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1 **BV EXPONENTIAL STABILITY FOR NETWORKS OF SCALAR CONSERVATION LAWS USING**
2 **LINEAR OR SATURATED CONTROLS**

3 MATHIAS DUS*

4 **Abstract.** In this paper, we investigate the *BV* exponential stability of general networks of scalar conservation laws with positive velocities
5 and under dissipative boundary conditions. The paper is divided in two parts, the first one focusing on linear controls while the last one deals
6 with saturated laws. For the linear case, the global exponential *BV* stability is proven. For the saturated case, we argue that we cannot expect
7 to have a basin of attraction larger than the region of linearity in a *BV* context. We rather prove an L^∞ local stability result. An explicit
8 estimate of the basin of attraction is given. The Lyapunov functional is inspired from Glimm's seminal work [13] reconsidered in [7].

9 **Key words.** Bounded variations, stabilization, feedback, saturation, wavefront tracking method.

10 **AMS subject classifications.** 93D05, 93D15, 93D20

11 **1. Introduction.** In this paper, we are interested in the exponential stabilization of 1D hyperbolic systems using
12 linear or saturated feedback control laws. The system we consider is of the form:

13 (1.1)
$$\begin{cases} R_t + [f(R)]_x & = 0 \\ R(t, 0) & = g(R(t, 1)) \\ R(0, x) & = R_0(x) \end{cases}$$

14 where $R : \mathbb{R}^+ \times [0, 1] \mapsto \mathbb{R}^d$ ($d \in \mathbb{N}^*$) and $f, g : \mathbb{R}^d \mapsto \mathbb{R}^d$.

15 For coherence, we impose that all characteristic velocities are positive and consequently, the boundary condition in
16 (1.1) is adapted.

17 In [3, Chapter 1], typical examples of systems modeled by hyperbolic PDEs with feedback boundary conditions are
18 cited; the telegrapher equations for electrical lines, the shallow water (Saint-Venant) equations for open channels, the
19 isothermal Euler equations for gas flow in pipelines or even the Aw-Rascle equations for road traffic.

20 More specifically, we are interested in the stabilization of (1.1) using feedback control laws at the boundary. The
21 problem is equivalent to find sufficient conditions on g such that for any R_0 initial data, the solution to (1.1) converges
22 exponentially fast toward zero in the sense that

$$\forall t \geq 0, \|R(t, \cdot)\|_X \leq C e^{-\gamma t} \|R_0\|_X$$

23 where $C, \gamma > 0$ are constants independent on t and $\|\cdot\|_X$ is a norm on a Banach space X .

24 Two types of feedback laws are analyzed in this article: the linear law and the saturated one.

25 **1.1. Linear feedback.** For the case where $g = H \in M_d(\mathbb{R})$ is a linear operator, the literature is quite rich.

26 **When the flux is linear;** $f(R) = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_d)$ with $\lambda_1, \dots, \lambda_d > 0$, the problem of stabilization can be
27 treated for the following classical functional spaces X :

- 28 a. Sobolev spaces $W^{m,p}([0, 1])$ for $m \in \mathbb{N}$ and $p \in [1, +\infty]$.
- 29 b. Spaces $C^m([0, 1])$ with ($m \in \mathbb{N}$).
- 30 c. $BV([0, 1])$.

31 Indeed, in [14, Theorem 3.5 p. 275], the authors prove that 0 is globally exponentially stable in spaces X defined
32 above if and only if there exists $\delta > 0$ such that

33 (1.2)
$$\left\{ z \in \mathbb{C} \mid \det(I_d - \text{diag}(e^{-z/\lambda_1}, \dots, e^{-z/\lambda_d})H) = 0 \right\} \subset \{z \in \mathbb{C} \mid \text{Re}(z) < -\delta\}.$$

34 However, the criteria (1.2) is not stable with respect to Λ . Indeed, when we take a H, Λ verifying (1.2), we cannot
35 guaranty that the same holds for $\tilde{\Lambda}$ with $\tilde{\Lambda}$ diagonal and arbitrarily close to Λ [14, p. 285].

36 In the same book, Silkowski [14, Theorem 6.1 p. 286] proves that for all Banach spaces X listed above, 0 is globally
37 exponentially stable and that this stability is robust with respect to Λ if and only if

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$$(1.3) \quad \rho_0(H) := \max \{ \rho(\text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})H) \mid \theta_i \in \mathbb{R} \} < 1$$

where ρ designates the usual spectral radius.

Condition (1.3) is stronger than (1.2). For some years, many results came out generalizing to nonlinear fluxes.

When the flux is not linear, only sufficient conditions of stability are given and most of the time this stability is only proven to be local:

a. For $X = C^m([0, 1])$ with $m \in \mathbb{N}^*$, a sufficient condition [10, 19, 21] is:

$$(1.4) \quad \rho_\infty(H) := \inf_{\Delta \in D_d^+(\mathbb{R})} |\Delta H \Delta^{-1}|_\infty < 1$$

where $|\cdot|_\infty$ is the canonical infinity norm of matrices and $D_d^+(\mathbb{R})$ is the set of diagonal positive matrices.

It should be mentioned that in [10, 19, 21], the stability was proven for $m = 1$ but the argument can be adapted for any integer $m > 0$.

b. For Sobolev space $W^{m,p}([0, 1])$ a sufficient condition for stability writes:

$$\rho_p(H) := \inf_{\Delta \in D_d^+(\mathbb{R})} |\Delta H \Delta^{-1}|_p < 1$$

where $|\cdot|_p$ is the canonical p norm of matrices.

The case $p = 2$ was treated in [6] and the general case $p \geq 1$ was treated in [8]. Also, it should be mentioned that in [6, 8], the stability was proven for $m = 2$ but the argument can be adapted for any integer $m > 0$.

c. For $BV([0, 1])$, few results are known. To the authors' knowledge only [7] deals with this case. They take a 2×2 system of conservation laws and give a sufficient condition on H to ensure the local BV stability.

In this article, we also place ourselves in a BV context and find a sufficient condition on H to ensure a global BV stability. Contrary to [7], we will not consider vectorial fluxes. In this case, solutions are only proven to exist for small initial data. This is why, we will rather consider scalar decentralized fluxes (see section 1.3) for which solutions exist for whatever initial data in BV . This hypothesis on the flux is all the more important that when we will study saturated feedback laws, we will also estimate the basin of attraction. This would not be possible with solutions defined only for small initial data.

1.2. Saturated control law. We take a matrix $H \in M_d(\mathbb{R})$ potentially unstable in the sense that $\rho_\infty(H) > 1$ (see (1.4)). Then we assume that there exist matrices $B, K \in M_d(\mathbb{R})$ such that $\rho_\infty(H + BK) < 1$. Finally, we consider the following system:

$$(1.5) \quad \begin{cases} R_t + [f(R)]_x & = 0 \\ R(t, 0) & = HR(t, 1) + B\sigma(KR(t, 1)) \\ R(0, x) & = R_0(x) \end{cases}$$

with σ defined as a saturation by component ie there exists a $\sigma_s > 0$ such that:

$$\forall i \in \llbracket 1, d \rrbracket, x \in \mathbb{R}, \begin{cases} \sigma_i(x) = x & \text{if } |x| \leq \sigma_s \\ \sigma_i(x) = \text{sign}(x)\sigma_s & \text{otherwise.} \end{cases}$$

From criterion (1.4), the system without saturation is locally stable in $C^m([0, 1])$ with $m \in \mathbb{N}^*$. It is natural to ask ourselves if this property of stability is conserved through the saturation. Apart from this theoretical interest, this problem has gained attention in the last few years because of the increasing need of precision for modeling real actuators. Physical controllers cannot provide infinite energy and sometimes, they saturate rendering classical unsaturated models restrictive. To avoid such situations, engineers choose controllers powerful enough to avoid saturation when the system operates in standard conditions. However, over-dimensioning actuators is not optimal in term of mass and cost of operation for many sophisticated systems as satellites for example. Moreover, in some exceptional configurations, actuators could saturate and lead to very dangerous situations; unpredictable via linear theory.

Very few papers consider the effect of saturation on hyperbolic systems. To our knowledge, only [18] deals with this question in an H^1 context and for the wave equation. Fortunately, the theory is much more developed for finite dimensional systems where polytopic and deadzone techniques were designed [22].

89 In this paper, we will argue that in a BV context, it is not possible to get a basin of attraction bigger than the region
 90 of linearity. We rather prove an L^∞ local stability result with an estimation of the basin of attraction. Then, we deduce
 91 the exponential decay of the BV norm for solutions whose initial data belongs to the L^∞ basin of attraction.

92 **1.3. Scalar conservation laws.** The feedback laws being presented, we can now focus on the partial differential
 93 equation in itself. In fact, we consider a particular form of systems of conservation laws (1.1). More precisely, we take a
 94 flux f verifying Hypothesis 1.1.

Hypothesis 1.1. The flux f is in $C^1(\mathbb{R}^d)$ and there exist scalar fluxes $f_i \in C^1(\mathbb{R})$ such that:

$$\forall R \in \mathbb{R}^d, \forall i \in \llbracket 1, d \rrbracket, [f(R)]_i = f_i(R_i).$$

95 Moreover, f_i is such that there exist $\alpha_i, \beta_i > 0$;

$$\forall i \in \llbracket 1, d \rrbracket, \alpha_i \leq f'_i \leq \beta_i.$$

96

97 Such hypothesis allows to define the maximal and the minimal velocity:

$$98 \quad (1.6) \quad \begin{cases} c_{\max} & := \max_{i \in \llbracket 1, d \rrbracket} \beta_i \\ c_{\min} & := \min_{i \in \llbracket 1, d \rrbracket} \alpha_i. \end{cases}$$

99 The aim of this section is to give a very short introduction to scalar conservation laws without giving any proof (see
 100 [4] for more details).

101 **1.3.1. The set of functions with bounded variations.** It is well-known that the space BV is well-adapted for
 102 conservation laws (see [4] for instance). This is why, we give the definition and main properties of such a space here:

103 **DEFINITION 1.2.** Let $R : [0, 1] \mapsto \mathbb{R}^d$ be a vector valued function. We say that R has bounded variations if

$$\forall n \in \mathbb{N}, \forall x_1 < \dots < x_n \in [0, 1], \sum_{i=1}^{n-1} |R(x_{i+1}) - R(x_i)| < \infty.$$

104 We denote $TV_{[0,1]}(R) = \sup_{n, (x_1, \dots, x_n)} \left\{ \sum_{i=1}^{n-1} |R(x_{i+1}) - R(x_i)| \right\}$ the total variation of R . $BV([0, 1])$ is the space of
 105 vector valued functions with bounded variations and it is a Banach space when $BV([0, 1])$ is embedded with the norm
 106 $\|\cdot\|_{BV([0,1])}$ defined as

$$107 \quad (1.7) \quad \forall R \in BV([0, 1]), \quad \|R\|_{BV([0,1])} = TV_{[0,1]}(R) + \|R\|_{L^1([0,1])}.$$

108 The reason why we consider this space is because any function with bounded variations has a left and a right limit at
 109 each point x of $[0, 1]$. Hence, it is easy to define the trace operator and impose a boundary condition. Moreover, $BV([0, 1])$
 110 has a very interesting property of compactness which will be very useful when we will pass to the limit in the Lyapunov
 111 analysis of approximating solutions. These properties are summed up in a lemma and a theorem:

112 **LEMMA 1.3.** Let $R : [0, 1] \mapsto \mathbb{R}^d$ with bounded variations. Then for all $x \in (0, 1)$, the left and right limit

$$R(x^-) = \lim_{y \rightarrow x^-} R(y), \quad R(x^+) = \lim_{y \rightarrow x^+} R(y)$$

113 exist.

114

115 Moreover, $R(0^+)$ and $R(1^-)$ are also well defined and R has at most countably many point of discontinuities.

116 *Proof.* This is an adaptation of [4, Lemma 2.1]. □

117 Defining the value of R at each jump by $R(x) = R(x^+)$, we can say that R is right continuous in the L^1 equivalence
 118 class. The following theorem is from Helly and states the compactness of $BV([0, 1])$ in $L^1_{loc}(\mathbb{R}^+, L^1([0, 1]))$.

119 **THEOREM 1.4.** [4, Theorem 2.4] Let $(R_\nu)_\nu$ be a sequence of functions from $\mathbb{R}^+ \times [0, 1]$ into \mathbb{R}^d such that there exist
 120 constants C, M and L satisfying

$$121 \quad (1.8) \quad \forall \nu > 1, \forall x \in [0, 1], \forall t \geq 0, TV_{[0,1]}(R_\nu(t, \cdot)) \leq C, |R_\nu(t, x)| \leq M,$$

122 *and*

123 (1.9)
$$\forall 0 \leq t, s \leq T, \|R_\nu(t, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} \leq L|t - s|.$$

124 *Then there exists a subsequence $(R_\mu)_\mu$ converging strongly toward a certain R in $L^1_{loc}(\mathbb{R}^+, L^1([0, 1]))$ and this limit*
 125 *satisfies (1.8)-(1.9) with R_ν replaced by R .*

126 **1.3.2. Entropy.** The concept of entropy is primordial in order to guaranty uniqueness of solutions to conservation
 127 laws. This is why we recall some basic definitions in this section.

