M. CORON, B. NOVEL, AND G. BASTIN, *Boundary feedback control in networks of open channels*, Automatica, 39 (2003), pp. 1365–1376, [https://doi.org/10.1016/S0005-1098\(03\)00109-2](https://doi.org/10.1016/S0005-1098(03)00109-2).
- [11] C. DONADELLO AND A. MARSON, *Stability of front tracking solutions to the initial and boundary value problem for systems of conservation laws*, Nonlinear Differential Equations and Applications NoDEA, 14 (2007), pp. 569–592, <https://doi.org/10.1007/s00030-007-5010-7>, <https://doi.org/10.1007/s00030-007-5010-7>.
- [12] M. DUS, F. FERRANTE, AND C. PRIEUR, *On L^2 stabilization of diagonal semilinear hyperbolic systems by saturated boundary control*, Submitted, (2019).
- [13] J. GLIMM, *Solutions in the large for nonlinear hyperbolic systems of equations*, Communications on Pure and Applied Mathematics, 18 (1965), pp. 697–715, <https://doi.org/10.1002/cpa.3160180408>, <https://onlinelibrary.wiley.com/doi/abs/10.1002/cpa.3160180408>, <https://arxiv.org/abs/https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160180408>.
- [14] J. K. HALE AND M. VERDUYN LUNEL, *Introduction To Functional Differential Equations*, vol. 99 of Applied Mathematical Sciences, Springer Verlag, 1993.
- [15] R. J. DIPERNA, *Global existence of solutions to nonlinear hyperbolic systems of conservation laws*, Journal of Differential Equations, 20 (1976), [https://doi.org/10.1016/0022-0396\(76\)90102-9](https://doi.org/10.1016/0022-0396(76)90102-9).
- [16] P. LAX, *Hyperbolic Systems of Conservation Laws*, Communications on Pure and Applied Mathematics, 10 (1957), pp. 537–566.
- [17] V. PERROLLAZ, *Asymptotic stabilization of entropy solutions to scalar conservation laws through a stationary feedback law*, Annales de l'Institut Henri Poincaré (C) Non Linear Analysis, 30 (2013), pp. 879 – 915, <https://doi.org/https://doi.org/10.1016/j.anihpc.2012.12.003>, <http://www.sciencedirect.com/science/article/pii/S0294144913000036>.
- [18] C. PRIEUR, S. TARBOURIECH, AND J. M. GOMES DA SILVA JR., *Wave equation with cone-bounded control laws*, IEEE Transactions on Automatic Control, 61 (2016), pp. 3452–3463.
- [19] T. H. QIN, *Global smooth solutions of dissipative boundary value problems for first order quasilinear hyperbolic systems*, Chinese Ann. Math. Ser. B, 6 (1985), pp. 289 – 298.
- [20] M. SABLÉ-TOUGERON, *Méthode de Glimm et problème mixte*, Annales de l'I.H.P. Analyse non linéaire, 10 (1993), pp. 423–443, http://www.numdam.org/item/AIHPC_1993__10_4_423_0.
- [21] L. TA-TSIEN, *Global classical solutions for quasilinear hyperbolic systems*, RAM: Research in Applied Mathematics, 32 (1994).
- [22] S. TARBOURIECH, G. GARCIA, J. M. GOMES DA SILVA JR., AND I. QUEINNEC, *Stability and Stabilization of Linear Systems with Saturating Actuators*, Springer London, 2011.