128

129 If one considers the conservation law $R_t + [f(R)]_x = 0$ in the usual weak sense:

$$\forall \phi \in C_c^1((0, T) \times (0, 1); \mathbb{R}^d), \int_0^T \int_0^1 (\phi_t R + \phi_x f(R)) = 0,$$

130 it is commonly known that this PDE (associated with fixed boundary and initial conditions) can have several weak
 131 solutions (see Example 4.3 from [4]). In order to restrain the set of solutions, an entropy functional was introduced ([4],
 132 [9], [16]) and is defined as follows:

133 **DEFINITION 1.5.** *A continuously differentiable convex function $\eta : \mathbb{R}^d \mapsto \mathbb{R}$ is called an entropy for the conservation*
 134 *law $R_t + [f(R)]_x = 0$ with entropy flux $q : \mathbb{R}^d \mapsto \mathbb{R}$, if*

$$\forall R \in \mathbb{R}^d, D\eta(R) \cdot Df(R) = Dq(R).$$

135 For scalar conservation laws of the form $u_t + [f_1(u)]_x = 0$, the usual choice of entropy is $\eta(u) := |u - k|$ with flux
 136 $q(u) := (f_1(u) - f_1(k))\text{sign}(u - k)$ where k is an arbitrary real. Knowing this, we introduce the notion of entropy solution
 137 to (1.1).

138 **DEFINITION 1.6.** *Under Hypothesis 1.1, we say that $R \in L^\infty_{loc}(\mathbb{R}^+, BV([0, 1]))$ is an entropy solution on $[0, T]$ to the*
 139 *system*

140 (1.10)
$$\begin{cases} R_t + [f(R)]_x = 0 \\ R(\cdot, 0) = g(R(\cdot, 1)) \\ R(0, \cdot) = R_0 \in BV([0, 1]), \end{cases}$$

141 *if:*

•

142 (1.11)
$$\forall k \in \mathbb{R}^d, \sum_{i=1}^d \int_0^T \int_0^1 \{|R_i - k_i| \phi_t + (f_i(R_i) - f_i(k_i)) \text{sign}(R_i - k_i) \phi_x\} dx dt \geq 0$$

143 *for all $\phi \geq 0$ and $\phi \in C_c^1((0, T) \times (0, 1); \mathbb{R})$.*

144 • *$R(0, \cdot) = R_0$ in the almost everywhere sense.*

145 • *$R(\cdot, 0^+) = g(R(\cdot, 1^-))$ in the almost everywhere sense.*

146 **Remark 1.7.** Here the entropy functional and its flux are defined for all k in \mathbb{R}^d by

147 (1.12)
$$\forall R \in \mathbb{R}^d, \eta_k(R) = \sum_{i=1}^d |R_i - k_i|, q_k(R) = \sum_{i=1}^d (f_i(R_i) - f_i(k_i)) \text{sign}(R_i - k_i).$$

148 Moreover, equation (1.11) can be rewritten as

$$\eta_k(R)_t + q_k(R)_x \leq 0$$

149 in a weak sense. Hence entropy solutions are the solutions of (1.1) which make the entropy η decrease.

150 **Remark 1.8.** Equation (1.11) is stronger than the usual definition of weak solutions. Indeed, if one takes k such that
 151 $k_i < \text{ess inf}(R_i)$ and $k_j > \text{ess sup}(R_j)$ for $j \neq i$, then

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt - \sum_{j \neq i} \int_0^T \int_0^1 \{R_j \phi_t + f_j(R_j) \phi_x\} dx dt \geq 0.$$

152 Also, taking $k_j < \text{ess inf}(R_j)$ for $j \neq i$, one gets:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt + \sum_{j \neq i} \int_0^T \int_0^1 \{R_j \phi_t + f_j(R_j) \phi_x\} dx dt \geq 0.$$

153 Summing the last two inequalities, one gets:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt \geq 0$$

154 Similarly, one can obtain:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt \leq 0.$$

155 Hence, for all $\phi \in C_c^1((0, T) \times (0, 1))$ with $\phi \geq 0$:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt = 0.$$

156 Replacing ϕ with $-\phi$, it is also true for $\phi \in C_c^1((0, T) \times (0, 1))$ with $\phi \leq 0$. Now take a $\phi \in C_c^1((0, T) \times (0, 1))$,
 157 $\phi = \phi^+ + \phi^-$ where ϕ^+, ϕ^- are respectively the positive and negative parts of ϕ . As $\phi^+, \phi^- \in C_c^1((0, T) \times (0, 1))$, we get:

$$\int_0^T \int_0^1 R_i \phi_t + f_i(R_i) \phi_x dx dt = 0.$$

158 As a consequence, each entropy solution is also a weak one.

159

160 **1.4. The contribution.** Now that all the notions have been introduced, we can be more specific concerning the
 161 main contributions of this paper:

162 • State and prove a well-posedness result of (1.1) in a BV context.

163 To help us in the task, we will use front tracking techniques from DiPerna [15] and Bressan [4] to get an entropy
 164 solution in the domain considered. To deal with the boundary condition, the article [7] will be the reference work.

165 • State and prove a global exponential stability result for linear feedback laws.

166 This is the first key result. To our knowledge, no global stabilization result holds for feedback laws of the
 167 form $R(t, 0) = HR(t, 1)$ in a BV entropy context. The article [17] proposes also a feedback law of the form
 168 $R(t, 0) = g(\|R(t, \cdot)\|_{L^1})$. However, in physical systems the L^1 norm of the solution is not always accessible by
 169 observations. Additionally, the article [7] which considers a 2×2 system of conservation laws gives only a local
 170 stabilization result for an entropy solution.

171 • State and prove a local exponential stability result for saturated feedback laws. We will see that this is not possible
 172 in a BV context.

173 To our knowledge, only [12] has studied this kind of saturated feedback laws in an L^∞ context and for the case
 174 of constant characteristic velocities.

175 **1.5. Outline.** In Section 2, we will present and prove an approximation and a well-posedness result for the entropy
 176 BV solution to (1.1). The technique of front tracking will be mainly used. Then in Section 3, a sufficient condition for
 177 global BV stability will be given in the case of a linear feedback. Additionally, we give a sufficient condition for the local
 178 L^∞ stability in the case of a saturated feedback with an estimation of the basin of attraction. Finally, Section 4 is devoted
 179 to concluding remarks and perspectives.

180

181 **Notation:** For all $R \in \mathbb{R}^d$, $|R|$ designates the canonical euclidean norm of R . For matrices $M \in M_d(\mathbb{R})$, $|M| =$
 182 $\sup_{|R|=1, R \in \mathbb{R}^d} |MR|$. For all matrices $M \in M_d(\mathbb{R})$, $|M|_\infty := \max_{i=1..d} \sum_{j=1}^d |M_{i,j}|$. $D_d^+(\mathbb{R})$ is the set of diagonal strictly positive

183 matrices. The value $\rho_\infty(M)$ for matrices $M \in M_d(\mathbb{R})$ is defined by $\rho_\infty(M) := \inf_{\Delta \in D_d^+(\mathbb{R})} |\Delta M \Delta^{-1}|_\infty$. L^p spaces
 184 on $[0, 1]$ ($1 \leq p \leq \infty$, $p \in \mathbb{N}$) are embedded with their canonical norms $\|\cdot\|_{L^p}$. For all matrices $P \in D_d^+(\mathbb{R})$ and
 185 $R \in L^\infty([0, 1])$, $\|R\|_{\infty, P} := \|PR\|_{L^\infty}$. The function $E : \mathbb{R} \mapsto \mathbb{N}$ is the integer part function and the function sign is the
 186 usual sign function with $\text{sign}(0) = 0$.

187 **2. Well-posedness and approximation results.** This section is devoted to the well-posedness of (1.10). Addi-
 188 tionally, we prove the existence of an approximation by piecewise constant functions of the solution to (1.10).

2.1. Piecewise constant entropy solutions. Piecewise constant functions play an important role in the theory of BV solutions to conservation laws. Let us recall the definition of what a piecewise constant function is in our context.

DEFINITION 2.1. *An element R of $L_{loc}^\infty(\mathbb{R}^+, BV([0, 1]))$ is piecewise constant if for all $T > 0$, R viewed as a function defined on $[0, T] \times [0, 1]$ is constant on a finite number of polyhedra. The edges of such polyhedra are called the fronts of R . Additionally, the absolute value of the jump across the front is called the intensity of the front.*

In this paper, we use the concept of approximating sequence of piecewise constant functions (PCF).

DEFINITION 2.2. $(R_\nu)_\nu$ is an approximating sequence of PCFs of an entropy solution R to (1.10) if:

- For $\nu > 1$ fixed, R_ν is piecewise constant in the sense of Definition 2.1 and takes its values in $2^{-(n+1)\nu}\mathbb{Z}$ on strips $\{(x, t) \mid 0 \leq x \leq 1, \max\{(x+n-1)/c_{\max}, 0\} \leq t \leq (x+n)/c_{\max}\}$ for $n \in \mathbb{N}$. The velocities of fronts are all bounded from below by c_{\min} and from above by c_{\max} (see (1.6) for the definition of c_{\min} and c_{\max}).
- For $\nu > 1$ fixed, no more than one front at a time can interact with the right boundary.
- For $\nu > 1$ fixed, if at a time $t \geq 0$ several fronts interact, the sum of intensities of outgoing fronts is inferior to the sum of intensities of ingoing fronts.
- The sequence $(R_\nu(0, \cdot))_\nu$ converges toward R_0 in $BV([0, 1])$.
- The approximated boundary condition is verified:

$$(2.1) \quad \forall n \in \mathbb{N}, \forall t \text{ s.t. } \frac{n}{c_{\max}} \leq t \leq \frac{n+1}{c_{\max}}, R_\nu(t, 0^+) = g_{(n+2)\nu}(R_\nu(t, 1^-))$$

where:

$$(2.2) \quad \forall R \in \mathbb{R}^d, \forall \nu > 1, g_\nu(R) = 2^{-\nu}(E(2^\nu g(R))).$$

- $\forall t \geq 0, \Delta t > 0,$

$$TV_{[0,1]}(R(t, \cdot)) \leq \limsup_{\nu \rightarrow +\infty} \sup_{s \in [t, t+\Delta t]} TV_{[0,1]}(R_\nu(s, \cdot))$$

and

$$(2.3) \quad \|R(t, \cdot)\|_{L^\infty([0,1])} \leq \limsup_{\nu \rightarrow +\infty} \sup_{s \in [t, t+\Delta t]} \|R_\nu(s, \cdot)\|_{L^\infty([0,1])}.$$

2.2. The result of well-posedness and approximation. Now we give the first result of this paper:

THEOREM 2.3. *Under Hypothesis 1.1 and for all $R_0 \in BV([0, 1])$, $g \in Lip(\mathbb{R}^d, \mathbb{R}^d)$ there exists a unique entropy solution $R \in L_{loc}^\infty(\mathbb{R}^+, BV([0, 1]))$ to (1.10). Moreover, there exists an approximating sequence of PCF $(R_\nu)_\nu$ of the entropy solution R .*

Proof. This theorem is proven in Appendix A for the existence and Appendix B for the uniqueness. \square

3. Lyapunov analysis. Before going into the stability analysis, we introduce the functional TV_H on the space BV. For all matrices H in $M_d(\mathbb{R})$, it is defined as follows:

$$(3.1) \quad \forall R \in BV([0, 1]), TV_H(R) = TV_{[0,1]}(R) + |HR(1^-) - R(0^+)|,$$

where $R(1^-)$ and $R(0^+)$ has to be understood as the left and right limits of the function R at $x = 1$ and $x = 0$.

Moreover, we introduce Hypothesis 3.1:

Hypothesis 3.1. The feedback matrix H verifies:

$$\rho_\infty(H) < 1.$$

The following lemma ensures the equivalence between TV_H and $\|\cdot\|_{BV([0,1])}$.

LEMMA 3.2. *Assume Hypothesis 3.1. The functional TV_H defined in (3.1) is a norm on $BV([0, 1])$ equivalent to the norm $\|\cdot\|_{BV([0,1])}$ defined in (1.7). Moreover, there exists a constant $C > 0$ such that*

$$(3.2) \quad \forall R \in BV([0, 1]), \|R\|_{L^\infty([0,1])} \leq C TV_H(R).$$

226 *Proof.* We first prove the following claim:

$$227 \quad (3.3) \quad \forall R \in \mathbb{R}^d, \quad |R| \leq C|R - HR|.$$

228 Let $P \in D_d^+(\mathbb{R})$ such that

$$|PHP^{-1}|_\infty < 1.$$

229 The map $\|\cdot\|_\infty: \begin{cases} M_d(\mathbb{R}) & \rightarrow \mathbb{R}^+ \\ M & \mapsto |PMP^{-1}|_\infty \end{cases}$ defines an algebra norm on $M_d(\mathbb{R})$ and $\|H\|_\infty < 1$. Hence, $I_d - H$ is
230 invertible, which gives (3.3) with $C := |(I - H)^{-1}|$.

$$\begin{aligned} TV_H(R) &= TV_{[0,1]}(R) + |HR(1^-) - R(0^+)| \\ &\leq TV_{[0,1]}(R) + |HR(1^-) - HR(0^+)| + |HR(0^+) - R(0^+)| \\ &\leq TV_{[0,1]}(R) + |H||R(1^-) - R(0^+)| + |H - I_d||R(0^+)| \\ &\leq (1 + |H|)TV_{[0,1]}(R) + |H - I_d||R(0^+)|. \end{aligned}$$

231 Take $x \in [0, 1]$, by the triangle inequality,

$$\begin{aligned} TV_H(R) &\leq (1 + |H|)TV_{[0,1]}(R) + |H - I_d||R(0^+) - R(x)| + |H - I_d||R(x)| \\ &\leq (1 + |H| + |H - I_d|)TV_{[0,1]}(R) + |H - I_d||R(x)|. \end{aligned}$$

232 Integrating with respect to x on $[0, 1]$, one obtains:

$$\begin{aligned} TV_H(R) &\leq (1 + |H| + |H - I_d|)TV(R) + |H - I_d|||R||_{L^1([0,1])} \\ &= C||R||_{BV([0,1])}. \end{aligned}$$

233 where $C = 1 + |H| + |H - I_d|$.

234

235 To get the converse inequality, we remark that by (3.3),

$$|R(1^-)| \leq C|HR(1^-) - R(1^-)|.$$

236 As a consequence,

$$\begin{aligned} ||R||_{BV([0,1])} &= TV_{[0,1]}(R) + ||R||_{L^1([0,1])} \\ &\leq TV_{[0,1]}(R) + |R(1^-)| + ||R - R(1^-)||_{L^1([0,1])} \\ &\leq 2TV_{[0,1]}(R) + C|HR(1^-) - R(1^-)| \\ &\leq 2TV_{[0,1]}(R) + C|HR(1^-) - R(0^+)| + C|R(0^+) - R(1^-)| \\ &\leq (2 + C)TV_{[0,1]}(R) + C|HR(1^-) - R(0^+)| \end{aligned}$$

237 and both norms are equivalent. Concerning the L^∞ estimate (3.2), take a couple $(x, y) \in [0, 1]^2$ and using again the
238 triangle inequality

$$|R(x)| \leq |R(x) - R(y)| + |R(y)| \leq TV_{[0,1]}(R) + |R(y)|.$$

239 Integrating with respect to y on $[0, 1]$, one gets

$$|R(x)| \leq TV_{[0,1]}(R) + ||R||_{L^1([0,1])} = ||R||_{BV([0,1])}.$$

240 And as this is true for all x in $[0, 1]$,

$$||R||_{L^\infty([0,1])} \leq ||R||_{BV([0,1])}.$$

241 The equivalence between the norms $\|\cdot\|_{BV([0,1])}$ and TV_H proven earlier allows to get (3.2). \square

242 **3.1. Lyapunov analysis for the unsaturated system.** In this section, we consider the following system

$$243 \quad (3.4) \quad \begin{cases} R_t + [f(R)]_x & = 0 \\ R(., 0) & = HR(., 1) \\ R(0, .) & = R_0 \in BV([0, 1]) \end{cases}$$

244 where the feedback operator g presented in the introduction is replaced by a matrix $H \in M_d(\mathbb{R})$.

245

246 The main theorem of this section is presented here:

247 **THEOREM 3.3.** *Under Hypothesis 3.1 and if $0 < \gamma < -\log(\rho_\infty(H))$, then the unique entropy solution of (3.4) satisfies*

$$\forall t \geq 0, \|R\|_{BV([0,1])} \leq C e^{-\gamma c_{\min} t} \|R_0\|_{BV([0,1])}$$

248 where $C > 0$ is a constant which does not depend on R_0 and t .

249 A candidate Lyapunov functional first introduced by Glimm [13] and then by Coron et al [7] applies well to piecewise
250 constant functions and is defined by:

251 **DEFINITION 3.4.** *Let R be a piecewise constant function on $[0, 1]$ and taking its values in \mathbb{R}^d . Take $i \in \llbracket 1, d \rrbracket$:*

252 • *We denote $x_{i,1} < x_{i,2} < \dots < x_{i,n_i}$ the discontinuities of R_i (n_i being the number of discontinuities).*

253 • *For all $j \in \llbracket 1, n_i \rrbracket$, $r_{i,j}^l, r_{i,j}^r$ designate the respective left and right state of R_i around $x_{i,j}$.*

254 *The Lyapunov functional \mathcal{L} evaluated at R writes*

$$255 \quad (3.5) \quad \mathcal{L}(R) = \sum_{i=1}^d P_i \sum_{j=1}^{n_i} |r_{i,j}^r - r_{i,j}^l| e^{-\gamma x_{i,j}} + \sum_{i=1}^d P_i |[HR]_i(1^-) - R_i(0^+)|$$

256 where $\gamma > 0$ and $P = \text{diag}\{P_i, i \in \llbracket 1, d \rrbracket\} \in D_d^+(\mathbb{R})$ will be selected later.

257 **Remark 3.5.** Obviously, there exists a constant $C(H, P, \gamma) > 1$ such that for all R piecewise constant, we have:

$$258 \quad (3.6) \quad \frac{\mathcal{L}(R)}{C(H, P, \gamma)} \leq TV_H(R) \leq C(H, P, \gamma) \mathcal{L}(R).$$

259 Theorem 3.3 will be proven using a piecewise approximation of the solution for which the exponential decay of the
260 Lyapunov functional \mathcal{L} will be proven. As a last step, we will pass to the limit.

261 *Proof.* We consider $(R_\nu)_\nu$ an approximating sequence of PCFs of the entropy solution R in the sense of Definition
262 2.2. Such a sequence exists by Theorem 2.3. The following lemma asserts the exponential stability of the approximation:

263 **LEMMA 3.6.** *If $0 < \gamma < -\log(\rho_\infty(H))$. Then, for all $P \in D_d^+(\mathbb{R}^d)$ such that $|PHP^{-1}|_\infty < e^{-\gamma}$, there exists $\tilde{\nu}(P, H, \gamma)$
264 such that*

$$265 \quad (3.7) \quad \forall \nu > \tilde{\nu}, \forall t \geq 0, \mathcal{L}(R_\nu) \leq e^{-\gamma c_{\min} t} \mathcal{L}(R_{0,\nu}) + \frac{1}{2^\nu} \sum_{i=1}^d P_i.$$

266 *Proof.* Fix $\nu > 1$, $P \in D_d^+(\mathbb{R}^d)$ such that $|PHP^{-1}|_\infty < e^{-\gamma}$ and time $0 \leq t \leq 1/c_{\max}$.

267

268 Three cases are to be considered:

269 • (Case 1) If at time t there is no interaction between two fronts nor between a front and the boundary, then $\mathcal{L}(R_\nu)$
270 is differentiable and

$$\begin{aligned} \frac{d\mathcal{L}(R_\nu(t, \cdot))}{dt} &= -\gamma \sum_{i=1}^d P_i \sum_{j=1}^{n_i} \frac{dx_{i,j}}{dt} |r_{i,j}^r - r_{i,j}^l| e^{-\gamma x_{i,j}} \\ &\leq -\gamma c_{\min} \sum_{i=1}^d P_i \sum_{j=1}^{n_i} |r_{i,j}^r - r_{i,j}^l| e^{-\gamma x_{i,j}}. \end{aligned}$$

271 Here, we used the fact that for all integers $i \in \llbracket 1, d \rrbracket$, characteristic velocities $\frac{dx_{i,j}}{dt}$ are bounded from below by
272 $c_{\min} > 0$. Finally, by the definition of $\mathcal{L}(R_\nu(t, \cdot))$

273

$$(3.8) \quad \begin{aligned} \frac{d\mathcal{L}(R_\nu(t, \cdot))}{dt} &\leq -\gamma c_{\min} \mathcal{L}(R_\nu(t, \cdot)) + \gamma c_{\min} \sum_{i=1}^d P_i | [HR_\nu]_i(t, 1^-) - R_{\nu,i}(t, 0^+) | \\ &\leq -\gamma c_{\min} \mathcal{L}(R_\nu(t, \cdot)) + \frac{\gamma c_{\min}}{2^\nu} \sum_{i=1}^d P_i \end{aligned}$$

274

where we used (2.1) with g replaced by H to get last equation.

275

- (Case 2) When a front interaction happens, the total variation is non increasing by construction and as a consequence

276

$$\mathcal{L}(R_\nu(t^+, \cdot)) - \mathcal{L}(R_\nu(t^-, \cdot)) \leq 0.$$

277

Here we used the third point of Definition 2.2.

278

- (Case 3) When an interaction of a front with the boundary happens, computations are a bit more difficult. Suppose that such a front is of type $i \in \llbracket 1, d \rrbracket$ and has $(R_{i,l}, R_{i,r})$ as respective left and right state (see Figure 1). We note its intensity by $I_i := |R_{i,l} - R_{i,r}|$. Note that as R_ν takes its values in $2^{-\nu}\mathbb{Z}$ on the triangle $\{(x, t) \mid 0 < t < x/c_{\max}\}$, we have:

279

280

281

282

$$(3.9) \quad I_i \geq 2^{-\nu}.$$

283

Moreover, recall that simultaneous interactions of fronts with the boundary are forbidden by construction. Using the approximate boundary condition (2.1) with g replaced by the linear operator H , we get

284

285

$$(3.10) \quad \mathcal{L}(R_\nu(t^+, \cdot)) - \mathcal{L}(R_\nu(t^-, \cdot)) \leq \sum_{j=1}^d P_j |H_{j,i}(R_{i,r} - R_{i,l})| - e^{-\gamma} I_i P_i + 2^{-2\nu+2} \sum_{j=1}^d P_j.$$

286

The second term on the right-hand side of (3.10) corresponds to the leaving front (which is of type i). The first term results from the entering fronts at the left boundary. Note that an entering front of type $j \in \llbracket 1, d \rrbracket$ may rather be a fan of fronts (see Figure 1). This is not problematic because the sum of the intensities of the fronts composing the fan is equal to the difference of extremal states of the fan by construction (see Appendix A.2 for details). The last term in (3.10) corresponds to the approximation of the boundary condition (2.1).

287

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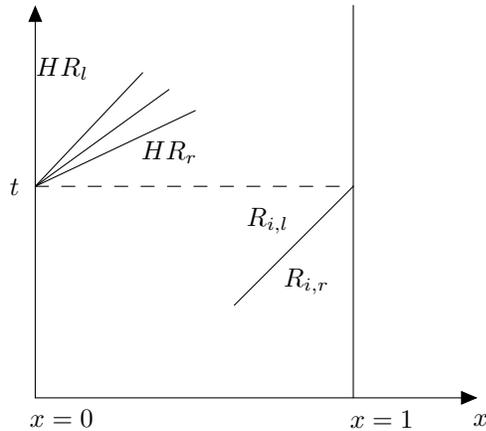


Figure 1: Case 3

291

Then, using the definition of $|\cdot|_\infty$ and (3.9), one gets:

$$\begin{aligned} \mathcal{L}(R_\nu(t^+, \cdot)) - \mathcal{L}(R_\nu(t^-, \cdot)) &\leq \left(\sum_{j=1}^d \frac{P_j}{P_i} |H_{j,i}| - e^{-\gamma} \right) P_i I_i + 2^{-\nu+2} \sum_{j=1}^d P_j I_i \\ &\leq \left(|PH^T P^{-1}|_\infty + 2^{-\nu+2} \sum_{j=1}^d P_j / P_i - e^{-\gamma} \right) P_i I_i. \end{aligned}$$

292 *Remark 3.7.* Here we see why the approximated boundary condition (2.1) is essential. Thanks to it, the error
 293 term $2^{-2\nu+2} \sum_{j=1}^d P_j$ coming from the approximation of g by g_ν can be bounded by the intensity $I_i \geq 2^{-\nu}$ of the
 294 front hitting the right boundary.

295 As $|PH^T P^{-1}|_\infty - e^{-\gamma} < 0$ by assumption, we can take ν sufficiently large say $\nu \geq \tilde{\nu}(P, H, \gamma)$ such that

$$\mathcal{L}(R_\nu(t^+, \cdot)) - \mathcal{L}(R_\nu(t^-, \cdot)) \leq 0$$

296 (Case 2) and (Case 3) can occur only a finite number of times on finite time intervals because R_ν is piecewise constant
 297 in the sense of Definition 2.1. Consequently, one can integrate (3.8) with respect to time to get:

$$\forall 0 \leq t \leq 1/c_{\max}, \mathcal{L}(R_\nu(t, \cdot)) \leq e^{-\gamma c_{\min} t} \mathcal{L}(R_{0,\nu}) + \frac{1}{2^\nu} \sum_{i=1}^d P_i.$$

298 The proof for time $n/c_{\max} \leq t \leq (n+1)/c_{\max}$ where n is an integer can be dealt in a similar way. This ends the proof
 299 of Lemma 3.6. \square

300 Now, we conclude on the proof of Theorem 3.3 taking $t \geq 0$ fixed. By (3.7) and (3.6), there exists a constant $C > 0$
 301 such that

$$\forall \nu > 0, TV_H(R_\nu(t, \cdot)) \leq C \left(e^{-\gamma c_{\min} t} TV_H(R_{0,\nu}) + \frac{1}{2^\nu} \sum_{i=1}^d P_i \right).$$

302 Using the equivalence between the norm TV_H and the norm $\|\cdot\|_{BV([0,1])}$,

$$303 \quad (3.11) \quad \forall \nu > 0, \|R_\nu(t, \cdot)\|_{BV([0,1])} \leq C \left(e^{-\gamma c_{\min} t} \|R_{0,\nu}\|_{BV([0,1])} + \frac{1}{2^\nu} \sum_{i=1}^d P_i \right)$$

304 where we may have changed the constant $C > 0$.

305 As $(R_\nu)_\nu$ is an approximating sequence of PCFs of R , one has:

$$\left\{ \begin{array}{l} \lim_{\nu \rightarrow \infty} R_\nu(0, \cdot) = R_0 \in BV([0, 1]) \\ \forall \tau \geq 0, d\tau > 0, TV_{[0,1]}(R(\tau, \cdot)) \leq \limsup_{\nu \rightarrow \infty} \sup_{s \in [\tau, \tau + d\tau]} TV_{[0,1]}(R_\nu(s, \cdot)). \end{array} \right.$$

307 Moreover, by Remark A.4,

$$\forall \tau \geq 0, \lim_{\nu \rightarrow \infty} \|R_\nu(\tau, \cdot) - R(\tau, \cdot)\|_{L^1([0,1])} = 0.$$

308 We have for all $dt > 0$,

$$\begin{aligned} \|R(t, \cdot)\|_{BV([0,1])} &\leq \limsup_{\nu \rightarrow \infty} \left(\sup_{s \in [t, t+dt]} TV_{[0,1]}(R_\nu(s, \cdot)) + \|R_\nu(t, \cdot)\|_{L^1([0,1])} \right) \\ &\leq \limsup_{\nu \rightarrow \infty} \sup_{s \in [t, t+dt]} \left(TV_{[0,1]}(R_\nu(s, \cdot)) + \|R_\nu(s, \cdot)\|_{L^1([0,1])} \right) \\ &= \limsup_{\nu \rightarrow \infty} \sup_{s \in [t, t+dt]} \|R_\nu(s, \cdot)\|_{BV([0,1])} \\ &\leq C \limsup_{\nu \rightarrow \infty} \left(e^{-\gamma c_{\min} t} \|R_{0,\nu}\|_{BV([0,1])} + \frac{1}{2^\nu} \sum_{i=1}^d P_i \right) \\ &= C e^{-\gamma c_{\min} t} \|R_0\|_{BV([0,1])} \end{aligned}$$

309 where we have used (3.11) to get the fourth equation.

310

311 This finishes the proof of Theorem 3.3. \square

312 **3.2. Stability analysis for the saturated system.** In this section, we consider the following system

$$313 \quad (3.12) \quad \begin{cases} R_t + [f(R)]_x &= 0 \\ R(\cdot, 0) &= [H \cdot + B\sigma(K \cdot)]R(\cdot, 1) \\ R(0, \cdot) &= R_0 \in BV([0, 1]). \end{cases}$$

314 We introduce the deadzone function defined by

315 (3.13) $\forall R \in \mathbb{R}^d, \phi(R) = \sigma(R) - R$

316 and Hypothesis 3.8:

317 *Hypothesis 3.8.* The matrices H, B, K are chosen such that:

$$\rho_\infty(H + BK) < 1.$$

318

319 Here the main result is different since we prove local exponential stability (Proposition 3.10). We cannot directly
320 study the problem of BV stability because of the lack of contractivity of the saturation σ .

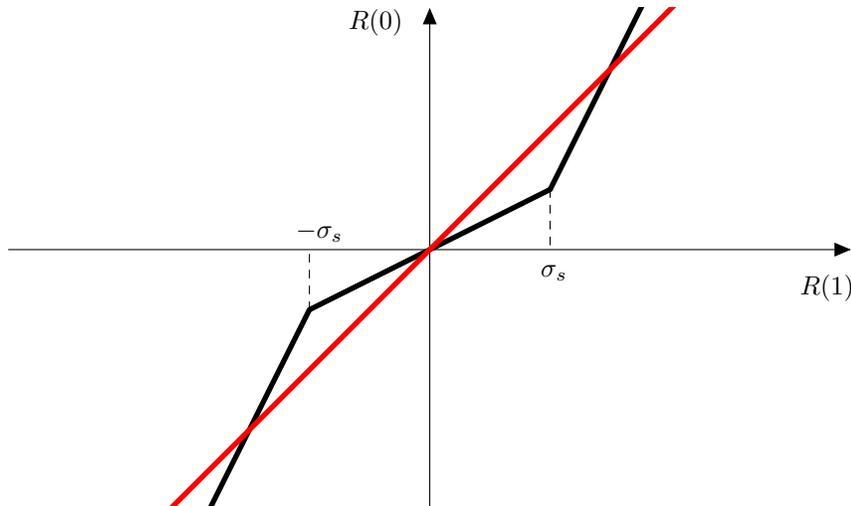


Figure 2: The feedback operator (black line) compared with the graph of the function $R(0) = R(1)$ (red line)

321 *Motivating example 3.9.* In Figure 2, we represent the boundary operator $H \cdot + B\sigma(K \cdot)$ for $d = 1, H = 2, B = 1,$
322 $K = -1.5$ and $\sigma_s = 2$. Except for the zone of linearity, the boundary operator is only 2-Lipschitz. As a consequence,
323 it is possible to construct a front whose left/right states are arbitrary close to the zone of linearity and whose intensity
324 increases after a passage through the feedback operator. This is why it is not possible to get a basin of attraction in BV
325 norm larger than the zone of linearity. We will rather prove the L^∞ local stability with a basin of attraction in L^∞ .

326 This section is devoted to the proof of the following proposition and theorem (the definition of $\|\cdot\|_{\infty, P}$ is given in the
327 section notation):

328 **PROPOSITION 3.10.** *Under Hypothesis 3.8, if $0 < \gamma < -\log(\rho_\infty(H + BK))$. Then, for all $P \in D_d^+(\mathbb{R}^d)$ such that
329 $|P(H + BK)P^{-1}|_\infty \leq e^{-\gamma}$, there exists a constant C depending on (H, B, K, P, γ) such that if*

330 (3.14)
$$\|R_0\|_{\infty, P} < \frac{|PBP^{-1}|_\infty P_{\min} \sigma_s}{|P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma}}.$$

331 *Then, the unique entropy solution $R \in L_{loc}^\infty(\mathbb{R}^+, BV([0, 1]))$ of (3.12) satisfies,*

332 (3.15)
$$\forall t \geq 0, \|R(t, \cdot)\|_{L^\infty([0, 1])} \leq Ce^{-\gamma e_{\min} t} \|R_0\|_{L^\infty([0, 1])}$$

333 *where C depends on the parameters of the problem but not R_0 .*

334 For cases where $\rho_\infty(H) > 1$, the denominator in (3.14) is not zero:

335 *Remark 3.11.* If $\rho_\infty(H) > e^{-\gamma}$, then we claim that for all $P \in D_d^+(\mathbb{R})$:

$$|P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma} > 0.$$

336 **Proof of the claim of Remark 3.11**

337

338 Let P be in $D_d^+(\mathbb{R})$. As $\rho_\infty(H) > e^{-\gamma}$,

$$|PHP^{-1}|_\infty > e^{-\gamma}.$$

339 This gives by the triangle inequality:

$$|P(H + BK)P^{-1}|_\infty + |PBKP^{-1}|_\infty > e^{-\gamma}.$$

Finally, by the fact that

$$\forall A, B \in M_d(\mathbb{R}), |PABP^{-1}|_\infty \leq |PAP^{-1}|_\infty |PBP^{-1}|_\infty,$$

we have:

$$|P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PBP^{-1}|_\infty > e^{-\gamma}$$

340 and the claim is proven.

341 The following theorem is a consequence of Proposition 3.10.

342 **THEOREM 3.12.** *Under the conditions of Proposition 3.10,*

$$\forall t \geq 0, \|R(t, \cdot)\|_{BV([0,1])} \leq Ce^{-\gamma c_{\min} t} \|R_0\|_{BV([0,1])}$$

343 where C depends on the parameters of the problem but not R_0 .

344 Let us assume for the time being Proposition 3.10 and prove Theorem 3.12:

345 *Proof of Theorem 3.12.* Equation (3.15) implies that at a certain time denoted t^* depending on $\|R_0\|_{L^\infty([0,1])}$, the
346 solution enters in the zone of linearity and stays in it. Then, Theorem 3.3 implies:

$$347 \quad (3.16) \quad \forall t \geq t^*, \|R(t, \cdot)\|_{BV([0,1])} \leq Ce^{-\gamma c_{\min}(t-t^*)} \|R(t^*, \cdot)\|_{BV([0,1])}$$

348 where C depends on $H, B, K, P, \gamma, \sigma_s$.

349

350 Then, for $t \leq t^*$, one can prove using the same techniques from Section 3.1 that:

$$351 \quad (3.17) \quad \forall 0 \leq t \leq t^*, \|R(t, \cdot)\|_{BV([0,1])} \leq e^{\nu t} \|R_0\|_{BV([0,1])}$$

352 where $\nu > 0$ is a constant depending on c_{\max}, γ and a Lipschitz constant of the feedback operator $H + B\sigma(K)$.

353

354 From (3.17) and (3.16), one gets:

$$\forall t \geq 0, \|R(t, \cdot)\|_{BV([0,1])} \leq Ce^{-\gamma c_{\min} t} \|R_0\|_{BV([0,1])}$$

355 where C depends on the parameters of the problem and on $\|R_0\|_{L^\infty([0,1])}$.

356

357 As we have the bound (3.14), we can conclude that C does not depend on $\|R_0\|_{L^\infty([0,1])}$ and the corollary is proven. \square

358 The following lemma will be useful for the proof of Proposition 3.10.

359 **LEMMA 3.13.** *Let $R \in \mathbb{R}^d$ be such that:*

$$360 \quad (3.18) \quad |PR|_\infty \leq \frac{|PBP^{-1}|_\infty P_{\min} \sigma_s}{\left| |P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma} \right|}.$$

361 Then,

$$|P(HR + B\sigma(KR))|_\infty \leq e^{-\gamma} |PR|_\infty.$$

362 *Proof.* Let i be in $\llbracket 1, d \rrbracket$. If $\text{sat}_i(R) := \{j \in \llbracket 1, d \rrbracket, \mid |[KR]_j| > \sigma_s \text{ and } B_{i,j} \neq 0\}$ is empty, then:

$$\begin{aligned} P_i |HR + B\sigma(KR)|_i &= P_i |(H + BK)R|_i \\ &\leq |P(H + BK)P^{-1}|_\infty |PR|_\infty \\ &\leq e^{-\gamma} |PR|_\infty. \end{aligned}$$

If the set $\text{sat}_i(R)$ is not empty, then:

$$\begin{aligned}
P_i|HR + B\sigma(KR)|_i &= P_i|(H + BK)R + B\phi(KR)|_i \\
&\leq \sum_{j=1}^d P_i|(H + BK)_{i,j}R_j| + \sum_{j \in \text{sat}_i(R)} P_i|B_{i,j}|(|[KR]_j| - \sigma_s) \\
&\leq \sum_{j=1}^d P_i|(H + BK)_{i,j} \frac{1}{P_j} P_j R_j| + \sum_{j \in \text{sat}_i(R)} P_i|B_{i,j}| \frac{P_j}{P_j} (|[KR]_j| - \sigma_s) \\
&\leq |P(H + BK)P^{-1}|_\infty |PR|_\infty + |PBP^{-1}|_\infty (|PKP^{-1}|_\infty |PR|_\infty - P_{\min}\sigma_s) \\
&\leq e^{-\gamma} |PR|_\infty
\end{aligned}$$

363 where we have used the hypothesis (3.18) to get the last inequality. □

364 Now we can focus on the proof of Proposition 3.10.

365 *Proof of Proposition 3.10.* We take $P \in D_d^+(\mathbb{R})$ such that $|P(H + BK)P^{-1}|_\infty < e^{-\gamma}$ and $R_0 \in BV([0, 1])$ satisfying
366 (3.14). We consider $(R_\nu)_\nu$ an approximating sequence of PCFs of the entropy solution R in the sense of Definition 2.2.
367 Such a sequence exists because of Theorem 2.3. Then, we analyze the exponential damping of R_ν for a fixed $\nu > 1$. As
368 $(R_{0,\nu})_\nu$ converges towards R_0 in $BV([0, 1])$, we have for ν sufficiently large:

$$369 \quad (3.19) \quad \|R_{0,\nu}\|_{\infty,P} \leq \frac{|PBP^{-1}|_\infty P_{\min}\sigma_s}{|P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma}}$$

370 We first recall the definition of c_{\min}, c_{\max} the respective minimum and maximum velocity, in (1.6). Let $t \leq 1/c_{\min}$
371 and $x > Lt$ be in $[0, 1]$. Constructing the light cone enclosed by line with slopes $1/c_{\min}$ and $1/c_{\max}$ and passing through
372 (t, x) , we get that:

$$373 \quad (3.20) \quad |PR_\nu(t, x)|_\infty \leq \|R_{0,\nu}\|_{\infty,P}$$

374 The argument of the light cone can be justified by the fact that the L^∞ norm does not increase by fronts interaction (see
375 Appendix A.2.3) and because fronts velocities belongs to $[c_{\min}, c_{\max}]$.

376

377 When $x \leq Lt$, constructing the light cone enclosed by line with slopes $1/c_{\min}$ and $1/c_{\max}$ and passing through (t, x) ,
378 one gets:

$$|PR_\nu(t, x)|_\infty \leq \max\{\|R_{0,\nu}\|_{\infty,P}, \sup_{t \in [0, 1/c_{\min}]} |PR_\nu(t, 0)|\}.$$

379 The boundary condition gives:

$$|PR_\nu(t, x)|_\infty \leq \max\{\|R_{0,\nu}\|_{\infty,P}, \sup_{t \in [0, 1/c_{\min}]} |P[H \cdot + B\sigma(K \cdot)]R_\nu(t, 1)|\}.$$

380 By (3.20) applied to $x = 1$ and (3.19), hypothesis of Lemma 3.13 are verified and we have:

$$|PR_\nu(t, x)|_\infty \leq \max\{\|R_{0,\nu}\|_{\infty,P}, e^{-\gamma}\|R_{0,\nu}\|_{\infty,P}\} \leq \|R_{0,\nu}\|_{\infty,P}.$$

381 Next we proceed by induction on intervals of the form $t \in [n/c_{\min}, (n+1)/c_{\min}]$ with $n \in \mathbb{N}$. Suppose that:

$$\forall t \in [n/c_{\min}, (n+1)/c_{\min}], \|R_\nu(t, \cdot)\|_{\infty,P} \leq e^{-\gamma n} \|R_{0,\nu}\|_{\infty,P}.$$

382 Let $(n+1)/c_{\min} \leq t \leq (n+2)/c_{\min}$ and x be in $[0, 1]$. Constructing the light cone enclosed by lines with slopes $1/c_{\min}$
383 and $1/c_{\max}$ and passing through (t, x) , one gets the existence of a $t^* \in [n/c_{\min}, (n+2)/c_{\min}]$ such that:

$$384 \quad (3.21) \quad |PR_\nu(t, x)|_\infty \leq |PR_\nu(t^*, 0)|_\infty \leq |P[H \cdot + B\sigma(K \cdot)]R_\nu(t^*, 1)|.$$

385 Using same reasoning as in the case $n = 0$, one proves that:

$$\|R_\nu(t^*, \cdot)\|_{\infty,P} \leq \|R_\nu(n/c_{\min}, \cdot)\|_{\infty,P}.$$

Hence, by the hypothesis of induction:

$$|PR_\nu(t^*, 1)|_\infty \leq \|R_\nu(t^*, \cdot)\|_{\infty,P} \leq e^{-\gamma n} \|R_{0,\nu}\|_{\infty,P} \leq \|R_{0,\nu}\|_{\infty,P}.$$

386 As a consequence, by (3.19):

$$|PR_\nu(t^*, 1)|_\infty \leq \frac{|PBP^{-1}|_\infty P_{\min} \sigma_s}{|P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma}}$$

387 Thus, we can use Lemma 3.13 in (3.21) to get:

$$\begin{aligned} |PR_\nu(t, x)|_\infty &\leq e^{-\gamma} |PR_\nu(t^*, 1)|_\infty \\ &\leq e^{-\gamma} \|R_\nu(t^*, \cdot)\|_{\infty, P}. \end{aligned}$$

388 Hence,

$$\|R_\nu(t, \cdot)\|_{\infty, P} \leq e^{-\gamma} \|R_\nu(t^*, \cdot)\|_{\infty, P} \leq e^{-\gamma(n+1)} \|R_{0, \nu}\|_{\infty, P}$$

389 where we have used the induction hypothesis.

390

391 To conclude, we have:

$$\forall t \geq 0, \|R_\nu(t, \cdot)\|_{\infty, P} \leq \max\{1, e^{-\gamma(c_{\min} t - 1)}\} \|R_{0, \nu}\|_{\infty, P}.$$

392 It remains to prove the exponential decay for the solution R . It suffices to use property (2.3) and to take a sequence
393 of initial data piecewise constant such that:

$$\forall \nu > 1, \|R_{0, \nu}\|_{\infty, P} \leq \|R_0\|_{\infty, P}.$$

394 Owing this, we can pass to the limit as ν goes to infinity to get:

$$\forall t \geq 0, \|R(t, \cdot)\|_{\infty, P} \leq \max\{1, e^{-\gamma(c_{\min} t - 1)}\} \|R_0\|_{\infty, P}.$$

395 This ends the proof of Proposition 3.10. □

396 **3.3. Numerical results.** Here, we study a numerical example with saturation and show the relevance of the esti-
397 mation of the region of attraction (3.14).

398 **3.3.1. Relevance of the estimation of the basin of attraction.** In this section, we analyze an example of
399 network of scalar conservation laws for $d = 2$ with saturated feedback control law. Matrices are defined as follows.

$$H = \begin{pmatrix} 0 & 1.1 \\ 1 & 0 \end{pmatrix}, \quad B = I_2, \quad K = \begin{pmatrix} 0 & -0.1050 \\ -0.1045 & 0 \end{pmatrix}.$$

400 We take a linear flux $f(R) = \Lambda R + 0.2(\arctan(R_1), \arctan(R_2))$ with

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

401 We recall the estimation of the basin of attraction for $\gamma > 0$ and $P \in D_d^+(\mathbb{R})$:

$$(3.22) \quad \|R_0\|_{\infty, P} \leq \frac{|PBP^{-1}|_\infty \sigma_s}{|P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - e^{-\gamma}}.$$

403 We calculate $P \in D_d^+(\mathbb{R})$ such that $|P(H + BK)P^{-1}|_\infty$ is minimal. We obtain:

$$P = \begin{pmatrix} 0.974 & 0 \\ 0 & 1.026 \end{pmatrix}.$$

404 To estimate the largest region of attraction, we take $\gamma = 0$ in (3.22) which gives the following criteria of stability:

$$(3.23) \quad \|R_0\|_{\infty, P} \leq \frac{|PBP^{-1}|_\infty \sigma_s}{|P(H + BK)P^{-1}|_\infty + |PBP^{-1}|_\infty |PKP^{-1}|_\infty - 1}.$$

406 **3.3.2. Numerical simulations.** Still keeping the matrices from previous section, we take a certain range of initial
 407 data R_0 constant on $[0, 1]$ belonging to the estimated region of attraction and simulate the behavior of the solution. For
 408 example, one can take R_0 constant with value in $(-40, 40)^2$ and look if the solution does not blow up at infinite time in
 409 BV norm. For information, we have used the classical superbee limiter scheme for our simulations. In the same graph,
 410 we plot the estimated region of attraction given by (3.23) and the region where the solution saturates at $t = 0$.

411 In Figure 3, contours correspond to the rate of exponential decay of the numerical solution. If it is negative, the
 412 solution decays exponentially in norm. If it is positive, we have exponential divergence. The largest square is the estimated
 413 region of attraction while the smallest one is the zone where saturation does not occur at $t = 0$. We also pick a initial
 414 data R_0 in the estimated region of attraction and observe the dynamic of the solution. For example, one can take
 415 $R_0(x) = (15, -15)$ on $[0, 1]$. The black dot in Figure 3 corresponds to this initial data. The value of the control is plotted
 416 in Figure 5 while the BV norm of the solution is given in Figure 4. We see that there is indeed saturation from $t = 0$
 417 until time $t \approx 13$, then we enter in the zone of linearity.

418 The initial jump of the BV norm is due to boundary condition which immediately creates a jump at $x = 0$ when
 419 $t = 0^+$. Here the BV norm behaves well in the zone of saturation *i.e* it does not exponentially blow and even decay quite
 420 fast. Then, in the zone of linearity we recover the exponential decay pattern.

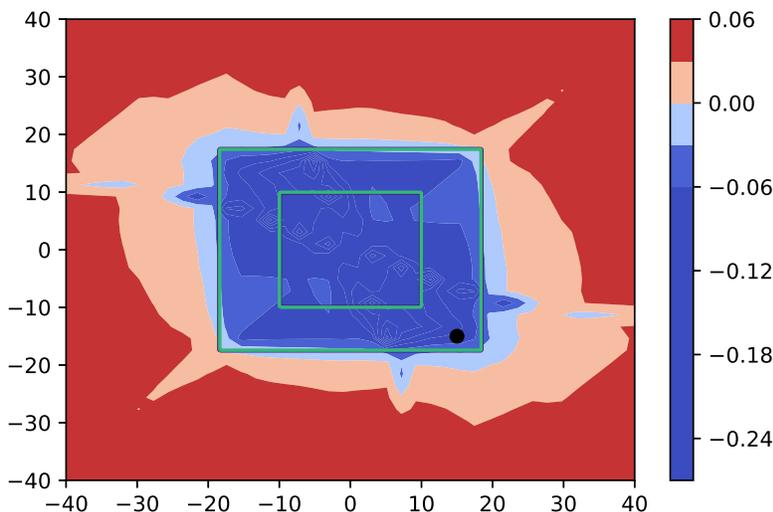


Figure 3: The basin of attraction

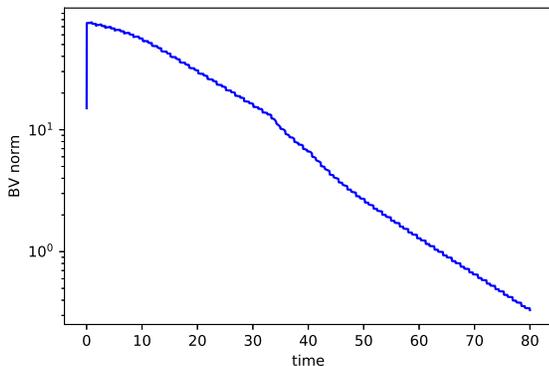


Figure 4: The BV norm of the solution

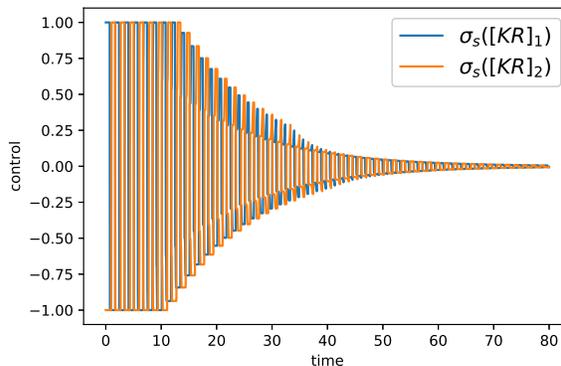


Figure 5: The control

421 **4. Conclusion.** The well-posedness for a wide class of systems of scalar conservation laws with boundary unsaturated
 422 and saturated feedback laws was established. The ρ_∞ criteria was established in the BV context for linear feedback laws.

423 Then, for saturated feedback laws, we proved with an example that estimating a basin of attraction in BV was not
424 relevant. We rather gave an estimation of the basin of attraction in L^∞ and deduce the exponential decay of the BV
425 norm of solutions whose initial data belongs to this basin of attraction. Some questions remain open. The estimation
426 (3.14) may not be optimal. Moreover, a method of maximizing the basin of attraction where the matrix K is the variable
427 of optimization is not given in this article. This is not an easy task since criterion (3.14) is not convex with respect to
428 K . Finally, the other big gap to bridge is the stabilization of general systems of conservation laws the main difficulty
429 coming from the well-posedness. The initial-boundary value problem for hyperbolic systems of conservation laws is indeed
430 a very delicate matter, even when no characteristic speed vanishes. We refer to [20], [1], [5], [2], [11], [7] and the references
431 therein.

432 **Appendix A. Existence of a solution.** All this section is dedicated to the proof of the existence result of Theorem
 433 **2.3.**
 434

A.1. The approximated problem. Let $\nu \geq 1$, i in $\llbracket 1, d \rrbracket$ and define $f_{\nu,i}$ the piecewise affine approximation of f_i coinciding with f_i at all $2^{-\nu}j$ nodes ($j \in \mathbb{Z}$) by:

$$f_{\nu,i}(s) = \frac{s - 2^{-\nu}j}{2^{-\nu}} f_i(2^{-\nu}(j+1)) + \frac{2^{-\nu}(j+1) - s}{2^{-\nu}} f_i(2^{-\nu}j) \text{ for } s \in [2^{-\nu}j, 2^{-\nu}(j+1)].$$

435 The sequence $(f_\nu)_\nu$ is introduced in order to construct a piecewise constant entropy solution. The following lemma
 436 gives its main properties:

437 **LEMMA A.1.** *For all $T > 0$, there exists a constant $C(g, T)$ such that for all $\nu > 1$ and $R_{0,\nu}$ piecewise constant taking
 438 its values in $2^{-\nu}\mathbb{Z}$, there exists R_ν piecewise constant in the sense of Definition 2.1 verifying the following assertions:*

- 439 • *The approximated boundary condition (2.1) is verified.*
- 440 • *Two fronts cannot interact simultaneously with the right boundary.*
- 441 • $\forall k \in \mathbb{R}^d, \phi \in C_c^1((0, T) \times (0, 1); \mathbb{R})$:

$$(A.1) \quad \int_0^T \int_0^1 \eta_k(R_\nu) \phi_t + q_k(R_\nu) \phi_x dx dt \geq -C(g, T) \frac{TV(R_{0,\nu})}{\nu} \|\phi\|_{L^\infty(\mathbb{R}^+ \times [0,1])}.$$

443 where :

$$(A.2) \quad \begin{cases} \eta_k(R_\nu(t, x)) &= \sum_{i=1}^d |R_{\nu,i}(t, x) - k_i| \\ q_k(R_\nu(t, x)) &= \sum_{i=1}^d |f_i(R_{\nu,i}(t, x)) - f_i(k_i)|. \end{cases}$$

- 445 • *The following bounds hold:*

$$\forall t \leq T, TV_{[0,1]}(R_\nu(t, \cdot)) \leq C(g, T) TV_{[0,1]}(R_{0,\nu}).$$

$$(A.3) \quad \forall t \leq T, \|R_\nu(t, \cdot)\|_{L^\infty([0,1])} \leq C(g, T) \|R_{0,\nu}(t, \cdot)\|_{L^\infty([0,1])}.$$

447 *Proof.* See Appendix A.2 □

448 It is relatively easy to construct piecewise constant functions that make the entropy decrease. The main rules of
 449 construction are presented in the following lemma.

450 **LEMMA A.2** (Characterization of entropy piecewise constant functions).

451 *A piecewise constant R in the sense of Definition 2.1 verifies the condition of entropy decay (1.11) if and only if for
 452 all integers i in $\llbracket 1, d \rrbracket$ and all fronts $\gamma(t)$ of R_i ,*

- 453 • *The Rankine-Hugoniot condition holds for R_i^l the left state and R_i^r the right state:*

$$(A.4) \quad \dot{\gamma}(t)[R_i^r(t) - R_i^l(t)] = f_i(R_i^r(t)) - f_i(R_i^l(t)).$$

- 455 • *If $R_i^l(t) < R_i^r(t)$ then:*

$$(A.5) \quad \forall \alpha \in [0, 1], f_i(\alpha R_i^r(t) + (1 - \alpha)R_i^l(t)) \geq \alpha f_i(R_i^r(t)) + (1 - \alpha)f_i(R_i^l(t)).$$

- 457 • *If $R_i^l(t) > R_i^r(t)$ then:*

$$(A.6) \quad \forall \alpha \in [0, 1], f_i(\alpha R_i^r(t) + (1 - \alpha)R_i^l(t)) \leq \alpha f_i(R_i^r(t)) + (1 - \alpha)f_i(R_i^l(t)).$$

459 *Proof.* This corresponds to [4, Theorem 4.4]. □

460 We call conditions (A.5)-(A.6), the entropy decay conditions of fronts; it selects values before and after the front such
 461 that the entropy of the solution decreases with time. If a front verifies such conditions, we say that the front is **entropic**.
 462

463 **A.2. Proof of Lemma A.1.** Now we prove Lemma A.1 constructing step by step a piecewise constant solution. We
 464 begin by solving a Riemann problem to get a solution near $t = 0$.

465 **A.2.1. The Riemann problem.** Let i be an integer of $\llbracket 1, d \rrbracket$, $\nu > 1$ and $R_i^l, R_i^r \in \mathbb{R}$ be two states. We recall
 466 techniques from [4, pp .108-113] to solve the Riemann problem associated to (R_i^l, R_i^r) when taking f_ν as flux. There are
 467 two cases to consider:

- 468 • If $R_i^l < R_i^r$. Then, we consider f_i^* the largest convex function inferior to $f_{\nu,i}$ on $[R_i^r, R_i^l]$. Denote also $w_0 := R_i^l <$
 469 $w_1 < w_2 < \dots < w_n := R_i^r$ the states where $f_i^{*,\prime}$ jumps. We give an example for $n = 2$ on Figure 6:

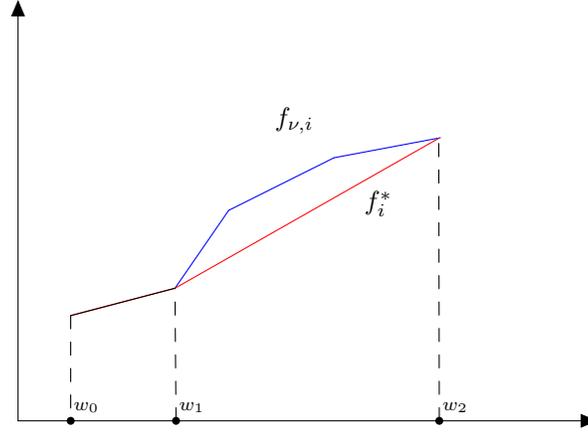


Figure 6: The case $R_i^l < R_i^r$

470 Introducing the speeds

471 (A.7)
$$\lambda_l = \frac{f_{\nu,i}(w_l) - f_{\nu,i}(w_{l-1})}{w_l - w_{l-1}}, \quad l \in \llbracket 1, n \rrbracket,$$

472 we define the solution to the Riemann problem as:

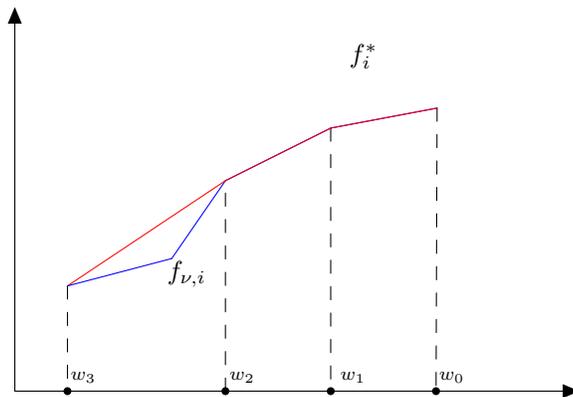
473 (A.8)
$$R_{\nu,i}(t, x) = \begin{cases} w_0 & \text{if } x < t\lambda_1 \\ w_l & \text{if } t\lambda_l < x < t\lambda_{l+1}, \quad l \in \llbracket 1, n-1 \rrbracket \\ w_n & \text{if } x > t\lambda_n. \end{cases}$$

474 This solution is entropic because it is piecewise constant, all fronts are entropic (A.5) and satisfy the Rankine-
 475 Hugoniot condition (A.4).

- 476 • If $R_i^l > R_i^r$. Then, we consider f_i^* the smallest concave function larger than $f_{\nu,i}$. Denote also $w_0 := R_i^l > w_1 >$
 477 $w_2 > \dots > w_n := R_i^r$ the states where $f_i^{*,\prime}$ jumps. We give an example for $n = 2$ on the Figure 7.

478 Defining velocities $(\lambda_l)_{l \in \llbracket 1, q \rrbracket}$ as in (A.7), we define the local solution also as:

479 (A.9)
$$R_{\nu,i}(t, x) = \begin{cases} w_0 & \text{if } x < t\lambda_1 \\ w_l & \text{if } t\lambda_l < x < t\lambda_{l+1}, \quad l \in \llbracket 1, n-1 \rrbracket \\ w_n & \text{if } x > t\lambda_n. \end{cases}$$

Figure 7: The case $R_i^l > R_i^r$

480 **A.2.2. Local in time solution.** Take a fixed $\nu > 1$. Let us define what we will call the limit line $t \mapsto c_{\max}t$ with
 481 maximal speed. Thanks to the Riemann solver defined in the previous section, we can find an entropy solution until a
 482 front interaction happens. The corresponding picture is given in Figure 8.

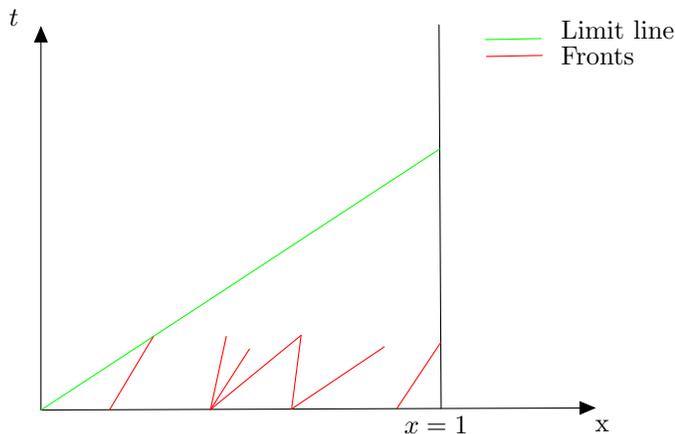


Figure 8: The local in time solution

483 **A.2.3. Dealing with shock interactions.** We recall the method described in [4, pp. 111-112]. Two cases have to
 484 be considered:

- 485 • (Case 1) All the incoming jumps have the same sign. Suppose they are all positive and let us denote $w_0 < w_1 <$
 486 $\dots < w_n$ ($n \in \mathbb{N}$) the consecutive “incoming” states. As all incoming fronts are entropic, we have:

$$487 \quad (\text{A.10}) \quad \forall \alpha \in [0, 1], f(\alpha w_i + (1 - \alpha)w_{i+1}) \geq \alpha f(w_i) + (1 - \alpha)f(w_{i+1}).$$

488 The fact that we have converging fronts gives that the function h built from lines passing through points
 489 $(w_i, f(w_i))_{i \in \llbracket 0, n \rrbracket}$ is concave. Moreover, by (A.10):

$$490 \quad (\text{A.11}) \quad \forall w \in [w_0, w_n], h(w) \leq f(w).$$

491 Hence, by the concavity of h and (A.11):

$$\forall \alpha \in [0, 1], f(\alpha w_0 + (1 - \alpha)w_n) \geq h(\alpha w_0 + (1 - \alpha)w_n) \geq \alpha f(w_0) + (1 - \alpha)f(w_n).$$

492
493
494

Thus, it is possible to link the extremal (w_0 and w_n) states by a unique entropic front whose jump intensity is strictly equal to the sum of the intensities of incoming jumps. Hence, in this case the total variation is conserved and so is the L^∞ norm.

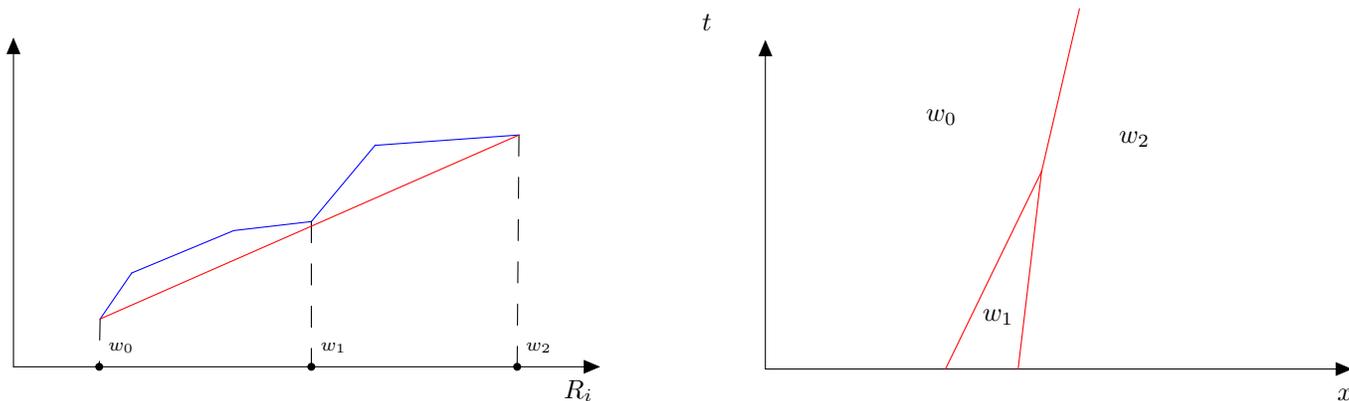


Figure 9: All jumps have the same sign

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499

- (Case 2) Not all jumps have the same sign. Let us denote w_0, w_1, \dots, w_n ($n \in \mathbb{N}$) the consecutive “incoming” states. It is possible to link the extremal (w_0 and w_n) states using fronts whose jumps have the same sign. To do so, it suffices to solve a Riemann problem between extremal states w_0 and w_n as in Section A.2.1. Moreover, by the triangle inequality, the total variation decreases at least by $2 \times 2^{-\nu}$. Concerning the L^∞ norm, it is conserved. This is because the $f_{\nu,i}$ s are non decreasing.

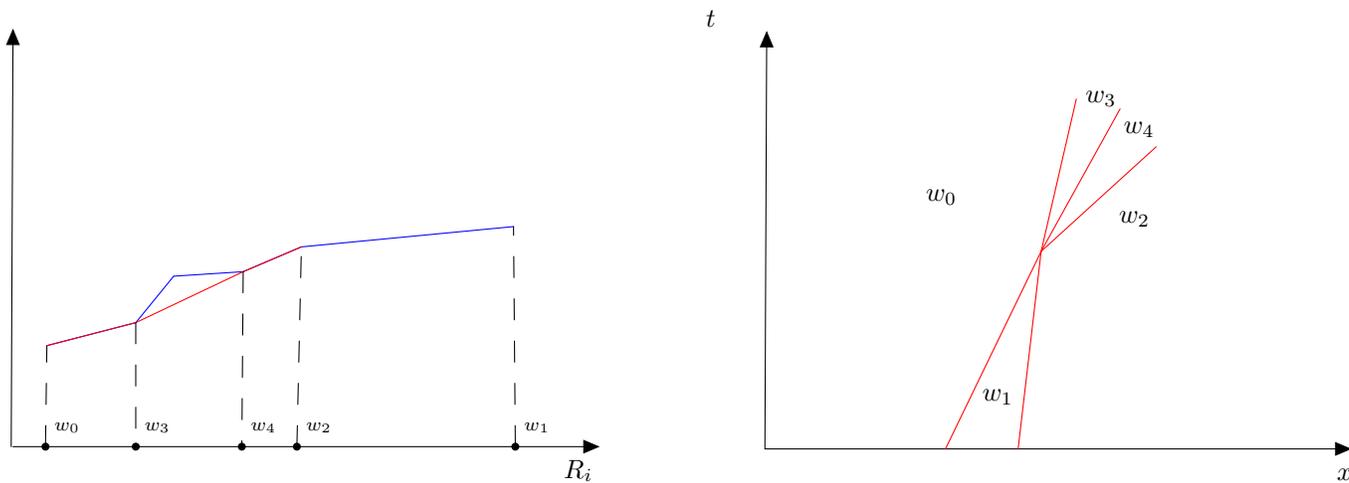


Figure 10: Not all jumps have the same sign

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503

(Case 1) creates a unique front and let the total variation unchanged whereas (Case 2) can create several fronts but the total variation decreases by at least by $2 \times 2^{-\nu}$. Consequently, (Case 2) can happen only a finite number of times and the number of fronts remains bounded as time evolves. As a consequence, it is possible to construct an entropy piecewise constant approximate solution under the limit line verifying

$$(A.12) \quad \begin{cases} \forall 0 \leq t \leq 1/c_{\max}, TV_{[c_{\max}t, 1]}(R_{\nu,i}(t, \cdot)) \leq TV_{[0, 1]}(R_{0,\nu,i}) \\ \forall x \in [0, 1], TV_{[0, x/c_{\max}]}(R_{\nu,i}(\cdot, x)) \leq TV_{[0, 1]}(R_{0,\nu,i}) \end{cases}$$

505

The corresponding picture is given in Figure 11:

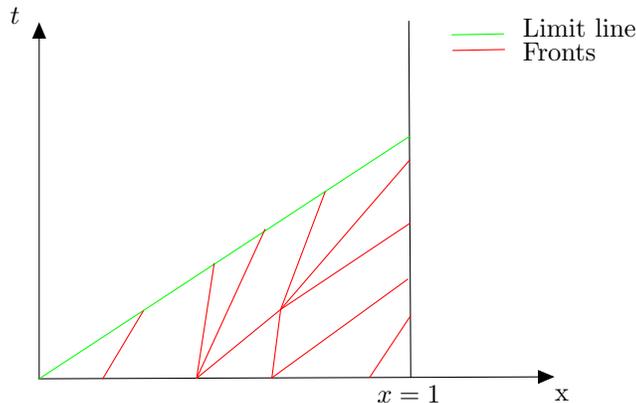


Figure 11: The solution under the limit line

506 *Remark A.3.* If $n \geq 2$ ($n \in \mathbb{N}$) fronts interacts exactly at $x = 1$ for some time $t > 0$. Then, we modify a bit the
 507 velocity of $n - 1$ fronts to prevent this situation. Taking one of such fronts, we denote λ and $\tilde{\lambda}$ the respective former and
 508 new velocities. We can choose them such that $|\lambda - \tilde{\lambda}| \leq \frac{1}{\nu}$.

509 **A.2.4. Finishing the construction.** To construct locally the solution above the limit line, we impose the boundary
 510 condition

$$\forall 0 \leq t \leq 1/c_{\max}, R_\nu(t, 0^+) = g_{2\nu}(R_\nu(t, 1^-))$$

511 where we recall that:

$$\forall R \in \mathbb{R}^d, \nu > 1, i \in \llbracket 1, d \rrbracket, g_{\nu,i}(R) = 2^{-\nu}(E(2^\nu g_i(R))).$$

512 Then, to construct a local solution, we solve the different Riemann problems as in section A.2.2 this time using
 513 the approximated flux $f_{2\nu}$. More precisely if at a time t , $R_i(t^-, 0^+) \neq R_i(t^+, 0^+)$, we solve the Riemann problem with
 514 $R_i(t^+, 0^+)$ as left state and $R_i(t^-, 0^+)$ as right state. Hence we are able to get a solution locally above the limit line taking
 515 its values in $2^{-2\nu}\mathbb{Z}$.

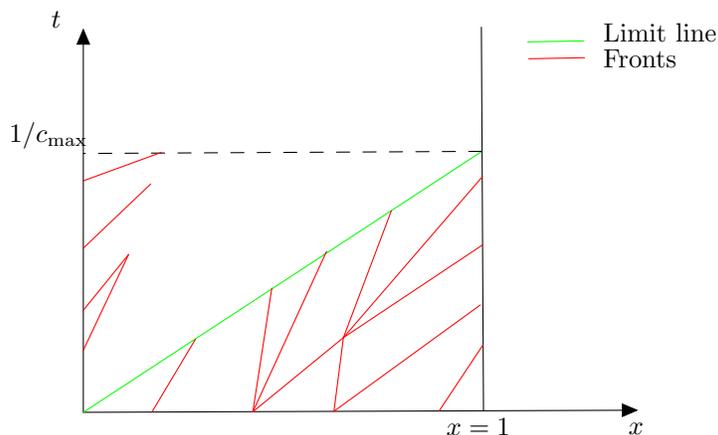


Figure 12: The local solution above the limit line

516 Finally, we extend fronts coming from the zone under the limit line and deal with front interactions as in section A.2.3
 517 this time using the approximated flux $f_{2\nu}$. The final picture is given in Figure 13. This is very important to remark that
 518 the picture under the limit line cannot be modified by fronts coming from the left boundary. This is because the limit
 519 line has maximal velocity.
 520

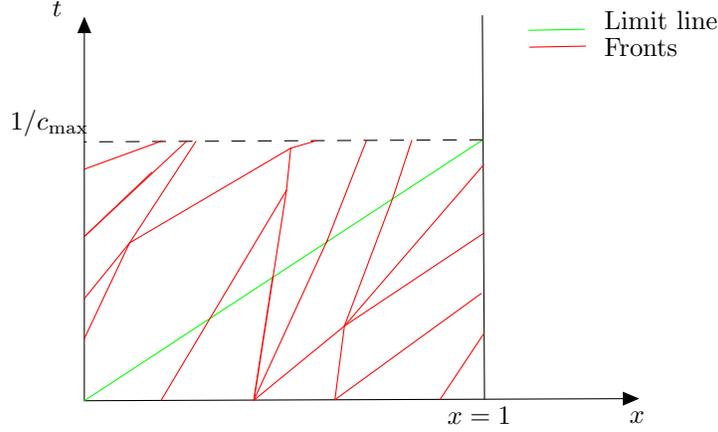


Figure 13: The final solution

521 Concerning the total variation, the way we dealt with front interactions prevents the total variation from increasing
 522 when we compare the total strength of ongoing fronts with the one of outgoing fronts. As a consequence,

$$523 \quad (\text{A.13}) \quad \begin{cases} \forall 0 \leq t \leq 1/c_{\max}, \quad TV_{[0, c_{\max}t]}(R_{\nu, i}(t, \cdot)) \leq TV_{[0, 1]}(R_{0, \nu, i}) + TV_{[0, 1/c_{\max}]}(R_{\nu, i}(\cdot, 0^+)) \\ \forall x \in [0, 1], \quad TV_{[x/c_{\max}, 1/c_{\max}]}(R_{\nu, i}(\cdot, x)) \leq TV_{[0, 1]}(R_{0, \nu, i}) + TV_{[0, 1/c_{\max}]}(R_{\nu, i}(\cdot, 0^+)) \end{cases}$$

524 **A.2.5. Conclusion.** All previous steps can be repeated on intervals $[k/c_{\max}, (k+1)/c_{\max}]$ and a solution defined for
 525 all time is built. Now let $T > 0$. There are several points to verify:

- 526 • (Boundary condition) The approximated boundary condition (2.1) is satisfied by construction.
- 527 • (Boundary interactions) Two fronts cannot interact simultaneously at the right boundary by construction.
- 528 • (Estimate on the total variation). Using (A.12), (A.13) and the fact that R_ν satisfies the approximated boundary
 529 condition (2.1), one can deduce that there exists a constant $C(g, T)$ (depending on the Lipschitz constant of g
 530 and T) such that

$$531 \quad (\text{A.14}) \quad \forall 0 \leq t \leq T, \quad TV_{[0, 1]}(R_\nu(t, \cdot)) \leq C(g, T)TV_{[0, 1]}(R_{0, \nu}(t, \cdot)).$$

- 532 • (Estimate on the entropy) Take a positive test function $\phi \in C_c^1((0, T) \times (0, 1))$, $T > 0$ and $k \in \mathbb{R}^d$. Then, by
 533 integration by parts, one obtains:

$$\int_0^T \int_0^1 [\eta_k(R_\nu)\phi_t + q_k(R_\nu)\phi_x] dx dt = \sum_\alpha [\dot{\gamma}_\alpha(\eta_k(R_{\nu, \alpha}^r) - \eta_k(R_{\nu, \alpha}^l)) - (q_k(R_{\nu, \alpha}^r) - q_k(R_{\nu, \alpha}^l))] \phi(t, \gamma_\alpha)$$

534 where α runs over the discontinuities $t \rightarrow (t, \gamma_\alpha(t))$ of R_ν .

535 We denote P the set of physical fronts *ie* the fronts for which the velocity has not been modified. NP designates
 536 the complement of P . As fronts of P are entropic by construction, we have:

$$\begin{aligned} \int_0^T \int_0^1 \eta_k(R_\nu)\phi_{i,t} + q_k(R_\nu)\phi_{i,x} dx dt &\geq 0 + \sum_{\alpha \in NP} [\dot{\gamma}_\alpha(\eta_k(R_{\nu, \alpha}^r) - \eta_k(R_{\nu, \alpha}^l)) - (q_k(R_{\nu, \alpha}^r) - q_k(R_{\nu, \alpha}^l))] \phi(t, \gamma_\alpha) \\ &= \sum_{\alpha \in NP} [\tilde{\lambda}_\alpha(\eta_k(R_{\nu, \alpha}^r) - \eta_k(R_{\nu, \alpha}^l)) - (q_k(R_{\nu, \alpha}^r) - q_k(R_{\nu, \alpha}^l))] \phi(t, \gamma_\alpha) \end{aligned}$$

537 where we replaced the notation $\dot{\gamma}$ by $\tilde{\lambda}$ to emphasize the fact that it corresponds to a modified velocity (see
 538 Remark A.3); the unmodified "entropic" velocity being denoted λ . Hence,

$$\begin{aligned}
& \int_0^T \int_0^1 \eta_k(R_\nu) \phi_t + q_k(R_\nu) \phi_x dx dt \\
& \geq \sum_{\alpha \in NP} [\tilde{\lambda}_\alpha (\eta_k(R_{\nu,\alpha}^r) - \eta_k(R_{\nu,\alpha}^l)) - (q_k(R_{\nu,\alpha}^r) - q_k(R_{\nu,\alpha}^l))] \phi(t, \gamma_\alpha) \\
& = \sum_{\alpha \in NP} [\lambda_\alpha (\eta_k(R_{\nu,\alpha}^r) - \eta_k(R_{\nu,\alpha}^l)) - (q_k(R_{\nu,\alpha}^r) - q_k(R_{\nu,\alpha}^l)) + (\tilde{\lambda}_\alpha - \lambda_\alpha) (\eta_k(R_{\nu,\alpha}^r) - \eta_k(R_{\nu,\alpha}^l))] \phi(t, \gamma_\alpha) \\
& \geq 0 - \frac{TV(R_\nu(t, \cdot))}{\nu} \|\phi\|_{L^\infty(\mathbb{R}^+ \times [0,1])} \\
& \geq -C(g, T) \frac{TV(R_{0,\nu})}{\nu} \|\phi\|_{L^\infty(\mathbb{R}^+ \times [0,1])}
\end{aligned}$$

539

where we have used chronologically:

540

– The fact that an unmodified velocity corresponds to an entropy front

541

– The equation $|\lambda - \tilde{\lambda}| \leq \frac{1}{\nu}$ from Remark A.3

542

– The fact that η_k is 1-Lipschitz

543

– The estimate (A.14) proven before.

544

- (L^∞ estimate). Remark that when we solved Riemann problem, the L^∞ norm did not increase. This is mainly because we are dealing with non decreasing fluxes. The only way for the L^∞ norm to increase is through the boundary condition. As a consequence, the estimate (A.3) holds.

545

546

547

This finishes the proof of Lemma A.1.

548

A.3. End of the proof of the existence result. To conclude on the existence, we will use Lemma A.1 and Helly's

549

Theorem 1.4. There are several points to prove:

550

- (Entropy decay) Take $T > 0$, $R_0 \in BV([0, 1])$ and a sequence $(R_{0,\nu})_\nu$ of piecewise constant functions converging to R_0 in BV (such a sequence exists by [4, Lemma 2.2]). For all $\nu > 1$, we denote $(R_\nu)_\nu$ the sequence of piecewise constant functions of Lemma A.1.

551

552

553

554

By Lemma A.1, there exists a $C(g, T) > 0$ such that

$$\forall 0 \leq t \leq T, \quad TV_{[0,1]}(R_\nu(t, \cdot)) \leq C(g, T) TV_{[0,1]}(R_{0,\nu}).$$

555

As $\lim_{\nu \rightarrow \infty} R_{0,\nu} = R_0 \in BV([0, 1])$,

556

$$(A.15) \quad \forall 0 \leq t \leq T, \quad TV_{[0,1]}(R_\nu(t, \cdot)) \leq C(g, T, R_0).$$

557

Next by (A.3) and the fact that the L^∞ norm of the elements of $(R_{0,\nu})_\nu$ are bounded, we have

558

$$(A.16) \quad \forall 0 \leq t \leq T, \quad \|R_\nu(t, \cdot)\|_{L^\infty([0,1])} \leq C(g, T, R_0).$$

559

Finally for all $0 \leq s, t \leq T$ and by the finiteness of the speed of propagation:

560

$$(A.17) \quad \begin{aligned} \|R_\nu(t, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} &\leq c_{\max}(t-s) \max_{u \in [s,t]} TV_{[0,1]}(R_\nu(u, \cdot)) \\ &\leq c_{\max}(t-s) C(g, T, R_0) \end{aligned}$$

561

where we have used (A.15).

562

563

By Helly's Theorem (Theorem 1.4), there exists a subsequence of $(R_\nu)_\nu$ still denoted $(R_\nu)_\nu$ converging in $L^1_{loc}(\mathbb{R}^+, L^1([0, 1]))$ to an element $R \in L^1_{loc}(\mathbb{R}^+, BV([0, 1]))$. Moreover,

564

565

$$(A.18) \quad \forall 0 \leq s, t \leq T, \quad \|R(t, \cdot) - R(s, \cdot)\|_{L^1([0,1])} \leq c_{\max}(t-s) C(g, T, R_0).$$

566

As $(f_\nu)_\nu$ converges uniformly towards f on bounded intervals, we can pass to the limit in (A.1) to get (1.11).

567

- (Initial condition). Let $\varepsilon > 0$ and $s > 0$

$$\begin{aligned}
\|R(0, \cdot) - R_\nu(0, \cdot)\|_{L^1([0,1])} &\leq \|R(0, \cdot) - R(s, \cdot)\|_{L^1([0,1])} + \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} \\
&\quad + \|R_\nu(s, \cdot) - R_\nu(0, \cdot)\|_{L^1([0,1])} \\
&\leq 2C(g, R_0)s + \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])}.
\end{aligned}$$

568

where we have used (A.18).

569 Integrating with respect to s on an interval $[0, t]$ for $0 \leq t \leq 1/c_{\max}$, one gets

$$\begin{aligned} \|R(0, \cdot) - R_\nu(0, \cdot)\|_{L^1([0,1])} &\leq C(g, R_0)t + \frac{1}{t} \int_0^t \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} ds \\ &\leq C(g, R_0)t + \frac{1}{t} \int_0^{1/c_{\max}} \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} ds \end{aligned}$$

570 Taking $t = \frac{\varepsilon}{2C(g, R_0)}$ and ν sufficiently large such that $\int_0^{1/c_{\max}} \|R(s, \cdot) - R_\nu(s, \cdot)\|_{L^1([0,1])} ds \leq \frac{\varepsilon^2}{4C(g, R_0)}$, one finally
571 obtains:

$$\|R(0, \cdot) - R_\nu(0, \cdot)\|_{L^1([0,1])} \leq \varepsilon.$$

572 By the fact that $(R_{0,\nu})_\nu$ converges towards R_0 in $L^1([0,1])$, we deduce that $R(0, \cdot) = R_0$ in a L^1 sense and
573 $R(0, \cdot) = R_0$ almost everywhere.

574 *Remark A.4.* We can repeat the same procedure for any $t \geq 0$ and

$$\forall t \geq 0, \lim_{\nu \rightarrow \infty} \|R_\nu(t, \cdot) - R(t, \cdot)\|_{L^1([0,1])} = 0.$$

575

576 • (Boundary condition). For the boundary condition, it suffices to consider the variable x as a time variable.

577

578 Using (A.12), (A.13) and the approximated boundary condition (2.1), one can easily prove that

$$\forall x \in [0, 1], TV_{[0,1/c_{\max}]}(R_\nu(\cdot, x)) \leq C(g)TV_{[0,1]}(R_{0,\nu}(\cdot)).$$

579

As $(R_{0,\nu})_\nu$ is bounded in BV ,

580

$$(A.19) \quad \forall x \in [0, 1], TV_{[0,1/c_{\max}]}(R(\cdot, x)) \leq C(g, R_0).$$

581

Additionally, with (A.16) we get the L^∞ estimate

582

$$(A.20) \quad \forall \nu > 1, \forall x \in [0, 1], \|R_\nu(\cdot, x)\|_{L^\infty([0,1/c_{\max}])} \leq C(g, R_0).$$

583

Finally, using (A.19) and recalling the definition $c_{\min} := \min_i \alpha_i$ of front velocities, we have for $0 \leq x, y \leq 1$:

584

$$(A.21) \quad \forall \nu > 1, \|R_\nu(\cdot, x) - R_\nu(\cdot, y)\|_{L^1([0,1/c_{\max}])} \leq \frac{|x - y|}{c_{\min}} C(g, R_0).$$

585

By Helly's Theorem (Theorem 1.4), $(R_\nu)_\nu$ tends towards R in $L_{loc}^\infty([0, 1], L^1([0, 1/c_{\max}]))$. Using a similar argu-
586 ment as in the previous item of the proof, one shows that $R(\cdot, 0^+) = g(R(\cdot, 1^-))$ in the almost everywhere sense
587 on $[0, 1/c_{\max}]$. We repeat the argument to get the same conclusion for all time.

588

589 Hence, R is a solution of (1.10) in the sense of Definition 1.6. It remains to prove that $(R_\nu)_\nu$ is an approximating
590 sequence of PCFs of the entropy solution R in the sense of Definition 2.2. By construction, $(R_\nu)_\nu$ satisfies the first five
591 points of Definition 2.2. It remains only to prove the bound

592

$$(A.22) \quad \forall t \geq 0, \delta t > 0, TV_{[0,1]}(R(t, \cdot)) \leq \limsup_{\nu \rightarrow \infty} \sup_{s \in [t, t + \delta t]} TV_{[0,1]}(R_\nu(s, \cdot)).$$

593

This is a consequence of Helly's Theorem. Indeed, take $t \geq 0$, $\delta t > 0$ and $n \in \mathbb{N}^*$. Instead of applying Helly's Theo-
594 rem on an interval of the form $[0, T]$ for the sequence $(R_\nu)_{\nu > 1}$, we apply it on the interval $[t, t + \delta t]$ for the sequence $(R_\nu)_{\nu > n}$.

595

596 As

$$\forall s \in [t, t + \delta t], \forall \nu > n, TV_{[0,1]}(R_\nu(s, \cdot)) \leq \sup_{\substack{u \in [t, t + \delta t] \\ \nu > n}} TV_{[0,1]}(R_\nu(u, \cdot)),$$

597

we deduce by Helly's Theorem that

$$\forall s \in [t, t + \delta t], TV_{[0,1]}(R(s, \cdot)) \leq \sup_{\substack{u \in [t, t + \delta t] \\ \nu > n}} TV_{[0,1]}(R_\nu(u, \cdot)).$$

598 Passing to the limit as n goes to infinity gives (A.22). To get the estimate (2.3), the proof is similar. The existence
599 part of Theorem 2.3 is proven.

600 **Appendix B. Uniqueness.** We will adapt the method of doubling variables of Kruzhkov to our boundary value
601 problem. Let u, v be two entropy solutions of (1.10) with their respective initial data u_0, v_0 .

602

603 We will first show the uniqueness on the triangle T_1 :

$$T_1 := \{(t, x) \mid Lt \leq x \leq 1, 0 \leq t \leq 1/c_{\max}\}$$

604 To do so, let $0 < t \leq 1/c_{\max}$ and define the domain Ω_t by:

$$\Omega_t := \{(s, x); 0 \leq s \leq t, Ls \leq x \leq 1\}.$$

605 We give a graphical representation of Ω_t in Figure 14:

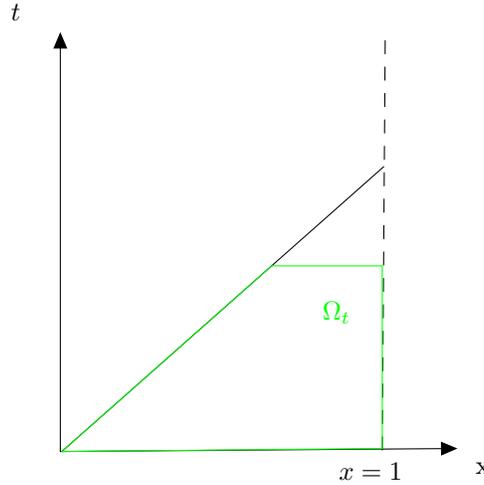


Figure 14: The domain Ω_t

606 Formally, as u in entropy on Ω_t , we have for all $k \in \mathbb{R}^d$

$$\begin{aligned} 0 &\geq \int \int_{\Omega_t} \eta_k(u)_t + q_k(u)_x dx dt \\ &= \int_{1-Lt}^1 \eta_k(u) dx - \int_0^1 \eta_k(u_0) dx \\ &\quad + \int_0^t c_{\max} \eta_k(u(\tau, c_{\max}\tau)) - q_k(u(\tau, c_{\max}\tau)) d\tau + \int_0^t q_k(u(1, s)) ds. \end{aligned}$$

607 The third term is positive because c_{\max} is superior to all the Lipschitz constants of the f_i s. The last term is positive
608 since all the f_i s are non decreasing. Hence,

$$609 \quad (B.1) \quad \int_{1-Lt}^1 \eta_k(u) dx \leq \int_0^1 \eta_k(u_0) dx.$$

610 It is equivalent to:

$$\forall k \in \mathbb{R}^d, \sum_{i=1}^d \int_{1-Lt}^1 |u_i(t, x) - k_i| dx \leq \sum_{i=1}^d \int_0^1 |u_{0,i}(x) - k_i| dx.$$

611 Kruzhkov's doubling variable method allows to replace the k_i by the v_i to give:

$$\sum_{i=1}^d \int_{1-Lt}^1 |u_i(t, x) - v_i(t, x)| dx \leq \sum_{i=1}^d \int_0^1 |u_{0,i}(x) - v_{0,i}(x)| dx.$$

612 *Remark B.1.* Rigorous justifications of previous computations can be found in the proof of [4, Theorem 6.2].
 613 As a consequence, the solution is unique on the triangle T_1 .

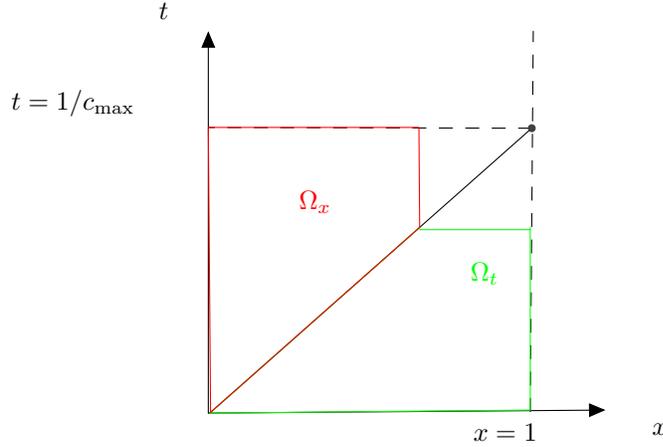


Figure 15: The domain Ω_x

614 Now let x be in $]0, 1[$, we apply the same strategy to the set

$$\Omega_x := \{(s, y); 0 \leq y \leq x, y/c_{\max} \leq s \leq 1/c_{\max}\}$$

615 represented in Figure 15.

616

617 Integrating $\eta_k(u)_t + q_k(u)_x \leq 0$ in Ω_x , one obtains:

$$\forall x \in [0, 1], k \in \mathbb{R}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} q_k(u(t, x)) dt \leq \int_0^{1/c_{\max}} q_k(u(t, 0)) dt + \int_0^{x/c_{\max}} c_{\max} \eta_k(u(t, c_{\max}t)) - q_k(u(t, c_{\max}t)) dt.$$

618 This is equivalent to: $\forall x \in [0, 1], k \in \mathbb{R}^d$,

$$\begin{aligned} \sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} (f_i(u_i) - f_i(k_i)) \text{sign}(u_i - k_i) dt &\leq \sum_{i=1}^d \int_0^{1/c_{\max}} (f_i(u_i(t, 0)) - f_i(k_i)) \text{sign}(u_i(t, 0) - k_i) dt \\ &+ \int_0^{x/c_{\max}} c_{\max} \eta_k(u(t, c_{\max}t)) - q_k(u(t, c_{\max}t)) dt. \end{aligned}$$

619 As all the f_i are non decreasing ($q_k \geq 0$) and all c_{\max} Lipschitz: $\forall x \in [0, 1], k \in \mathbb{R}^d$,

$$\begin{aligned} \sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} |f_i(u_i) - f_i(k_i)| dt &\leq c_{\max} \sum_{i=1}^d \int_0^{1/c_{\max}} |u_i(t, 0) - k_i| dt \\ &+ c_{\max} \int_0^{x/c_{\max}} \eta_k(u(t, c_{\max}t)) dt. \end{aligned}$$

620 Following Kruzhkov's method, the k_i s can be replaced by the v_i s and for all x in $[0, 1]$:

$$\begin{aligned} \sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} |f_i(u_i) - f_i(v_i)| dt &\leq c_{\max} \sum_{i=1}^d \int_0^{1/c_{\max}} |u_i(t, 0) - v_i(t, 0)| dt \\ &+ c_{\max} \int_0^{x/c_{\max}} |u(t, c_{\max}t) - v(t, c_{\max}t)| dt. \end{aligned}$$

621 As u, v satisfy the boundary condition on $[0, 1/c_{\max}]$, there exists a constant $C(g, c_{\max})$ depending on the Lipschitz
 622 constant of g such that for all x in $[0, 1]$:

$$\sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} |f_i(u_i) - f_i(v_i)| dt \leq C(g, c_{\max}) \sum_{i=1}^d \int_0^{1/c_{\max}} |u_i(t, 1) - v_i(t, 1)| dt \\ + c_{\max} \int_0^{x/c_{\max}} |u(t, c_{\max}t) - v(t, c_{\max}t)| dt.$$

623 If $u_0 = v_0$, we have seen that u and v coincide on T_1 . This implies that if $u_0 = v_0$, u and v coincide on the segment
624 $[0, 1/c_{\max}] \times \{1\}$ and on the line $(x = c_{\max}t, t)$ for $t \leq 1/c_{\max}$. As a consequence,

$$\forall x \in [0, 1], \sum_{i=1}^d \int_{1/c_{\max}-x/c_{\max}}^{1/c_{\max}} |f_i(u_i) - f_i(v_i)| dt = 0.$$

625 By the monotonicity of the f_i 's, u and v coincide on the triangle T_2 defined by

$$T_2 := \{(t, x) \mid 0 \leq x \leq 1, x/c_{\max} \leq t \leq 1/c_{\max}\}.$$

626 To conclude, u and v coincide for $t \leq 1/c_{\max}$ and repeating this argument, we can prove the uniqueness for all time.
627 This finishes the proof of the uniqueness.

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